Four solutions of an inhomogeneous elliptic equation with critical exponent and singular term

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Abstract

In this paper, we prove the existence of four nontrivial solutions of

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{2^*-2}u + \mu|x|^{\alpha-2}u + f(x), \qquad x \in \Omega\backslash\{0\}$$

and show that at least one of them is sign changing. Our results extend some previous works on the literature, as Tarantello(1993), Kang-Deng(2005) and Hirano-Shioji(2005).

Key words: Variational methods, inhomogeneous Laplacian equation, singular term, four solutions, sign changing solution.
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1 Introduction

In Chen-Li-Li [4], it has been showed the effect of suitable singular potential $V(x)$ on the existence of multiple solutions of

$$-\Delta u = \lambda V(x)u + |u|^{2^*-2}u, \quad u \in H_0^1(\Omega). \tag{E}$$

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Here, we will prove an additional inhomogeneous perturbation of (E) can produce more solutions. More precisely, we study the existence of four solutions of the following problem

$$\begin{cases} -\Delta u(x) - \frac{\lambda}{|x|^2}u(x) = |u(x)|^{2^*-2}u(x) + \mu|x|^{\alpha-2}u(x) + f(x) \text{ in } & \Omega, \\ \qquad\qquad u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and we prove that at least one of them is sign changing. We assume that $0 \in \Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $2^* \doteq 2N/(N-2)$ is the critical Sobolev exponent, $0 \leq \lambda < \Lambda \doteq ((N-2)/2)^2$ and $f \in L^\infty(\Omega)$.

Elliptic equations with a singular term have attracted great interests during the past several years, e.g. see Ferrero-Gazzola [8], Jannelli [10], Terracini [17], Smets [13]. Particularly, we point out that when $\alpha = 2$ and $N \geq 7$, Kang-Deng [11] proved the existence of two nontrivial solutions of Eq.(1) provided f satisfying some other conditions. The main result in this paper (Theorem 1.2) proves the existence of four solutions of Eq.(1), and also implies that suitable unbounded coefficients $|x|^{\alpha-2}$ can release the restriction of spatial dimension N.

In what follows, we state the main result (Theorem 1.2) but for the presentation coherence, we first prove an auxiliary result (Lemma 1.1). Assume that $\alpha > 0$. From the work of Chaudhuri-Ramaswamy [2], we know that

$$\mu_1 \doteq \inf\left\{ \int_\Omega \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right)dx : \int_\Omega |x|^{\alpha-2}|u|^2 dx = 1\right\} > 0.$$

Define $p \doteq 2^*$, $T(u) \doteq \int_\Omega (|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2 - \mu|x|^{\alpha-2}|u|^2)dx$ and

$$M \doteq \inf\left\{ \left(T(u)\right)^{\frac{1}{2}} : \int_\Omega |u|^p dx = 1\right\}.$$

Lemma 1.1 *If $0 \leq \lambda < \Lambda$, $\alpha > 0$ and $0 < \mu < \mu_1$, then $M > 0$.*

Proof. For any $u \neq 0$, we have from the assumption $0 < \mu < \mu_1$ and the Hardy inequality that

$$T(u) \geq \left(1 - \frac{\mu}{\mu_1}\right) \int_\Omega \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right)dx \geq \left(1 - \frac{\mu}{\mu_1}\right)\left(1 - \frac{\lambda}{\Lambda}\right) \int_\Omega |\nabla u|^2 dx.$$

Thus

$$\left(1 - \frac{\mu}{\mu_1}\right)\left(1 - \frac{\lambda}{\Lambda}\right) \int_\Omega |\nabla u|^2 dx \leq T(u) \leq \int_\Omega |\nabla u|^2 dx.$$

Note that the best Sobolev constant

$$S_0 \doteq \inf\left\{ \int_\Omega |\nabla u|^2 dx : \int_\Omega |u|^p dx = 1\right\} > 0.$$

We immediately have that $M > 0$. $\square$

Throughout this paper, we always assume that $0 \le \lambda < \Lambda$ and $0 < \mu < \mu_1$. We say that $u \in H_0^1(\Omega)$ is a solution of Eq.(1) if and only if u is a critical point of the Euler functional $I(u) \doteq \frac{1}{2}T(u) - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega fu\, dx$, i.e. for any $v \in H_0^1(\Omega)$ there holds

$$\int_\Omega \left(\nabla u \nabla v - \frac{\lambda}{|x|^2}uv - \mu|x|^{\alpha-2}uv - |u|^{p-2}uv - fv \right) dx = 0.$$

We will prove the following result:

Theorem 1.2 *Let $0 \le \lambda < \Lambda$, $0 < \mu < \mu_1$, $f \in L^\infty(\Omega)$ and satisfies*

$$|f|_{\frac{p}{p-1}} \doteq |f|_\sharp < \frac{p-2}{2(p-1)} M \left(\frac{M^p}{p-1} \right)^{\frac{1}{p-2}}. \tag{A}$$

If $0 < \alpha < \sqrt{\Lambda - \lambda}$, then Eq.(1) has at least four nontrivial solutions in $H_0^1(\Omega)$ and at least one of them is sign changing.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we obtain the necessary auxiliary results in order to prove the Theorem 1.2.

The proof of Theorem 1.2 is based on variational methods and it is inspired by Tarantello [15] and Hirano-Shioji [9]. The main strategy, based on four steps, is the following. In the first step, we define a Nehari type set $\mathscr{M}$ and use assumption (A) to divide $\mathscr{M}$ into three subsets $\mathscr{M}^+$, $\mathscr{M}^0$ and $\mathscr{M}^-$. In the second step, we solve two minimization problem $\inf_{\mathscr{M}} I$ and $\inf_{\mathscr{M}^-} I$ and get two solutions w_0, w_1 of Eq.(1). In the third step, we construct two subsets $\mathscr{M}_1^-$ and $\mathscr{M}_2^-$ of $\mathscr{M}^-$ and prove that the minimizer of $\inf\{I(u) : u \in \mathscr{M}_1^- \cap \mathscr{M}_2^-\}$ is a sign changing solution of Eq.(1). In the final step, we define a translated functional and get a fourth solution of Eq.(1).

Note that, although the proof is inspired by Tarantello [15] and Hirano-Shioji [9], the arguments used by them can not be directly applied here, since we are facing the singular term $\frac{\lambda}{|x|^2}u$ (see Remark 3.6, Remark 3.9 and Remark 3.12). In fact, we need to develop some techniques recently used in Chen [5, 6] and the exact local behavior for the solutions of Eq.(1), in order to overcome the difficulties created by the singular term $\frac{\lambda}{|x|^2}u$.

Notations. In what follows, we denote the norm in $H_0^1(\Omega)$ by $\|\cdot\|$, the integral $\int_\Omega \cdot\, dx$ by $\int \cdot$, and the ball in $\mathbb{R}^N$ with center at x and radius R by $B(x,R)$. We use $\doteq$ to emphasize a new definition. Different positive constants may be denoted by the same letter K or K_j. Additionally, $O(\varepsilon^\beta)$ means that $|O(\varepsilon^\beta)\varepsilon^{-\beta}| \le K$, $o(\varepsilon^\beta)$ is $|o(\varepsilon^\beta)\varepsilon^{-\beta}| \to 0$ as $\varepsilon \to 0$, and $o(1)$ is an infinitesimal value, and $\to$ (respectively, $\rightharpoonup$) will denote strongly (respectively, weakly) convergence.

2 Preliminaries

In this section, we give some preliminaries which play important roles in the variational methods used to solve Eq.(1). Namely, we briefly describe the solution of an auxiliary problem, the local behavior of the solutions of Eq.(1) and some integral estimates.

From Catrina-Wang [3], Terracini [17], and Chou-Chu [7], we have the following proposition:

Proposition 2.1 *For $0 < \lambda < \Lambda = (\frac{N-2}{2})^2$, equation*

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{2^*-2}u \qquad x \in \mathbb{R}^N\backslash\{0\}, \quad u(x) \to 0 \text{ as } |x| \to +\infty, \qquad (2)$$

has a family of solutions

$$U_\varepsilon(x) = \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}}}{[\varepsilon|x|^{\gamma'/\sqrt{\Lambda}} + |x|^{\gamma/\sqrt{\Lambda}}]^{\frac{N-2}{2}}}, \quad \varepsilon > 0.$$

where $\Lambda = (\frac{N-2}{2})^2$, $\gamma' = \sqrt{\Lambda} - \sqrt{\Lambda-\lambda}$,$\gamma = \sqrt{\Lambda} + \sqrt{\Lambda-\lambda}$. Moreover $U_\varepsilon(x)$ is the unique positive radial symmetric solution of Eq.(2) up to a dilation. And $U_\varepsilon(x)$ is the extremal function of the minimization problem

$$S_\lambda = \inf\left\{\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right)dx; \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}|u|^{2^*}dx = 1\right\}.$$

Clearly,

$$\int_{\mathbb{R}^N}|U_\varepsilon(x)|^{2^*}dx = \int_{\mathbb{R}^N}\left(|\nabla U_\varepsilon|^2 - \frac{\lambda}{|x|^2}U_\varepsilon^2\right)dx = S_\lambda^{\frac{N}{2}}.$$

We now recall some exact local behavior of the solutions of Eq.(1) (see Chen [5, Th.1.1]).

Proposition 2.2 *Let $0 \le \lambda < \Lambda$. We have that*

- *if $u \in H_0^1(\Omega)$ is a solution of Eq.(1), then there holds*

$$|u(x)| \le K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x \in B(0,r)\backslash\{0\} \qquad (3)$$

  for some positive constant K_1 and sufficiently small $r > 0$;

- *if $u \in H_0^1(\Omega)$ is a positive solution of Eq.(1), then there holds*

$$K_2|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})} \le |u(x)| \le K_1|x|^{-(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})}, \qquad x \in B(0,r)\backslash\{0\} \quad (4)$$

  for $r > 0$ sufficiently small and some positive constants $K_1,\ K_2$.

Remark 2.3 *Let u be a positive solution of Eq.(1).*

(i) When $\lambda = 0$, $u(0)$ is positive and we come back to the usual case.
(ii) When $0 < \lambda < \Lambda$, the singular order at $x = 0$ of u stated in Proposition 2.2 coincide with the singularity of the explicit form $U_\varepsilon(x)$.
(iii) When $\lambda \to \Lambda$, the singularity of the positive solutions become more and more stronger.

The following integral estimates are also relevant. Define a cut-off function $\phi(x) = 1$ if $|x| \le \delta$, $\phi(x) = 0$ if $|x| \ge 2\delta$, $\phi(x) \in C_0^1(\Omega)$ and $|\phi(x)| \le 1$, $|\nabla\phi(x)| \le C$. Let $v_\varepsilon(x) = \phi(x)U_\varepsilon(x)$.

Proposition 2.4 *Let $0 \le \lambda < \Lambda$ and $w \in H_0^1(\Omega)$ be a solution of Eq.(1). Then for $\varepsilon > 0$ small enough we have that*

$$\int w^{2^*-1}v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}}) \quad \text{and} \quad \int w v_\varepsilon^{2^*-1}dx = O(\varepsilon^{\frac{N-2}{4}}); \quad (5)$$

$$\int \left(|\nabla v_\varepsilon|^2 - \frac{\lambda}{|x|^2}v_\varepsilon^2\right) = S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N-2}{2}}); \quad (6)$$

$$\int v_\varepsilon^{2^*} = S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{N}{2}}); \quad (7)$$

$$\int |x|^{\alpha-2}v_\varepsilon^2 = O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}),\ \text{when } 0 < \alpha < 2\sqrt{\Lambda-\lambda}; \quad (8)$$

$$\int v_\varepsilon dx = O(\varepsilon^{\frac{N-2}{4}}); \quad (9)$$

Proof. For the proofs of (5), (6) and (7), see Chen [6]. Here we only prove (8) and (9). Recalling the definition of v_ε, we have that

$$\begin{aligned}\int |x|^{\alpha-2}v_\varepsilon^2 &= \int_{\Omega\backslash B(0,\delta)} |x|^{\alpha-2}v_\varepsilon^2 + \int_{B(0,\delta)} |x|^{\alpha-2}v_\varepsilon^2 \\ &= O(\varepsilon^{\frac{N-2}{2}}) + \int_{B(0,\delta)} |x|^{\alpha-2}v_\varepsilon^2.\end{aligned}$$

$$\begin{aligned}&\int_{B(0,\delta)} |x|^{\alpha-2}v_\varepsilon^2 dx = K\cdot\varepsilon^{\frac{(N-2)}{2}}\int_0^\delta \frac{\rho_{\alpha-2+N-1}d\rho}{[\varepsilon\rho^{\gamma'/\sqrt{\Lambda}} + \rho^{\gamma/\sqrt{\Lambda}}]^{N-2}} \\ &= K\cdot\varepsilon^{\frac{(N-2)}{2}}\int_0^\delta \frac{\rho_{\alpha-2+N-1}d\rho}{\varepsilon^{N-2}\rho^{2\gamma'}[1+\varepsilon^{-1}\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{N-2}} \\ &= K\cdot\varepsilon^{\frac{(N-2)}{2}}\int_0^{\delta\varepsilon^{-\frac{\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}} \frac{\varepsilon^{\frac{(\alpha-2+N)\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\rho_{\alpha-2+N-1}d\rho}{\varepsilon^{N-2}\varepsilon^{\frac{\gamma'}{2\sqrt{\Lambda-\lambda}}}\rho^{2\gamma'}[1+\rho^{2\sqrt{\Lambda-\lambda}/\sqrt{\Lambda}}]^{N-2}}\end{aligned}$$

Since

$$-1+\alpha-2+N-2(\sqrt{\Lambda}-\sqrt{\Lambda-\lambda})-4\sqrt{\Lambda-\lambda}=-1+\alpha-2\sqrt{\Lambda-\lambda}<-1,$$

$$\frac{(\alpha-2+N)\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-(N-2)-\frac{\gamma'\sqrt{\Lambda}}{\sqrt{\Lambda-\lambda}}=\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}-\sqrt{\Lambda}<0,$$

we get that

$$\int_{B(0,\delta)}|x|^{\alpha-2}v_\varepsilon^2dx=K\cdot\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}.$$

It follows from $\int_{\Omega\setminus B(0,\delta)}|x|^{\alpha-2}v_\varepsilon^2=O(\varepsilon^{\frac{N-2}{2}})$ and $0<\alpha<2\sqrt{\Lambda-\lambda}$ that

$$\int|x|^{\alpha-2}v_\varepsilon^2=O\left(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right),\ \text{for }\varepsilon>0\text{ small enough.}$$

This proves (8). The proof of (9) is similar but simpler than the proof of (8), so we omit the details. □

3 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. As described in the Introduction, the proof is divided into four steps. We start defining some Nehari type sets; we prove the existence of two solutions of Eq.(1); we prove the existence of a third solution which is a sign changing solution of Eq.(1); and we prove the existence of a fourth solution by a translated argument.

Firstly for $u\in H_0^1(\Omega)$, recall the definition of

$$T(u)\doteq\int\left(|\nabla u|^2-\frac{\lambda}{|x|^2}|u|^2-\mu|x|^{\alpha-2}|u|^2\right).$$

So, the Euler functional can be rewritten as $I(u)=\frac{1}{2}T(u)-\frac{1}{p}\int|u|^p-\int fu$. We define

$$Q(u)\doteq T(u)-\int|u|^p-\int fu\quad\text{and}\quad J(u)\doteq 2T(u)-p\int|u|^p-\int fu,$$

then Q and J are well defined C^1 functionals on $H_0^1(\Omega)$. Next, set

$$\mathscr{M}\doteq\{u\in H_0^1(\Omega):Q(u)=0\}.$$

Then for any $u\in\mathscr{M}$, $J(u)=T(u)-(p-1)\int|u|^p$. We also define several subsets of $\mathscr{M}$,

$$\mathscr{M}^+\doteq\{u\in\mathscr{M}:J(u)>0\},\quad\mathscr{M}^0\doteq\{u\in\mathscr{M}:J(u)=0\}$$

$$\text{and} \quad \mathscr{M}^- \doteq \{u \in \mathscr{M} : J(u) < 0\}.$$

Lemma 3.1 *Let $0 \le \lambda < \Lambda$ and $0 < \mu < \mu_1$. Then the following hold:*

(i) $(p-2)T(u)^{\frac{1}{2}} < (p-1)M^{-1}|f|_\sharp$ *for all* $u \in \mathscr{M}^+$*;*

(ii) $T(u) > \left(\frac{M^p}{p-1}\right)^{\frac{2}{p-2}}$ *for all* $u \in \mathscr{M}^-$*;*

(iii) if (A) holds, then $I(u) \ge 0$ *for all* $u \in \mathscr{M}^-$*.*

Proof. (i) For any $u \in \mathscr{M}^+$, using $Q(u) = 0$ we get that

$$\begin{aligned} 0 < J(u) &= T(u) - (p-1)\int |u|^p = T(u) - (p-1)\Big(T(u) - \int fu\Big) \\ &= (2-p)T(u) + (p-1)\int fu. \end{aligned}$$

Since $M|u|_p \le T(u)^{\frac{1}{2}}$, we get that

$$(p-2)T(u) < (p-1)|f|_\sharp |u|_p \le (p-1)|f|_\sharp M^{-1} T(u)^{\frac{1}{2}}.$$

Therefore

$$(p-2)T(u)^{\frac{1}{2}} < (p-1)M^{-1}|f|_\sharp.$$

(ii) For any $u \in \mathscr{M}^-$, we have that $J(u) < 0$. From the definition of M, we have that $\int |u|^p \le M^{-p}T(u)^{\frac{p}{2}}$. Therefore

$$T(u) < (p-1)\int |u|^p \le (p-1)M^{-p}T(u)^{\frac{p}{2}}.$$

Thus

$$T(u) > \left(\frac{M^p}{p-1}\right)^{\frac{2}{p-2}}.$$

(iii) For any $u \in \mathscr{M}^- \subset \mathscr{M}$, we obtain from $\int |u|^p = T(u) - \int fu$ that

$$\begin{aligned} I(u) &= \frac{1}{2}T(u) - \frac{1}{p}\Big(T(u) - \int fu\Big) - \int fu \\ &= \frac{p-2}{2p}T(u) - \frac{p-1}{p}\int fu \\ &\ge \frac{p-2}{2p}T(u) - \frac{p-1}{p}|f|_\sharp |u|_p \\ &\ge \frac{p-2}{2p}T(u) - \frac{p-1}{p}|f|_\sharp M^{-1}T(u)^{\frac{1}{2}} \\ &= \frac{1}{p}T(u)^{\frac{1}{2}}\Big(\frac{p-2}{2}T(u)^{\frac{1}{2}} - (p-1)|f|_\sharp M^{-1}\Big). \end{aligned} \tag{10}$$

On the other hand we get from (ii) that

$$T(u) > \left(\frac{M^p}{p-1}\right)^{\frac{2}{p-2}}. \tag{11}$$

It is deduced from (10) and (11) that

$$I(u) \geq \frac{1}{p} T(u)^{\frac{1}{2}} \left(\frac{p-2}{2} \left(\frac{M^p}{p-1}\right)^{\frac{1}{p-2}} - (p-1)|f|_\sharp M^{-1} \right). \tag{12}$$

Therefore if (A) holds, then $I(u) \geq 0$. The proof is complete. □

Lemma 3.2 *Let $0 \leq \lambda < \Lambda$, $0 < \mu < \mu_1$ and $f \not\equiv 0$ satisfy (A). Then for any $u \in H_0^1(\Omega)$ and $u \neq 0$, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+(u)u \in \mathscr{M}^-$ and*

$$t^+ > \left(\frac{T(u)}{(p-1)\int |u|^p} \right)^{\frac{1}{p-2}} \doteq t_{max}$$

and

$$I(t^+u) = \max_{t \geq t_{max}} I(tu).$$

Moreover if $\int fu > 0$, then there exists an unique $t^- = t^-(u) > 0$ such that $t^-(u)u \in \mathscr{M}^+$, $t^- < t_{max}$ and

$$I(t^-u) = \inf_{0 \leq t \leq t_{max}} I(tu).$$

Proof. The proof follows exactly the scheme in the proof of Lemma 2.1 in Tarantello [15]. □

Proposition 3.3 *Assume $0 \leq \lambda < \Lambda$, $0 < \mu < \mu_1$ and (A) hold. Let $\{u_n\} \subset \mathscr{M}^-$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $I(u_n) \to c$ but u_n does not converge strongly to u in $H_0^1(\Omega)$. Then the following holds:*

(1) $c > I(t^+(u)u)$ in the case $u \neq 0$ and $t^+(u) \leq 1$;
(2) $c \geq I(t^-(u)u) + \frac{1}{N} S_\lambda^{\frac{N}{2}}$ in the case $u \neq 0$ and $t^+(u) > 1$;
(3) $c \geq \frac{1}{N} S_\lambda^{\frac{N}{2}}$ in the case $u = 0$.

Proof. We use the methods employed in Hirano-Shioji [9]. Note that $u_n \rightharpoonup u$, $\int |x|^{\alpha-2}|u_n - u|^2 \to 0$ as $n \to \infty$. We may assume that

$$T(u_n - u) = \int \left(|\nabla u_n - \nabla u|^2 - \frac{\lambda}{|x|^2}|u_n - u|^2 \right) + 0(1) \to a^2$$

$$\text{and} \qquad \int |u_n - u|^p \to b^p.$$

Since u_n does not converge strongly to u, we have $a \neq 0$. We set

$$r(t) = I(tu), \quad \beta(t) = \frac{a^2}{2}t^2 - \frac{b^p}{p}t^p$$

and $\theta(t) = r(t) + \beta(t)$, then $I(tu_n) \to \theta(t)$ as $n \to +\infty$. We consider three situations:

(1) When $u \neq 0$ and $t^+(u) \leq 1$. Recall, that $t^+(u)$ is defined according to Lemma 3.2. We have that $r'(1) \leq 0$. Since $u_n \in \mathscr{M}^-$, we have that $\theta'(1) = 0$. Thus $\beta'(1) \geq 0$ and hence $a^2 - b^p \geq 0$. So we have that $\beta(t^+(u)) > 0$ and hence

$$c \geq \theta(1) \geq \theta(t^+(u)) = I(t^+(u)u) + \beta(t^+(u)) > I(t^+(u)u).$$

(2) When $u \neq 0$ and $t^+(u) > 1$. We have first from $t^+(u) > 1$ that $b \neq 0$. Indeed if $b = 0$, then from $\theta'(1) = 0$ and $\theta''(1) \leq 0$, we have that $r'(1) = -a^2 < 0$ and $r''(1) \leq -a^2 < 0$, which contradicts to $t^+(u) > 1$. So we have $b \neq 0$. We set $t_* = (a^2/b^p)^{\frac{1}{p-2}}$. We know that β attains its maximum at t_* and $\beta'(t) > 0$ for $0 < t < t_*$ and $\beta'(t) < 0$ for $t > t_*$. Therefore we have that $\beta(t_*) = (a^2/b^2)^{\frac{N}{2}}/N \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}$. Next, we will show that $t_* \leq t^+(u)$. Suppose this is not the case, i.e., $1 < t^+(u) < t_*$. As $0 > \theta'(t) = r'(t)+\beta'(t)$ for all $t > 1$, we have $r'(t) \leq -\beta'(t) < 0$ for $t \in (1, t_*)$, which contradicts to $1 < t^+(u) < t_*$ and $r'(t^+(u)) = 0$. So we have shown that $t_* \leq t^+(u)$. Hence we obtain

$$c = \theta(1) \geq \theta(t_*) = I(t_*u) + \beta(t_*) \geq I(t^-(u)u) + \frac{1}{N}S_\lambda^{\frac{N}{2}}.$$

This implies that (ii) holds.

(3) When $u = 0$. Since $u_n \in \mathscr{M}^- \subset \mathscr{M}$, we have that

$$\int (|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2) = \int |u_n|^p + 0(1).$$

Using the fact that $S_\lambda |v|_p^2 \leq \int(|\nabla v|^2 - \frac{\lambda}{|x|^2}|v|^2)$ for all $v \in H_0^1(\Omega)$ and $v \neq 0$, we obtain that

$$\begin{aligned} c &\geq \frac{1}{2}\int \left(|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) - \frac{1}{p}\int |u_n|^p + 0(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\int \left(|\nabla u_n|^2 - \frac{\lambda}{|x|^2}|u_n|^2\right) + 0(1) \geq \frac{1}{N}S_\lambda^{\frac{N}{2}}. \end{aligned}$$

The proof is complete. □

3.1 Existence of two solutions

In this subsection, we prove the existence of two solutions of Eq.(1). Set

$$c_0 \doteq \inf_{u\in\mathscr{M}} I(u) \quad \text{and} \quad c_1 \doteq \inf_{u\in\mathscr{M}^-} I(u).$$

Proposition 3.4 *If (A) holds, then $c_0 < 0$ and there is a critical point $w_0 \in \mathscr{M}^+$ of I such that $I(w_0) = c_0$ and w_0 is a local minimizer for I.*

Proof. The proof is exactly the same as Tarantello [15]. We omit the details here. □

Lemma 3.5 *Let $0 \le \lambda < \Lambda$ and $0 < \alpha < \sqrt{\Lambda - \lambda}$. Then $c_1 < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof. First using the same argument as Tarantello [15, Proposition 2.2], we know that there is $s_0 > 0$ and $\varepsilon > 0$ sufficiently small such that $w_0 + s_0 v_\varepsilon \in \mathscr{M}^-$. Next to prove $c_1 < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, we only need to prove that $\sup_{s>0} I(w_0 + s v_\varepsilon) < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$. Note that $I(w_0 + s v_\varepsilon) \to -\infty$ as $s \to +\infty$, we only estimate $I(w_0 + s v_\varepsilon)$ for bounded s. Since w_0 is a solution of Eq.(1), we have that

$$\int \left(\nabla w_0 \nabla v_\varepsilon - \frac{\lambda}{|x|^2} w_0 v_\varepsilon - \mu |x|^{\alpha-2} w_0 v_\varepsilon \right) = \int \left(w_0^{p-1} v_\varepsilon + f v_\varepsilon \right).$$

Hence

$$\begin{aligned} I(w_0 + s v_\varepsilon) &= \frac{1}{2} T(w_0 + s v_\varepsilon) - \frac{1}{p} \int |w_0 + s v_\varepsilon|^p - \int f(w_0 + s v_\varepsilon) \\ &= I(w_0) + I(s v_\varepsilon) + 2s \int (w_0^{p-1} v_\varepsilon + f v_\varepsilon) \\ &\quad + \frac{1}{p} \int \left(w_0^p + |s v_\varepsilon|^p - |w_0 + s v_\varepsilon|^p \right) \\ &\le I(w_0) + I(s v_\varepsilon) + C \int \left(v_\varepsilon + w_0^{p-1} v_\varepsilon + w_0 v_\varepsilon^{p-1} \right). \end{aligned}$$

Note that

$$\begin{aligned} \sup_{s>0} I(s v_\varepsilon) &= \sup_{s>0} \left(\frac{s^2}{2} T(v_\varepsilon) - \frac{s^p}{p} \int v_\varepsilon^p - s \int f v_\varepsilon \right) \\ &\le \sup_{s>0} \left(\frac{s^2}{2} T(v_\varepsilon) - \frac{s^p}{p} \int v_\varepsilon^p \right) + C \int v_\varepsilon \\ &\le \frac{1}{N} T(v_\varepsilon)^{\frac{N}{2}} \left(\int v_\varepsilon^p \right)^{1-\frac{N}{2}} + C \int v_\varepsilon. \end{aligned}$$

We obtain from Proposition 2.4 that

$$\begin{aligned}
\sup_{s>0} I(w_0+sv_\varepsilon) &< \frac{1}{N}\left(S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}}) - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}} \\
&\quad + c_0 + O(\varepsilon^{\frac{N-2}{4}}) \\
&= \frac{1}{N}S_\lambda^{\frac{N}{2}}\left(1 - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})\right)^{\frac{N}{2}}\left(1 - O(\varepsilon^{\frac{N}{2}})\right)^{1-\frac{N}{2}} + c_0 + O(\varepsilon^{\frac{N-2}{4}}) \\
&= c_0 + O(\varepsilon^{\frac{N-2}{4}}) + \frac{1}{N}S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) \\
&< c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}} \quad \text{since} \quad \alpha < \sqrt{\Lambda-\lambda}.
\end{aligned}$$

The proof is complete. □

Remark 3.6 We emphasize that in the estimate $\int w_0^{p-1}v_\varepsilon$ and $\int w_0 v_\varepsilon^{p-1}$, the exact local behavior of the solution of Eq.(1) (see Propositions 2.2 and 2.4) played an essential role. Indeed, without Propositions 2.2, estimates $\int w_0^{p-1}v_\varepsilon$ and $\int w_0 v_\varepsilon^{p-1}$ seem to be impossible.

Proposition 3.7 *Let* $0 \le \lambda < \Lambda$, $0 < \mu < \mu_1$ *and* $0 < \alpha < \sqrt{\Lambda-\lambda}$. *If* $f \in L^\infty(\Omega)$ *and satisfies (A), then there is a critical point* $w_1 \in \mathscr{M}^-$ *of* I *such that* $I(w_1) = c_1$.

Proof. First we will prove that there is $w_1 \in \mathscr{M}^-$ of I such that $I(w_1) = c_1$. Let $\{u_n\} \subset \mathscr{M}^-$ and $I(u_n) \to c_1$. Then by direct calculations we know that

$$0 < \inf T(u_n) \le \sup T(u_n) < \infty.$$

The definition of μ_1 and $0 < \mu < \mu_1$ implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. We may assume that $\{u_n\}$ converges weakly to some w_1. By Proposition 3.3 we have that $w_1 \ne 0$. Now suppose that $\{u_n\}$ does not converge to w_1. Then by (i) and (ii) of Proposition 3.3, we get that $c_1 > I(t^+(w_1)w_1)$ or $c_1 \ge I(t^-(w_1)w_1) + \frac{1}{N}S_\lambda^{\frac{N}{2}} \ge c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$. In any case we get a contradiction since $c_1 < c_0 + \frac{1}{N}S_\lambda^{\frac{N}{2}}$. Therefore $\{u_n\}$ converges strongly to w_1. This means $w_1 \in \mathscr{M}^-$ and $I(w_1) = c_1$.

Next we will show that such w_1 is a weak solution of Eq.(1). Choose any $v \in H_0^1(\Omega)$. For any $\rho \in (0,1)$ we set $t_\rho = t^+(w_1+\rho v)$ (where $t^+(w_1+\rho v)$ is defined according to Lemma 3.2). Since w_1, $t_\rho(w_1+\rho v) \in \mathscr{M}^-$ and $I(w_1) = \inf_{u\in\mathscr{M}^-} I(u)$, we have that

$$I(t_\rho(w_1+\rho v)) \ge I(w_1).$$

On the other hand from $w_1 \in \mathscr{M}^-$, we have that for any $t > 0$, $I(w_1) \geq I(tw_1)$. In particular, $I(w_1) \geq I(t_\rho w_1)$. Thus we have for any $\rho \in (0,1)$,

$$I(t_\rho(w_1 + \rho v)) \geq I(t_\rho w_1).$$

Hence we get that

$$\begin{aligned} 0 &\leq \frac{1}{\rho}\Big(I(t_\rho(w_1+\rho v)) - I(t_\rho w_1)\Big) \\ &= \frac{1}{\rho}\frac{t_\rho^2}{2}\Big(T(w_1+\rho v) - T(w_1)\Big) - \frac{1}{\rho}\frac{t_\rho^p}{p}\int\Big(|w_1+\rho v|^p - |w_1|^p\Big) \\ &\quad \frac{1}{\rho}t_\rho\int(f(w_1+\rho v) - fw_1). \end{aligned}$$

Since $t_\rho \to 1$ as $\rho \to 0+$, letting $\rho \to 0+$, we obtain

$$\int\Big(\nabla w_1 \nabla v - \frac{\lambda}{|x|^2}w_1 v - \mu|x|^{\alpha-2}w_1 v - |w_1|^{p-2}w_1 v - fv\Big) \geq 0.$$

As v is arbitrarily, we get that

$$\int\Big(\nabla w_1 \nabla v - \frac{\lambda}{|x|^2}w_1 v - \mu|x|^{\alpha-2}w_1 v - |w_1|^{p-2}w_1 v - fv\Big) = 0.$$

Which means that w_1 is a solution of Eq.(1). □

3.2 Existence of sign changing solution

This subsection is devoted to proving the existence of sign changing solution of Eq.(1). For $u \in H_0^1(\Omega)$, denote $u^+ = \max\{0, u\}$ and $u^- = \max\{0, -u\}$, then $u^+,\ u^- \in H_0^1(\Omega)$ and $u = u^+ - u^-$. Following Tarantello [15], we define

$$\mathscr{M}_1^- \doteq \{u \in \mathscr{M};\quad u^+ \in \mathscr{M}^-\} \quad \text{and} \quad \mathscr{M}_2^- \doteq \{u \in \mathscr{M};\quad -u^- \in \mathscr{M}^-\}.$$

Set

$$\mathscr{M}_*^- \doteq \mathscr{M}_1^- \cap \mathscr{M}_2^-.$$

Lemma 3.8 *Let $0 \leq \lambda < \Lambda$ and $0 < \alpha < \sqrt{\Lambda - \lambda}$ and (A) holds. Then $\mathscr{M}_*^- \neq \emptyset$.*

Proof. From the definition of $\mathscr{M}_*^-$, we only need to prove that there exist $s > 0$ and $t \in \mathbb{R}$ such that

$$s(w_1 - tU_\varepsilon)^+ \in \mathscr{M}^- \quad \text{and} \quad -s(w_1 - tU_\varepsilon)^- \in \mathscr{M}^-.$$

To this purpose, let

$$t_2 \doteq \max_{\bar{\Omega}\setminus\{0\}} \frac{w_1}{U_\varepsilon} \quad \text{and} \quad t_1 \doteq \min_{\bar{\Omega}\setminus\{0\}} \frac{w_1}{U_\varepsilon}.$$

For $t \in (t_1, t_2)$, $(w_1 - tU_\varepsilon)^+$ and $-(w_1 - tU_\varepsilon)^-$, denoted by $s^+(t)$ and $s^-(t)$ the positive values given by Lemma 3.2, according to which we have

$$s^+(t)(w_1 - tU_\varepsilon)^+ \in \mathscr{M}^- \quad \text{and} \quad -s^-(t)(w_1 - tU_\varepsilon)^- \in \mathscr{M}^-.$$

Note that $s^+(t)$ is continuous with respect to t satisfying

$$\lim_{t\to t_1+0} s^+(t) = t^+(w_1 - t_1 U_\varepsilon) < +\infty \quad \text{and} \quad \lim_{t\to t_2-0} s^+(t) = +\infty.$$

Similarly, $s^-(t)$ is continuous with respect to t and

$$\lim_{t\to t_1+0} s^-(t) = +\infty \quad \text{and} \quad \lim_{t\to t_2-0} s^-(t) = t^+(t_2 U_\varepsilon - w_1) < +\infty.$$

The continuity of $s^+(t)$ and $s^-(t)$ implies that there is $t_0 \in (t_1, t_2)$ such that

$$s^+(t_0) = s^-(t_0) = s_0 > 0.$$

This proves the Lemma. □

Remark 3.9 The exact local behavior for the solution w_1 of Eq.(1) stated in Proposition 2.2 is essential in the definition of t_2 and t_1. Indeed because of Proposition 2.2, both t_1 and t_2 are finite. It seems to be very difficult to prove this Lemma without Proposition 2.2.

Lemma 3.10 *If (A) holds, then* $\mathscr{M}_1^-$, $\mathscr{M}_2^- \subset \mathscr{M}^-$.

Proof. Let $u \in \mathscr{M}_1^-$, i.e. $u \in \mathscr{M}$ and $u^+ \in \mathscr{M}^-$. Then we obtain from $T(u) - \int |u|^p = \int fu$ that

$$\begin{aligned} J(u) &= T(u) - (p-1)\int |u|^p = T(u) - (p-1)\Big(T(u) - \int fu\Big) \\ &= -(p-2)T(u) + (p-1)\int fu \\ &\le -(p-2)T(u) + (p-1)|f|_\sharp |u|_p. \end{aligned}$$

By the definition of M, we get that $|u|_p \le M^{-1}T(u)^{\frac{1}{2}}$. Therefore

$$J(u) \le -T(u)^{\frac{1}{2}}\Big((p-2)T(u^+)^{\frac{1}{2}} - (p-1)|f|_\sharp M^{-1}\Big) < 0.$$

Now according to $u^+ \in \mathscr{M}^-$ and (2) of Lemma 3.1, $T(u^+) > \left(\frac{M^p}{p-1}\right)^{\frac{2}{p-2}}$. Combining this with assumption (A), $(p-2)T(u^+)^{\frac{1}{2}} - (p-1)|f|_\sharp M^{-1} > 0$. Thus $J(u) < 0$ and hence $u \in \mathscr{M}^-$. This proves that $\mathscr{M}_1^- \subset \mathscr{M}^-$. By a similar argument we can prove that $\mathscr{M}_2^- \subset \mathscr{M}^-$. □

Define

$$c_2 \doteq \inf_{u \in \mathscr{M}_*^-} I(u).$$

Lemma 3.11 *Let $0 \le \lambda < \Lambda$ and $0 < \alpha < \sqrt{\Lambda - \lambda}$ and (A) holds. Then $c_2 < c_1 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof. It suffices to estimate $I(sw_1 - tU_\varepsilon)$ for $s \ge 0$ and $t \in \mathbb{R}$. Since at this time, ε can be sufficiently small, we replace U_ε by $v_\varepsilon = \phi(x)U_\varepsilon$ defined in Section 2. From the structure of I, we find there is $R > 0$ possibly large such that $I(sw_1 - tv_\varepsilon) \le c_1$ for all $s^2 + t^2 \ge R^2$. Thus it suffices to estimate $I(sw_1 - tv_\varepsilon)$ for all $s^2 + t^2 \le R^2$. From an elementary inequality

$$|c + d|^q \ge |c|^q + |d|^q - K(|c|^{q-1}|d| + |c||d|^{q-1}), \quad \forall\ c,\ d \in \mathbb{R}, \quad q > 1$$

and w_1 is a solution of Eq.(1), we obtain that

$$\begin{aligned} I(sw_1 - tv_\varepsilon) &\le I(sw_1) + I(tv_\varepsilon) - st\int(|w_1|^{p-1}v_\varepsilon + fv_\varepsilon) \\ &\quad + K_4\Big(\int |sw_1|^{p-1}|tv_\varepsilon| + \int |sw_1||tv_\varepsilon|^{p-1}\Big) \\ &\le I(sw_1) + I(tv_\varepsilon) + K_5\int\Big(|w_1|^{p-1}|v_\varepsilon| + |w_1||v_\varepsilon|^{p-1} + fv_\varepsilon\Big). \end{aligned}$$

Since $w_1 \in \mathscr{M}$, we have that $I(sw_1) \le I(w_1)$ for all $s \ge 0$. Note that

$$\sup_{t \in \mathbb{R}} I(tv_\varepsilon) \le \frac{1}{N} S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + O(\varepsilon^{\frac{N-2}{4}}).$$

We obtain from Proposition 2.4 that

$$\begin{aligned} &\max_{s>0, t\in\mathbb{R}} I(sw_1 - tv_\varepsilon) \\ &\le \max_{s>0} I(sw_1) + \max_{t\in\mathbb{R}} I(tu_\varepsilon) + K_7\varepsilon^{(N-2)/4} + K_8\varepsilon^{(N-2)/4} \\ &\le I(w_1) + \frac{1}{N} S_\mu^{\frac{N}{2}} - O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}) + K_9\varepsilon^{(N-2)/4} \\ &< c_1 + \frac{1}{N} S_\mu^{\frac{N}{2}}, \end{aligned}$$

since $0 < \alpha < \sqrt{\Lambda - \lambda}$. □

Remark 3.12 Similar to those pointed out in Remark 3.6, we emphasize that the exact local behavior of the solution of Eq.(1) (see Propositions 2.2 and 2.4) played an essential role in estimate of c_2.

Theorem 3.13 *Let $0 \le \lambda < \Lambda$ and $0 < \alpha < \sqrt{\Lambda - \lambda}$ and (A) holds. Then there is a $w_2 \in \mathscr{M}_*^-$ such that $I(w_2) = c_2$ and w_2 is a sign changing solution of Eq.(1).*

Proof. In the first step, we will prove that there is $w_2 \in \mathscr{M}_*^-$ such that $I(w_2) = c_2$. Let $\{u_n\} \subset \mathscr{M}_*^-$ be such that $I(u_n) \to c_2$. Using the fact that $\{u_n^+\} \subset \mathscr{M}^-$ and Sobolev inequality, one can easily show that

$$0 < \inf \|u_n^+\| \le \sup \|u_n^+\| < +\infty.$$

Similarly we have that $\{u_n^-\}$ is bounded with respect to n. Going if necessary to a subsequence, we may assume that $u_n^+ \rightharpoonup u^+$ and $u_n^- \rightharpoonup u^-$ in $H_0^1(\Omega)$ and that $I(u_n^+) \to d_1$, $I(u_n^-) \to d_2$ and $c_2 = d_1 + d_2$. We claim that $u^+ \ne 0$ and $u^- \ne 0$. If $u^+ = 0$ and $u^- = 0$, then by Proposition 3.3, $d_1 \ge \frac{1}{N} S_\lambda^{\frac{N}{2}}$, $d_2 \ge \frac{1}{N} S_\lambda^{\frac{N}{2}}$ and hence $c_2 \ge \frac{2}{N} S_\lambda^{\frac{N}{2}}$. If $u^+ = 0$ and $u^- \ne 0$, then by Proposition 3.3, $d_1 \ge \frac{1}{N} S_\lambda^{\frac{N}{2}}$, $d_2 \ge c_1$ or $d_2 \ge c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, which implies that $c_2 \ge c_1 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$ or $c_2 \ge c_0 + \frac{2}{N} S_\lambda^{\frac{N}{2}}$. If $u^+ \ne 0$ and $u^- = 0$, then by Proposition 3.3, one gets $c_2 \ge c_1 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$ or $c_2 \ge c_0 + \frac{2}{N} S_\lambda^{\frac{N}{2}}$. All the above three cases contradict Lemma 3.5 and Lemma 3.11. Therefore $u^+ \ne 0$ and $u^- \ne 0$. According to (1) and (2) of Proposition 3.3, we have one of the following:

(i) $\{u_n^+\}$ converges strongly to u^+;
(ii) $d_1 > I(t^+(u^+)u^+)$;
(iii) $d_1 > I(t^-(u^+)u^+) + \frac{1}{N} S_\lambda^{\frac{N}{2}}$;

and we also have one of the following:

(iv) $\{u_n^-\}$ converges strongly to u^-;
(v) $d_2 > I(-t^+(-u^-)u^-)$;
(vi) $d_2 > I(-t^-(-u^-)u^-) + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.

We will prove that only cases (i) and (iv) hold. For example, in the case (ii) and (v), we have that $t^+(u^+)u^+ - t^+(-u^-)u^- \in \mathscr{M}_*^-$ and hence

$$c_2 \le I(t^+(u^+)u^+ - t^+(-u^-)u^-) = I(t^+(u^+)u^+) + I(-t^+(-u^-)u^-) < d_1 + d_2 = c_2,$$

which is a contradiction. In the case (iii) and (vi), we have that $t^-(u^+)u^+ - t^-(-u^-)u^- \in \mathscr{M}^+$ and hence

$$\begin{aligned} c_1 + \frac{1}{N} S_\lambda^{\frac{N}{2}} &< c_0 + \frac{2}{N} S_\lambda^{\frac{N}{2}} \le I(t^-(u^+)u^+ - t^-(-u^-)u^-) + \frac{2}{N} S_\lambda^{\frac{N}{2}} \\ &= I(t^-(u^+)u^+) + I(-t^-(u^-)u^-) + \frac{2}{N} S_\lambda^{\frac{N}{2}} \\ &\le d_1 + d_2 = c_2, \end{aligned}$$

which contradicts to Lemma 3.11. In the cases (ii) and (vi), we have that $t^+(u^+)u^+ - t^-(-u^-)u^- \in \mathscr{M}^-$ and hence

$$c_1 + \frac{1}{N} S_\lambda^{\frac{N}{2}} \le I(t^+(u^+)u^+ + t^-(u^-)u^-) + \frac{1}{N} S_\lambda^{\frac{N}{2}} < d_1 + d_2 = c_2,$$

which again contradicts to Lemma 3.11. In the case (i) and (v), we have $u^+ - t^+(-u^-)u^- \in \mathscr{M}_*^-$ and hence

$$c_2 \le I(u^+ - t^+(-u^-)u^-) < d_1 + d_2 = c_2,$$

which is a contradiction. Therefore we prove that only cases (i) and (iv) hold. Hence both $\{u_n^+\}$ and $\{u_n^-\}$ converge strongly to u^+ and u^-, respectively and we get that $u^+,\ u^- \in \mathscr{M}^-$. Denote $w_2 = u^+ - u^-$. We get that $I(w_2) = c_2$.

Next we show that w_2 is a critical point of I. Suppose that w_2 is not a critical point of I, i.e. $\nabla I(w_2) \ne 0$. Note that for $u \in \mathscr{M}^-$, we have that

$$\langle \nabla Q(u), u \rangle = J(u) < 0.$$

Hence we can define

$$V(u) = \nabla I(u) - \left\langle \nabla I(u), \frac{\nabla Q(u)}{\|\nabla Q(u)\|} \right\rangle \frac{\nabla Q(u)}{\|\nabla Q(u)\|}, \qquad u \in \mathscr{M}^-.$$

Choosing $\delta \in (0, \min\{\|u^+\|, \|u^-\|\}/3)$ such that $\|V(v) - V(w_2)\| \le \frac{1}{2}\|V(w_2)\|$ for each $v \in \mathscr{M}^-$ with $\|v - w_2\| \le 2\delta$. Let $\psi :\ \mathscr{M}^- \to [0,1]$ be a Lipschitz mapping such that

$$\psi(v) = \begin{cases} 1 & \text{for } v \in \mathscr{M}^- \quad \text{with} \quad \|v - w_2\| \le \delta, \\ 0 & \text{for } v \in \mathscr{M}^- \quad \text{with} \quad \|v - w_2\| \ge 2\delta. \end{cases}$$

Let $\eta :\ [0, s_0] \times \mathscr{M}^-$ be the solution of the differential equation

$$\eta(0,v) = v, \quad \frac{d}{ds}\eta(s,v) = -\psi(\eta(s,v))V(\eta(s,v)) \quad \text{for} \quad (s,v) \in [0,s_0] \times \mathscr{M}^-,$$

where s_0 is some positive number. We set

$$\chi(t) = t^+((1-t)u^+ - tu^-) \cdot ((1-t)u^+ - tu^-) \quad \text{and} \quad \xi(t) = \eta(s_0, \chi(t)) \text{ for } 0 \le t \le 1.$$

Keep the definition of $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ in mind. We have that if $t \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, then

$$I(\xi(t)) \leq I(\chi(t)) = I(\chi(t)^+) + I(\chi(t)^-) < I(u^+) + I(u^-) = I(w_2)$$

and $I(\xi(\frac{1}{2})) < I(\chi(\frac{1}{2})) = I(w_2)$. Therefore $I(\xi(t)) < I(w_2)$ for $t \in (0,1)$. Since $t^+(\xi(t)^+) - t^+(-\xi(t)^-) \to -\infty$ as $t \to 0+$ and $t^+(\xi(t)^+) - t^+(-\xi(t)^-) \to +\infty$ as $t \to 1-0$, we get a $t_1 \in (0,1)$ such that $t^+(\xi(t_1)^+) = t^+(-\xi(t_1)^-)$. So $\xi(t_1) = \xi(t_1)^+ - \xi(t_1)^- \in \mathscr{M}_*^-$ and $I(\xi(t_1)) < I(w_2)$, which is a contradiction. Hence we obtain $\nabla I(w_2) = 0$. □

3.3 A fourth solution

Up to now, we got three solutions w_0, w_1 and w_2. Next we will prove that there is another solution by a translated argument. We define a C^1 functional $\bar{I}: H_0^1(\Omega) \to \mathbb{R}$ by

$$\bar{I}(v) \doteq \frac{1}{2}T(v) - \frac{1}{p}\int \left(|v^+ + w_0|^p - |w_0|^p - p|w_0|^{p-2}w_0 v^+\right)$$

for $v \in H_0^1(\Omega)$. Consider the following minimax value

$$\bar{c} \doteq \inf_{\gamma \in \Gamma} \sup_{0 \leq t \leq 1} \bar{I}(\gamma(t)),$$

where

$$\Gamma \doteq \{\gamma \in C([0,1],\ H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = kv_\varepsilon\}$$

with suitable ε and k.

Lemma 3.14 *We have* $\bar{c} < \frac{1}{N}S_\lambda^{\frac{N}{2}}$.

Proof. The proof is almost the same as Chen [5, Lemma 5.2]. □

Lemma 3.15 $\bar{I}$ *satisfies the* $(PS)_c$ *condition for* $\bar{c} < \frac{1}{N}S_\lambda^{\frac{N}{2}}$.

Proposition 3.16 *There exists a critical point* $\bar{w}_1 \in H_0^1(\Omega)$ *of* I *such that* $\bar{w}_1 > w_0$ *in* Ω. *Moreover,* $w_2 \neq \bar{w}_1$.

Proof. Similar to those in the proof of Chen [5, Lemma 5.1], we know that 0 is a local minimizer of $\bar{I}$. By Lemmas 3.14 and 3.15 and a standard mountain pass theorem (see Rabinowitz [12], Struwe [14], Willem [18]), we obtain that there is a critical point $v \neq 0$ of $\bar{I}$. By standard argument and maximum principle we have that $v > 0$ in Ω. Set $\bar{w}_1 = v + w_0$. Then $\bar{w}_1$ is is a critical point of I and $\bar{w}_1 > w_0$ in Ω.

Next we show that $w_2 \neq \bar{w}_1$. Suppose that $w_2 = \bar{w}_1$. Then we have that $0 \geq -w_2^- \geq -w_0^-$. Since $-w_2^- \in \mathscr{M}^-$ and $w_0 \in \mathscr{M}^+$, we get that

$$T(-w_2^-) < (p-1)\int |-w_2^-|^p \leq (p-1)\int |-w_0^-|^p \leq (p-1)\int |w_0|^p < T(w_0).$$

On the other hand, using Lemma 3.1 and assumption (A),

$$T(w_0) < \left(\frac{p-1}{p-2}M^{-1}|f|_\sharp\right)^2 < \left(\frac{1}{2}\left(\frac{M^p}{p-1}\right)^{\frac{1}{p-2}}\right)^2 < T(-w_2^-),$$

which is a contradiction. Thus, we have proved $w_2 \neq \bar{w}_1$. □

Proposition 3.17 *There exists a critical point $\widehat{w}_1 \in H_0^1(\Omega)$ of I such that $\widehat{w}_1 < w_0$ in Ω. Moreover, $w_2 \neq \widehat{w}_1$.*

Proof. For $v \in H_0^1(\Omega)$, we define the following functional

$$\widehat{I}(v) \doteq \frac{1}{2}T(v) - \frac{1}{p}\int \left(|w_0 - v^-|^p - |w_0|^p + p|w_0|^{p-2}w_0 v^-\right).$$

Now using the same procedure as in getting the solution $\bar{w}_1$, we can easily get the existence of a critical point $\widehat{w}_1 \in H_0^1(\Omega)$ of I and $\widehat{w}_1$ satisfies all the requirement of Proposition 3.17. □

Proof of Theorem 1.2. From the previous three subsection, we have got five weak solutions of Eq.(1) w_0, w_1, w_2, $\bar{w}_1$ and $\widehat{w}_1$. Since w_1 may equal to $\bar{w}_1$ or $\widehat{w}_1$, we have obtained at least four solutions w_0, w_2, $\bar{w}_1$ and $\widehat{w}_1$ of Eq.(1), and we know that w_2 is sign changing. □

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