Laplacian spread of graphs: lower bounds and relations with invariant parameters

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Abstract

The spread of an $n \times n$ complex matrix B with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$ is defined by

$$s(B) = \max_{i,j} |\beta_i - \beta_j|,$$

where the maximum is taken over all pairs of eigenvalues of B. Let G be a graph on n vertices. The concept of Laplacian spread of G is defined by the difference between the largest and the second smallest Laplacian eigenvalue of G. In this work, by combining old techniques of interlacing eigenvalues and rank 1 perturbation matrices new lower bounds on the Laplacian spread of graphs are deduced, some of them involving invariant parameters of graphs, as it is the case of the bandwidth, independence number and vertex connectivity.

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1. Introduction

By an (n, m)-graph G we mean an undirected simple graph with a vertex set $\mathcal{V}(G)$ of cardinality n (the order of the graph) and an edge set $\mathcal{E}(G)$ of cardinality m (the size of the graph). If $e \in \mathcal{E}(G)$ has end vertices u and v we say that u and v are adjacent and we denote this edge by uv. For $u \in \mathcal{V}(G)$, the number of vertices adjacent to u is denoted by d_u and is called the vertex degree of u. The smallest and largest vertex degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A k-regular graph G is a graph where every vertex has degree k. The complement of a graph G is denoted by $\overline{G}$. A set of vertices that induces a subgraph with no edges is called an independent set and a maximum independent set of a graph G is an independent set of largest cardinality $\alpha(G)$ which is called the independence number of G. A maximal independent set is an independent set not included in an independent set of larger cardinality.

The adjacency matrix of graph G is $A(G)$ and its vertex degree matrix is the diagonal matrix of the vertex degrees $D(G)$. The Laplacian matrix of G is the positive semidefinite matrix given by $L(G) = D(G) - A(G)$. Note that 0 is always a Laplacian eigenvalue with $\mathbf{e}$, the all one vector, as an associated eigenvector and whose multiplicity corresponds to the number of connected components of G.

A symmetric matrix $M = (m_{ij})$ is said to have bandwidth ω if $m_{ij} = 0$ for all (i, j) satisfying $|i - j| > \omega$. The bandwidth $\omega(G)$ of the graph G is the smallest possible bandwidth for its adjacency matrix (or Laplacian matrix).

The spread of an $n \times n$ complex hermitian matrix M with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$ is defined by

$$s(M) = \max_{i,j} |\beta_i - \beta_j|,$$

where the maximum is taken over all pairs of eigenvalues of M. There are several papers devoted to this parameter, see for instance [8, 9, 13, 15].

For the remaining basic terminology and notation used throughout the paper we refer the book [4].

2. Some preliminary results

Among the results obtained for the spread of symmetric matrices $M = (m_{ij})$ we outline the following lower bound obtained in [2],

$$s(M) \geq \max_{i,j} \left((m_{jj} - m_{ii})^2 + 2 \sum_{s \neq j} |m_{js}|^2 + 2 \sum_{s \neq i} |m_{is}|^2 \right)^{1/2}. \qquad (1)$$

In [11], this lower bound was presented as the best lower bound for symmetric matrices.

On the other hand, the Laplacian spread of a graph G, $s_L(G)$, is the difference among the largest and the second smallest Laplacian eigenvalue of G. In [17] it was proven that among the trees of order n, the star, S_n, is the unique tree with maximal Laplacian spread, and the path, P_n is the unique tree with minimal Laplacian spread. In this work, using some known results on the Laplacian spectrum and a known theorem concerning a rank one perturbation of matrices [3], new lower bounds on the Laplacian spread of G are deduced.

For a real symmetric matrix M, denote by $\beta_i(M)$ and σ_M the i-th largest eigenvalue of M and the multiset of eigenvalues of M, respectively. If $M = L(G)$, then its multiset of eigenvalues, $\sigma_L(G)$, is called the Laplacian spectrum of G.

The Laplacian eigenvalues, $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$, $\mu_1(\overline{G}) \geq \mu_2(\overline{G}) \geq \cdots \geq \mu_n(\overline{G}) = 0$ of G and $\overline{G}$, respectively, are related by $\mu_j(\overline{G}) = n - \mu_{n-j}(G)$, for $j = 1, 2, \ldots, n-1$. As immediate consequence, we may conclude that for an arbitrary graph G, $\mu_1(G) \leq n$ and this upper bound is attained if and only if the complement of G is disconnected [1]. Moreover, both Laplacian spread coincide. A classical result in spectral graph theory, see [12], states that if G has at least one edge then $\Delta(G) + 1 \leq \mu_1(G)$. Furthermore, assuming that G is connected with n vertices, this inequality holds as equality if and only if $\Delta(G) = n - 1$.

The algebraic connectivity of a graph G, introduced in [5], is defined as the second smallest Laplacian eigenvalue $\mu_{n-1}(G)$. From the properties of this eigenvalue we may say that it measures the connectivity of the graph. In fact, a graph G is connected if and only if $\mu_{n-1}(G) > 0$, see [5].

It is known, see [5], that if G is a non-complete graph, then $\mu_{n-1}(G) \leq \upsilon(G)$, where $\upsilon(G)$ denotes the vertex connectivity of G (that is, the minimum number of vertices whose removal yields a disconnected graph). The graphs for which the algebraic connectivity attains the vertex connectivity are characterized in [10] by the following result.

Theorem 2.1. *[10] Let G be a non-complete connected graph on $n \geq 3$ vertices. Then $\mu_{n-1}(G) = \upsilon(G)$ if and only if G is isomorphic to the join operation of the graphs G_1 and G_2, $G_1 \vee G_2$, where G_1 is a disconnected graph on $n - \upsilon(G)$ vertices and G_2 is a graph on $\upsilon(G)$ vertices, with $\mu_{n-1}(G_2) \geq 2\upsilon(G) - n$.*

Since, $\upsilon(G) \leq \delta(G)$, it follows that $\mu_{n-1}(G) \leq \delta(G)$ and then

$$s_L(G) \geq \Delta(G) + 1 - \delta(G). \tag{2}$$

Regarding lower bounds on the Laplacian spread of a graph it is also worth to recall the following results.

Theorem 2.2. *[16] For a connected (n,m)-graph G, with $n \geq 3$, $s_L(G) \geq \frac{(n-1)\mu_1(G)-2m}{n-2}$. Equality holds if and only if $G \cong K_{1,n-1}$ or $G \cong K_{n/2,n/2}$, where $K_{r,s}$ denotes a complete bipartite graph.*

As immediate consequence, assuming the hypothesis of Theorem 2.2 and taking into account that $\mu_1(G) \geq \Delta(G) + 1$, we may conclude that

$$s_L(G) \geq \frac{(n-1)(\Delta(G)+1) - 2m}{n-2}. \tag{3}$$

Theorem 2.3. *[16] Let G be a connected non-complete graph with $n \geq 3$ vertices. Then $s_L(G) \geq \Delta(G) + 1 - \delta(G)$ and the inequality holds as equality if and only if $\frac{\mu_1(G)}{\mu_{n-1}(G)} = \frac{\Delta(G)+1}{\delta(G)}$.*

Remark 2.4. *From Theorem 2.3, a graph G of order n for which the equality $s_L(G) = \Delta(G) + 1 - \delta(G)$ holds is such that $\mu_{n-1}(G) = \upsilon(G) = \delta(G)$ and $\mu_1(G) = \Delta(G) + 1$ (notice that $\delta(G) \geq \upsilon(G) \geq \mu_{n-1}(G) = \frac{\delta(G)}{\Delta(G)+1}\mu_1(G) \geq \delta(G)$) and then $\Delta(G) = n - 1$. Therefore, according to Theorem 2.1, this graph is isomorphic to the graph $G = G_1 \vee G_2$, where G_1 is a disconnected graph on $n - \delta(G)$ vertices and G_2 is a graph on $\delta(G)$ vertices, with $\mu_{n-1}(G_2) \geq 2\delta(G) - n$.*

Concerning connected regular graphs, in [6] the following lower bound was obtained.

Theorem 2.5. *[6] Let G be a connected k-regular graph with $n \geq 3$ vertices. Then $s_L(G) \geq 2\sqrt{\frac{k(n-k-1)}{n}}$.*

Before we proceed to the next section, it is worth recalling the following lemma proved in [7].

Lemma 2.6. *[7] Let X and Y be vertex disjoint sets of a graph G of order n such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \left(\frac{\mu_1(G) - \mu_{n-1}(G)}{\mu_1(G) + \mu_{n-1}(G)}\right)^2.$$

3. Lower bounds obtained by interlacing and relations with the bandwidth

In what follows, based on Lemma 2.6, a few lower bounds on the Laplacian spread of a graph are obtained.

Theorem 3.1. *Let G be a graph of order n, maximum degree Δ and diameter d. If X and Y are disjoint vertex sets such that there is no edge between X and Y, then*

$$s_L(G) \geq \left(\Delta + 1 + \frac{4}{nd}\right)\sqrt{\frac{|X|\,|Y|}{(n-|X|)\,(n-|Y|)}}.$$

Proof. From Lemma 2.6, it follows that

$$\begin{aligned} \frac{\mu_1(G) - \mu_{n-1}(G)}{\mu_1(G) + \mu_{n-1}(G)} &\geq \sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}} \\ &\Updownarrow \\ s_L(G) &\geq (\mu_1(G) + \mu_{n-1}(G))\sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}}. \qquad (4) \end{aligned}$$

Since $\mu_1(G) \geq \Delta + 1$ and taking into account the inequality $\mu_{n-1}(G) \geq \frac{4}{nd}$ obtained in [14], the result follows. ■

By noticing that in $\overline{G}$ each vertex in X is connected to each vertex in Y, we obtain the following corollary.

Corollary 3.2. *Let X and Y be disjoint sets of vertices of a graph G, such that each vertex in X is connected to each one in Y. Then*

$$s_L(G) \geq (n - \delta(G))\sqrt{\frac{|X|\,|Y|}{(n-|X|)\,(n-|Y|)}}.$$

Proof. Since $\mu_j(\overline{G}) = n - \mu_{n-j}(G)$ for $1 \leq j \leq n-1$, $s_L(G) = s_L(\overline{G})$. Moreover, applying inequality (4) to $\overline{G}$ it follows that

$$\begin{aligned} s_L(G) &\geq (\mu_1(\overline{G}) + \mu_{n-1}(\overline{G}))\sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}} \\ &= (n - \mu_{n-1}(G) + n - \mu_1(G))\sqrt{\frac{|X||Y|}{(n-|X|)(n-|Y|)}}. \end{aligned}$$

Therefore, taking into account that $\mu_{n-1}(G) \leq \delta(G)$ and $\mu_1(G) \leq n$, the result follows. ■

As an application of the previous results, some relations between the Laplacian spread and the bandwidth of a graph are obtained.

Theorem 3.3. *Let $\omega = \omega(G)$ be the bandwidth of a graph G of order n, maximum degree Δ and diameter d, then*

$$s_L(G) \geq \begin{cases} \dfrac{\left(\Delta + 1 + \frac{4}{nd}\right)(n-\omega)}{n+\omega}, & \text{if } n-\omega \text{ is even;} \\[2ex] \dfrac{\left(\Delta + 1 + \frac{4}{nd}\right)(n-\omega-1)}{n+\omega+1}, & \text{if } n-\omega \text{ is odd.} \end{cases}$$

Proof. If $n-\omega$ is even, then none of the first $\frac{n-\omega}{2}$ vertices are linked by an edge to the last $\frac{n-\omega}{2}$ vertices. Applying Theorem 3.1, with $|X| = |Y| = \frac{n-\omega}{2}$, we get

$$\frac{\left(\Delta + 1 + \frac{4}{nd}\right)\frac{n-\omega}{2}}{\frac{n+\omega}{2}} \leq s_L(G)$$

and the first inequality holds. On the other hand, if $n-\omega$ is odd, then none of the first $\frac{n-\omega-1}{2}$ vertices are linked by an edge to the last $\frac{n-\omega-1}{2}$ vertices. Applying Theorem 3.1, with $|X| = |Y| = \frac{n-\omega-1}{2}$, we get

$$\frac{\left(\Delta + 1 + \frac{4}{nd}\right)\frac{n-\omega-1}{2}}{\frac{n+\omega+1}{2}} \leq s_L(G),$$

and the second inequality holds. ■

When $n-\omega$ is odd, the lower bound of Theorem 3.3 can be improved by applying Theorem 3.1 with $|X| = \frac{n-\omega+1}{2}$ and $|Y| = \frac{n-\omega-1}{2}$. Thus, we have

$$s_L(G) \geq \left(\Delta + 1 + \frac{4}{nd}\right)\sqrt{\frac{(n-\omega)^2 - 1}{(n+\omega)^2 - 1}}.$$

Regarding $\overline{G}$ it is immediate to obtain the following corollary.

Corollary 3.4. *Let G be a graph with minimum degree δ and such that $\overline{\omega} = \omega(\overline{G})$ is the bandwidth of the graph $\overline{G}$. Then*

$$s_L(G) \geq \begin{cases} \dfrac{(n-\delta)(n-\overline{\omega})}{n+\overline{\omega}}, & \text{if } n-\overline{\omega} \text{ is even;} \\[2ex] \dfrac{(n-\delta)(n-\overline{\omega}-1)}{n+\overline{\omega}+1}, & \text{if } n-\overline{\omega} \text{ is odd.} \end{cases}$$

4. Lower bounds obtained by rank one perturbations and relations with the independence number and vertex connectivity

The next theorem, due to Brauer [3], relates the eigenvalues of an arbitrary matrix with the eigenvalues of a matrix resulting from it after a rank one additive perturbation.

Theorem 4.1. *[3] Let M be an arbitrary $n \times n$ matrix with eigenvalues $\beta_1, \beta_2, \ldots, \beta_n$. Let $\mathbf{u}_k$ be an eigenvector of M associated with the eigenvalue β_k, and let $\mathbf{q}$ be an arbitrary n-dimensional vector. Then the matrix $M+\mathbf{u}_k\mathbf{q}^t$ has eigenvalues $\beta_1, \ldots, \beta_{k-1}, \beta_k + \mathbf{u}_k^t\mathbf{q}, \beta_{k+1}, \ldots, \beta_n$.*

Using Theorem 4.1 and the inequality (1) we obtain the following results:

Theorem 4.2. *Let G be a graph of order $n \geq 3$ with at least one edge, such that $\delta = \delta(G)$ and $\Delta = \Delta(G)$. Then*

$$s_L(G) \geq \sqrt{(\Delta-\delta)^2 + 2(\Delta+\delta) - \left(2\frac{\Delta+1}{n}\right)^2 \left(\frac{\delta-1}{\Delta+1}n+1\right)}. \qquad (5)$$

In particular, if G is k-regular, then

$$s_L(G) \geq \sqrt{4k - \left(2\frac{k+1}{n}\right)^2 \left(\frac{k-1}{k+1}n+1\right)}. \qquad (6)$$

Proof. Recall that $(0, \mathbf{e})$ is an eigenpair of $L(G)$. By Theorem 4.1 it should be noted that for any scalar α such that $\mu_{n-1} \leq \alpha \leq \mu_1$, setting $\xi = \frac{\alpha}{n}$, $s_L(G) = s(M_\xi)$, where $M_\xi = L(G)+\xi\mathbf{e}\mathbf{e}^t$. From the definition of $M_\xi = (m_{ij})$ it follows that

$$m_{ij} = \begin{cases} -1+\xi, & \text{if } v_iv_j \in \mathcal{E}(G); \\ \xi, & \text{if } v_iv_j \notin \mathcal{E}(G); \\ d_i+\xi, & \text{if } i=j. \end{cases} \qquad (7)$$

By using the inequality (1), we obtain

$$s(M_\xi) \geq \max_{i<j}\left\{\sqrt{(d_i-d_j)^2 + 4(n-1)\xi^2 + 2(1-2\xi)(d_i+d_j)}\right\}. \qquad (8)$$

Thus, considering $\xi = \frac{\alpha}{n}$, with $\mu_{n-1}(G) \leq \alpha \leq \mu_1(G)$

$$\begin{aligned} s_L(G) &= s\left(M_{\frac{\alpha}{n}}\right) \\ &\geq \max_{i<j}\left\{\sqrt{(d_i-d_j)^2 + 4(n-1)\frac{\alpha^2}{n^2} + 2(1-2\frac{\alpha}{n})(d_i+d_j)}\right\}. \quad (9) \end{aligned}$$

Now, let us define the function

$$\begin{aligned} f_{ij} : [0,n] &\rightarrow \mathbb{R} \\ \alpha &\mapsto f_{ij}(\alpha) = 4(n-1)\frac{\alpha^2}{n^2} + 2(1-\frac{2}{n}\alpha)(d_i+d_j). \qquad (10) \end{aligned}$$

Then the inequality (9) can be written as

$$s_L(G) \geq \max_{i<j}\sqrt{(d_i-d_j)^2 + f_{ij}(\alpha)}.$$

Since the second derivative of $f_{ij}(\alpha)$ is positive, this function is convex and thus, considering a closed interval $I \subset [0,n]$ its maximum in I is attained in one its extremes. If we consider $I = [\delta, \Delta+1]$ (notice that $I \subseteq [\mu_{n-1}(G), \mu_1(G)]$), it is immediate that $f_{ij}(\Delta+1) \geq f_{ij}(\delta)$. Then

$$\begin{aligned} s_L(G)^2 &\geq \max_{i<j}\{(d_i-d_j)^2 + f_{ij}(\Delta+1)\} \\ &\geq (\Delta-\delta)^2 + (n-1)\left(2\frac{\Delta+1}{n}\right)^2 + 2(1-\frac{2}{n}(\Delta+1))(\Delta+\delta) \\ &= (\Delta-\delta)^2 + 2(\Delta+\delta) - \left(2\frac{\Delta+1}{n}\right)^2\left(1+\frac{\delta-1}{\Delta+1}n\right). \end{aligned}$$

■

Remark 4.3. *When G is a graph of order $n > 1$ with at least one edge, such that $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we may conclude the following inequalities.*

(i) If $\Delta \leq \frac{3}{4}n - 1$, then

$$\sqrt{(\Delta-\delta)^2 + 2(\Delta+\delta) - \left(2\frac{\Delta+1}{n}\right)^2\left(\frac{\delta-1}{\Delta+1}n+1\right)} > \Delta - \delta + 1.$$

(ii) If G is k-regular, then

$$\sqrt{4k-\left(2\frac{k+1}{n}\right)^2\left(\frac{k-1}{k+1}n+1\right)}\geq 2\sqrt{\frac{k(n-k-1)}{n}}.$$

Therefore, the lower bound (5) for arbitrary graphs G of order $n \geq 3$ is better than the one obtained in [16] (see Theorem 2.3) when $\Delta(G)+1 \leq \frac{3}{4}n$, and the lower bound (6) obtained for k-regular graphs of order $n \geq 3$ is better than the one obtained in [6] (see Theorem 2.5).

Proof. The above inequalities can be proven as follows.

(i) Taking into account that $\Delta \leq \frac{3}{4}n-1$,

$$\begin{aligned}\left(2\frac{\Delta+1}{n}\right)^2\left(\frac{\delta-1}{\Delta+1}n+1\right) &= 4\frac{\Delta+1}{n}\left(\delta-1+\frac{\Delta+1}{n}\right)\\ &< 4\delta-4\left(1-\frac{\Delta+1}{n}\right)\\ &\leq 4\delta-1.\end{aligned}$$

Therefore, $(\Delta-\delta)^2+2(\Delta+\delta)-\left(2\frac{\Delta+1}{n}\right)^2\left(\frac{\delta-1}{\Delta+1}n+1\right) > (\Delta-\delta)^2+2(\Delta+\delta)-4\delta+1 = (\Delta-\delta)^2+2(\Delta-\delta)+1 = \left((\Delta-\delta)+1\right)^2$.

(ii) Since $k-(1-\frac{k+1}{n}) \leq k \Leftrightarrow \frac{k+1}{n}\left(k-(1-\frac{k+1}{n})\right) \leq \frac{k+1}{n}k$, we have

$$\begin{aligned}\left(\frac{k+1}{n}\right)^2\left(\frac{k-1}{k+1}n+1\right) &\leq \frac{k+1}{n}k\\ &\Updownarrow\\ k-\left(\frac{k+1}{n}\right)^2\left(\frac{k-1}{k+1}n+1\right) &\geq k-\frac{k+1}{n}k,\end{aligned}$$

and (4.3) follows.

■

By combining the techniques used in the proofs of Theorems 3.1, 3.3 and 4.2, we obtain the following lower bound for $s_L(G)$.

Theorem 4.4. *Let G be an (n,m)-graph with at least one edge and vertex connectivity $\upsilon = \upsilon(G)$. Assume that $T = \{v_1, v_2, \ldots, v_\alpha\}$ is a maximal independent set of G and the subgraph of G induced by the vertex subset $V(G) \setminus T$ has m' edges. Then*

$$s_L(G) \geq \left| \frac{n(m-m')}{\alpha(n-\alpha)} - \upsilon \right|.$$

The equality holds for $G = S_n$, the star of order n.

Proof. Denoting by $\ominus_{\alpha\times\alpha}$ the all zero square matrix of order α, the adjacency matrix of G is

$$A(G) = \begin{pmatrix} \ominus_{\alpha\times\alpha} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{22} is the adjacency matrix of the subgraph induced by the vertex subset $V(G) \setminus T$. Let us denote by I_p and $J_{p\times q}$ the identity matrix of order p and the all one $p \times q$ matrix, respectively. Considering the perturbed matrix $M_\xi(G) = L(G) + \xi \mathbf{e}\mathbf{e}^t = L(G) + \xi J_{n\times n}$, with $\xi = \frac{\upsilon}{n}$, it is immediate that $s_L(G) = s(M_\xi(G))$. Notice that

$$M_\xi(G) = \begin{pmatrix} D_\alpha + \xi J_{\alpha\times\alpha} & -A_{12} + \xi J_{\alpha\times(n-\alpha)} \\ -A_{21} + \xi J_{(n-\alpha)\times\alpha} & D_{(n-\alpha)} - A_{22} + \xi J_{(n-\alpha)\times(n-\alpha)} \end{pmatrix},$$

where $D = \begin{pmatrix} D_\alpha & \ominus_{\alpha\times(n-\alpha)} \\ \ominus_{(n-\alpha)\times\alpha} & D_{n-\alpha} \end{pmatrix}$ is the diagonal matrix of the vertex degrees of G. Then, the quotient matrix of $M_\xi(G)$ is

$$B = \begin{pmatrix} \frac{m-m'}{\alpha} + \alpha\xi & (n-\alpha)\xi - \frac{m-m'}{\alpha} \\ \alpha\xi - \frac{m-m'}{n-\alpha} & (n-\alpha)\xi + \frac{m-m'}{n-\alpha} \end{pmatrix}.$$

From $n\xi \in \sigma(B)$ we conclude that $s(B) = \left| n\xi - \frac{n(m-m')}{\alpha(n-\alpha)} \right| = \left| \frac{n(m-m')}{\alpha(n-\alpha)} - \upsilon \right|$. Therefore, since by the Interlacing Theorem $s(B) \leq s(M_\xi(G))$, the result follows. The second part is immediate, taking into account the Laplacian spectrum of a star. ∎

Considering a maximum independent set T of the graph G in the hypothesis of Theorem 4.4, we may conclude that

$$s_L(G) \geq \frac{n(m-m')}{\alpha(G)(n-\alpha(G))} - \upsilon(G). \tag{11}$$

In fact, since $m - \alpha(G)\Delta(G) \leq m' \leq m - \alpha(G)\delta(G)$, it follows that

$$\frac{n}{n-\alpha(G)}\delta(G) \leq \frac{n(m-m')}{\alpha(G)(n-\alpha(G))} \leq \frac{n}{n-\alpha(G)}\Delta(G),$$

and therefore $0 \leq \frac{n}{n-\alpha(G)}\delta(G) - \upsilon(G) \leq \frac{n(m-m')}{\alpha(G)(n-\alpha(G))} - \upsilon(G)$. Furthermore, from the inequality (11), we also have

$$s_L\left(G\right) \geq \frac{n}{n-\alpha(G)}\delta(G) - \upsilon(G) \tag{12}$$

and, as immediate consequence, $\alpha(G) \leq n\frac{s_L(G)}{s_L(G)+\delta(G)}$. Notice that this upper bound on the independence number is attained when G is a star.

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