

Chapter 11

Burst Erasure Correction of 2D convolutional codes*

Joan-Josep Climent, Diego Napp, Raquel Pinto, and Rita Simões

Abstract In this paper we address the problem of decoding 2D convolutional codes over the erasure channel. In particular, we present a procedure to recover bursts of erasures that are distributed in a diagonal line. To this end we introduce the notion of balls around a burst of erasures which can be considered an analogue of the notion of sliding window in the context of 1D convolutional codes. The main result reduces the decoding problem of 2D convolutional codes to a problem of decoding a set of associated 1D convolutional codes.

Key words 2D convolutional codes, erasure channel

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1 Introduction

When transmitting over an erasure channel the symbol sent either arrive correctly or they are erased. Internet is an important instance of such a channel. One of the problems that arises in this channel is that some packets get lost and the receiver experience it as a delay on the received information. The solutions proposed to deal with this problem are commonly based on the use of block codes. However, in recent years, there has been an increased interest in the study of one-dimensional (1D) convolutional codes over the erasure channel [2, 8–10] as a possible alternative for the widely use of block codes. Due to their rich structure 1D convolutional codes have an interesting property called sliding window property that allows adaptation to the correction process to the distribution of the erasure pattern. In the recent paper [10] it has been shown how it is possible to exploit this property in order to easily recover erasures which are uncorrectable by any other kind of (block) codes. The codes proposed in this paper are codes with strong distance properties, called Maximal Distance Profile (MDP), reverse-MDP and complete-MDP, and simulations results have shown that they can decode extremely efficiently when compared to MDS block codes.

In the 1D case, if the received codeword is viewed as a finite sequence $v = (v_0, v_1, \dots, v_\ell)$, then the sliding windows is given by selecting a subsequence of v , (v_i, \dots, v_{i+N}) , where $i, N \in \mathbb{N}$ depend on the erasure burst pattern. However, when considering two-dimensional (2D) convolutional codes [4–7, 11] the information is distributed in two dimensions and therefore there is not an obvious way to extend the idea of sliding window to the 2D case. In this work we propose several solutions for dealing with this problem by introducing the notion of *balls* around an erasure. We show that when considering these particular balls one reduces the problem of decoding 2D convolutional codes over the erasure channel to a problem related to decoding of 1D convolutional codes.

2 2D convolutional codes

In this section we recall the basic background on 2D finite support convolutional codes. Denote by $\mathbb{F}[z_1, z_2]$ the ring of polynomials in the two variables, z_1 and z_2 , with coefficients in the finite field \mathbb{F} .

Definition 1. A 2D finite support convolutional code \mathcal{C} of rate k/n is a free $\mathbb{F}[z_1, z_2]$ -submodule of $\mathbb{F}[z_1, z_2]^n$ with rank k .

A full column rank polynomial matrix $\widehat{G}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n \times k}$ whose columns constitute a basis for \mathcal{C} , i.e., such that

$$\begin{aligned} \mathcal{C} &= \text{im}_{\mathbb{F}[z_1, z_2]} \widehat{G}(z_1, z_2) \\ &= \left\{ \widehat{v}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n \mid \widehat{v}(z_1, z_2) = \widehat{G}(z_1, z_2) \widehat{u}(z_1, z_2) \text{ with } \widehat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^k \right\}, \end{aligned}$$

is called an *encoder* of \mathcal{C} . The elements of \mathcal{C} are called *codewords*.

If the code \mathcal{C} admits a right factor prime encoder [3], then it can be equivalently described using an $(n-k) \times n$ full rank polynomial matrix $\widehat{H}(z_1, z_2)$, called *parity-check matrix* of \mathcal{C} , as

$$\mathcal{C} = \ker_{\mathbb{F}[z_1, z_2]} \widehat{H}(z_1, z_2) = \left\{ \widehat{v}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n \mid \widehat{H}(z_1, z_2) \widehat{v}(z_1, z_2) = 0 \right\}.$$

We denote by \mathbb{N}_0 the set of nonnegative integers, and define an ordering in \mathbb{N}_0^2 as

$$(a, b) \prec (c, d) \text{ if and only if } a + b < c + d, \text{ or } a + b = c + d \text{ and } b < d. \quad (1)$$

For a polynomial vector $\widehat{v}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n$, we write

$$\widehat{v}(z_1, z_2) = v(0, 0) + v(1, 0)z_1 + v(0, 1)z_2 + \cdots + v(0, \gamma)z_2^\gamma = \sum_{0 \leq a+b \leq \gamma} v(a, b)z_1^a z_2^b,$$

(with $\gamma \geq 0$) and we define its support as the set

$$\text{supp}(\widehat{v}(z_1, z_2)) = \{(a, b) \in \mathbb{N}_0^2 \mid v(a, b) \neq 0\}.$$

Moreover, we represent a polynomial matrix $\widehat{H}(z_1, z_2)$ as

$$\widehat{H}(z_1, z_2) = H(0, 0) + H(1, 0)z_1 + H(0, 1)z_2 + \cdots + H(0, \delta)z_2^\delta = \sum_{0 \leq i+j \leq \delta} H(i, j)z_1^i z_2^j, \quad (2)$$

where $H(i, j) \neq 0$ for some (i, j) with $i + j = \delta$. We call δ the degree of $\widehat{H}(z_1, z_2)$.

The *weight* of $\widehat{v}(z_1, z_2)$ is defined as

$$\text{wt}(\widehat{v}(z_1, z_2)) = \sum_{(a, b) \in \mathbb{N}_0^2} \text{wt}(v(a, b))$$

where $\text{wt}(v(a, b))$ is the number of nonzero entries of $v(a, b)$ and the *distance* of a code is

$$\text{dist}(\mathcal{C}) = \min \{ \text{wt}(\widehat{v}(z_1, z_2)) \mid \widehat{v}(z_1, z_2) \in \mathcal{C}, \text{ with } \widehat{v}(z_1, z_2) \neq 0 \}.$$

We can expand the kernel representation

$$\widehat{H}(z_1, z_2) \widehat{v}(z_1, z_2) = \sum_{0 \leq a+b \leq \gamma} \left[\sum_{0 \leq i+j \leq \delta} H(i, j)v(a-i, b-j) \right] z_1^a z_2^b = 0$$

as

$$\mathbf{H}\mathbf{v} = 0 \quad (3)$$

where \mathbf{H} , for $\delta = 3$, and \mathbf{v} are given in Figure 1, where O denotes the $(n-k) \times n$ zero matrix. To understand the structure of matrix \mathbf{H} , note that for $t = 0, 1, 2, \dots$ in the columns corresponding to the block indices $\frac{t(t+1)}{2} + 1, \frac{t(t+1)}{2} + 2, \dots, \frac{t(t+1)}{2} + t + 1$ appear all the coefficient matrices of $\widehat{H}(z_1, z_2)$ ordered according to \prec with the

were the components of the vector $\mathbf{v}_{\mathcal{E}}$ that are considered to the indeterminates to be determined. Thus, in order to decode $\mathbf{v}_{\mathcal{E}}$ we need to solve system (4).

The next lemma shows the importance of the distance of a code when transmitting over the erasure channel.

Lemma 2. Let $\mathcal{C} = \ker_{\mathbb{F}[z_1, z_2]} \widehat{H}(z_1, z_2)$ be given. The following are equivalent:

1. $\text{dist}(\mathcal{C}) \geq d$.
2. Any $d - 1$ erasures can be recovered.
3. Any $d - 1$ columns of $\mathbf{H}_{\mathcal{E}}$ are linearly independent.

In the context of 1D convolutional codes the analogous set of homogeneous equations of (3) is

$$\begin{bmatrix} H_0 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ H_{\alpha} & \cdots & H_0 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & H_{\alpha} & \cdots & H_0 \\ & & & & & & \ddots & \\ & & & & & & & H_{\alpha} \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\gamma} \end{bmatrix} = 0, \quad (5)$$

where $\mathcal{C} = \ker \widehat{H}(z)$ with $\widehat{H}(z) = H_0 + H_1 z + \cdots + H_{\alpha} z^{\alpha}$.

In this case every component of the received codeword $v = (v_0, v_1, \dots, v_{\gamma})$ depends on the previous α components. In order to find the values of a burst of erasures occurring in v , we can use the so-called *sliding window*, that is, we can select a suitable interval of consecutive components of v , say (v_i, \dots, v_{i+N}) , and solve the corresponding system of equations (see [10]).

In the 2D case each component of \mathbf{v} , say $v(a, b)$, depends on components which support lie in the *triangle* $\{(a - i, b - j) \mid 0 \leq i + j \leq \delta\}$, where δ is the degree of $\widehat{H}(z_1, z_2)$ of the given 2D code $\mathcal{C} = \ker_{\mathbb{F}[z_1, z_2]} \widehat{H}(z_1, z_2)$. It is not straightforward to extend the notion of the sliding window in this context in order to correct burst of 2D erasures. A particular case is treated in the following section.

3 Decoding burst of erasures on lines

It is well-known that a phenomena observed in many channels modeled via the erasure channel is that errors tend to occur in bursts. This point is important to keep in mind when designing codes which are capable of correcting many errors over the erasure channel. In this preliminary work we aim at decoding burst of erasures that are distributed in a diagonal. We present a notion that can be considered as the analogue of the notion of sliding window, called *ball around a burst of erasures*, that will reduce the problem of decoding a 2D convolutional code to the problem of decoding a set of associated 1D convolutional codes.

Let us first suppose that the set of erasures of $\widehat{v}(z_1, z_2)$ contains a burst of erasures which support lie in a diagonal, i.e., given by

$$\mathcal{E}'(\widehat{v}(z_1, z_2)) = \{(r+t, s), (r+t-1, s+1), \dots, (r, s+t)\} \subset \mathcal{E}(\widehat{v}(z_1, z_2)). \quad (6)$$

Hence, equation (3) can be divided as

$$\mathbf{H}_{\mathcal{E}'} \mathbf{v}_{\mathcal{E}'} = -\mathbf{H}_{\bar{\mathcal{E}}'} \mathbf{v}_{\bar{\mathcal{E}}'} \quad (7)$$

where $\mathbf{H}_{\mathcal{E}'}$ and $\mathbf{H}_{\bar{\mathcal{E}}'}$ are submatrices of \mathbf{H} whose block columns are indexed by $\mathcal{E}'(\widehat{v}(z_1, z_2))$ and $\bar{\mathcal{E}}'(\widehat{v}(z_1, z_2)) = \text{supp}(\widehat{v}(z_1, z_2)) \setminus \mathcal{E}'(\widehat{v}(z_1, z_2))$, respectively, and $\mathbf{v}_{\mathcal{E}'}$ and $\mathbf{v}_{\bar{\mathcal{E}}'}$ are defined accordingly. If no confusion arises we use \mathcal{E} and $\bar{\mathcal{E}}$ for $\mathcal{E}'(\widehat{v}(z_1, z_2))$ and $\bar{\mathcal{E}}'(\widehat{v}(z_1, z_2))$, respectively.

Definition 3. Let \mathcal{E}' be given with $(r^f, s^f) = (r+t, s)$ and $(r^l, s^l) = (r, s+t)$ being the first and last position (ordered by \prec) in this set. We define $\delta + 1$ different balls around \mathcal{E}' as

$$\Omega_{j,\delta}(\mathcal{E}') = \{(a, b) \mid a \leq r^f + j, b \leq s^l + j, r^f + s^f + j - \delta \leq a + b \leq r^f + s^f + j\}$$

for $j = 0, 1, 2, \dots, \delta$.

Example 4. Consider the burst of erasures given by

$$\mathcal{E}' = \{(8, 5), (7, 6), (6, 7), (5, 8), (4, 9)\},$$

then, $(r^f, s^f) = (8, 5)$ and $(r^l, s^l) = (4, 9)$ and for $\delta = 3$. Figure 2 shows the set $\Omega_{0,3}(\mathcal{E}')$ and $\Omega_{1,3}(\mathcal{E}')$.

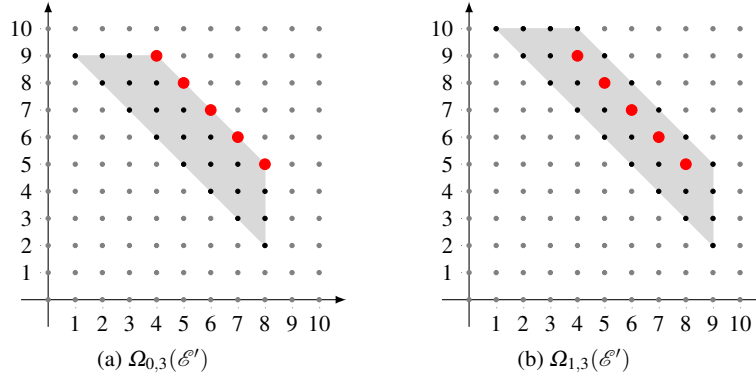


Fig. 2 Balls around the erasure given by $\mathcal{E}' = \{(8, 5), (7, 6), (6, 7), (5, 8), (4, 9)\}$

By definition, the vector $\mathbf{v}_{\mathcal{E}'}$ contains a burst of erasures in a diagonal and $\mathbf{v}_{\bar{\mathcal{E}}'}$ may contain erasures as well. Depending on the structure of $\mathbf{H}_{\mathcal{E}'}$ and $\mathbf{H}_{\bar{\mathcal{E}}'}$ these errors may appear together in some of the equations of (7).

We have shown that if \mathcal{E}' does not have more erasures *close* (meaning in a ball centered around \mathcal{E}') then it is possible to consider \mathcal{E}' as a burst of erasures of a set of 1D convolutional codes. Decoding these 1D convolutional codes would immediately imply the recovery of the \mathcal{E}' .

This procedure is far from solving all the possible erasure patterns but it represents the first step toward the development of an effective approach to solve more general patterns of erasures.

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