View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Repositório Institucional da Universidade de Aveiro

Upper bounds on the Laplacian spread of graphs

Enide Andrade[a,*], Helena Gomes[a,b], María Robbiano[c], Jonnathan Rodríguez[c]

[a] Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Aveiro, Portugal
[b] Departamento de Ciências Exatas e Naturais, Escola Superior de Educação de Viseu, Instituto Politécnico de Viseu, Portugal.
[c] Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Chile

Abstract

The Laplacian spread of a graph G is defined as the difference between the largest and the second smallest eigenvalue of the Laplacian matrix of G. In this work, an upper bound for this graph invariant, that depends on first Zagreb index, is given. Moreover, another upper bound is obtained and expressed as a function of the nonzero coefficients of the Laplacian characteristic polynomial of a graph.

Keywords:
Graphs, Laplacian matrix, matrix spread, Laplacian spread.
2000 MSC: 05C50, 15A18

1. Notation and Preliminares

By an (n, m)-graph $G = (\mathcal{V}(G), \mathcal{E}(G))$, for short $G = (\mathcal{V}, \mathcal{E})$, we mean an undirected simple graph on $|\mathcal{V}| = n$ vertices and $m = |\mathcal{E}|$ edges. If $e \in \mathcal{E}$ is the edge connecting vertices u and v we say that u and v are adjacent and the edge is denoted by $\{u, v\}$. The notation $u \sim v$ means that $\{u, v\} \in \mathcal{E}$. For $u \in \mathcal{V}$ the set of neighbors of u, $N_G(u)$, is the set of vertices adjacent to u. The cardinality of $N_G(u)$, d_u, is called the vertex degree of u. The smallest and largest vertex degree of G are denoted by δ and Δ, respectively.

*Corresponding author.
Email addresses: enide@ua.pt (Enide Andrade), hgomes@ua.pt (Helena Gomes), mrobbiano@ucn.cl (María Robbiano), jrodriguez01@ucn.cl (Jonnathan Rodríguez)

A graph in which all vertex degrees are equal to p is regular of degree p (or p-regular). The path and the star with n vertices are denoted by P_n and S_n, respectively. A *caterpillar graph* is a tree of order $n \geq 5$ such that by removing all the pendant vertices one obtains a path with at least two vertices. In this context the caterpillar, $T(q_1, \ldots, q_k)$ is obtained from a path P_k, with $k \geq 2$, by associating the central vertex of the star S_{q_i} $(1 \leq i \leq k)$ to the i-th vertex of the path P_k. The adjacency matrix $A(G)$ of a graph G with $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the square matrix of order n, whose (i, i)-entry is equal to 1 if v_i and v_j are adjacent, and 0 otherwise. The Laplacian matrix of G is given by $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix whose (i, i)-entry is equal to the degree of $v_i \in \mathcal{V}$. This matrix is positive semidefinite and 0 is always a Laplacian eigenvalue whose multiplicity corresponds to the number of connected components of G with $\mathbf{e}$, the all ones vector, as an associated eigenvector. For spectral results on this matrix see, for instance, [6, 8]. There are numerous results in the literature concerning upper and lower bounds on the largest eigenvalue of $L(G)$, see [15, 19].

If B is a real symmetric matrix, $\beta_i(B)$ (or simply β_i) and σ_B denote the i -th largest eigenvalue of B and the set of eigenvalues of B, respectively. The set of eigenvalues of $L(G)$ is denoted by $\sigma_L(G)$ and called the Laplacian spectrum of G. If β is an eigenvalue of B and $\mathbf{x}$ one of its eigenvectors the pair $(\beta, \mathbf{x})$ is an eigenpair of B. From now on we represent the Laplacian eigenvalues of G by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. An important result in graph theory, see [19], states that if G has at least one edge, $\Delta + 1 \leq \mu_1$, and if G is connected the equality is attained if and only if $\Delta = n - 1$. Considering G, an upper bound on Laplacian eigenvalues can be easily obtained, $\mu_1 \leq n$, see [2]. Among the most important Laplacian eigenvalues is the algebraic connectivity of G, defined as the second smallest Laplacian eigenvalue μ_{n-1}, [10]. Recently, the algebraic connectivity has received much attention, see [1, 19, 22, 23] and the references cited therein. A graph is connected if and only if $\mu_{n-1} > 0$, [10].

2. The Laplacian spread of an arbitrary graph and some motivation

Let B be an $n \times n$ complex matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. The spread of B (or matricial spread) is defined by

$$s(B) = \max_{i,j} |\beta_i - \beta_j|,$$

where the maximum is taken over all pairs of eigenvalues of B. There is a considerable literature on this parameter, see for instance [13, 18, 21]. Suppose that $B \in \mathbb{C}^{n\times n}$ is a Hermitian complex matrix. For $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we denote by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \mathbf{y}$, the inner product in $\mathbb{C}^n$ and by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ the norm of $\mathbf{x}$. Here, $|B| = \sqrt{trace(B^*B)}$ is the Frobenius matrix norm of B, where B^* represents the transconjugate of B. Since the Frobenius matrix norm is a unitarily invariant matrix norm,

$$|B| = \sqrt{|\beta_1|^2 + |\beta_2|^2 + \cdots + |\beta_n|^2}, \tag{1}$$

the following upper bound for the spread of a square matrix B was given in [18]

$$s^2(B) \leq 2|B|^2 - \frac{2}{n}(trace(B))^2. \tag{2}$$

The Laplacian spread of G, [26], is defined as

$$spr_L(G) = \max\{|\mu_i - \mu_j| : \mu_i, \mu_j \in \sigma_L(G) \setminus \{0\}\}. \tag{3}$$

Note that in this definition we avoid the eigenvalue 0, so the spread becomes equal to the largest minus the second smallest eigenvalue. There are several results in literature related to this graph invariant. For instance, Fan et al. [26] showed that the star S_n and the path P_n are, respectively, the trees with the maximal Laplacian spread and the minimal Laplacian spread among all trees of order n. Recently the unicyclic graphs with maximum and minimum value of the Laplacian spread were studied in [5, 16] and [24], respectively. The maximum Laplacian spread of bicyclic graphs and tricyclic graphs of a given order were presented in [9, 17] and [7], respectively. In [27] bounds for the Laplacian spread of graphs were obtained. In particular, for an (n, m)-graph with minimum and maximum degree δ and Δ, the authors presented an upper bound that depends on the clique number of the graph and its complement and of the previous parameters (see [27, Theorem 4.3]). By definition of Laplacian spread the next upper bound was also presented in [27].

Proposition 2.1. *[27, Proposition 1.1] Let G be a simple undirected graph having Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Then $spr_L(G) < n$.*

Here, for a *non-trivial* upper bound on $spr_L(G)$ we mean any upper bound $l \geq 0$, such that $l < n$. On the contrary case we call *trivial* upper bound.

In this paper we present two upper bounds for the Laplacian spread of connected graphs. In the next section, using a theorem due to Brauer we obtain a new matrix whose matricial spread coincides with the Laplacian spread of G. Regarding this fact, we obtain an upper bound for the Laplacian spread that depends on the first Zagreb index. Also, we present in Section 4 an upper bound in function of the nonzero coefficients of the Laplacian characteristic polynomial of a graph. In Section 5 some computational experiments to verify the behavior of the obtained upper bounds (compared with the published ones) are presented.

3. An upper bound for the Laplacian spread using a rank one perturbation on the Laplacian matrix

We start this section recalling a well known theorem due to Brauer, (see [4]) that relates the eigenvalues of an arbitrary matrix and the matrix resulting from it by a rank one additive perturbation. This theorem plays an important role in the study of the nonnegative eigenvalue problem.

Theorem 3.1. *[4] Let B be an arbitrary $n\times n$ matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. Let $\mathbf{x}_k$ be an eigenvector of B associated with the eigenvalue β_k, and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $B+\mathbf{x}_k\mathbf{q}^t$ has eigenvalues $\beta_1, \ldots, \beta_{k-1}, \beta_k + \mathbf{x}_k^t\mathbf{q}, \beta_{k+1}, \ldots, \beta_n$.*

Using this theorem, a new matrix whose matricial spread coincides with the Laplacian spread of G can be obtained.

Remark 3.2. *Let*

$$M = L(G) + \xi \mathbf{e}\mathbf{e}^t. \tag{4}$$

If $\xi = \frac{\alpha}{n}$, for any given value α such that

$$\mu_{n-1} \leq \alpha \leq \mu_1, \tag{5}$$

using Theorem 3.1, we can conclude

$$spr_L\left(G\right) = s\left(M\right). \tag{6}$$

Using a rank one perturbation on the Laplacian matrix we present an upper bound for the Laplacian spread that depends on first Zagreb index.

Let us recall the number

$$Z_g(G) = \sum_{i=1}^{n} d_i^2,$$

also known as the first Zagreb index of G, [11].

If $M = (m_{ij})$ is the matrix defined in (4), then

$$m_{ij} = \begin{cases} -1+\xi & \text{if } i \sim j \\ \xi & \text{if } i \nsim j \\ d_i + \xi & \text{if } i = j. \end{cases} \tag{7}$$

Using inequality in (2) we get the following result.

Theorem 3.3. *Let G be a graph on n vertices with m edges and M the matrix defined in (4). Then*

$$s(M) \leq \sqrt{2Z_g(G) + 4m(1-2\xi) + 2\xi^2 n(n-1) - \frac{8m^2}{n}}. \tag{8}$$

If $\xi = \frac{\alpha}{n}$, with $\mu_{n-1} \leq \alpha \leq \mu_1$, we have

$$s(M) \leq \sqrt{2Z_g(G) + \frac{4m}{n}(n - 2\alpha - 2m) + \frac{2\alpha}{n}^2 (n-1)}. \tag{9}$$

Proof. We shall use the upper bound (2) for the matrix M in (4). Taking into account the entries m_{ij} in (7), and that $|M|^2 = trace(M^*M)$ we obtain

$$|M|^2 = |L(G)|^2 + \xi^2 n^2 = Z_g(G) + 2m + \xi^2 n^2$$

and by a direct computation we have

$$trace(M) = 2m + \xi n.$$

Applying the lower bound (2) for M we have

$$s^2(M) \leq 2Z_g(G) + 4m + 2\xi^2 n^2 - \frac{2}{n}(2m + \xi n)^2. \tag{10}$$

By elementary algebra, the term on the right hand side of the previous inequality can be written as the upper bound in (8).

If $\xi = \frac{\alpha}{n}$, the next equality

$$2Z_g(G) + 4m + 2\xi^2 n^2 - \frac{2}{n}(2m + \xi n)^2 =$$

$$2Z_g(G) + \frac{4m}{n}(n - 2\alpha - 2m) + \frac{2\alpha^2}{n}(n-1),$$

is obtained from (10). Thus the result follows. ∎

Remark 3.4. *In [25] it was conjectured that for a graph G with n vertices, $spr_L(G) \leq n-1$ with equality if and only if G or $\overline{G}$ are isomorphic to the join of an isolated vertex and a disconnected graph with $n-1$ vertices. If G is a p-regular graph with $n \geq 2$ vertices then $\mu_{n-1} \leq p \leq \mu_1$, see e.g. [3]. Considering $\alpha = p$ and $m = \frac{np}{2}$ in (9) one obtains*

$$spr_L(G) \leq \sqrt{2pn - 2p^2 - \frac{2p^2}{n}} < n-1.$$

The last inequality is a direct consequence of replacing $x = \frac{p}{n}$ as an argument into the function $f(x) = -2\left(1+\frac{1}{n}\right)x^2 + 2x + \frac{2}{n} - \left(1+\frac{1}{n^2}\right)$ and $f(p/n) < 0$.

Remark 3.5. *Using (6) and $\alpha = \mu_1$ the upper bound in (9) becomes*

$$spr_L(G) \leq \sqrt{2Z_g(G) + \frac{4m}{n}(n - 2\mu_1 - 2m) + \frac{2{\mu_1}^2}{n}(n-1)}. \qquad (11)$$

For $\alpha = \mu_{n-1}$ the upper bound in (9) becomes

$$spr_L(G) \leq \sqrt{2Z_g(G) + \frac{4m}{n}(n - 2\mu_{n-1} - 2m) + \frac{2{\mu_{n-1}}^2}{n}(n-1)}. \qquad (12)$$

Remark 3.6. *For the graph G with 7 vertices $G = (K_2UK_2UK_2)vK_1$, that is, the join with $K2UK2UK2$ and K_1, $\sigma_L(G) = \{7,3,3,3,1,1,0\}$, so $spr_L(G) = 7-1 = 6$. However, the upper bounds in (11) and (12) are 8.685 and 7.4066, respectively. Therefore one can see that the upper bound obtained in Theorem 3.3 is tighter in some cases (as in the cases of regular graphs) and in other cases it appears as a trivial bound. Then it make sense to continue searching for non-trivial bounds.*

4. An upper bound as a function of the nonzero coefficients of characteristic polynomial of Laplacian matrix

The next upper bound for the Laplacian spread uses the coefficients of the Laplacian characteristic polynomial of a graph. We present here some remarks based on the existing literature, see, for instance, [8].

Remark 4.1. *It is helpful to express the characteristic polynomial of any square matrix in terms of its principal minors. Recall that an $r \times r$ principal submatrix of an $n \times n$ matrix B is a submatrix of B that lies on the same set of r rows and columns, and an $r \times r$ principal minor is the determinant of an $r \times r$ principal submatrix. There are $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ such minors. If*

$$\det(B - \beta I) = \beta^n + b_1\beta^{n-1} + b_2\beta^{n-2} + \cdots + b_{n-1}\beta + b_n,$$

then

$$b_k = (-1)^k \sum (\textit{all } k \times k \textit{ principal minors}), \qquad 1 \leq k \leq n.$$

Remark 4.2. *One of the earliest uses of the matrix $L(G)$ is the Matrix-Tree Theorem, due to Kirchhoff (see [8]). It states that any $(n-1) \times (n-1)$ cofactor (minor with corresponding signs) of $L(G)$ gives the number of spanning trees of the graph. For notation, let $L_{[i,j]}$ be the submatrix of $L(G)$ with the i^{th} row and j^{th} column removed. Denote the number of spanning trees of G by $t(G)$. Then the next lemma can be established.*

Lemma 4.3. *[8] $t(G) = (-1)^{i+j} \det L_{[i,j]}$.*

Remark 4.4. *Let $q(\lambda) = \det(L(G) - \lambda I)$ be the characteristic polynomial of $L(G)$ and suppose that*

$$q(\lambda) = \lambda^n + f_1\lambda^{n-1} + f_2\lambda^{n-2} + \cdots + f_{n-1}\lambda. \tag{13}$$

By using Remarks 4.1 and 4.2 together with Lemma 4.3 it can be concluded (see [8]) that

$$(-1)^{n-1} f_{n-1} = nt(G) = \mu_1\mu_2 \cdots \mu_{n-1},$$

In particular when G is a tree we obtain $t(G) = 1$ which implies $(-1)^{n-1} f_{n-1} = n$.

In [8, 12] there is a more general result concerning the coefficients of the characteristic polynomial of $L(G)$.

Proposition 4.5. *[8] Let* $q(\lambda) = \lambda^n + f_1\lambda^{n-1} + f_2\lambda^{n-2} + \cdots + f_{n-1}\lambda$ *be the characteristic polynomial of* $L(G)$*. Then, for* $k = 1, 2, \ldots, n-1$*,*

$$f_k = (-1)^k \sum_{J \subset V(G),\ |J|=n-k} t(G_J),$$

where $t(G_J)$ *stands for the number of spanning trees of a graph* G_J *obtained from* G*.*

We now present a lower bound for the algebraic connectivity of G in order to establish an upper bound on the Laplacian spread of a graph.

Proposition 4.6. *Let* B *be an* $n \times n$ *positive definite matrix with eigenvalues* $\beta_1(B) \geq \beta_2(B) \geq \cdots \geq \beta_n(B) > 0$*. Then*

$$\frac{1}{trace\,(B^{-1})} < \beta_n(B).$$

Proof. The proof is an immediate consequence of

$$trace\left(B^{-1}\right) = \frac{1}{\beta_1(B)} + \cdots + \frac{1}{\beta_n(B)} > \frac{1}{\beta_n(B)}.$$

■

From now on we let G be a connected graph.

Remark 4.7. *Let* ξ *be equal to* $\frac{\alpha}{n}$*, where* α *satisfies the condition in Remark 3.2. If* M *is the matrix defined in (4), then it is an* $n \times n$ *symmetric matrix with (only) positive eigenvalues* $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M) > 0$*, where* $\lambda_1(M) = \mu_1$ *and* $\lambda_n(M) = \mu_{n-1}$*. By Proposition 4.6 we conclude that*

$$\frac{1}{trace\,(M^{-1})} < \lambda_n(M) = \mu_{n-1}. \tag{14}$$

Remark 4.8. *To compute the left hand term in (14) we use the coefficients in the characteristic equation of* M*. Therefore, in the characteristic polynomial of* M

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n, \tag{15}$$

we consider $\chi_k(M)$, *the* k^{th} *symmetric function of the eigenvalues* $\lambda_1(M)$, $\lambda_2(M), \ldots, \lambda_n(M)$ *of* M, *that is*

$$\chi_k(M) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}(M)\, \lambda_{i_2}(M) \cdots \lambda_{i_k}(M). \tag{16}$$

Then

$$c_k = (-1)^k \chi_k(M), \qquad 1 \leq k \leq n. \tag{17}$$

Since the eigenvalues $\lambda_1(M),\ \lambda_2(M), \ldots, \lambda_n(M)$ *are positive we check that the signs of* $c_1, c_2, \ldots, c_n$ *alternate.*

Remark 4.9. *Similarly, replacing* M *by* $L(G)$ *in the previous considerations and taking into account the polynomial in (13) we have*

$$f_k = (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mu_{i_1} \cdots \mu_{i_k}, \qquad 1 \leq k \leq n. \tag{18}$$

From both relations (17), (18) and defining $f_0 \equiv 1,\ f_n \equiv 0$ *one can see that,*

$$c_k = f_k - (n\xi) f_{k-1} = f_k - \alpha f_{k-1}, \qquad 1 \leq k \leq n. \tag{19}$$

Proposition 4.10. *Let* μ_{n-1} *be the algebraic connectivity of* G. *Let* ξ *be as in Remark 3.2. If* M *is as in (4), with characteristic polynomial* $p(\lambda)$ *as in (15), then*

$$-\frac{c_n}{c_{n-1}} \leq \mu_{n-1}.$$

Proof. Using Eqs. (16) and (17) we obtain

$$\frac{1}{\lambda_n(M)} \leq \frac{1}{\lambda_1(M)} + \cdots + \frac{1}{\lambda_n(M)} = -\frac{c_{n-1}}{c_n}.$$

From Remark 4.7 the result follows. ■

From Proposition 4.10 and (19) the next result follows.

Corollary 4.11. *Let* μ_{n-1} *be the algebraic connectivity of* G. *Let* $\mu_{n-1} \leq \alpha \leq \mu_1$. *Let* $q(\lambda) = \det(L(G) - \lambda I)$ *be the characteristic polynomial of* $L(G)$ *and suppose that*

$$q(\lambda) = \lambda^n + f_1 \lambda^{n-1} + f_2 \lambda^{n-2} + \cdots + f_{n-1}\lambda. \tag{20}$$

Then

$$\frac{\alpha f_{n-1}}{f_{n-1} - \alpha f_{n-2}} \leq \mu_{n-1}. \tag{21}$$

Let G be a graph with vertices $v_1, \ldots, v_n$ and $d_1, d_2, \ldots, d_n$ the corresponding degrees. Let $m_1, m_2, \ldots, m_n$ be define as

$$m_k = \tfrac{1}{d_k} \sum_{v_j \sim v_k} d_j, \qquad 1 \leq k \leq n.$$

Let $C = \max\{d_k + m_k : v_k \in \mathcal{V}(G)\}$.

The following result is due to R. Merris and gives an upper bound for μ_1.

Lemma 4.12. *[20] Let G be a graph with vertices $v_1, \ldots, v_n$. Let $d_1, d_2, \ldots, d_n$ be the corresponding degrees of the vertices. Let C be as above. Then*

$$\mu_1 \leq C. \tag{22}$$

The equality holds if and only if G is bipartite regular or semiregular.

Theorem 4.13. *Let G be a connected graph with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. Then*

$$spr_L(G) \leq \mu_1 - \frac{\mu_1 f_{n-1}}{f_{n-1} - \mu_1 f_{n-2}} = \mu_1 - \frac{(-1)^{n-1} \mu_1 nt(G)}{(-1)^{n-1} nt(G) - \mu_1 f_{n-2}} < n, \tag{23}$$

where $f_1, f_2, \ldots, f_{n-1}$ are the nonzero coefficients of the Laplacian characteristic polynomial.

Proof. By a direct consequence of Proposition 4.10 we have

$$spr_L(G) = \mu_1 - \mu_{n-1} \leq \mu_1 + \frac{c_n}{c_{n-1}}.$$

Taking into account (19), and setting $\alpha = \mu_1$, (23) holds. ■

Corollary 4.14. *Let G be a connected graph with Laplacian eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. Consider $\mu_{n-1} \leq \alpha \leq \mu_1$. Then*

$$spr_L(G) \leq \min\left\{n + \frac{\alpha f_{n-1}}{\alpha f_{n-2} - f_{n-1}}, C + \frac{\alpha f_{n-1}}{\alpha f_{n-2} - f_{n-1}}\right\},$$

where the sequence $f_1, f_2, \ldots, f_n$ is defined as in Remark 4.8. Moreover,

$$spr_L(G) \leq \min\left\{n + \frac{\mu_1 f_{n-1}}{\mu_1 f_{n-2} - f_{n-1}}, C + \frac{\mu_1 f_{n-1}}{\mu_1 f_{n-2} - f_{n-1}}\right\}. \tag{24}$$

Proof. Note that the upper bound to be obtained is nontrivial. By a direct consequence of Proposition 4.10 and Lemma 4.12 we have

$$spr_L(G) = \mu_1 - \mu_{n-1} \leq C + \frac{c_n}{c_{n-1}}.$$

On the other hand, $\mu_1 \leq n$ implies that

$$spr_L(G) = \mu_1 - \mu_{n-1} \leq n + \frac{c_n}{c_{n-1}}.$$

Taking into account (19), and setting $\alpha = \mu_1$, (24) holds. ■

Remark 4.15. *In Corollary 4.14 the parameter α can be equal to μ_{n-1} however, the function $\varphi(x) = \dfrac{xf_{n-1}}{xf_{n-2} - f_{n-1}}$ is non-increasing, as $\varphi'(x) = -\dfrac{(f_{n-1})^2}{(xf_{n-2} - f_{n-1})^2}$. Thus, by considering the largest known α in $[\mu_{n-1}, \mu_1]$ (for example $\alpha = \Delta + 1$ or $\alpha = \mu_1$) it is possible to get a tight upper bound for the Laplacian spread.*

5. Comparing bounds and final remarks

In this section we present some computational experiments to compare our new upper bounds to previously published upper bounds for certain connected graphs. We compare the estimates obtained by Theorem 3.3 (Th. 3.3), Theorem 4.13 (Th. 4.13), Corollary 4.14 (Cor. 4.14), Proposition 1 in [27] (Prop. 1 [27]), Theorem 4.3 in [27] (Th. 4.3 [27]), Theorem 4.7 in [27] (Th. 4.7 [27]) and Theorem 2.5 in [25] (Th. 2.5 [25]), with the actual Laplacian spread.

In the following tables, *ee* is the relative error, defined as usual:

$$ee = \frac{bound - spr_L(G)}{spr_L(G)}$$

G	$spr_L(G)$	$Th.$ 3.3	ee	$Th.$ 4.13	ee	$Cor.$ 4.14	ee
P_6	3.4642	4.5826	0.3228	3.8960	0.1246	**3.8361**	0.1074
P_7	3.6038	5.0563	0.4030	**3.6809**	0.0214	3.8790	0.0764
P_8	3.6956	5.4750	0.4815	3.9407	0.0663	**3.9071**	0.0572
P_{13}	3.8838	7.1388	0.8381	**3.9065**	0.0058	3.9646	0.0208
P_{18}	3.9392	8.4456	1.1440	3.9881	0.0124	**3.9815**	0.0107
P_{19}	3.9454	8.6818	1.2005	**3.9561**	0.0027	3.9834	0.0096
P_{20}	3.9508	8.9115	1.2556	3.9904	0.0100	**3.9850**	0.0087
S_9	8	10.6667	0.3333	8.8615	0.1077	**8.8615**	0.1077
S_{12}	11	14.8941	0.3540	12.0984	0.0999	**11.9016**	0.0820
S_{15}	14	19.1276	0.3663	14.9239	0.0660	**14.9239**	0.0660
S_{16}	15	20.5396	0.3693	16.0708	0.0714	**15.9292**	0.0619
S_{17}	16	21.9518	0.3720	16.9339	0.0584	**16.9339**	0.0584
$T(0,0,2)$	3.6513	4.4170	0.2097	**3.9097**	0.0708	4.0729	0.1155
$T(0,0,0,2)$	3.8894	4.9676	0.2772	4.3938	0.1297	**4.3205**	0.1108
$T(1,1,1)$	3.9208	4.9359	0.2589	4.4880	0.1447	**4.4814**	0.1430
$T(1,1,2)$	4.3070	5.7296	0.3303	**4.4814**	0.0405	4.8527	0.1267
$T(2,2,2)$	5.1816	7.3030	0.4094	**5.3444**	0.0314	5.8949	0.1377

G	$spr_L(G)$	$Prop.$1 [27]	ee	$Th.$4.3 [27]	ee	$Th.$4.7 [27]	ee	$Th.$2.5 [25]	ee
P_6	3.4642	6	0.7320	5.9161	0.7078	5.7321	0.6547	3.8667	0.1162
P_7	3.6038	7	0.9424	7.2457	1.0106	7.8019	1.1649	3.9048	0.0835
P_8	3.6956	8	1.1647	8.3666	1.2639	7.8478	1.1235	3.9286	0.0630
P_{13}	3.8838	13	2.3472	14.5504	2.7464	13.9419	2.5898	3.9744	0.0233
P_{18}	3.9392	18	3.5695	20.6155	4.2334	17.9696	3.5617	3.9869	0.0121
P_{19}	3.9454	19	3.8157	21.8815	4.5461	19.9727	4.0623	3.9883	0.0109
P_{20}	3.9508	20	4.0623	23.0651	4.8381	19.9754	4.0560	3.9895	0.0098
S_9	8	9	0.1250	15.9499	0.9937	15.8794	0.9849	−−	−
S_{12}	11	12	0.0909	22.6382	1.0580	21.9319	0.9938	−−	−
S_{15}	14	15	0.0714	29.3205	1.0943	27.9563	0.9969	−−	−
S_{16}	15	16	0.0667	31.5472	1.1031	29.9616	0.9974	−−	−
S_{17}	16	17	0.0625	33.7737	1.1109	31.9659	0.9979	−−	−
$T(0,0,2)$	3.6513	5	0.3694	5.8305	0.5968	5.6180	0.5386	−−	−
$T(0,0,0,2)$	3.8894	6	0.5427	7.1237	0.8316	7.7321	0.9880	4.8333	0.2427
$T(1,1,1)$	3.9208	6	0.5303	6.9161	0.7639	5.7321	0.4620	4.8333	0.2327
$T(1,1,2)$	4.3070	7	0.6253	8.2457	0.9145	7.8019	0.8115	5.8571	0.3599
$T(2,2,2)$	5.1816	9	0.7369	11.7980	1.2769	11.8794	1.2926	7.8889	0.5225

Analyzing the above examples, we observe the following:

- The upper bound in [27, Theorem 4.3] depends on the computation the clique number of the graph (and of its complement) which is an NP- hard problem. Therefore these are theoretical bounds, but for the graphs in the DIMACS collection, [14], these clique numbers have been computed.
- In all our test cases our bounds were better than existing bounds in the second table.
- For paths, the upper bound given by Corollary 4.14, is the best when n is even and for n odd, the upper bound given by Theorem 4.13 is the best.
- For stars, the best results are obtained by Corollary 4.14.
- For caterpillars, the best results are obtained by Theorem 4.13 and by Corollary 4.14.

Acknowledgement. We are grateful to the anonymous referee for all the corrections that improved the text. Enide Andrade and Helena Gomes are partially supported by CIDMA - Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), within project PEst-OE/MAT/UI4106/2013. M. Robbiano was supported by Project VRIDT-UCN 2014 N. 220202-10301403 and J. Rodríguez was supported by CONICYT-PCHA/Doctorado Nacional/2015-21150477.

[1] N. M. M. de Abreu. Old and new results on algebraic connectivity of graphs. Lin. Algebra Appl. 423 (2007): 53–73.

[2] W. N. Anderson, T. D. Morely. Eigenvalues of the Laplacian of a graph. Linear Multilin. Algebra 18 (1985): 141–145.

[3] E. Andrade, D. M. Cardoso, M. Robbiano, J. Rodríguez. Laplacian spread of graphs: Lower bounds and relations with invariant parameters. Lin. Algebra Appl. 486 (2015): 494–503.

[4] A. Brauer. Limits for the characteristic roots of a matrix. IV. Applications to stochastics matrices, Duke Math. J., 19-1, (1952): 75-91.

[5] Y. H. Bao, Y.Y. Tan, Y. Z. Fan. The Laplacian spread of unicyclic graphs. Appl. Math. Lett. 22 (2009): 1011–1015.

[6] A. Brouwer, W. Haemers. Spectra of Graphs. Springer-Verlag, Berlin, 2012.

[7] Y. Chen, L. Wang. The Laplacian spread of tricyclic graphs. Electron. J. Comb. 16 (2009) R(80).

[8] D. Cvetković, M. Doob and H. Sachs. Spectra of Graphs - Theory and Application. Academic Press, New York, 1980.

[9] Y. Fan, S. Li, Y. Tan. The Laplacian spread of bicyclic graphs. J. Math. Res. Expo. 30 (2010): 17–28.

[10] M. Fiedler. Algebraic connectivity of graphs. Czechoslovak Mathematical Journal 23 (1973): 298–305.

[11] I. Gutman, K. C. Das. The first Zagreb index 30 years after. MATCH Comm. Math. Comput. Chem. 50 (2004): 82–92.

[12] A. K. Kel'mans. Properties of the characteristic polynomial of a graph. Kibernetiky-na sluzbu kommunizmu 4 Energija, Moskvz-Leningrad (1967): 27-41 (in Russian)

[13] C. R. Johnson, R. Kumar, H. Wolkowicz. Lower bounds for the spread of a matrix. Linear Algebra Appl. 71 (1985): 161–173.

[14] D. S. Johnson, M. A. Trick. Cliques, coloring, and satisfiability: second DIMACS challenge. DIMACS Series in Discrete Mathematics and Theoretical Computer Science, American Mathematical Society, Providence, 1996.

[15] J. S. Li, X. D. Zhang. On the Laplacian eigenvalues of a graph. Lin. Algebra Appl. 285 (1998): 305–307.

[16] J. Li, W. C. Shiu, W. H. Chan. Some results on the Laplacian eigenvalues of unicyclic graphs. Lin. Algebra Appl. 430 (2009): 2080-2093.

[17] P. Li, J. S. Shi, R. L. Li. Laplacian spread of bicyclic graphs. J. East China Norm. Univ. (Nat. Sci. Ed.) 1 (2010): 6-9. (In Chinese).

[18] Mirsky. The spread of a matrix. Mathematika 3 (1956): 127–130.

[19] R. Merris. Laplacian matrices of a graph: A survey. Lin. Algebra Appl. 197&198 (1994): 143–176.

[20] R. Merris. A note on Laplacian graph eigenvalue. Linear Algebra Appl. 285 (1998): 33-35.

[21] P. Nylen, T. Y. Tam. On the spread of a hermitian matrix and a conjecture of Thompson. Linear and Multilinear Algebra 37 (1994): 3–11.

[22] O. Rojo, L. Medina, N. M. M. Abreu, C. Justel. Extremal algebraic connectivities of certain caterpillar classes and symmetric caterpillars. Electron. J. Linear Algebra 20 (2010): 136–157.

[23] O. Rojo, L. Medina, N. M. M. Abreu, C. Justel. On the algebraic connectivity of some caterpillars: a sharp upper bound and total ordering. Lin. Algebra Appl. 432 (2010): 586–605.

[24] Z. You, B. Liu. The minimum Laplacian spread of unicyclic graphs. Linear Algebra Appl. 432 (2010): 499-504.

[25] M. Zhai, J. Shu, Y. Hong. On the Laplacian spread of graphs. Appl. Math. Lett. 24(12) (2011): 2097-2101.

[26] Y. Zheng Fan, X. Jing, Y. Wang, D. Liang. The Laplacian spread of a tree. Discrete Mathematics and Theoretical Computer Science, 10:1 (2008): 79-86.

[27] Z. You, B. Liu, Guangzhou. The Laplacian spread of graphs. Czechoslovak Mathematical Journal, 62 (137) (2012): 155-168.