Sobolev Type Fractional Dynamic Equations and Optimal Multi-Integral Controls with Fractional Nonlocal Conditions*

Amar Debbouche[a]
amar_debbouche@yahoo.fr

Delfim F. M. Torres[b]
delfim@ua.pt

[a]Department of Mathematics, Guelma University, 24000 Guelma, Algeria

[b]Center for Research and Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

Abstract

We prove existence and uniqueness of mild solutions to Sobolev type fractional nonlocal dynamic equations in Banach spaces. The Sobolev nonlocal condition is considered in terms of a Riemann–Liouville fractional derivative. A Lagrange optimal control problem is considered, and existence of a multi-integral solution obtained. Main tools include fractional calculus, semigroup theory, fractional power of operators, a singular version of Gronwall's inequality, and Leray–Schauder fixed point theorem. An example illustrating the theory is given.

2010 Mathematics Subject Classification: 26A33, 49J15.

Keywords: Sobolev type equations, fractional evolution equations, optimal control, nonlocal conditions, mild solutions.

1 Introduction

Fractional differential equations have attracted the attention of scientists, in reason to their accurate, helpful, and successful results in fields such as mathematical modelling of physical, engineering, and biological phenomena. Both theoretical and practical aspects of the subject are being explored. In particular, fractional differential equations provide an excellent tool to describe hereditary properties of various materials and processes, finding numerous applications in viscoelasticity, electrochemistry, porous media, and electromagnetism. The reader interested in the development of the theory, methods, and applications of fractional calculus is referred to the books [1–9] and to the papers [10–17]. For recent developments in the area of nonlocal fractional differential equations and inclusions see [18–24] and references therein.

The study of fractional control systems and fractional optimal control problems is under intense investigation [25–27]. Those control systems are most often based on the principle of feedback, whereby the signal to be controlled is compared to a desired reference signal and the discrepancy used to compute corrective control actions [28]. The fractional optimal control of a distributed system is an optimal control problem for which the system dynamics is defined by means of fractional differential equations [29]. In our previous work [22], we introduced multi-delay controls and we investigated a nonlocal condition for fractional semilinear control systems. The existence of optimal pairs for systems governed by fractional evolution equations with initial and nonlocal conditions, is also presented by Wang et al. [24] and Wang and Zhou [30]. Here we are concerned with the study of fractional nonlinear evolution equations subject to fractional Sobolev

*This is a preprint of a paper whose final and definite form will appear in *Fract. Calc. Appl. Anal.* Paper submitted 10/April/2014; accepted for publication 21/Sept/2014.

nonlocal conditions. Sobolev type semilinear equations serve as an abstract formulation of partial differential equations, which arise in various applications, such as in the flow of fluid through fissured rocks, thermodynamics, and shear in second order fluids. Moreover, fractional differential equations of Sobolev type appear in the theory of control of dynamical systems, when the controlled system and/or the controller is described by a fractional differential equation of Sobolev type [31]. The mathematical modeling and simulations of such systems and processes are based on the description of their properties in terms of fractional differential equations of Sobolev type. These new models are claimed to be more adequate than previously used integer order models, so fractional order differential equations of Sobolev type have been investigated by many researchers, e.g., in [32–35]. Motivated by these facts, we introduce here a new nonlocal fractional condition of Sobolev type and we present the optimal control of multiply integrated Sobolev type nonlinear fractional evolution equations. The problem requires to formulate a new solution operator and its properties, such as boundedness and compactness. Further, we present a class of admissible multi-integral controls and we prove, under an appropriate set of sufficient conditions, an existence result of optimal multi-integral controls for a Lagrange optimal control problem, denoted in the sequel by (LP). More precisely, we are concerned with the study of fractional nonlinear evolution equations

$$ {}^{C}D_t^{\alpha}[Lu(t)] = Eu(t) + f(t, W(t)) \tag{1} $$

subject to fractional Sobolev nonlocal conditions

$$ {}^{L}D_t^{1-\alpha}[Mu(0)] = u_0 + h(u(t)), \tag{2} $$

where ${}^{C}D_t^{\alpha}$ and ${}^{L}D_t^{1-\alpha}$ are, respectively, Caputo and Riemann–Liouville fractional derivatives with $0 < \alpha \leq 1$ and $t \in J = [0, a]$. Let X and Y be two Banach spaces such that Y is densely and continuously embedded in X, the unknown function $u(\cdot)$ takes its values in X and $u_0 \in X$. We consider the operators $L : D(L) \subset X \to Y$, $E : D(E) \subset X \to Y$ and $M : D(M) \subset X \to X$, $W(t) = (B_1(t)u(t), \ldots, B_r(t)u(t))$, such that $\{B_i(t) : i = 1, \ldots, r,\ t \in J\}$ is a family of linear closed operators defined on dense sets $S_1, \ldots, S_r$ in X with values in Y. It is also assumed that $f : J \times X^r \to Y$ and $h : C(J : X) \to X$ are given abstract functions satisfying some conditions to be specified later. In Section 2 we present some essential notions and facts that will be used in the proof of our results, such as, fractional operators, fractional powers of the generator of an analytic compact semigroup, and the form of mild solutions of (1)–(2). In Section 3, we prove existence (Theorem 1) and uniqueness (Theorem 2) of mild solutions to system (1)–(2). Then, in Section 4, we prove existence of optimal pairs for the (LP) Lagrange optimal control problem (Theorem 3). We end with Section 5, where an example illustrating the application of the abstract results (Theorems 1, 2 and 3) is given.

2 Preliminaries

In this section we introduce some basic definitions, notations and lemmas, which will be used throughout the work. In particular, we give main properties of fractional calculus [3, 4] and well known facts in semigroup theory [36–38].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a, b], \mathbb{R}^+)$ is given by

$$ I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, $$

where Γ is the classical gamma function. If $a = 0$, we can write $I^{\alpha} f(t) = (g_{\alpha} * f)(t)$, where

$$ g_{\alpha}(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases} $$

and, as usual, $*$ denotes convolution of functions. Moreover, $\lim_{\alpha \to 0} g_{\alpha}(t) = \delta(t)$, with δ the delta Dirac function.

Definition 2. The Riemann–Liouville fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is given by

$$ {}^{L}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0, $$

where function f has absolutely continuous derivatives up to order $n-1$.

Definition 3. The Caputo fractional derivative of order $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, is given by

$$ {}^{C}D^{\alpha} f(t) = {}^{L}D^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, $$

where function f has absolutely continuous derivatives up to order $n-1$.

Remark 1. Let $n-1 < \alpha < n$, $n \in \mathbb{N}$. The following properties hold:

(i) If $f \in C^n([0,\infty))$, then

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0. $$

(ii) The Caputo derivative of a constant function is equal to zero.

(iii) The Riemann–Liouville derivative of a constant function C is given by

$$ {}^{L}D^{\alpha}_{a^+} C = \frac{C}{\Gamma(1-\alpha)} (x-a)^{-\alpha}. $$

If f is an abstract function with values in X, then the integrals which appear in Definitions 1–3 are taken in Bochner's sense.

Let $(X, \|\cdot\|)$ be a Banach space, $C(J,X)$ denotes the Banach space of continuous functions from J into X with the norm $\|u\|_J = \sup\{\|u(t)\| : t \in J\}$, and let $\mathcal{L}(X)$ be the Banach space of bounded linear operators from X to X with the norm $\|G\|_{\mathcal{L}(X)} = \sup\{\|G(u)\| : \|u\| = 1\}$. We make the following assumptions:

(H_1) $E : D(E) \subset X \to Y$ is linear, closed, and $L : D(L) \subset X \to Y$ and $M : D(M) \subset X \to X$ are linear operators.

(H_2) $D(M) \subset D(L) \subset D(E)$ and L and M are bijective.

(H_3) $L^{-1} : Y \to D(L) \subset X$ and $M^{-1} : X \to D(M) \subset X$ are linear, bounded, and compact operators.

Note that (H_3) implies L to be closed. Indeed, if L^{-1} is closed and injective, then its inverse is also closed. From (H_1)–(H_3) and the closed graph theorem, we obtain the boundedness of the linear operator $EL^{-1} : Y \to Y$. Consequently, EL^{-1} generates a semigroup $\{Q(t), t \geq 0\}$, $Q(t) := e^{EL^{-1}t}$. We suppose that $M_0 := \sup_{t \geq 0} \|Q(t)\| < \infty$ and, for short, we denote $C_1 = \|L^{-1}\|$ and $C_2 = \|M^{-1}\|$.

According to previous definitions, it is suitable to rewrite problem (1)–(2) as the equivalent integral equation

$$ Lu(t) = Lu(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Eu(s) + f(s, W(s))] ds, \tag{3} $$

provided the integral in (3) exists for a.a. $t \in J$.

Remark 2. Note that:

(i) For the nonlocal condition, the function $u(0)$ is dependent on t.

(ii) ${}^{L}D_t^{1-\alpha}[Mu(0)]$ is well defined, i.e., if $\alpha = 1$ and M is the identity, then (2) reduces to the usual nonlocal condition.

(iii) Function $u(0)$ takes the form

$$M^{-1}v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{M^{-1}[u_0 + h(u(s))]}{(t-s)^\alpha} ds,$$

where $Mu(0)|_{t=0} = v_0$.

(iv) The explicit and implicit integrals given in (3) exist (taken in Bochner's sense).

Throughout the paper, $A = EL^{-1} : D(A) \subset Y \to Y$ will be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $Q(\cdot)$. Then, there exists a constant $M_0 \geq 1$ such that $\|Q(t)\| \leq M_0$ for $t \geq 0$. Without loss of generality, we assume that $0 \in \rho(A)$, the resolvent set of A. Then it is possible to define the fractional power A^q, $0 < q \leq 1$, as a closed linear operator on its domain $D(A^q)$ with inverse A^{-q}. Furthermore, the subspace $D(A^q)$ is dense in X and the expression $\|u\|_q = \|A^q u\|, u \in D(A^q)$ defines a norm on $D(A^q)$. Hereafter, we denote by X_q the Banach space $D(A^q)$ normed with $\|u\|_q$.

Lemma 1 (See [37])**.** *Let A be the infinitesimal generator of an analytic semigroup $Q(t)$. If $0 \in \rho(A)$, then*

(a) $Q(t) : X \to D(A^q)$ for every $t > 0$ and $q \geq 0$.

(b) For every $u \in D(A^q)$, we have $Q(t)A^q u = A^q Q(t)u$.

(c) For every $t > 0$, the operator $A^q Q(t)$ is bounded and $\|A^q Q(t)\| \leq M_q t^{-q} e^{-\omega t}$.

(d) If $0 < q \leq 1$ and $u \in D(A^q)$, then $\|Q(t)u - u\| \leq C_q t^q \|A^q u\|$.

Remark 3. Note that:

(i) $D(A^q)$ is a Banach space with the norm $\|u\|_q = \|A^q u\|$ for $u \in D(A^q)$.

(ii) If $0 < p \leq q \leq 1$, then $D(A^q) \hookrightarrow D(A^p)$.

(iii) A^{-q} is a bounded linear operator in X with $D(A^q) = Im(A^{-q})$.

Remark 4. Observe, as in [39], that by Lemma 1 (a) and (b), the restriction $Q_q(t)$ of $Q(t)$ to X_q is exactly the part of $Q(t)$ in X_q. Let $u \in X_q$. Since $\|Q(t)u\|_q \leq \|A^q Q(t)u\| = \|Q(t)A^q u\| \leq \|Q(t)\|\|A^q u\| = \|Q(t)\|\|u\|_q$, and as t decreases to 0^+, $\|Q(t)u - u\|_q = \|A^q Q(t)u - A^q u\| = \|Q(t)A^q u - A^q u\| \to 0$ for all $u \in X_q$, it follows that $\{Q(t), t \geq 0\}$ is a family of strongly continuous semigroups on X_q and $\|Q_q(t)\| \leq \|Q(t)\| \leq M_0$ for all $t \geq 0$.

In the sequel, we will also use $\|\phi\|_{L^p(J,\mathbb{R}^+)}$ to denote the $L^p(J, \mathbb{R}^+)$ norm of ϕ whenever $\phi \in L^p(J, \mathbb{R}^+)$ for some p with $1 < p < \infty$. We will set $q \in (0, 1)$ and denote by Ω_q the Banach space $C(J, X_q)$ endowed with supnorm given by $\|u\|_\infty = \sup_{t \in J} \|u\|_q$ for $u \in \Omega_q$.

Motivated by [22, 32, 40], we give the definition of mild solution to (1)–(2).

Definition 4. A function $u \in \Omega_q$ is called a mild solution of system (1)–(2) if it satisfies the following integral equation:

$$u(t) = S_\alpha(t)LM^{-1}\left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u_0 + h(u(s))]}{(t-s)^\alpha} ds\right] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, W(s)) ds,$$

where

$$S_\alpha(t) = \int_0^\infty L^{-1}\zeta_\alpha(\theta) Q(t^\alpha \theta) d\theta, \quad T_\alpha(t) = \alpha \int_0^\infty L^{-1}\theta\zeta_\alpha(\theta) Q(t^\alpha\theta) d\theta,$$

$$\zeta_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0, \quad \varpi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \theta \in (0, \infty),$$

with ζ_α the probability density function defined on $(0, \infty)$, that is, $\zeta_\alpha(\theta) \geq 0, \theta \in (0, \infty)$ and $\int_0^\infty \zeta_\alpha(\theta) d\theta = 1$.

Remark 5. For $v \in [0,1]$, ones has

$$\int_0^\infty \theta^v \zeta_\alpha(\theta) d\theta = \int_0^\infty \theta^{-\alpha v} \varpi_\alpha(\theta) d\theta = \frac{\Gamma(1+v)}{\Gamma(1+\alpha v)}$$

(see [41]).

Lemma 2 (See [32, 40, 41])**.** *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties:*

(a) For any fixed $t \geq 0$, the operators $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded, i.e., for any $u \in X$, $\|S_\alpha(t)u\| \leq C_1 M_0 \|u\|$ and $\|T_\alpha(t)u\| \leq \frac{C_1 M_0}{\Gamma(\alpha)} \|u\|$.

(b) $\{S_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t),\, t \geq 0\}$ are strongly continuous, i.e., for $u \in X$ and $0 \leq t_1 < t_2 \leq a$, we have $\|S_\alpha(t_2)u - S_\alpha(t_1)u\| \to 0$ and $\|T_\alpha(t_2)u - T_\alpha(t_1)u\| \to 0$ as $t_1 \to t_2$.

(c) For every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are compact operators.

(d) For any $u \in X$, $p \in (0,1)$ and $q \in (0,1)$, we have $AT_\alpha(t)u = A^{1-p} T_\alpha(t) A^p u$, $t \in J$, and $\|A^q T_\alpha(t)\| \leq \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} t^{-q\alpha}$, $0 < t \leq a$.

(e) For fixed $t \geq 0$ and any $u \in X_q$, we have $\|S_\alpha(t)u\|_q \leq C_1 M_0 \|u\|_q$ and $\|T_\alpha(t)u\|_q \leq \frac{C_1 M_0}{\Gamma(\alpha)} \|u\|_q$.

(f) $S_\alpha(t)$ and $T_\alpha(t)$, $t > 0$, are uniformly continuous, that is, for each fixed $t > 0$ and $\epsilon > 0$ there exists $g > 0$ such that $\|S_\alpha(t+\epsilon) - S_\alpha(t)\|_q < \epsilon$ for $t+\epsilon \geq 0$ and $|\epsilon| < g$, $\|T_\alpha(t+\epsilon) - T_\alpha(t)\|_q < \epsilon$ for $t + \epsilon \geq 0$ and $|\epsilon| < g$.

Lemma 3 (See [42])**.** *For each $\psi \in L^p(J, X)$ with $1 \leq p < \infty$,*

$$\lim_{g \to 0} \int_0^a \|\psi(t+g) - \psi(t)\|^p dt = 0,$$

where $\psi(s) = 0$ for $s \notin J$.

Lemma 4 (See [41])**.** *A measurable function $G : J \to X$ is a Bochner integral if $\|G\|$ is Lebesgue integrable.*

3 Main results

Our first result provides existence of mild solutions to system (1)–(2). To prove that, we make use of the following assumptions:

(F_1) The linear closed operators $\{B_i(t)\}_{i=\overline{1,r}}$ are defined on dense sets $S_1, \ldots, S_r \supset D(A)$, respectively from X_q into Y.

(F_2) The function $f : J \times X_q^r \to Y$ satisfies: for each $W \in X_q^r$, in particular, for every element $u \in \cap_i S_i$, $i = 1, \ldots, r$, the function $t \to f(t, W(t))$ is measurable.

(F_3) For arbitrary $u, u^* \in X_q$ satisfying $\|u\|_q, \|u^*\|_q \leq \rho$, there exists a constant $L_f(\rho) > 0$ and functions $m_i \in L^1(J, \mathbb{R}^+)$ such that

$$\|f(t, W) - f(t, W^*)\| \leq L_f(\rho)[m_1(t) + \cdots + m_r(t)] \|u - u^*\|_q$$

for almost all $t \in J$. Here, $W^*(t) = (B_1(t)u^*(t), \ldots, B_r(t)u^*(t))$, $i = 1, \ldots, r$.

(F_4) There exists a constant $a_f > 0$ such that

$$\|f(t, W)\| \leq a_f(1 + r\|u\|_q) \;\text{ for all }\; W \in X_q^r \;\text{ and }\; t \in J.$$

(F_5) The function $h : C(J : X_q) \to X_q$ is Lipschitz continuous and bounded in X_q, i.e., for all $u, v \in C(J, X_q)$ there exist constants $k_1, k_2 > 0$ such that

$$\|h(u) - h(v)\|_q \leq k_1 \|u - v\|_q \quad \text{and} \quad \|h(u)\|_q \leq k_2.$$

Theorem 1. *Assume hypotheses (F_1)–(F_5) are satisfied. If $u_0 \in X_q$ and $\alpha q < 1$ for some $\frac{1}{2} < \alpha < 1$, then system (1)–(2) has a mild solution on J.*

The following lemmas are used in the proof of Theorem 1.

Lemma 5. *Let operator $P : \Omega_q \to \Omega_q$ be given by*

$$(Pu)(t) = S_\alpha(t) L M^{-1} \left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u_0 + h(u(s))]}{(t-s)^\alpha} ds \right] + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, W(s)) ds. \quad (4)$$

Then, the operator P satisfies $Pu \in \Omega_q$.

Proof. Let $0 \leq t_1 < t_2 \leq a$ and $\alpha q < \frac{1}{2}$. We have

$$\begin{aligned}
&\|(Pu)(t_1) - (Pu)(t_2)\|_q \\
&= \left\| [S_\alpha(t_1) - S_\alpha(t_2)] L M^{-1} \left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} (t_1 - s)^{-\alpha} [u_0 + h(u(s))] ds \right] \right\|_q \\
&\quad + \left\| S_\alpha(t_2) L M^{-1} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^{t_1} [(t_1 - s)^{-\alpha} - (t_2 - s)^{-\alpha}] [u_0 + h(u(s))] ds \right] \right\|_q \\
&\quad + \left\| S_\alpha(t_2) L M^{-1} \left[\frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} [u_0 + h(u(s))] ds \right] \right\|_q \\
&\quad + \int_0^{t_1} (t_1 - s)^{\alpha-1} \|T_\alpha(t_1 - s) f(s, W(s)) - T_\alpha(t_2 - s) f(s, W(s))\|_q ds \\
&\quad + \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \|T_\alpha(t_2 - s) f(s, W(s))\|_q ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|T_\alpha(t_2 - s) f(s, W(s))\|_q ds.
\end{aligned}$$

We use Lemma 2, and fractional power of operators, to get

$$\begin{aligned}
\|(Pu)(t_1) - (Pu)(t_2)\|_q \leq & C_2 \|L\| \left[\|v_0\|_q + (k_2 + \|u_0\|_q) \frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} \right] \|S_\alpha(t_1) - S_\alpha(t_2)\|_q \\
& + C_1 C_2 M_0 \|L\| \left[(k_2 + \|u_0\|_q) \frac{(t_2 - t_1)^{1-\alpha} + t_1^{1-\alpha} - t_2^{1-\alpha}}{\Gamma(2-\alpha)} \right] \\
& + C_1 C_2 M_0 \|L\| \left[(k_2 + \|u_0\|_q) \frac{(t_2 - t_1)^{1-\alpha}}{\Gamma(2-\alpha)} \right] \\
& + \int_0^{t_1} (t_1 - s)^{\alpha-1} \|A^q [T_\alpha(t_1 - s) - T_\alpha(t_2 - s)]\| \|f(s, W(s))\| ds \\
& + \int_0^{t_1} |(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}| \|A^q T_\alpha(t_2 - s)\| \|f(s, W(s))\| ds \\
& + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|A^q T_\alpha(t_2 - s)\| \|f(s, W(s))\| ds
\end{aligned}$$

$$
\begin{aligned}
\leq & C_2\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
&+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}+t_1^{1-\alpha}-t_2^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)}\int_0^{t_1}(t_1-s)^{\alpha-1}|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|ds \\
&+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|(t_2-s)^{-q\alpha}\|f(s,W(s))\|ds \\
&+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_{t_1}^{t_2}(t_2-s)^{-q\alpha+\alpha-1}\|f(s,W(s))\|ds.
\end{aligned}
$$

From Lemma 2 and Hölder's inequality, one can deduce the following inequality:

$$
\begin{aligned}
&\|(Pu)(t_1)-(Pu)(t_2)\|_q \\
&\quad\leq C_2\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
&\qquad+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}+t_1^{1-\alpha}-t_2^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&\qquad+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&\qquad+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)}\left[\left(\int_0^{t_1}|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|^2ds\right)^{\frac{1}{2}}\right. \\
&\qquad\times\left(\int_0^{t_1}(t_1-s)^{2(\alpha-1)}ds\right)^{\frac{1}{2}}+\left(\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|^2ds\right)^{\frac{1}{2}} \\
&\qquad\left.\times\left(\int_0^{t_1}(t_2-s)^{-2q\alpha}ds\right)^{\frac{1}{2}}+\frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}\right] \\
&\quad\leq C_2\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\right]\|S_\alpha(t_1)-S_\alpha(t_2)\|_q \\
&\qquad+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}+t_1^{1-\alpha}-t_2^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&\qquad+C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}}{\Gamma(2-\alpha)}\right] \\
&\qquad+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)}\left[\sqrt{\frac{1}{2\alpha-1}}t_1^{\alpha-\frac{1}{2}}\left(\int_0^a|(t_1-s)^{-q\alpha}-(t_2-s)^{-q\alpha}|^2ds\right)^{\frac{1}{2}}\right. \\
&\qquad+\left(\int_0^a|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|^2ds\right)^{\frac{1}{2}}\sqrt{\frac{1}{1-2q\alpha}}\left(t_2^{1-2q\alpha}-(t_2-t_1)^{1-2q\alpha}\right)^{\frac{1}{2}} \\
&\qquad\left.+\frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}\right],
\end{aligned}
$$

which means that $Pu\in\Omega_q$. □

Lemma 6. *The operator P given by (4) is continuous on Ω_q.*

Proof. Let $u, u^* \in \Omega_q$ and $\|u - u^*\|_\infty \le 1$. Then, $\|u\|_\infty \le 1 + \|u^*\|_\infty = \rho$ and

$$\begin{aligned}
\|(Pu)(t) - (Pu^*)(t)\|_q &= \left\| S_\alpha(t) L M^{-1} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} [h(u) - h(u^*)] ds \right] \right\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s)[f(s, W(s)) - f(s, W^*(s))]\|_q ds \\
&\le \|S_\alpha(t) L M^{-1}\| \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \left\| A^q [h(u) - h(u^*)] \right\| ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|A^q T_\alpha(t-s)\| \|f(s, W(s)) - f(s, W^*(s))\| ds \\
&\le C_1 C_2 k_1 M_0 \|L\| \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} \|u - u^*\|_q \\
&\quad + L_f(\rho) \sum_{i=1}^r m_i(t) \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u - u^*\|_q ds \\
&\le C_1 C_2 k_1 M_0 \|L\| \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} \|u - u^*\|_\infty \\
&\quad + L_f(\rho) \sum_{i=1}^r m_i(t) \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \frac{1}{\alpha(1-q)} t^{\alpha(1-q)} \|u - u^*\|_\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(Pu)(t) - (Pu^*)(t)\|_\infty &\le C_1 C_2 k_1 M_0 \|L\| \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} \|u - u^*\|_\infty \\
&\quad + L_f(\rho) \sum_{i=1}^r m_i(t) \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \frac{1}{\alpha(1-q)} t^{\alpha(1-q)} \|u - u^*\|_\infty
\end{aligned}$$

and we conclude that P is continuous. □

Lemma 7. *The operator P given by* (4) *is compact.*

Proof. Let Σ be a bounded subset of Ω_q. Then there exists a constant η such that $\|u\|_\infty \le \eta$ for all $u \in \Sigma$. By (F_4), there exists a constant τ such that $\|f(t, W(t))\| \le a_f(1 + r\eta) = \tau$. Then $P\Sigma$ is a bounded subset of Ω_q. In fact, let $u \in \Sigma$. Using Lemma 2 (a) and (d), we get

$$\begin{aligned}
\|(Pu)(t)\|_q &\le \left\| S_\alpha(t) L M^{-1} \left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u_0 + h(u(s))]}{(t-s)^\alpha} ds \right] \right\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|T_\alpha(t-s) f(s, W(s))\|_q ds \\
&\le C_1 C_2 M_0 \|L\| \left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} (k_2 + \|u_0\|_q) \right] \\
&\quad + \int_0^t (t-s)^{\alpha-1} \|A^q T_\alpha(t-s)\| \|f(s, W(s))\| ds \\
&\le C_1 C_2 M_0 \|L\| \left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} (k_2 + \|u_0\|_q) \right] \\
&\quad + \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \tau \int_0^t (t-s)^{-q\alpha+\alpha-1} ds \\
&\le C_1 C_2 M_0 \|L\| \left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} (k_2 + \|u_0\|_q) \right] \\
&\quad + \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \tau \frac{1}{\alpha(1-q)} t^{\alpha(1-q)}.
\end{aligned}$$

Then, we obtain

$$\|(Pu)(t)\|_\infty \le C_1C_2M_0\|L\|\left[\eta+\frac{a^{1-\alpha}}{\Gamma(2-\alpha)}(k_2+\eta)\right]+\frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{\tau a^{\alpha(1-q)}}{\alpha(1-q)}.$$

We conclude that $P\Sigma$ is bounded. Define $\Pi = P\Sigma$ and $\Pi(t) = \{(Pu)(t)|u \in \Sigma\}$ for $t \in J$. Obviously, $\Pi(0) = \{(Pu)(0)|u \in \Sigma\}$ is compact. For each $g \in (0,t)$, $t \in (0,a]$, and arbitrary $\delta > 0$, let us define $\Pi_{g,\delta}(t) = \{(P_{g,\delta}u)(t)|u \in \Sigma\}$, where

$$\begin{aligned}(P_{g,\delta}u)(t) &= Q(g^\alpha\delta)\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta-g^\alpha\delta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t-g}\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\\
&\quad+Q(g^\alpha\delta)\int_0^{t-g}(t-s)^{\alpha-1}\left(\alpha\int_\delta^\infty L^{-1}\theta\zeta_\alpha(\theta)Q((t-s)^\alpha\theta-g^\alpha\delta)d\theta\right)f(s,W(s))ds\\
&=\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t-g}\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\\
&\quad+\alpha\int_0^{t-g}\int_\delta^\infty\theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,W(s))d\theta ds.\end{aligned}$$

Then, since the operator $Q(g^\alpha\delta)$, $g^\alpha\delta > 0$, is compact in X_q, the sets $\{(P_{g,\delta}u)(t)|u \in \Sigma\}$ are relatively compact in X_q. This comes from the following inequalities:

$$\begin{aligned}&\|(Pu)(t)-(P_{g,\delta}u)(t)\|_q\\
&\le\left\|\int_0^\delta L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\right\|_q\\
&\quad+\left\|\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_{t-g}^t\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\right\|_q\\
&\quad+\left\|\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t-g}\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\right.\\
&\quad-\left.\int_\delta^\infty L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^{t-g}\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]d\theta\right\|_q\\
&\quad+\alpha\left\|\int_0^t\int_0^\delta\theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,W(s))d\theta ds\right\|_q\\
&\quad+\alpha\left\|\int_0^t\int_\delta^\infty\theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,W(s))d\theta ds\right.\\
&\quad-\left.\int_0^{t-g}\int_\delta^\infty\theta(t-s)^{\alpha-1}L^{-1}\zeta_\alpha(\theta)Q((t-s)^\alpha\theta)f(s,W(s))d\theta ds\right\|_q\\
&\le\int_0^\delta\|L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\|\left\|A^q\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]\right\|d\theta\\
&\quad+\int_\delta^\infty\|L^{-1}\zeta_\alpha(\theta)Q(t^\alpha\theta)LM^{-1}\|\left\|A^q\left[v_0+\frac{1}{\Gamma(1-\alpha)}\int_{t-g}^t\frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]\right\|d\theta\\
&\quad+\alpha\int_0^t\int_0^\delta\theta(t-s)^{\alpha-1}\|L^{-1}\|\zeta_\alpha(\theta)\|A^qQ((t-s)^\alpha\theta)\|\|f(s,W(s))\|d\theta ds\\
&\quad+\alpha\int_{t-g}^t\int_\delta^\infty\theta(t-s)^{\alpha-1}\|L^{-1}\|\zeta_\alpha(\theta)\|A^qQ((t-s)^\alpha\theta)\|\|f(s,W(s))\|d\theta ds\end{aligned}$$

$$
\begin{aligned}
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{g^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_\delta^\infty \zeta_\alpha(\theta)d\theta\\
&\quad + C_1M_q\alpha\tau\int_0^t\int_0^\delta \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)(t-s)^{-\alpha q}\theta^{-q}d\theta ds\\
&\quad + C_1M_q\alpha\tau\int_{t-g}^t\int_\delta^\infty \theta(t-s)^{\alpha-1}\zeta_\alpha(\theta)(t-s)^{-\alpha q}\theta^{-q}d\theta ds\\
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{g^{1-\alpha}}{\Gamma(2-\alpha)}\right]\\
&\quad + C_1M_q\alpha\tau\int_0^t\int_0^\delta \theta^{1-q}(t-s)^{-\alpha q+\alpha-1}\zeta_\alpha(\theta)d\theta ds\\
&\quad + C_1M_q\alpha\tau\int_{t-g}^t\int_\delta^\infty \theta^{1-q}(t-s)^{-\alpha q+\alpha-1}\zeta_\alpha(\theta)d\theta ds\\
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
&\quad + C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{g^{1-\alpha}}{\Gamma(2-\alpha)}\right]\\
&\quad + C_1M_q\alpha\tau\left(\int_0^t (t-s)^{-\alpha q+\alpha-1}ds\right)\int_0^\delta \theta^{1-q}\zeta_\alpha(\theta)d\theta\\
&\quad + C_1M_q\alpha\tau\frac{\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\left(\int_{t-g}^t (t-s)^{-\alpha q+\alpha-1}ds\right)
\end{aligned}
$$

and

$$
\int_0^t (t-s)^{-\alpha q+\alpha-1}ds \leq \frac{1}{\alpha(1-q)}t^{\alpha(1-q)},\;\; \int_{t-g}^t (t-s)^{-\alpha q+\alpha-1}ds \leq \frac{1}{\alpha(1-q)}g^{\alpha(1-q)},
$$

so that

$$
\begin{aligned}
\|(Pu)(t)-(P_{g,\delta}u)(t)\|_q \leq & C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{a^{1-\alpha}}{\Gamma(2-\alpha)}\right]\int_0^\delta \zeta_\alpha(\theta)d\theta\\
& + C_1C_2M_0\|L\|\left[\|v_0\|_q+(k_2+\|u_0\|_q)\frac{g^{1-\alpha}}{\Gamma(2-\alpha)}\right]\\
& + \frac{C_1M_q\alpha\tau}{\alpha(1-q)}a^{\alpha(1-q)}\int_0^\delta \theta^{1-q}\zeta_\alpha(\theta)d\theta\\
& + \frac{C_1M_q\alpha\tau\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{1}{\alpha(1-q)}g^{\alpha(1-q)}.
\end{aligned}
$$

Therefore, $\Pi(t)=\{(Pu)(t)|u\in\Sigma\}$ is relatively compact in X_q for all $t\in(0,a]$ and, since it is compact at $t=0$, we have relatively compactness in X_q for all $t\in J$.

Next, let us prove that $\Pi = P\Sigma$ is equicontinuous. For $g\in[0,a)$,

$$
\begin{aligned}
\|(Pu)(g)-(Pu)(0)\|_q \leq & C_2\|v_0\|_q\|S_\alpha(g)L-I\|_q\\
& + C_1C_2M_0\|L\|\left[(k_2+\|u_0\|_q)\frac{g^{1-\alpha}}{\Gamma(2-\alpha)}\right]\\
& + \frac{\alpha C_1M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{\tau}{\alpha(1-q)}g^{\alpha(1-q)},
\end{aligned}
$$

and for $0 < s < t_1 < t_2 \leq a$, $\|(Pu)(t_1) - (Pu)(t_2)\|_q \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6$, where

$$
\begin{aligned}
I_1 &= C_2\|L\| \left[\|v_0\|_q + (k_2 + \|u_0\|_q)\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)}\right] \|S_\alpha(t_1) - S_\alpha(t_2)\|_q, \\
I_2 &= C_1C_2M_0\|L\| \left[(k_2 + \|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha} + t_1^{1-\alpha} - t_2^{1-\alpha}}{\Gamma(2-\alpha)}\right], \\
I_3 &= C_1C_2M_0\|L\| \left[(k_2 + \|u_0\|_q)\frac{(t_2-t_1)^{1-\alpha}}{\Gamma(2-\alpha)}\right], \\
I_4 &= \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)}\sqrt{\frac{1}{2\alpha-1}}t_1^{\alpha-\frac{1}{2}} \left(\int_0^a |(t_1-s)^{-q\alpha} - (t_2-s)^{-q\alpha}|^2 ds\right)^{\frac{1}{2}}, \\
I_5 &= \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)} \left(\int_0^a |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|^2 ds\right)^{\frac{1}{2}} \\
&\quad \times\sqrt{\frac{1}{1-2q\alpha}}\left(t_2^{1-2q\alpha} - (t_2-t_1)^{1-2q\alpha}\right)^{\frac{1}{2}}, \\
I_6 &= \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\|f\|_{C(J,X)}\frac{1}{\alpha(1-q)}(t_2-t_1)^{\alpha(1-q)}.
\end{aligned}
$$

Now, we have to verify that I_j, $j = 1, \ldots, 6$, tend to 0 independently of $u \in \Sigma$ when $t_2 \to t_1$. Let $u \in \Sigma$. By Lemma 2 (c) and (f), we deduce that $\lim_{t_2 \to t_1} I_1 = 0$ and $\lim_{t_2 \to t_1} I_4 = 0$. Moreover, using the fact that $|(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}| \to 0$ as $t_2 \to t_1$, we obtain from Lemma 3 that

$$\int_0^a |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}|^2 ds \to 0 \text{ as } t_2 \to t_1.$$

Thus, $\lim_{t_2 \to t_1} I_5 = 0$ since $q\alpha < \frac{1}{2}$. Also, it is clear that $\lim_{t_2 \to t_1} I_2 = I_3 = I_6 = 0$. In summary, we have proven that $P\Sigma$ is relatively compact for $t \in J$ and $\Pi(t) = \{Pu | u \in \Sigma\}$ is a family of equicontinuous functions. Hence, by the Arzela–Ascoli theorem, P is compact. □

Proof of Theorem 1. We shall prove that the operator P has a fixed point in Ω_q. According to Leray–Schauder fixed point theory (and from Lemmas 5–7), it suffices to show that the set $\Delta = \{u \in \Omega_q | u = \beta P u, \beta \in [0,1]\}$ is a bounded subset of Ω_q. Let $u \in \Delta$. Then,

$$
\begin{aligned}
\|u(t)\|_q &= \|\beta(Pu)(t)\|_q \\
&\leq \left\| S_\alpha(t) L M^{-1}\left[v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{[u_0 + h(u(s))]}{(t-s)^\alpha} ds\right]\right\|_q \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s) f(s, W(s))\|_q ds \\
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)}(k_2 + \|u_0\|_q)\right] \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|A^q T_\alpha(t-s)\|\|f(s, W(s))\| ds \\
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)}(k_2 + \|u_0\|_q)\right] \\
&\quad + \frac{a_f \alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^t (t-s)^{-q\alpha+\alpha-1}(1 + r\|u\|_q) ds \\
&\leq C_1C_2M_0\|L\|\left[\|v_0\|_q + \frac{a^{1-\alpha}}{\Gamma(2-\alpha)}(k_2 + \|u_0\|_q)\right] \\
&\quad + \frac{a_f \alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\frac{a^{\alpha(1-q)}}{\alpha(1-q)} + \frac{a_f \alpha r C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^t (t-s)^{-q\alpha+\alpha-1}\|u\|_q ds.
\end{aligned}
$$

Based on the well known singular version of Gronwall inequality, we can deduce that there exists a constant $R > 0$ such that $\|u\|_\infty \leq R$. Thus, Δ is a bounded subset of Ω_q. By Leray–Schauder fixed point theory, P has a fixed point in Ω_q. Consequently, system (1)–(2) has at least one mild solution u on J. $\square$

Theorem 2. *Mild solution $u(\cdot)$ of system* (1)–(2) *is unique.*

Proof. Let $u^*(\cdot)$ be another mild solution of system (1)–(2) with Sobolev–fractional nonlocal initial value $M^{-1}\left[v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{[u_0+h(u(s))]}{(t-s)^\alpha}ds\right]$. It is not difficult to verify that there exists a constant $\rho > 0$ such that $\|u\|_q, \|u^*\|_q \leq \rho$. From

$$\|u(t) - u^*(t)\|_q \leq \left\| S_\alpha(t)LM^{-1}\left\{[v_0 - v_0^*] + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{[u_0 - u_0^*] + [h(u) - h(u^*)]}{(t-s)^\alpha}ds\right\}\right\|_q$$
$$+ \int_0^t (t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,W(s)) - f(s,W^*(s))]\|_q ds,$$

we get

$$\|u(t) - u^*\|_q \leq C_1C_2M_0\|L\|\left\{\|v_0 - v_0^*\|_q + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\|u_0 - u_0^*\|_q + k_1\|u(s) - u^*(s)\|_q}{(t-s)^\alpha}ds\right\}$$
$$+ L_f(\rho)\sum_{i=1}^r m_i(t)\frac{\alpha C_1 M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\int_0^t (t-s)^{-q\alpha+\alpha-1}\|u(s) - u^*(s)\|_q ds.$$

Again, by the singular version of Gronwall's inequality, there exists a constant $R^* > 0$ such that

$$\|u(t) - u^*(t)\|_q \leq C_1C_2M_0\|L\|R^*\|u_0 - u_0^*\|_q,$$

which gives the uniqueness of u. Thus, system (1)–(2) has a unique mild solution on J. $\square$

4 Optimal multi-integral controls

Let Z be another separable reflexive Banach space from which the controls $\mathfrak{u}_1, \ldots, \mathfrak{u}_k$ take their values. We denote by $V_f(Z)$ a class of nonempty closed and convex subsets of Z. The multifunction $\omega : J \to V_f(Z)$ is measurable, $\omega(\cdot) \subset \Lambda$, where Λ is a bounded set of Z. The admissible control set is $U_{ad} = S_\omega^p = \{\mathfrak{u}_j \in L^p(\Lambda) | \mathfrak{u}_j(t) \in \omega(t)\ a.e.\}$, $j = \overline{1,k}$, $1 < p < \infty$. Then, $U_{ad} \neq \emptyset$ [43].

Consider the following Sobolev type fractional nonlocal multi-integral-controlled system:

$$^CD_t^\alpha[Lu(t)] = Eu(t) + f(t,W(t)) + \int_0^t [\mathcal{B}_1\mathfrak{u}_1(s) + \cdots + \mathcal{B}_k\mathfrak{u}_k(s)]ds, \tag{5}$$

$$^LD_t^{1-\alpha}[Mu(0)] = u_0 + h(u(t)). \tag{6}$$

Besides the sufficient conditions (F_1)–(F_5) of the last section, we assume:

(F_6) $\mathcal{B}_j \in L^\infty(J, L(Z, X_q))$, which implies that $\mathcal{B}_j\mathfrak{u}_j \in L^p(J, X_q)$ for all $\mathfrak{u}_j \in U_{ad}$.

Corollary 1. *In addition to assumptions of Theorem 1, suppose (F_6) holds. For every $\mathfrak{u}_j \in U_{ad}$ and $p\alpha(1-q) > 1$, system* (5)–(6) *has a mild solution corresponding to $\mathfrak{u}_j$ given by*

$$u^{\mathfrak{u}_j}(t) = S_\alpha(t)LM^{-1}\left[v_0 + \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{[u_0 + h(u(s))]}{(t-s)^\alpha}ds\right]$$
$$+ \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[f(s,W(s)) + \int_0^s [\mathcal{B}_1\mathfrak{u}_1(\eta) + \cdots + \mathcal{B}_k\mathfrak{u}_k(\eta)]d\eta\right]ds.$$

Proof. Based on our existence result (Theorem 1), it is required to check the term containing multi-integral controls. Let us consider

$$\varphi(t)=\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\left[\int_0^s [\mathcal{B}_1\mathfrak{u}_1(\eta)+\cdots+\mathcal{B}_k\mathfrak{u}_k(\eta)]d\eta\right]ds.$$

Using Lemma 2 (d) and Hölder inequality, we have

$$\begin{aligned}
\|\varphi(t)\|_q &\le \left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s [\mathcal{B}_1\mathfrak{u}_1(\eta)+\cdots+\mathcal{B}_k\mathfrak{u}_k(\eta)]d\eta ds\right\|_q\\
&\le \int_0^t (t-s)^{\alpha-1}\|A^qT_\alpha(t-s)\|[\|\mathcal{B}_1\mathfrak{u}_1(s)\|a+\cdots+\|\mathcal{B}_k\mathfrak{u}_k(s)\|a]ds\\
&\le \frac{\alpha a C_1 M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\Bigg[\|\mathcal{B}_1\|_\infty\int_0^t (t-s)^{-q\alpha+\alpha-1}\|\mathfrak{u}_1(s)\|_Z ds\\
&\quad +\cdots+\|\mathcal{B}_k\|_\infty\int_0^t (t-s)^{-q\alpha+\alpha-1}\|\mathfrak{u}_k(s)\|_Z ds\Bigg]\\
&\le \frac{\alpha a C_1 M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\Bigg[\|\mathcal{B}_1\|_\infty\left(\int_0^t (t-s)^{\frac{p}{p-1}(-q\alpha+\alpha-1)}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|\mathfrak{u}_1(s)\|_Z^p ds\right)^{\frac{1}{p}}\\
&\quad +\cdots+\|\mathcal{B}_k\|_\infty\left(\int_0^t (t-s)^{\frac{p}{p-1}(-q\alpha+\alpha-1)}ds\right)^{\frac{p-1}{p}}\left(\int_0^t \|\mathfrak{u}_k(s)\|_Z^p ds\right)^{\frac{1}{p}}\Bigg]\\
&\le \frac{\alpha a C_1 M_q\Gamma(2-q)}{\Gamma(1+\alpha(1-q))}\Bigg[\|\mathcal{B}_1\|_\infty\left(\frac{p-1}{p\alpha(1-q)-1}\right)^{\frac{p-1}{p}} a^{\frac{p\alpha(1-q)-1}{p-1}}\|\mathfrak{u}_1\|_{L^p(J,Z)}\\
&\quad +\cdots+\|\mathcal{B}_k\|_\infty\left(\frac{p-1}{p\alpha(1-q)-1}\right)^{\frac{p-1}{p}} a^{\frac{p\alpha(1-q)-1}{p-1}}\|\mathfrak{u}_k\|_{L^p(J,Z)}\Bigg],
\end{aligned}$$

where $\|\mathcal{B}_1\|_\infty,\dots,\|\mathcal{B}_k\|_\infty$ are the norm of operators $\mathcal{B}_1,\dots,\mathcal{B}_k$, respectively, in the Banach space $L_\infty(J,L(Z,X_q))$. Thus,

$$\left\|(t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s [\mathcal{B}_1\mathfrak{u}_1(\eta)+\cdots+\mathcal{B}_k\mathfrak{u}_k(\eta)]d\eta\right\|_q$$

is Lebesgue integrable with respect to $s\in[0,t]$ for all $t\in J$. It follows from Lemma 4 that

$$(t-s)^{\alpha-1}T_\alpha(t-s)\int_0^s [\mathcal{B}_1\mathfrak{u}_1(\eta)+\cdots+\mathcal{B}_k\mathfrak{u}_k(\eta)]d\eta$$

is a Bochner integral with respect to $s\in[0,t]$ for all $t\in J$. Hence, $\varphi(\cdot)\in\Omega_q$. The required result follows from Theorem 1. □

Furthermore, let us now assume

- (F_7) The functional $\mathcal{L}: J\times X_q\times Z^k\to\mathbb{R}\cup\{\infty\}$ is Borel measurable.
- (F_8) $\mathcal{L}(t,\cdot,\dots,\cdot)$ is sequentially lower semicontinuous on $X_q\times Z^k$ for almost all $t\in J$.
- (F_9) $\mathcal{L}(t,u,\cdot,\dots,\cdot)$ is convex on Z^k for each $u\in X_q$ and almost all $t\in J$.
- (F_{10}) There exist constants $d\ge 0$, $c_1,\dots,c_k>0$, such that ψ is nonnegative and $\psi\in L^1(J,\mathbb{R})$ satisfies

$$\mathcal{L}(t,u,\mathfrak{u}_1,\dots,\mathfrak{u}_k)\ge\psi(t)+d\|u\|_q+c_1\|\mathfrak{u}_1\|_Z^p+\cdots+c_k\|\mathfrak{u}_k\|_Z^p.$$

We consider the following Lagrange optimal control problem:

$$\begin{cases} \text{Find } (u^0, \mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0) \in C(J, X_q) \times U_{ad}^k \\ \text{such that } \mathcal{J}(u^0, \mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0) \leq \mathcal{J}(u^{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}, \mathfrak{u}_1, \ldots, \mathfrak{u}_k) \text{ for all } \mathfrak{u}_j \in U_{ad}, \end{cases} \tag{LP}$$

where

$$\mathcal{J}(u^{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}, \mathfrak{u}_1, \ldots, \mathfrak{u}_k) = \int_0^a \mathcal{L}(t, u^{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}, \mathfrak{u}_1(t), \ldots, \mathfrak{u}_k(t))dt$$

with $u^{\mathfrak{u}_j}$ denoting the mild solution of system (5)–(6) corresponding to the multi-integral controls $\mathfrak{u}_j \in U_{ad}$. The following lemma is used to obtain existence of a fractional optimal multi-integral control (Theorem 3).

Lemma 8. *Operators $\Upsilon_j : L^p(J, Z) \to \Omega_q$ given by*

$$\begin{cases} (\Upsilon_1 \mathfrak{u}_1)(\cdot) = \int_0^\cdot \int_0^s T_\alpha(\cdot - s)\mathcal{B}_1 \mathfrak{u}_1(\eta) d\eta ds, \\ \quad \vdots \\ (\Upsilon_1 \mathfrak{u}_k)(\cdot) = \int_0^\cdot \int_0^s T_\alpha(\cdot - s)\mathcal{B}_k \mathfrak{u}_k(\eta) d\eta ds, \end{cases}$$

where $p\alpha(1-q) > 1$ and $j = \overline{1,k}$, are strongly continuous.

Proof. Suppose that $\{\mathfrak{u}_j^n\}_{j=\overline{1,k}} \subseteq L^p(J, Z)$ are bounded. Define $\Theta_{j,n}(t) = (\Upsilon_j \mathfrak{u}_j^n)(t)$, $t \in J$. Similarly to the proof of Corollary 1, we can conclude that for any fixed $t \in J$ and $p\alpha(1-q) > 1$, $\|\Theta_{j,n}(t)\|_q$, $j = \overline{1,k}$, are bounded. By Lemma 2, it is easy to verify that $\Theta_{j,n}(t)$, $j = \overline{1,k}$, are compact in X_q and are also equicontinuous. According to the Ascoli–Arzela theorem, $\{\Theta_{j,n}(t)\}$ are relatively compact in Ω_q. Clearly, Υ_j, $j = \overline{1,k}$, are linear and continuous. Hence, Υ_j are strongly continuous operators (see [43, p. 597]). ☐

Now we are in position to give the following result on existence of optimal multi-integral controls for the Lagrange problem (LP).

Theorem 3. *If the assumptions (F_1)–(F_{10}) hold, then the Lagrange problem (LP) admits at least one optimal multi-integral pair.*

Proof. Assume that $\inf\{\mathcal{J}(u^{\mathfrak{u}_1,\ldots,\mathfrak{u}_k}, \mathfrak{u}_1, \ldots, \mathfrak{u}_k) | u^{\mathfrak{u}_j} \in U_{ad}\} = \epsilon < +\infty$. Using assumptions (F_7)–(F_{10}), we have $\epsilon > -\infty$. By definition of infimum, there exists a minimizing feasible multi-pair $\{(u^m, \mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m)\} \subset \mathcal{U}_{ad}$ sequence, where $\mathcal{U}_{ad} = \{(u, \mathfrak{u}_1, \ldots, \mathfrak{u}_k) | u$ is a mild solution of system (5)–(6) corresponding to $\mathfrak{u}_1, \ldots, \mathfrak{u}_k \in U_{ad}\}$, such that $\mathcal{J}(u^m, \mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m) \to \epsilon$ as $m \to +\infty$. Since $\{(\mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m)\} \subseteq U_{ad}$, $m = 1, 2, \ldots, \{(\mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m)\}$ is bounded in $L^p(J, Z)$ and there exists a subsequence, still denoted by $\{(\mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m)\}$, $\mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0 \in L^p(J, Z)$, such that

$$(\mathfrak{u}_1^m, \ldots, \mathfrak{u}_k^m) \overset{\text{weakly}}{\longrightarrow} (\mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0)$$

in $L^p(J, Z)$. Since U_{ad} is closed and convex, by Marzur lemma $\mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0 \in U_{ad}$. Suppose $u^m(u^0)$ is the mild solution of system (5)–(6) corresponding to $\mathfrak{u}_1^m(\mathfrak{u}_1^0), \ldots, \mathfrak{u}_k^m(\mathfrak{u}_k^0)$. Functions u^m and u^0 satisfy, respectively, the following integral equations:

$$\begin{aligned} u^m(t) = S_\alpha(t) L M^{-1} &\left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u_0 + h(u^m(s))]}{(t-s)^\alpha} ds \right] \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[f(s, W^m(s)) + \int_0^s [\mathcal{B}_1 \mathfrak{u}_1^m(\eta) + \cdots + \mathcal{B}_k \mathfrak{u}_k^m(\eta)] d\eta \right] ds, \end{aligned}$$

$$\begin{aligned} u^0(t) = S_\alpha(t) L M^{-1} &\left[v_0 + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[u_0 + h(u^0(s))]}{(t-s)^\alpha} ds \right] \\ &+ \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \left[f(s, W^0(s)) + \int_0^s [\mathcal{B}_1 \mathfrak{u}_1^0(\eta) + \cdots + \mathcal{B}_k \mathfrak{u}_k^0(\eta)] d\eta \right] ds. \end{aligned}$$

It follows from the boundedness of $\{\mathfrak{u}_1^m\}, \ldots, \{\mathfrak{u}_k^m\}, \{\mathfrak{u}_1^0\}, \ldots, \{\mathfrak{u}_k^0\}$ and Theorem 1 that there exists a positive number ρ such that $\|u^m\|_\infty, \|u^0\|_\infty \leq \rho$. For $t \in J$, we have

$$\|u^m(t) - u^0(t)\|_q \leq \|\xi_m^{(1)}(t)\|_q + \|\xi_m^{(2)}(t)\|_q + \|\xi_m^{(3)}(t)\|_q + \cdots + \|\xi_m^{(k+2)}(t)\|_q,$$

where

$$\begin{aligned}
\xi_m^{(1)}(t) &= S_\alpha(t) L M^{-1} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{[h(u^m(s)) - h(u^0(s))]}{(t-s)^\alpha} ds, \\
\xi_m^{(2)}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) [f(s, W^m(s)) - f(s, W^0(s))] ds, \\
\xi_m^{(3)}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \int_0^s \mathcal{B}_1 [\mathfrak{u}_1^m(\eta) - \mathfrak{u}_1^0(\eta)] d\eta ds, \\
&\vdots \\
\xi_m^{(k+2)}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \int_0^s \mathcal{B}_k [\mathfrak{u}_k^m(\eta) - \mathfrak{u}_k^0(\eta)] d\eta ds.
\end{aligned}$$

The assumption (F_5) gives

$$\|\xi_m^{(1)}(t)\|_q \leq C_1 C_2 M_0 k_1 \|L\| \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} \|u^m - u^0\|_q.$$

Using Lemma 2 (d) and (F_3),

$$\|\xi_m^{(2)}(t)\|_q \leq L_f(\rho) \sum_{i=1}^r m_i(t) \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u^m(s) - u^0(s)\|_q ds.$$

From Lemma 8, we get

$$\xi_m^{(j+2)}(t) \overset{\text{strongly}}{\longrightarrow} 0 \quad \text{in} \quad X_q \text{ as } m \to \infty, \quad j = \overline{1,k}.$$

Thus,

$$\begin{aligned}
\|u^m(t) - u^0(t)\|_q \leq{}& \sum_{j=1}^k \|\xi_m^{(j+2)}(t)\|_q + C_1 C_2 M_0 k_1 \|L\| \frac{a^{1-\alpha}}{\Gamma(2-\alpha)} \|u^m - u^0\|_q \\
&+ L_f(\rho) \sum_{i=1}^r m_i(t) \frac{\alpha C_1 M_q \Gamma(2-q)}{\Gamma(1+\alpha(1-q))} \int_0^t (t-s)^{-q\alpha+\alpha-1} \|u^m(s) - u^0(s)\|_q ds.
\end{aligned}$$

By virtue of the singular version of Gronwall's inequality, there exists $M_* > 0$ such that

$$\|u^m(t) - u^0(t)\|_q \leq M_* \sum_{j=1}^k \|\xi_m^{(j+2)}(t)\|_q,$$

which yields that

$$u^m \to u^0 \text{ in } C(J, X_q) \text{ as } m \to \infty.$$

Because $C(J, X_q) \hookrightarrow L^1(J, X_q)$, using the assumptions (F_7)–(F_{10}) and Balder's theorem, we obtain that

$$\begin{aligned}
\epsilon &= \lim_{m \to \infty} \int_0^a \mathcal{L}(t, u^m(t), \mathfrak{u}_1^m(t), \ldots, \mathfrak{u}_k^m(t)) dt \\
&\geq \int_0^a \mathcal{L}(t, u^0(t), \mathfrak{u}_1^0(t), \ldots, \mathfrak{u}_k^0(t)) dt \\
&= \mathcal{J}(u^0, \mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0) \\
&\geq \epsilon.
\end{aligned}$$

This shows that $\mathcal{J}$ attains its minimum at $\mathfrak{u}_1^0, \ldots, \mathfrak{u}_k^0 \in U_{ad}$. □

5 An example

Consider the following fractional nonlocal multi-controlled system of Sobolev type:

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[u(t,x)-u_{xx}(t,x)\right]+\frac{\partial^2}{\partial x^2}u(t,x)=\int_0^t[\mathfrak{u}_1(s,x)+\cdots+\mathfrak{u}_k(s,x)]ds+F(t,D_x^r u(x,t)), \tag{7}$$

$$u(0,x)=\frac{\partial^2}{\partial x^2}\left[v_0(x)+\sum_{\eta=1}^{m}\frac{c_\eta}{\Gamma(1-\alpha)}\int_0^{t_\eta}\frac{u_0(x)+u(s_\eta,x)}{(t_\eta-s_\eta)^\alpha}ds_\eta\right],\ x\in[0,\pi], \tag{8}$$

$$u(t,0)=u(t,\pi)=0,\quad 0<t\leq 1, \tag{9}$$

where $0<\alpha\leq 1$, $0<t_1<\cdots<t_m<1$ and c_η are positive constants, $\eta=1,\ldots,m$; the functions $u(t)(x)=u(t,x), f(t,\cdot)=F(t,\cdot), W(t)(x)=D_x^r u(x,t)$ and $h(u(t))(x)=\sum_{\eta=1}^m c_\eta u(t_\eta,x)$. Let us take $\mathcal{B}_j\mathfrak{u}_j(t)(x)=\mathfrak{u}_j(t,x)$, $j=\overline{1,k}$, and the operator D_x^r as follows:

$$D_x^r u(x,t)=\left(\partial_x u(x,t),\partial_x^2 u(x,t),\ldots,\partial_x^r u(x,t)\right).$$

Let $X=Y=Z=L^2[0,\pi]$. Define the operators L, E, and M on domains and ranges contained in $L^2[0,\pi]$ by $Lw=w-w''$, $Ew=-w''$ and $M^{-1}w=w''$, where the domains $D(L)$, $D(E)$ and $D(M)$ are given by

$$\{w\in X: w,w' \text{ are absolutely continuous, } w''\in X, w(0)=w(\pi)=0\}.$$

Then L and E can be written, respectively, as

$$Lw=\sum_{n=1}^{\infty}(1+n^2)(w,w_n)w_n \text{ and } Ew=\sum_{n=1}^{\infty}-n^2(w,w_n)w_n,$$

where $w_n(t)=(\sqrt{2/\pi})\sin nt$, $n=1,2,\ldots$, is the orthogonal set of eigenfunctions of E. Furthermore, for any $w\in X$, we have

$$L^{-1}w=\sum_{n=1}^{\infty}\frac{1}{1+n^2}(w,w_n)w_n,\quad EL^{-1}w=\sum_{n=1}^{\infty}\frac{-n^2}{1+n^2}(w,w_n)w_n,$$

and

$$Q(t)x=\sum_{n=1}^{\infty}\exp\left(\frac{-n^2t}{1+n^2}\right)(w,w_n)w_n.$$

It is easy to see that L^{-1} is compact, bounded, with $\|L^{-1}\|\leq 1$, and $A=EL^{-1}$ generates the above strongly continuous semigroup $Q(t)$ on $L^2[0,\pi]$ with $\|Q(t)\|\leq e^{-t}\leq 1$. If $\mathcal{B}_j=0$, $j=\overline{1,k}$, then, with the above choices, system (7)–(9) can be written in the form (1)–(2). Therefore, Theorems 1 and 2 can be applied to guarantee existence and uniqueness of a mild solution to (7)–(9).

Let the admissible control set be

$$U_{ad}=\left\{\mathfrak{u}_j\in Z \mid \sum_{j=1}^{k}\int_0^t\|\mathfrak{u}_j(s,x)\|_{L^2([0,1],Z)}ds\leq 1\right\}.$$

Choose $\alpha=\frac{4}{5}, p=2$ and $q=\frac{1}{4}$. Find the controls $\mathfrak{u}_1(t,x),\ldots,\mathfrak{u}_k(t,x)$ that minimize the functional

$$\mathcal{J}(u,\mathfrak{u}_1,\ldots,\mathfrak{u}_k)=\int_0^1\int_0^\pi|u(t,x)|^2dxdt+\sum_{j=1}^{k}\int_0^1\int_0^t\int_0^\pi|\mathfrak{u}_j(s,x)|^2dxdsdt$$

subject to system (7)–(9). If $\mathcal{B}_j\mathfrak{u}_j(t)(x)=\mathfrak{u}_j(t,x)$, $j=\overline{1,k}$, then system (7)–(9) can be transformed into (5)–(6) with the cost function

$$\mathcal{J}(\mathfrak{u}_1,\ldots,\mathfrak{u}_k)=\int_0^1\left[\|u(t)\|^2+\int_0^t\{\|\mathfrak{u}_1(s)\|_Z^2+\cdots+\|\mathfrak{u}_k(s)\|_Z^2\}ds\right]dt.$$

We can check that $\alpha q=\frac{4}{5}\times\frac{1}{4}=\frac{1}{5}<1$ and $p\alpha(1-q)=2\frac{4}{5}\frac{3}{4}=\frac{6}{5}>1$. Then all assumptions of Theorem 3 are satisfied and we conclude that the optimal control problem has an optimal pair.

Acknowledgments

This work was partially supported by CIDMA and FCT project PEst-OE/MAT/UI4106/2014. The first author is very grateful to CIDMA and the Department of Mathematics of University of Aveiro, for the hospitality and the excellent working conditions.

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