Maximum principle for the regularized Schrödinger operator*

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Abstract

In this paper we present analogues of the maximum principle and of some parabolic inequalities for the regularized time-dependent Schrödinger operator on open manifolds using Günter derivatives. Moreover, we study the uniqueness of bounded solutions for the regularized Schrödinger-Günter problem and obtain the corresponding fundamental solution. Furthermore, we present a regularized Schrödinger kernel and prove some convergence results. Finally, we present an explicit construction for the fundamental solution to the Schrödinger-Günter problem on a class of conformally flat cylinders and tori.

Keywords: Clifford analysis; Time dependent operators; Schrödinger equation; Günter derivatives; Boundary problems on manifolds.

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1 Introduction

Time evolution problems are of extreme importance in Mathematical Physics. However, there is still a strong need to develop further special techniques in order to get explicit representation formulas or particular existence and uniqueness results for the solutions for those problems addressing particular geometric settings.

Unfortunately, there is no simple way to extend the stationary theory directly to the framework of non-stationary problems. Clifford analysis techniques turned out to be a very useful toolkit to address these problems. First steps in this direction have been made for instance in [8] where the non-stationary Navier-Stokes equation over time-varying domains have been successfully treated with Clifford analysis methods. After publication of that paper, other authors used these ideas to develop a continuous and a discrete operator function theory to deal with the time-dependent Schrödinger equation for several type of domains, see for instance [3, 4, 5, 6, 7, 14, 15, 16]).

The study of boundary value problems for partial differential equations over more general surfaces and manifolds have a lot of applications. For example, it is applied in the description of heat conduction problems over surfaces. Furthermore, it is used in the treatment of equations of surface flow, of shell problems in elasticity, of the vacuum Einstein equations describing gravitational fields, of the Navier-Stokes equations in spherical domains and of many other problems, see e.g. [11]. However, the generalization of the well-known results for partial differential equations in the Euclidean case to more general geometric settings is not immediate. We also need to handle geometric characteristics of the considered generic surface such as curvature.

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The main goal of this paper is to present an analogue of the maximum principle and of some parabolic inequalities for the regularized time-dependent Schrödinger operator on open manifolds. Our approach uses Günter derivatives. The main results presented here are based on Dodziuk's ideas (see [10]). This paper is structured as follows. In the preliminary section we recall some basic notions about Clifford analysis, the Laplacian and Günter derivatives. We also introduce the regularization procedure. In Section 3 we present a generalization of the classical maximum principle for a generic manifold. Section 4 is dedicated to the study of the uniqueness of solutions for the regularized Schrödinger-Günter problem over a Riemannian manifold. In the following section we construct the fundamental solution for the regularized Schrödinger-Günter equation. In Section 6 and Section 7 we study the regularized Schrödinger-Günter kernel and prove some important fundamental convergence results. In Section 8 we round off by presenting fully explicit representation formulas for the fundamental solution to the Schrödinger-Günter equation on a class of conformally flat cylinders and tori with particular spin structures.

2 Preliminaries

2.1 Clifford analysis

We consider the n-dimensional vector space $\mathbb{R}^n$ endowed with an orthonormal basis $\{e_1, \cdots, e_n\}$. We define the universal real Clifford algebra $C\ell_{0,n}$ as the 2^n-dimensional associative algebra which obeys the multiplication rules $e_ie_j + e_je_i = -2\delta_{i,j}$. A vector space basis for $C\ell_{0,n}$ is generated by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \dots, h_k\} \subset M = \{1, \dots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Each element $x \in C\ell_{0,n}$ can be represented by $x = \sum_A x_A e_A$, $x_A \in \mathbb{R}$. The Clifford conjugation is defined by $\overline{1} = 1$, $\overline{e_j} = -e_j$ for all $j = 1, \dots, n$, and we have $\overline{ab} = \overline{b}\,\overline{a}$. We introduce the complexified Clifford algebra $\mathbb{C}_n$ as the tensor product

$$\mathbb{C} \otimes C\ell_{0,n} = \left\{ w = \sum_A w_A e_A, \ w_A \in \mathbb{C}, A \subset M \right\}.$$

In this context, the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \dots, n$. To avoid ambiguities with the Clifford conjugation, we denote the complex conjugation which maps a complex scalar $w_A = a_A + ib_A$ with real components a_A and b_A onto the complex scalar $\overline{w}_A = a_A - ib_A$ by $\sharp$. The complex conjugation leaves the elements e_j invariant, i.e., $e_j^\sharp = e_j$ for all $j = 1, \dots, n$. We also have a pseudonorm on $\mathbb{C}$ viz $|w| := \sum_A |w_A|$ where $w = \sum_A w_A e_A$, as usual. Notice also that for $a, b \in \mathbb{C}_n$ we only have $|ab| \leq 2^n |a||b|$. The other norm criteria are fulfilled.

A function $u : U \mapsto \mathbb{C}_n$ has a representation $u = \sum_A u_A e_A$ with $\mathbb{C}$-valued components u_A. Properties such as continuity will be understood component-wisely. Next, we introduce the Euclidean Dirac operator $D = \sum_{j=1}^n e_j \partial_{x_j}$. This first order operator factorizes the n-dimensional Euclidean Laplacian. We have $D^2 = -\Delta = -\sum_{j=1}^n \partial x_j^2$. A $\mathbb{C}_n$-valued function defined on an open set $U \subseteq \mathbb{R}^n$, $u : U \mapsto \mathbb{C}_n$, is called *left-monogenic* if it satisfies $Du = 0$ on U (resp. *right-monogenic* if it satisfies $uD = 0$ on U).

One possibility to address in-stationary problems consists of considering the square root of the heat operator in the context of the above-defined Clifford algebra. This approach, however, would demand that one has to deal with fractional derivatives. An elegant alternative way to avoid fractional derivatives consists of adding two more basis elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$, which act in the following way:

$$\mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} = 1, \qquad \mathfrak{f}^2 = (\mathfrak{f}^\dagger)^2 = 0, \qquad \mathfrak{f}e_i + e_i\mathfrak{f} = 0, \qquad \mathfrak{f}^\dagger e_j + e_j\mathfrak{f}^\dagger = 0.$$

This treatment allows us to define a suitable factorization of the heat [8] and Schrödinger [7] operators, where only partial derivatives are used.

From now until the end of the paper, we will consider functions in the variables $(x_1, x_2, \dots, x_n, t)$ where $(x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$ for $i = 1, \dots, n$, $t \in I = [0, T[$. The functions take values in the complexified Clifford algebra $\mathbb{C}_n$ generated by the extended Witt basis $e_1, \dots, e_n, \mathfrak{f}, \mathfrak{f}^\dagger$. For the sake of readability we abbreviate the space-time tuple $(x_1, x_2, \dots, x_n, t)$ simply by (x, t), assigning $x = x_1e_1 + \cdots x_ne_n$. For additional details on Clifford analysis, we refer the interested reader for instance to [9, 13].

The space $L^2(\Omega)$ can be endowed with the structure of a Hilbert $\mathbb{C}_n$−module by introducing the following inner product

$$< f, g >:= \int_\Omega Sc\left(\overline{f(x,t)}^\sharp g(x,t)\right)\ dx\ dt, \qquad f,\, g \in L^2(\Omega).$$

2.2 Laplace operator and Günter derivatives

One possible extension of the most basic partial differential operators from the flat Euclidean setting to a curved manifold $S \subset \mathbb{R}^n$ consists of expressing them globally in terms of the standard spatial coordinates in $\mathbb{R}^n$. It turns out that a convenient way to carry out this program is to employ the so-called Günter derivatives (for more details see [11, 12])

$$\mathcal{D} := (\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n), . \tag{1}$$

In the latter equation, for each $1 \leq j \leq n$ the first-order differential operator $\mathcal{D}_j$ is the directional derivative along ψe_j. Here $\psi : \mathbb{R}^n \to TS$ is the orthogonal projection onto the tangent plane to S. Further, as usual, $e_j = (\delta_{j,k})_{1\leq k\leq n} \in \mathbb{R}^n$, where δ_{jk} simply represents the usual Kronecker symbol. The operator $\mathcal{D}$ is globally defined on S by means of the unit normal vector field. It has a relatively simple structure. In terms of (1), the Laplace operator defined via Günter derivatives becomes

$$\Delta_G = \mathcal{D}^2 = \sum_{j=1}^{n} \mathcal{D}_j^2 = \sum_{j=1}^{n} (\partial_j - \nu\partial_\nu)(\partial_j - \nu\partial_\nu), \tag{2}$$

with $\nu(x) := \frac{x}{||x||}$, $x \in \mathbb{R}^n \setminus \{0\}$. The expression $\partial_\nu = \sum_{j=1}^{n} \nu_i(x)\ \partial_j$ is the radial derivative in $\mathbb{R}^n$. This Laplacian is related to the Euclidean Laplacian by the following identity

$$\Delta = \psi\mathcal{D}^2 + 2R^2 - \mathcal{G}R, \tag{3}$$

where $R(x) = \nabla\nu(x)$ and $\mathcal{G}(x) = \text{div}\nu(x)$. For more details about the relations between Günter derivatives and the Laplace operator we refer the reader for instance to [11]. This decomposition allows us to treat solutions of the second order operator Δ_G in terms of the first order operator $\mathcal{D}$ which acts on spinor valued sections. That is the point where Clifford analysis techniques can be applied to describe the solutions.

2.3 Regularization procedure

A fundamental solution e_- for the Schrödinger operator has a singularity at all the points of the hyperplane $t = 0$,. This is an important difference to the context of hypoelliptic operators, where the fundamental solution only has a 1-point singularity, see [1]. Moreover, all these singularities are not removable by standard calculation methods. As a consequence we cannot guarantee the convergence (in the classical sense) of the arising integral operators. To overcome this problem, we need to regularize the fundamental solution as well as the arising operators (see [4, 14, 20]). This process of regularization creates a sequence of operators and corresponding fundamental solutions, which are locally integrable in $\mathbb{R}^n \times \mathbb{R}_0^+ \setminus \{(0,0)\}$. Moreover, this family will converge to the modified original operators and fundamental solutions when $\epsilon \to 0^+$.

To this end, we will replace the imaginary unit appearing in the Schrödinger operator by the constant $\mathbf{k} = \frac{\epsilon+i}{\epsilon^2+1}$, and we obtain the modified operator $-\Delta - \mathbf{k}\partial_t$. For each $\epsilon > 0$, the operator $-\Delta - \mathbf{k}\partial_t$ is a hypoelliptic operator (see [4]). In the context of using this operator the good convergence behavior of the associated integral operators is guaranteed.

3 The maximum principle

Let u be a $\mathbb{C}_n$-valued function. Further, we look at the regularized Schrödinger-Günter operator $\Delta_G - \mathbf{k}\partial_t$ defined on a open subset $V \subset (M \times \mathbb{R}_+)$, where M is a Riemannian manifold in $\mathbb{R}^n$. Additionally we require that for a neighborhood of each point in V there exists a real non-zero constants C such that

$$C^{-1} \sum_j \xi_j^2 \leq \sum_{j,k} \xi_j\xi_k \leq C \sum_j \xi_j^2, \tag{4}$$

for every choice of real constants $\xi_1, \xi_2, \ldots, \xi_n$. Following the ideas of Calabi [2] and Dodziuk [10], we introduce the following definition:

Definition 3.1 *Consider the regularized Schrödinger-Günter operator $\Delta_G - \mathbf{k}\partial_t$ satisfying (4). Moreover, let u be a continuous function on an open domain $U \subset (M \times \mathbb{R})$. Consider a function φ on U with no restriction.*

We say that $(\Delta_G - \mathbf{k}\partial_t)u \succ \varphi$ holds on an open subset K of U if, for every $(x_0,t_0) \in K$ and every $\lambda > 0$, there exists a neighborhood $V = V(\lambda, x_0, t_0) \subset U$ of (x_0,t_0) and a function $u^{\epsilon}_{\lambda,x_0,t_0}$ on V which is supposed to be C^2 in the space coordinates and C^1 in the time coordinate, such that

$$\left|u^{\epsilon}(x,t) - u^{\epsilon}_{\lambda,x_0,t_0}\right| \geq \left|u^{\epsilon}(x_0,t_0) - u^{\epsilon}_{\lambda,x_0,t_0}\right| \tag{5}$$

for $(x,t) \in V$ and at (x_0,t_0)

$$\left|(\Delta_G - \mathbf{k}\partial_t)u^{\epsilon}_{\lambda,x_0,t_0}\right| \geq |\varphi - \epsilon| \,. \tag{6}$$

Remark 3.2 *In a similar way we define $(\Delta_G - \mathbf{k}\partial_t)u \prec \varphi$, if $(\Delta_G - \mathbf{k}\partial_t)(-u) \succ -\varphi$.*

Following the ideas presented in [10] we can immediately establish the following result:

Lemma 3.3 *If u is sufficiently smooth, i.e. C^2 in the spatial coordinates and C^1 in the time coordinate, then $|(\Delta_G - \mathbf{k}\partial_t)u| \geq |\varphi|$ if and only if $(\Delta_G - \mathbf{k}\partial_t)u \succ \varphi$.*

As presented in [10], we can now formulate our extension of the classical (weak) maximum principle.

Theorem 3.4 *Let u be a continuous function in a cylinder $C = U \times [t_1,t_2]$, where U is an open relatively compact subset of M, such that $(\Delta_G - \mathbf{k}\partial_t)u \succ 0$ in $U \times]t_1,t_2[$. Then*

$$\sup_{C} |u^{\epsilon}(x,t)| = \sup_{U\times\{0\}\ \cup\ \partial U\times[t_1,t_2]} |u^{\epsilon}(x,t)|.$$

Proof: We define $v^{\delta} := u - \delta t$, $\delta > 0$. It follows immediate that

$$\left|(\Delta_G - \mathbf{k}\partial_t)v^{\delta}\right| > \delta > 0 \tag{7}$$

on $U\times]t_1,t_2[$. Moreover, let $C_\lambda = U \times [t_1, t_2 - \lambda]$. We claim that

$$\sup_{C_\lambda} |v^{\delta}| = \sup_{U\times\{0\}\ \cup\ \partial U\times[t_1,t_2]} |v^{\delta}(x,t)|. \tag{8}$$

The conclusion of the proof follows from (8) by considering λ and δ arbitrarily small. Suppose that (8) is not true. Then, we can find an $x_0 \in U$ and a $t_0 \in]t_1, t_2-\lambda[$ such that v^{δ} restricted to C_λ has a maximum at (x_0,t_0). From Definition 3.1 and in view of (7), there is a function $\tilde{v}$ in a neighborhood of (x_0,t_0), which is C^2 in the spatial coordinates and C^1 in the time coordinate, such that

$$\left|v^{\delta}(x,t) - \tilde{v}(x,t)\right| \geq \left|v^{\delta}(x_0,t_0) - \tilde{v}(x_0,t_0)\right|, \tag{9}$$

and

$$|(\Delta_G - \mathbf{k}\partial_t)\tilde{v}| \geq \frac{\delta}{2} > 0, \qquad \text{at } (x_0,t_0). \tag{10}$$

We rewrite (9) as

$$|\tilde{v}(x_0,t_0)| \geq \left|\tilde{v}(x,t) + v^{\delta}(x_0,t_0) - v^{\delta}(x,t)\right|.$$

Since v^{δ} has a maximum at (x_0,t_0), so does $\tilde{v}$. However, from (10) we conclude that this is impossible and, hence, the validity of (8) is established.

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As it occurs in [10], the maximum principle presented in the previous theorem is not of the strong form. Notice that it permits the situation that the maximum may occur both on the boundary and in the interior points. The following strong form also holds.

Theorem 3.5 *Let u be a continuous function defined on an open space $U \subset (M\times\mathbb{R}_+)$ satisfying $(\Delta_G - \mathbf{k}\partial)u \succ 0$. Suppose that the maximum of u is attained at a point $(x_0,t_0) \in U$. Then u is constant along every curve in U beginning at (x_0,t_0).*

The proof is a direct adaptation of the proof presented in Chapter 3 of [19].

4 Uniqueness of bounded solutions

Let us again suppose that u is a $\mathbb{C}_n$-valued function and consider the regularized Schrödinger-Günter operator $\Delta_G - \mathbf{k}\partial_t$ defined on an open subset $V \subset (M \times \mathbb{R}_+)$, where M is a Riemannian manifold in $\mathbb{R}^n$. Following the lines of [10], we introduce the following definition:

Definition 4.1 *A $\mathbb{C}_n$-valued function u defined on $V = M \times [0, T[)$ is a solution of the regularized Schrödinger-Günter problem on V with initial boundary data u_0^ϵ*

$$\begin{cases} (\Delta_G - \mathbf{k}\partial_t)u = 0 & \text{on } V \\ \\ u^\epsilon(x,0) = u_0^\epsilon(x) & \text{on } M \end{cases}, \tag{11}$$

if u satisfies the following three conditions simultaneously:

- *u is continuous, C^2 in the spatial coordinates, C^1 in $t \in]0, T[$;*
- *u satisfies the regularized Schrödinger-Günter equation $(\Delta_G - \mathbf{k}\partial_t)u = 0$ on $M \times]0, T[$;*
- *u satisfies the initial condition $u^\epsilon(x, 0) = u_0^\epsilon(x)$ for all $x \in M$.*

We will concentrate on describing the conditions that imply the uniqueness of the solution of problem (11). To proceed in this direction we will consider the following preparatory lemma, in which we state an important property of $|x - x_0|^2 = r(x)$ from a fixed point $x_0 \in M$:

Lemma 4.2 *Let φ be a nondecreasing function of the class C^2 in the spatial coordinate defined on the half-line $\mathbb{R}_+$. If M is complete and if $2R^2 - \mathcal{G}R > 0$, then the function $f = \varphi(r)$ satisfies*

$$\Delta_G f \prec \varphi''(r) + \left(\frac{n-1}{r} + C\right)\varphi'(r),$$

where C is a constant that only depends on the lower value of $2R^2 - \mathcal{G}R$. The inequality is understood in the sense of Definition 3.1.

The proof of this result follows along the same lines as the proof presented in [10]. The main difference is that we are dealing now with $2R^2 - \mathcal{G}R$ instead of dealing with the Ricci curvature. We now present the uniqueness's result:

Theorem 4.3 *Let M be a complete Riemannian manifold where $2R^2 - \mathcal{G}R > 0$. The initial data determines the uniqueness of every bounded solution of the regularized Schrödinger-Günter problem (11).*

Proof: For the proof we rely on Lemma 4.2. Consider a non-decreasing function φ of the class C^2 on $\mathbb{R}_+$ such that $\varphi(s) = 0$ for $s \in \left]0, \frac{1}{2}\right[$ and $\varphi(s) = s$ for $s \geq 1$. Let $\rho(x) = \varphi(r(x)) = \varphi(d(x_0, s))$. Then $\rho : M \to \mathbb{R}$ is a continuous function. Further, relying on Lemma 4.2, there exists a constant K which only depends on φ and on the lower value of $2R^2 - \mathcal{G}R$, but not on x_0, such that $\Delta_G \rho < K$. Now consider the function $v(x,t) = u^\epsilon(x,t) - N_0 - \frac{N}{R}(\rho + Kt)$, where $N_0 = \sup_M |u^\epsilon(x,0)|$, $N = \sup_{M \times [0,T[} |u^\epsilon(x,0)|$ and where R is a sufficient large positive constant. Moreover, let $v(x,t)$ be defined in the compact ball $B_R(x_0) = \{x \in M : \; d(x, x_0) \leq R\}$. Suppose that $v(x,t)$ is C^2 in the spatial coordinates and C^1 in the time coordinate. It is clear that $v \leq 0$ on the set $B_R(x_0) \times \{0\} \; \cup \; \partial B_R(x_0) \times]0, T[$, and

$$(\Delta_G - \mathbf{k}\partial_t)\, v = \frac{N}{R}(K - \Delta_G \rho) \succ 0.$$

Theorem 3.4 implies the following inequality

$$|u^\epsilon(x,t)| \leq N_0 + \frac{N}{R}(\rho(x) + Kt), \qquad \text{on } B_R(x_0) \times [0, T[.$$

If $|x - x_0|^2 \leq R$, then we apply the same argument to $-u$ and obtain

$$|u^\epsilon(x,t)| \leq N_0 + \frac{N}{R}(\rho(x) + Kt) \tag{12}$$

for $x \in B_R(x_0)$, $t \in [0, T[$. In the limit case where R tends to infinity one has

$$|u^\epsilon(x,t)| \leq N_0 = \sup_M |u^\epsilon(x,0)|, \tag{13}$$

for any $R > 0$, x_0, and $x \in B_R(x_0)$. As a consequence of (13) we observe that $N = N_0$, and that the estimate (12) remains true for every choice of x_0.

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Remark 4.4 *From the arguments of the previous proof, we can deduce the following additional estimate*

$$|u^\epsilon(x,t)| \leq \sup_{B_R(x_0)} |u^\epsilon(x,0)| + \frac{N_0}{R}(\rho(x) + Kt), \tag{14}$$

for arbitrary $R > 0$ and x_0, $x \in B_R(x_0)$.

5 Fundamental solution

In this section we present a method that allows us to construct a fundamental solution for the regularized Schrödinger-Günter equation. We begin by introducing the concept of a fundamental solution that we will consider in this paper.

Definition 5.1 *A continuous function $p^\epsilon(x,y,t)$ on $M \times M \times \mathbb{R}_+$ is called fundamental solution of the regularized Schrödinger-Günter problem if, for every bounded continuous function u_0^ϵ on M, the function*

$$u^\epsilon(x,t) = \begin{cases} \displaystyle\int_M p^\epsilon(x,y,t)\ dV_y & \textit{for } t > 0 \\ \\ u_0^\epsilon(x) & \textit{for } t = 0 \end{cases} \tag{15}$$

is a solution of the regularized Schrödinger-Günter problem (11) with initial data u_0^ϵ.

Usually, the fundamental solution is not uniquely determined. However, if M is complete and if $2R^2 - \mathcal{G}R > 0$, then the fundamental solution becomes unique. In this particular case, we shall refer to $p^\epsilon(x,y,t)$ as the regularized Schrödinger-Günter kernel.

Now suppose that D is a relatively compact open subset of M with C^∞ boundary. $p_D^\epsilon(x,y,t)$ stands for the regularized Schrödinger-Günter kernel for the regularized Schrödinger-Günter equation on D with Dirichlet boundary condition $p_D^\epsilon(x,y,t) = 0$ if either x or y belongs to ∂D. The kernel $p_D^\epsilon(x,y,t)$ can be constructed by applying the method of the double-layer potential (for more details see [18]). Some of its properties are presented in the following result:

Lemma 5.2 *$p_D^\epsilon(x,y,t)$ is C^∞ on $D \times D \times \mathbb{R}_+$ and vanishes if either x or y is a point of ∂D. Moreover,*

(i) $p_D^\epsilon(x,y,t) = p_D^\epsilon(y,x,t)$, *for $t > 0$ and $x, y \in D$;*

(ii) $(\Delta_G - \mathbf{k}\partial_t) p_D^\epsilon \equiv 0$;

(iii) $\int_D p_D^\epsilon(x,z,t)\ p_D^\epsilon(z,y,t)\ dV_z = p_D^\epsilon(x,y,t+s)$, *for $s, t > 0$ and $x, y \in \tilde{D}$;*

(iv) For every $D \subset M$ there exists a C^∞ function $\Phi(x,y)$ on $D \times D$ such that $\Phi(x,x) \equiv 1$ and for $x, y \in D$

$$p_D^\epsilon(x,y,t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\mathbf{k}\frac{|x-y|^2}{4t}\right)\Phi(x,y) + O\left(t^{-\frac{n}{2}-1}\right)\exp\left(-\mathbf{k}\frac{|x-y|^2}{4t}\right), \qquad t \to 0. \tag{16}$$

The previous estimate holds uniformly on every compact subset of $D \times D$;

(v) Let D_1 and D_2 be two subdomains of M. By $p_1^\epsilon, p_2^\epsilon$ we denote the associated regularized Schrödinger-Günter kernels. Then for $x, y \in D_1 \cap D_2$ we have

$$p_1^\epsilon(x,y,t) - p_2^\epsilon(x,y,t) = O(t^{\mathbf{N}}), \tag{17}$$

as t tends to zero, for all $\mathbf{N} > 0$. Again estimate (17) holds uniformly if x, y range over compact subsets of $D_1 \cap D_2$.

The proof of the previous lemma follows along the same lines as presented in the proof for the corresponding result presented in [10]. Moreover, this result implies the uniqueness of the solution of the initial boundary value problem

$$\begin{cases} (\Delta_G - \mathbf{k}\partial_t)u^\epsilon = 0 & \text{on } D \times \mathbb{R}_+ \\ \\ u^\epsilon|_{\partial D\times\mathbb{R}_+} = 0 & \\ \\ \lim\limits_{t\to 0^+} u^\epsilon(x,t) = u_0^\epsilon(x) & \text{for } x \in D \end{cases}, \tag{18}$$

where u_0^ϵ is continuous on D and vanishes at each point of ∂D, and

$$u^\epsilon(x,t) = \int_D p_D^\epsilon(x,y,t)\ u_0^\epsilon(y)\ dV_y.$$

In what follows we shall regard $p_D^\epsilon(x,y,t)$ as a function on $M \times M \times \mathbb{R}_+$ by defining it to be zero if either x or y lies outside of D.

Theorem 5.3 *The function $p_D^\epsilon(x,y,t)$ has the following properties:*

(i) For every $x \in D$, $t > 0$

$$\int_D |p_D^\epsilon(x,y,t)|\ dV_y < 1;$$

(ii) If $D_1 \subset D_2$, then

$$|p_{D_1}^\epsilon(x,y,t)| \leq |p_{D_2}^\epsilon(x,y,t)|$$

for $x, y \in D_1$ and $t > 0$.

Proof: Let us start with addressing property (i). We have

$$|\mathbf{k}|\partial_t \int_D |p_D^\epsilon(x,y,t)|\ dV_y = \int_D \Delta_G^x |p_D^\epsilon(x,y,t)|\ dV_y = \int_D \Delta_G^y |p_D^\epsilon(x,y,t)|\ dV_y = \int_{\partial D} \partial_{\vec{n}_y} |p_D^\epsilon(x,y,t)|\ dS_y$$

where dS is the surface area element of ∂D. $\partial_{\vec{n}}$ is the derivative in the direction of the outward unit normal. We recall that $p_D^\epsilon(x,y,t) = 0$ for $y \in \partial D$. From Lemma 5.2-(iv) we may conclude

$$\lim_{t\to 0^+} \int_D p_D^\epsilon(x,y,t)\ dV_y = 1, \qquad \text{for } x \in D. \tag{19}$$

This proves property (i). Next we look at property (ii). Now we apply the maximum principle to the function

$$u^\epsilon(x,t) = p_{D_2}^\epsilon(x,y,t) - p_{D_1}^\epsilon(x,y,t)$$

restricted to D_1 for a fixed $y \in D_1$. By Lemma 5.2, we infer that u has a continuous extension to $\tilde{D}_1 \times \mathbb{R}_+$ which vanishes on $\tilde{D}_1 \times \{0\}$. This establishes the inequality for $x, y \in D_1$.

■

We now pass to the construction of our fundamental solution. We choose a sequence $\{D_l\}_{l=1}^\infty$ such that $\tilde{D}_l \subset D_{l+1}$ and $\bigcup_l D_l = M$. Next consider

$$p^\epsilon(x,y,t) = \lim_{l\to+\infty} p_l^\epsilon(x,y,t), \tag{20}$$

where p_l^ϵ is the regularized Schrödinger-Günter kernel for D_l. This limit exists in view of Theorem 5.3. However, we cannot guarantee that this limit is finite. Before we study some properties of $p^\epsilon(x,y,t)$, we need to consider the following auxiliar result:

Lemma 5.4 *Let M be a Riemannian manifold and let $a, b \in \mathbb{R}$. Suppose that $\{u_l^\epsilon\}_{l=1}^\infty$ is a sequence of solutions of the regularized Schrödinger-Günter equation on $M\times]a,b[$ such that*

$$\int_N |u_l^\epsilon(x,t)|\ dV_x \leq C,$$

where the constant C is independent of l and $t \in]a,b[$. Then $u = \lim_{l\to\infty} u_l^\epsilon$ is a smooth solution of the regularized Schrödinger-Günter equation and $u_l^\epsilon \to u$ holds uniformly on a compact set together with the derivatives of all orders.

The proof of this lemma is an immediate adaptation of the correspondent result in [10]. We now continue with the study of the function $p^\epsilon(x,y,t)$.

Theorem 5.5 *$p^\epsilon(x,y,t)$ is C^∞ on $M \times M \times \mathbb{R}_+$. It is a fundamental solution of the regularized Schrödinger-Günter equation, and has the following properties*

(i) $p^\epsilon(x,y,t) = p^\epsilon(y,x,t)$ *for* $t > 0$, $x,y \in M$;

(ii) $(\Delta_G - \mathbf{k}\partial_t)p^\epsilon \equiv 0$;

(iii) $\int_M p^\epsilon(x,y,t)\ p^\epsilon(z,y,s)\ dV_z = p^\epsilon(x,y,t+s)$ *for* $t,s > 0$ *and* $x,y \in M$;

(iv) $p^\epsilon(x,y,t)$ *is independent of the exhaustion used to define it. As a matter of fact*

$$p^\epsilon(x,y,t) = \sup_{D\subset M} p_D^\epsilon(x,y,t);$$

Proof: The properties (i) and (iii) follow from the properties of the regularized kernels $p_l^\epsilon(x,y,t)$ presented in Lemma 5.2 as soon as we have proved that p_l^ϵ converges to p in an appropriately strong sense. To this end, consider the sequence of functions $u_l^\epsilon(x,t) = p_l^\epsilon(x,y,t)$ for a fixed $y \in M$. For every relatively compact subset $D \subset M$ with a smooth boundary we will show that the sequence u_l^ϵ, $l = 1, 2, \ldots$, converges uniformly on every cylinder $D \times [t_1, t_2]$, $0 < t_1 < t_2$ to a C^∞ solution of the regularized Schrödinger-Günter equation. From Theorem 5.3-(i) the sequence $u_l^\epsilon(x,t) = p_l^\epsilon(x,y,t)$ for a fixed $y \in M$ satisfies the conditions of Lemma 5.4. Hence, its limit $p^\epsilon(x,y,t)$ satisfies the regularized Schrödinger-Günter equation in the variables (x,t). Consider $p^\epsilon(x,y,t)$ as a function on $M \times M \times \mathbb{R}_+$. The equation

$$(\Delta_G^x + \Delta_G^y - 2\mathbf{k}\partial_t)\, u^\epsilon(x,y,t) = 0 \tag{21}$$

becomes the regularized Schrödinger-Günter equation on $M \times M$ after a change of time coordinate. For a fixed $D \subset M$ and large l, $p_l^\epsilon(x,y,t)$ satisfies the previous equation on $D \times D\times]0,+\infty[$, and

$$\int_{D\times D} |p_l^\epsilon(x,y,t)|\ dV_x\ dV_y \leq vol(D).$$

Using Lemma 5.4, we conclude that $p_l^\epsilon(x,y,t) = \lim_{l\to+\infty} p_l^\epsilon(x,y,t)$ is a C^∞ solution of equation (21). Now we show that $p^\epsilon(x,y,t)$ is a fundamental solution. First observe that for every open subset U of M and every $x \in U$

$$\lim_{t\to 0^+} \int_U p^\epsilon(x,y,t)\ dV_y = 1. \tag{22}$$

From(19), Theorem 5.3-(i), we conclude that

$$\begin{aligned} 1 &\geq \overline{\lim_{t\to 0^+}} \int_M |p^\epsilon(x,y,t)|\ dV_y \geq \underline{\lim}_{t\to 0^+} \int_M |p^\epsilon(x,y,t)|\ dV_y \\ &\geq \underline{\lim}_{t\to 0^+} \int_U |p_D^\epsilon(x,y,t)|\ dV_y \geq \lim_{t\to 0^+} \int_D |p_D^\epsilon(x,y,t)|\ dV_y = 1 \end{aligned}$$

where D is an arbitrary relatively compact, open subset of U with smooth boundary and $x \in D$. Moreover,

$$\int_M |p^\epsilon(x,y,t)|\ dV_y \leq 1, \qquad \text{for } x \in M, \tag{23}$$

as a consequence of Theorem 5.3-(i). From (22) and (23) it follows that for every bounded continuous function u_0^ϵ on M, the function

$$u^\epsilon(x,t) = \begin{cases} \displaystyle\int_M p^\epsilon(x,y,t)\ dV_y & \text{for } t>0 \\ \\ u_0^\epsilon(x) & \text{for } t=0 \end{cases}$$

is continuous and bounded. Assuming, without lost of generality, that $u_0 \geq 0$, we have that

$$u^\epsilon(x,t) = \lim_{l\to 0^+} \int_M p_l^\epsilon(x,y,t)\ u_0^\epsilon(y)\ dV_y$$

is a limit of a sequence of solutions of the regularized Schrödinger-Günter equation with local L^1 bounds which are independent of l. From Theorem 5.5 we conclude that $u^\epsilon(x,t)$ satisfies the regularized Schrödinger-Günter equation. Property (iv) follows from an immediate application of the maximum principle.

■

6 The regularized Schrödinger-Günter kernel

In this section we impose some additional conditions on the manifold M and we study their implications. One reasonable additional property consists of demanding that the fundamental solutions should satisfy the following conservation law

$$\int_M p^\epsilon(x,y,t)\ dV_y = 1, \qquad t>0, \quad x\in M. \tag{24}$$

The physical interpretation of this conservation law means that the total amount of energy in M is conserved. In general, this fact is not true in view of Theorem 5.3-(i). Nevertheless, we may derive the following consequence of Theorem 4.3.

Theorem 6.1 *A complete Riemannian manifold where $2R^2 - \mathcal{G}R > 0$ has a uniquely defined fundamental solution of the regularized Schrödinger-Günter equation (in the following called regularized Schrödinger-Günter kernel). This kernel satisfies the properties stated in Theorem 5.5 and satisfies the conservation law (24).*

Proof: From the results of the previous section we may conclude that the function $u^\epsilon(x,t) = \int_M p^\epsilon(x,y,t)\ dV_y$ is a solution of the regularized Schrödinger-Günter problem with initial boundary data $u_0^\epsilon(x,t) \equiv 1$. The fact that $u^\epsilon(x,t) \equiv 1$ follows from the uniqueness property.

■

Let $\{P_t^{\mathbf{k}}\}_{t\in\mathbb{R}^+}$ be the semigroup of operators defined on the space of bounded continuous functions on M by

$$(P_t^{\mathbf{k}})(x) = \int_M p^\epsilon(x,y,t)\ u^\epsilon(y)\ dV_y. \tag{25}$$

In [15] some properties of a particular case of these operators were studied.

Theorem 6.2 *Let M be a complete Riemannian manifold with $2R^2 - \mathcal{G}R > 0$, and let u_0^ϵ be a continuous function on M which vanishes at infinity. Then $P_t^{\mathbf{k}} u_0^\epsilon$ vanishes at infinity for every $t > 0$.*

Proof: Let $u^\epsilon(x,t) = P_t^{\mathbf{k}} u_0^\epsilon(x)$. For every $\lambda > 0$, $|u^\epsilon(x,0)| = |u_0^\epsilon(x)| < \lambda$ whenever x is in the complement of a compact set K_λ. Applying (14) with $R = d(x_0, K_\lambda)$, $x_0 \notin K_\lambda$, $x = x_0$, and since $\rho(x_0) = \varphi(r(x_0)) = \varphi(0) = 0$ we get

$$|u^\epsilon(x_0,t)| \leq \lambda + \frac{N_0 K t}{d(x_0, K_\lambda)}.$$

Hence, $|u^\epsilon(x,t)| \leq 2\lambda$ outside the compact set $\{x :\ d(x,K_\lambda) \leq N_0 \frac{Kt}{\lambda}\}$, i.e., $u^\epsilon(\cdot,t) = P_t^{\mathbf{k}} u_0^\epsilon$ vanishes at infinity for every fixed $t > 0$.

■

For an arbitrary open Riemannian manifold M we can define the regularized Schrödinger-Günter kernel $q^\epsilon(x,y,t)$ for the generalized Dirichlet boundary conditions as the kernel generating the semigroup $\Gamma_t^{\mathbf{k}}$ on $L^2(M)$.

Proposition 6.3 *The fundamental solution $p^\epsilon(x,y,t)$ constructed in Section 5 is equal to $q^\epsilon(x,y,t)$.*

Proof: It suffices to show that $P_t^{\mathbf{k}}u = \Gamma_t^{\mathbf{k}}u$ for every function $u \in C_0^\infty(M)$. Our first step will be to prove that $P_t^{\mathbf{k}}u \in L^2(M)$ for every $t > 0$. Consider the sequence $\{D_l\}_{l=1}^\infty$, such that $\tilde{D}_l \subset D_{l+1}$ and $\bigcup_l D_l = M$, used in the construction of $p^\epsilon(x,y,t)$. For sufficiently large l, we have $\operatorname{supp} u \subset M$. A standard energy argument shows that

$$\left|\left|\int_M p_l^\epsilon(\cdot,y,t)\ u^\epsilon(y)\ dV_y\right|\right|_{L^2} < ||u||_{L^2}.$$

Since $P_t^{\mathbf{k}}u^\epsilon(x) = \lim_{l\to+\infty}\int_M P_l(x,y,t)u^\epsilon(y)\ dV_y$, Fatou's lemma implies that $P_t^{\mathbf{k}}u \in L^2(M)$ and $||P_t^{\mathbf{k}}u||_{L^2} \le ||u||_{L^2}$. Differentiation under the integral shows that $\Delta_G P_t^{\mathbf{k}}u \in L^2(M)$ for $t > 0$. Consider the function $v(x,t) = (P_t^{\mathbf{k}} - \Gamma_t^{\mathbf{k}})u$. We have to show that $v(x,t) \equiv 0$. However,

$$\frac{\mathbf{k}}{2}\partial_t \int_M v^2(x,t)\ dV_x = \mathbf{k}\int_M v\ \partial_t v\ dV_x = \mathbf{k}\int_M v\Delta_G v\ dV_x.$$

It is clear that $P_t^{\mathbf{k}}u$ lies inside of the domain of $\overline{\Delta_G}$, because it is a limit of functions with compact support. We get

$$\frac{|\mathbf{k}|}{2}\partial_t \int_M v^2(x,t)\ dV_x = -\int_M |\mathcal{D}_x v|^2\ dV_x \le 0. \tag{26}$$

Since $v(\cdot,0) \equiv 0$, $P_t^{\mathbf{k}}u = \Gamma_t^{\mathbf{k}}$, i.e., $p^\epsilon(x,y,t) = q^\epsilon(x,y,t)$.

■

Another application of our results is formulated in the following theorem:

Theorem 6.4 *Let M be complete Riemannian such that $2R^2 - \mathcal{G}R > 0$. For every function $u_0^\epsilon \in C_0^\infty(M)$, the bounded solution $u^\epsilon(x,t)$ of the regularized Schrödinger-Günter equation with initial data u_0^ϵ satisfies the following conservation law*

$$\int_M u^\epsilon(x,t)\ dV_x = \int_M u_0^\epsilon(x,t)\ dV_x.$$

Proof: From the uniqueness property of the solutions we know that

$$u^\epsilon(x,t) = \int_M p^\epsilon(x,y,t)\ u_0^\epsilon(y)\ dV_y.$$

Hence, by Fubini's theorem and Theorem 6.1 we obtain

$$\int_M u^\epsilon(x,t)\ dV_x = \int_{M\times M} p^\epsilon(x,y,t)\ u_0^\epsilon(y)\ dV_y\ dV_x = \int_M \left(\int_M p^\epsilon(x,y,t)\ dV_x\right) u_0^\epsilon(y)\ dV_y = \int_M u_0^\epsilon(y)\ dV_y. \tag{27}$$

■

We finish this section by presenting an application to the regularized Schrödinger-Günter equation, where the Laplacian is defined via Günter derivatives.

Theorem 6.5 *Suppose that M is a complete Riemannian manifold and that $2R^2 - \mathcal{G}R > 0$. For every $T > 0$, bounded solutions of the regularized Schrödinger-Günter problem on $L^2(M)$ are uniquely determined by their initial value. Moreover, if the initial data vanishes at infinity, then the solution vanishes at infinity for every $t \in]0,T[$.*

Proof: From our assumption on M, $< (2R^2 - \mathcal{G}R)u, u > \geq c||u||_{L^2}$ for all the functions $u \in L^2(M)$ at every point $x \in M$, where $< \cdot, \cdot >$ denotes the pointwise inner product and $| \cdot |$ is the corresponding norm. Let $u \in L^2(M)$ depending on a parameter $t \in [0, T]$ which is a bounded solution of the regularized Schrödinger-Günter problem with the initial value u_0^ϵ. Consider the function $\tilde{u} = e^c u$. Relying on relation (3) and after some calculations we obtain

$$(\Delta_G - \mathbf{k}\partial_t)|\tilde{u}|^2 = e^{2c}(2 < (2R^2 - \mathcal{G}R)u, u > +2 < \mathcal{D}u, \mathcal{D}u > -2c < u, u >).$$

Now we can repeat the argument used in the proof of Theorem 4.3 with $u = |\tilde{u}|^2$ and we conclude that (12), (13) and (14) are valid, and therefore we prove the uniqueness. The second part follows from (14) using the same arguments of the proof Theorem 6.2.

■

7 Convergence results

We start this section by studying the behavior of the regularized Schrödinger-Günter kernel (16), when $\epsilon \to 0^+$. Moreover, we will prove that (16) converges to

$$p_D(x, y, t) = (4\pi t)^{-\frac{n}{2}} \exp\left(-i\frac{|x-y|^2}{4t}\right)\Phi(x, y) + O\left(t^{-\frac{n}{2}-1}\right)\exp\left(-i\frac{|x-y|^2}{4t}\right). \tag{28}$$

In this sense, we have the following result

Theorem 7.1 *We have the following weak convergence in* $W_2^{-\frac{n}{2}-1}(M)$

$$< p^\epsilon, \varphi > \longrightarrow < p, \varphi >, \qquad \varphi \in W_2^{-\frac{n}{2}+1}(M)$$

when $\epsilon \to 0^+$.

Proof: Suppose that $\varphi \in W_2^{-\frac{n}{2}+1}(M)$. Then

$$\begin{aligned}
&| < p^\epsilon, \varphi > - < p, \varphi > | \\
&\qquad = \left| \int_M (p^\epsilon(x, t) - p(x, t))\varphi(x, t)\ dxdt \right| \\
&\qquad \leq \left| \int_M \left[\left(\exp\left(-\mathbf{k}\frac{d^2(x, y)}{4t}\right) - \exp\left(-i\frac{d^2(x, y)}{4t}\right) \right) (4\pi t)^{-\frac{n}{2}} \right] \varphi(x, t)\Phi(x, y)\ dxdt \right| \\
&\qquad\quad + \left| \int_M \left[\left(\exp\left(-\mathbf{k}\frac{d^2(x, y)}{4t}\right) - \exp\left(-i\frac{d^2(x, y)}{4t}\right) \right) (4\pi t)^{-\frac{n}{2}} \right] O\left(t^{-\frac{n}{2}-1}\right)\ dxdt \right|
\end{aligned} \tag{29}$$

The latter expression converges to zero when $\epsilon \to 0$ since $\varphi \in W_2^{-\frac{n}{2}+1}(M)$.

■

From this theorem we get the following two corollaries:

Corollary 7.2 *The sequence of the semigroup operators* $\{P_t^{\mathbf{k}}\}_{t \in \mathbb{R}^+}$*, defined in (25), converges weakly in* $W_2^{-\frac{n}{2}+1}(M)$ *to the following semigroup operator*

$$P_t(x) = \int_M p(x, y, t)\ u^\epsilon(y)\ dV_y.$$

Corollary 7.3 *The sequence of regularized fundamental solutions*

$$u^\epsilon(x, t) = \begin{cases} \displaystyle\int_M p^\epsilon(x, y, t)\ dV_y & \textit{for } t > 0 \\ u_0^\epsilon(x) & \textit{for } t = 0 \end{cases},$$

converges weakly in $W_2^{-\frac{n}{2}+1}(M)$ to the following fundamental solution

$$u(x,t) = \begin{cases} \int_M p(x,y,t)\; dV_y & \textit{for } t>0 \\ \\ u_0(x) & \textit{for } t=0 \end{cases}.$$

Moreover, we can guarantee that this sequence of solutions is a fundamental sequence in $W_2^{-\frac{n}{2}+1}(M)$.

The conclusion of Corollary 7.3 can be refined. In fact, consider $u \in W_2^{-\frac{n}{2}+1}(M)$ the function limit of the Cauchy family studied. In view of Definition 5.1 and (18), we can guarantee that $(\Delta_G - i\partial)u = 0$ and $(\Delta_G - i\partial_t)u^\epsilon = 0$, with $u^\epsilon|_{\partial D\times\mathbb{R}_+} = 0 = u|_{\partial D\times\mathbb{R}_+}$.

Since $(\Delta_G - i\partial_t)^{-1}$ exists and since it is unique (for more details see [21]), we can establish the following equality:

$$u - u^\epsilon = (\Delta_G - i\partial_t)^{-1}((\Delta_G - \mathbf{k}\partial_t) - (\Delta_G - i\partial_t))u^\epsilon$$

which implies that

$$||u-u^\epsilon||_{L^2(M)} = ||(\Delta_G - i\partial_t)^{-1}||_{L^1(M)}||(\Delta_G - \mathbf{k}\partial_t) - (\Delta_G - i\partial_t)||_{L^1(M)}||u^\epsilon||_{L^2(M)}.$$

Since $||(\Delta_G - \mathbf{k}\partial_t) - (\Delta_G - i\partial_t)||_{L^1(M)}$ converges to zero when $\epsilon \to 0^+$, we conclude that the right-hand side of the last expression also converges to zero. This fact implies that $u \in L^2(M)$.

8 The Schrödinger-Günter problem on conformally flat cylinders and the n-torus

In this section we briefly outline how we can construct fundamental solutions of the Schrödinger-Günter problem on a class of higher dimensional conformally flat cylinders and the n-torus with different spin structures embedded in the space $\mathbb{R}^n \oplus \mathbb{R}^+$. The explicit knowledge of the fundamental solution allows us to directly carry over the results of the previous sections to the setting of these particular manifolds.

To make the paper self-contained, we recall from previous works that a conformally flat spin manifold in n real variables is a Riemannian manifold with a well-defined spin structure possessing an atlas whose transition functions are Möbius transformations.

As explained for example in [17] and [15], we obtain a class of higher dimensional conformally flat spin cylinders C_k in n real variables by forming the topological quotient $C_k := \mathbb{R}^n/\mathbb{Z}^k$ where $\mathbb{Z}^k = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$ is the k-dimensional orthonormal lattice; k stands for a positive integer from the set $\{1,\ldots,n\}$. In the case $k=n$ we obtain a flat n-torus. In another interesting subcase represented by $n=2, k=1$ one re-obtains the classical infinite cylinder of radius 1 embedded in the three-dimensional Euclidean space.

Since $\mathbb{R}^n$ is the universal covering space of all these generalized cylinders C_k there exists a well defined projection map $p_k : \mathbb{R}^n \to C_k, x \mapsto x \bmod \mathbb{Z}^k$. One has $p_k(x) = p_k(y)$ if and only if there exists an $\omega \in \mathbb{Z}^k$ such that $x = y + \omega$.

Next, every subset $U \subset \mathbb{R}^n$ that has the property that $x \in U$ also implies that $x + \omega \in U$, for all $\omega = \sum_{i=1}^k \omega_i e_i \in \mathbb{Z}^k$, gives rise to an open subset U' on C_k defined by $U' := p_k(U)$.

More generally, on C_k one can consider 2^k inequivalent different spinor bundles. To construct them, we decompose the lattice $\mathbb{Z}^k$, as suggested in [17], into the direct sum of the sublattices $\mathbb{Z}^l := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_l$ and $\mathbb{Z}^{k-l} := \mathbb{Z}e_{l+1} + \cdots + \mathbb{Z}e_k$ where l is some integer from $\{1,\ldots,k\}$. We now obtain 2^k different spinor bundles $E^{(l)}$ on C_k by making the identification $(x,X) \Leftrightarrow (x+\omega, (-1)^{\omega_1+\cdots+\omega_l}X)$ with $x \in \mathbb{R}^n$ and $X \in \mathbb{C}_n$.

The basic idea to construct fundamental solutions of the regularized Schrödinger-Günter equation on these manifolds is to periodize the fundamental solutions described earlier in Section 5, i.e.

$$p_D^\epsilon(x,t) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{\mathbf{k}|x|^2}{4t}}\tilde{\Phi}(x) + O(t^{-n/2-1})e^{-\frac{\mathbf{k}|x|^2}{4t}}$$

over the period lattice $\mathbb{Z}^k$ and adding a further parity factor that addresses the particular chosen spinor bundle. Here in this setting the domain D simply is supposed to be the whole $\mathbb{R}^n$.

Notice that the function Φ has the structure $\Phi(x,y) = \tilde{\Phi}(x-y)$. The condition $\Phi(x,x) = 1$ then is re-phrased in the form $\tilde{\Phi}(0) = 1$.

To get concrete examples, let us assume that the C^∞ function $\tilde{\Phi}$ is a polynomial of some arbitrary degree in the variables $x_1, \ldots, x_n$ and that $\tilde{\Phi}(0) = 1$ in order to meet the above mentioned requirement.

Under these conditions the function series

$$\begin{aligned} \wp_D^\epsilon(x,t) &:= \sum_{m\in\mathbb{Z}^l} \sum_{n\in\mathbb{Z}^{k-l}} (-1)^{m_1+\cdots+m_l} p_D^\epsilon(x+m+n,t) \\ &= \sum_{m\in\mathbb{Z}^l} \sum_{n\in\mathbb{Z}^{k-l}} (-1)^{m_1+\cdots+m_l} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\mathbf{k}|x+m+n|^2}{4t}} \tilde{\Phi}(x+m+n) + O(t^{-n/2-1}) e^{-\frac{\mathbf{k}|x+m+n|^2}{4t}} \end{aligned}$$

converges normally due to the dominance of the decay of the exponential terms in these series expression. This is done by applying the same arguments as in the convergence proof of [15]. Basically the new item in this expression is the appearance of the function $\tilde{\Phi}$ which did not appear in the context of our previous work [15]. In the context of [15] we simply dealt with the special situation where the function $\Phi(x) \equiv 0$ while the second term involving a further function of growth $O(t^{-n/2-1})$ did not appear at all. However, as the latter function of the order $O(t^{-n/2-1})$ only depends on the time coordinate t, it has no influence on the convergence behavior of the series, since the summation is only extended over a period lattice in the spatial coordinates.

However, the presence of the function $\tilde{\Phi}$ appearing in the first term may significantly affect the convergence behavior of the whole function series. If we only requires as in Section 5 that $\tilde{\Phi}$ is a C^∞ function, then the series will not converge in general. To guarantee the convergence we have to put additional restrictions on $\tilde{\Phi}$.

As long as the functions $\tilde{\Phi}$ are supposed to be only polynomials, the exponential decay from $e^{-\frac{\mathbf{k}|x+m+n|^2}{4t}}$ dominates the polynomial increase of $\tilde{\Phi}$ such that the complete series expression remains convergent under this particular condition. Then we obtain a fundamental solution on the associated manifold C_k for the regularized Schrödinger-Günter problem by applying the projection map p_k to this convergent series, i.e., $e(x',t) := p_k(\wp_D^\epsilon(x,t))$ and $E(x',y',t) := e(x'-y',t)$, then serves as integral kernel for the regularized Schrödinger-Günter problem on C_k. Instead of polynomials only one can admit for $\tilde{\Phi}$ more generally also some transcendental functions that still have a growth behavior that is dominated by the terms $e^{-\frac{\mathbf{k}|x|^2}{4t}}$.

Next, in order to address the limit case $\epsilon \to 0^+$ which refers to the usual Schrödinger-Günter problem one can consider the construction

$$\begin{aligned} \wp_D(x,t) &:= \lim_{\epsilon\to 0} \sum_{m\in\mathbb{Z}^l} \sum_{n\in\mathbb{Z}^{k-l}} (-1)^{m_1+\cdots+m_l} p_D^\epsilon(x+m+n,t) \\ &= \lim_{\epsilon\to 0} \sum_{m\in\mathbb{Z}^l} \sum_{n\in\mathbb{Z}^{k-l}} (-1)^{m_1+\cdots+m_l} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\frac{\varepsilon+1}{\varepsilon^2+1}|x+m+n|^2}{4t}} \tilde{\Phi}(x+m+n) + O(t^{-n/2-1}) e^{-\frac{\frac{\varepsilon+1}{\varepsilon^2+1}|x+m+n|^2}{4t}} \end{aligned}$$

where we again impose additional restrictional growth conditions on $\tilde{\Phi}$ that guarantee the convergence of the series. As simplest choice we again may admit every polynomial for $\tilde{\Phi}(0) = 1$. Applying the same arguments as in [15] one can conclude that the projection of this expression then yields the fundamental solution of the Schrödinger-Günter problem on C_k.

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