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Constant sign and nodal solutions for a class of nonlinear Dirichlet problems

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ABSTRACT

We consider a nonlinear Dirichlet problem with a Carathéodory reaction which has arbitrary growth from below. We show that the problem has at least three nontrivial smooth solutions, two of constant sign and the third nodal. In the semilinear case (i.e., $p = 2$), with the reaction $f(z, .)$ being C^1 and with subcritical growth, we show that there is a second nodal solution, for a total of four nontrivial smooth solutions. Finally, when the reaction has concave terms and is subcritical and for the nonlinear problem (i.e., $1 < p < \infty$) we show that again we can have the existence of three nontrivial smooth solutions, two of constant sign and a third nodal.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 boundary $\partial\Omega$. We study the following nonlinear Dirichlet problem

$$-\Delta_p u(z) = f\big(z, u(z)\big) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.1}$$

where Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u(z) = div\big(\big\|Du(z)\big\|^{p-2} Du(z)\big) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

$1 < p < \infty$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous).

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Our aim is to prove multiplicity theorems for problem (1.1), providing precise sign information for all the solutions. The interesting feature of our analysis is that $f(z,\cdot)$ can have unrestricted growth from below.

Our work extends the semilinear (i.e. $p=2$) ones by Ambrosetti and Lupo [3], Ambrosetti and Mancini [4] and Struwe [19,20] and the nonlinear work of Papageorgiou and Papageorgiou [18].

In fact, in all the aforementioned works (with the exception of Papageorgiou and Papageorgiou [18]) the problem is semilinear and parametric and the authors produce three nontrivial solutions for certain values of the parameter. The hypotheses on the reaction are more restrictive and they do not prove the existence of nodal solutions.

Here the equation is nonlinear driven by the p-Laplacian and our multiplicity theorem provides sign information for all the solutions.

The parametric equation of the works mentioned earlier is a particular case of our problem here. Moreover, in the semilinear case ($p=2$), using Morse theory, we generate a second nodal solution, for a total of four nontrivial solutions. In addition, other cases are also studied.

We should also mention the more recent work of Bonanno and Molica Bisci [6] and Marano, Molica Bisci and Motreanu [15], which prove three solutions theorems for semilinear problems using different methods. Note that in Marano, Molica Bisci and Motreanu [15], the potential is nonsmooth.

Our approach is variational, based on the critical point theory. The variational methods are coupled with suitable truncation and comparison techniques. For the semilinear problem (i.e. $p=2$), we also use tools from Morse theory (critical groups). In the next section, for the convenience of the reader, we recall the main mathematical definitions and facts which we will need in the sequel.

2. Mathematical background

We start with the critical point theory. So let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Also, by $\xrightarrow{w}$ we will designate the weak convergence in X.

Definition. A map $A: X\to X^*$ is said to be of type $(S)_+$, if for every sequence $\{u_n\}_{n\geq 1}\subseteq X$ such that $u_n\xrightarrow{w} u$ in X and

$$\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\leq 0,$$

one has $u_n\to u$ in X as $n\to\infty$.

Let $\varphi\in C^1(X)$. A number $c\in\mathbb{R}$ is said to be a critical value of φ if there exists $x^*\in X$ such that $\varphi'(x^*)=0$ and $\varphi(x^*)=c$. We say that φ satisfies the *Palais–Smale condition* (the *PS-condition* for short), if the following holds:

"Every sequence $\{x_n\}_{n\geq 1}\subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}\subseteq R$ is bounded and $\varphi'(x_n)\to 0$ in $W_0^{1,p}(\Omega)^*=W^{-1,p'}(\Omega)$ ($\frac{1}{p}+\frac{1}{p'}=1$), admits a strongly convergent subsequence."

Using this compactness-type condition, we can have the following minimax theorem known in the literature as the *mountain pass theorem*.

Theorem 1. *If $\varphi\in C^1(X)$ satisfies the PS-condition, $x_0,x_1\in X$, $r>0$, $\|x_1-x_0\|>r$, $\max\{\varphi(x_0),\varphi(x_1)\}<\inf\{\varphi(x):\|x-x_0\|=r\}=\eta_r$ and $c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t))$ with $\Gamma=\{\gamma\in C^1([0,1],X):\gamma(0)=0,\gamma(1)=x_1\}$, then $c\geq\eta_r$ and c is a critical value of φ.*

For $\varphi \in C^1(X)$ we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \le c\},$$
$$K_\varphi = \{x \in X : \varphi'(x) = 0\},$$
$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The next result is known in the literature as the *second deformation theorem* (see, for example, Gasinski and Papageorgiou [12, p. 628]).

Theorem 2. *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \le +\infty$, φ satisfies the PS-condition, $K_\varphi \cap (a,b) = \varnothing$ and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ, then there exists a deformation $h : [0,1] \times (\varphi^b \backslash K_\varphi^b) \to \varphi^b$ such that*

(a) $h(1, \varphi^b \backslash K_\varphi^b) \subseteq \varphi^a$*;*
(b) $h(t,x) = x$ *for all* $(t,x) \in [0,1] \times \varphi^a$*;*
(c) $\varphi(h(t,x)) \le \varphi(h(s,x))$ *for all* $t, s \in [0,1]$, $s \le t$, *all* $x \in \varphi^b \backslash K_\varphi^b$.

Remark. Theorem 2 implies that φ^a is a strong deformation retract of $\varphi^b \backslash K_\varphi^b$.

Throughout this work by $\|\cdot\|$ we denote the norm for the Sobolev space $W_0^{1,p}(\Omega)$, i.e., $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$ (by the Poincaré inequality). By $\|\cdot\|$ we will also denote the $\mathbb{R}^N$-norm. No confusion is possible, since it will always be clear from the context which norm is used. For $1 < p < \infty$ we define

$$p^* = \begin{cases} \frac{pN}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N. \end{cases}$$

The study of problem (1.1) relies on some basic facts about the spectrum of the negative Dirichlet p-Laplacian, hereafter denoted by $-\Delta_p^D$. So, let $m \in L^\infty(\Omega)_+$, $m \ne 0$ and consider the following nonlinear weighted eigenvalue problem:

$$-\Delta_p u(z) = \widehat{\lambda} m(z) |u(z)|^{p-2} u(z) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{2.1}$$

By an eigenvalue of $-\Delta_p^D$ we mean a number $\widehat{\lambda}(m) \in \mathbb{R}$ such that (2.1) has a nontrivial solution $\widehat{u}$. The nonlinear regularity theory (see Gasinski and Papageorgiou [12, pp. 737–738]) implies that $\widehat{u} \in C_0^1(\overline{\Omega})$.

The least number $\widehat{\lambda} \in \mathbb{R}$ for which (2.1) has a nontrivial solution is the *first eigenvalue of* $-\Delta_p^D$ and it is denoted by $\widehat{\lambda}_1(m)$. We recall the following well-known properties of $\widehat{\lambda}_1(m)$:

- $\widehat{\lambda}_1(m) > 0$;
- $\widehat{\lambda}_1(m)$ is isolated, i.e., we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(m), \widehat{\lambda}_1(m) + \varepsilon)$ contains no eigenvalues;
- $\widehat{\lambda}_1(m)$ is simple, i.e., if $\widehat{u}$, $\widehat{v}$ are eigenfunctions for $\widehat{\lambda}_1(m)$, then $\widehat{u} = \xi \widehat{v}$ for some $\xi \in \mathbb{R} \backslash \{0\}$.

The first eigenvalue $\widehat{\lambda}_1(m) > 0$ has the following variational characterization:

$$\widehat{\lambda}_1(m) = \inf \left\{ \frac{\|Du\|_p^p}{\int_\Omega m|u|^p dz} : u \in W_0^{1,p}(\Omega),\ u \ne 0 \right\}. \tag{2.2}$$

The infimum in (2.2) is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1(m)$.

Recall that $C_0^1(\overline{\Omega})$ is an ordered Banach space with order cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$int\, C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$.

It is clear from (2.2) that any eigenfunction $\widehat{u}$ corresponding to $\widehat{\lambda}_1(m)$ does not change sign. So $\widehat{u} \in C_+\backslash\{0\}$ and by virtue of the nonlinear strong maximum principle of Vasquez [21], we have $\widehat{u} \in int\, C_+$.

An eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$ is nodal. If $m \equiv 1$, then we set $\widehat{\lambda}_1 := \widehat{\lambda}_1(1)$ and by $\widehat{u}_1$ we denote the L^p-normalized (i.e., $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1$. We have just seen that

$$\widehat{u}_1 \in int\, C_+.$$

Since the set of eigenvalues of (2.1) is closed and $\widehat{\lambda}_1(m) > 0$ is isolated, the second eigenvalue

$$\widehat{\lambda}_2^*(m) = \inf\{\widehat{\lambda} : \widehat{\lambda} \text{ is an eigenvalue of (2.1)},\ \widehat{\lambda} > \widehat{\lambda}_1(m)\}$$

is also well-defined.

If $N = 1$ (ordinary differential equations), then the set of eigenvalues of (2.1) is a sequence

$$\{\widehat{\lambda}_k(m)\}_{k\geq 1} \subseteq (0, +\infty)$$

of simple eigenvalues such that $\widehat{\lambda}_k(m) \to +\infty$ as $k \to +\infty$ and the corresponding eigenfunction $\{\widehat{u}_k(m)\}_{k\geq 1}$ has exactly $k - 1$ zeros (see, for example, Gasinski and Papageorgiou [12, p. 761]).

If $N \geq 2$ (partial differential equations), then using the Ljusternik–Schnirelmann minimax scheme, we obtain an increasing sequence $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ of eigenvalues such that $\widehat{\lambda}_k(m) \to +\infty$ as $k \to +\infty$.

If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues of $-\Delta_p^D$. If $p \neq 2$, then we do not know if this is the case. However we know that $\widehat{\lambda}_2^*(m) = \widehat{\lambda}_2(m)$ and so the second eigenvalue admits a minimax characterization provided by the Ljusternik–Schnirelmann theory.

However, for our purpose, this characterization is not convenient. Instead we will use an alternative one due to Cuesta, de Figueiredo and Gossez [8]. So, let

$$\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}, \qquad M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}$$

and

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1(m),\ \widehat{\gamma}(1) = \widehat{u}_1(m)\}$$

(recall $\widehat{u}_1(m) \in M \cap int\, C_+$ is the L^p-normalized eigenfunction corresponding to $\widehat{\lambda}_1(m) > 0$).

Proposition 1. $\widehat{\lambda}_2(m) = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leq t\leq 1} \|D\widehat{\gamma}(t)\|_p^p$.

Viewed as functions of the weight $m \in L^\infty(\Omega)_+$, the eigenvalues $\widehat{\lambda}_1(m)$ and $\widehat{\lambda}_2(m)$ exhibit certain monotonicity properties.

(a) If m, $m' \in L^\infty(\Omega)_+$, $m(z) \le m'(z)$ a.e. in Ω, $m \ne m'$, then $\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)$.
(b) If m, $m' \in L^\infty(\Omega)_+$, $m(z) < m'(z)$ a.e. in Ω, then $\widehat{\lambda}_2(m') < \widehat{\lambda}_2(m)$.

Another spectrum of $-\Delta_p^D$ that we will use is the so-called *Fucik spectrum*. This is the set $\sigma_F(p)$ of all $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$-\Delta_p u(z) = \lambda u^+(z)^{p-1} - \mu u^-(z)^{p-1} \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0$$

admits a nontrivial solution.

Evidently $\sigma_F(p)$ contains the two lines $\{\widehat{\lambda}_1\} \times \mathbb{R}$ and $\mathbb{R}\times\{\widehat{\lambda}_1\}$ and the pairs $\{(\widehat{\lambda}_k, \widehat{\lambda}_k)\}_{k\ge1}$. The first nontrivial curve $C_1 \subseteq \sigma_F(p)$ through $(\widehat{\lambda}_2, \widehat{\lambda}_2)$, which is asymptotic to the lines $\{\widehat{\lambda}_1\} \times \mathbb{R}$ and $\mathbb{R}\times\{\widehat{\lambda}_1\}$, was constructed and characterized variationally by Cuesta, de Figueiredo and Gossez [8].

Next we recall some basic definitions and facts from Morse theory. Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \ge 0$, by $H_k(Y_1, Y_2)$ we denote the kth-relative singular homology group with integer coefficients for the pair (Y_1, Y_2).

Let $\varphi \in C^1(X)$. The critical groups of φ, at an isolated critical point $x \in X$ with $\varphi(x) = c$, are defined by

$$C_k(\varphi, x) = H_k\big(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U}\backslash\{x\}\big)$$

for all $k \ge 0$, where $\mathcal{U}$ is a neighborhood of x, such that $K_\varphi \cap \mathcal{U} = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the neighborhood $\mathcal{U}$.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k\big(X, \varphi^c\big), \quad \text{for all } k \ge 0.$$

Remark. The second deformation theorem implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We set

$$M(t, x) = \sum_{k\ge0} \mathit{rank}\, C_k(\varphi, x) t^k \quad \text{for all } t \in \mathbb{R},\ \text{all } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k\ge0} \mathit{rank}\, C_k(\varphi, \infty) t^k \quad \text{for all } t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{x\in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.3}$$

where $Q(t) = \sum_{k\ge0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

Let H be a Hilbert space, $x \in H$, let $\mathcal{U}$ be a neighborhood of x and $\varphi \in C^2(\mathcal{U})$. For $x \in K_\varphi$, the *Morse index of* x is defined to be the maximum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite.

We say that $x \in K_\varphi$ is *nondegenerate* if $\varphi''(x)$ is invertible. The critical groups of $\varphi \in C^2(\mathcal{U})$, at a nondegenerate critical point $x \in H$ with Morse index m, are given by

$$C_k(\varphi, x) = \delta_{k,m}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{2.4}$$

Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear map defined by

$$\langle A(u), y \rangle = \int_\Omega \|Du\|^{p-2}(Du, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } u, y \in W_0^{1,p}(\Omega). \tag{2.5}$$

The following result concerning the map A is well-known (see, for example, Gasinski and Papageorgiou [12]).

Proposition 2. *If $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the nonlinear map defined by (2.5), then A is continuous, bounded, strictly monotone (strongly monotone if $p \geq 2$), hence maximal monotone too and of type $(S)_+$.*

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and for $u \in W_0^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in W_0^{1,p}(\Omega)$ and $|u| = u^+ + u^-$, $u = u^+ - u^-$. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Finally, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we set $N_h(u)(\cdot) = h(\cdot, u(\cdot))$ for all $u \in W_0^{1,p}(\Omega)$.

3. Three solutions theorem

In this section, we prove a three solutions theorem for problem (1.1), providing precise sign information for all of them. First we produce two constant sign smooth solutions. To this end we introduce the following hypotheses on the reaction $f(z, x)$:

$\mathbf{H}_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ a.e. in Ω and
- (i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|f(z, x)| \leq a_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leq \rho$;
- (ii) $\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = -\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exists $\eta \in L^\infty(\Omega)_+$, $\eta(z) \geq \widehat{\lambda}_1$ a.e. in Ω, $\eta \neq \widehat{\lambda}_1$ such that $\liminf_{x\to 0} \frac{pF(z,x)}{|x|^p} \geq \eta(z)$ uniformly for a.a. $z \in \Omega$;
- (iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that $f(z, x)x + \xi_\rho|x|^p \geq 0$ a.e. in Ω, for all $|x| \leq \rho$.

Remark. The above hypotheses allow for arbitrary growth of $f(z, \cdot)$ from below.

Proposition 3. *If hypotheses $\mathbf{H}_1$ hold, then problem (1.1) has at least two constant sign smooth solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$.*

Proof. We do the proof for the positive solution, since the proof for the negative solution is similar.

By virtue of the hypotheses $\mathbf{H}_1$(i), (ii), we can find $c_1 > 0$ such that

$$f(z, x) \leq -x^p + c_1 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3.1}$$

Since A is maximal monotone (see Proposition 2) and it is clearly coercive, it is surjective (see, for example, Gasinski and Papageorgiou [12, p. 320]). So, we can find $e \in W_0^{1,p}(\Omega)$ such that $A(e) = 1$.

Acting with $-e^- \in W_0^{1,p}(\Omega)$, we show that $e \geq 0$. Nonlinear regularity theory (see, for example, Gasinski and Papageorgiou [12, pp. 737–738]) and the nonlinear strong maximum principle of Vasquez [21] imply that $e \in int\, C_+$. We have

$$-\Delta_p e(z) = 1 \quad \text{a.e. in } \Omega, \qquad e|_{\partial\Omega} = 0. \tag{3.2}$$

Let $\xi^{p-1} > c_1$ (see (3.1)) and set $\bar{u} = \xi e \in int\, C_+$. Then

$$\begin{aligned} -\Delta_p \bar{u}(z) - f\big(z, \bar{u}(z)\big) &= \xi^{p-1}\big(-\Delta_p e(z)\big) - f\big(z, \xi e(z)\big) \\ &\geq \xi^{p-1} + \xi^{p-1} e(z) - c_1 > 0 \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{3.3}$$

(see (3.1), (3.2) and recall $\xi^{p-1} - c_1 > 0$ and $e \in int\, C_+$).

We consider the following truncation of the reaction $f(z, \cdot)$

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z,x) & \text{if } 0 \leq x \leq \bar{u}(z) \\ f(z, \bar{u}(z)) & \text{if } x > \bar{u}(z). \end{cases} \tag{3.4}$$

This is a Carathéodory function. We set $\widehat{F}_+(z,x) = \int_0^x \widehat{f}_+(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \widehat{F}_+\big(z, u(z)\big)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (3.4) that $\widehat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we can easily show that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. Thus by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf\big\{\widehat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\big\} = \widehat{m}_+. \tag{3.5}$$

By virtue of hypothesis $\mathbf{H}_1$(iii), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \geq \frac{1}{p}\big(\eta(z) - \varepsilon\big)|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta. \tag{3.6}$$

Let $t \in (0,1)$ be small such that $t\widehat{u}_1 \leq \bar{u}$ (recall $\bar{u} \in int\, C_+$ and see Kyritsi and Papageorgiou [14, Lemma 2.1]) and $0 \leq t\widehat{u}_1(z) \leq \delta$ for all $z \in \overline{\Omega}$. Then

$$\begin{aligned} \widehat{\varphi}_+(t\widehat{u}_1) &= \frac{t^p}{p}\|D\widehat{u}_1\|_p^p - \int_\Omega F(z, t\widehat{u}_1)dz \\ &\leq \frac{t^p}{p}\widehat{\lambda}_1 - \frac{t^p}{p}\int_\Omega \eta\widehat{u}_1^p dz + \varepsilon\frac{t^p}{p} \quad \big(\text{see (3.6), recall that } \|\widehat{u}_1\|_p = 1\big) \\ &= \frac{t^p}{p}\left[\varepsilon - \int_\Omega \big(\eta(z) - \widehat{\lambda}_1\big)\widehat{u}_1(z)^p dz\right]. \end{aligned} \tag{3.7}$$

Since $\widehat{u}_1(z) > 0$ for all $z \in \Omega$ and $\eta(z) \geq \widehat{\lambda}_1$ a.e. in Ω, $\eta \neq \widehat{\lambda}_1$, we have

$$\xi = \int_\Omega \big(\eta(z) - \widehat{\lambda}_1\big)\widehat{u}_1(z)^p dz > 0.$$

So, if we choose $\varepsilon \in (0, \xi)$, then from (3.7) it follows that $\widehat{\varphi}_+(t\widehat{u}_1) < 0$, which implies that

$$\widehat{\varphi}_+(u_0) = \widehat{m}_+ < 0 = \widehat{\varphi}_+(0) \quad \bigl(\text{see (3.5)}\bigr)$$

hence

$$u_0 \neq 0.$$

From (3.5) we have $\widehat{\varphi}'_+(u_0) = 0$, hence

$$A(u_0) = N_{\widehat{f}_+}(u_0). \tag{3.8}$$

On (3.8) we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then $\|Du_0^-\|_p^p = 0$ (see (3.4)), hence $u_0 \geq 0$, $u_0 \neq 0$.

Also, on (3.8) we act with $(u_0 - \bar{u})^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
\bigl\langle A(u_0), (u_0 - \bar{u})^+ \bigr\rangle &= \int\limits_{\Omega} \widehat{f}_+(z, u_0)(u_0 - \bar{u})^+ dz \\
&= \int\limits_{\Omega} f(z, \bar{u})(u_0 - \bar{u})^+ dz \quad \bigl(\text{see (3.4)}\bigr) \\
&\leq \bigl\langle A(\bar{u}), (u_0 - \bar{u})^+ \bigr\rangle \quad \bigl(\text{see (3.3)}\bigr),
\end{aligned}$$

which implies that

$$\int\limits_{\{u_0 > \bar{u}\}} \bigl(\|Du_0\|^{p-2} Du_0 - \|D\bar{u}\|^{p-2} D\bar{u}, Du_0 - D\bar{u}\bigr)_{\mathbb{R}^N} dz \leq 0,$$

hence

$$\bigl|\{u_0 > \bar{u}\}\bigr|_N = 0, \quad \text{i.e.} \quad u_0 \leq \bar{u}.$$

Therefore we have proved that

$$u_0 \in [0, \bar{u}] = \bigl\{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq \bar{u}(z) \text{ a.e. in } \Omega\bigr\}$$

and so from (3.4) and (3.8), we have $A(u_0) = N_f(u_0)$, then $-\Delta_p u_0(z) = f(z, u_0(z))$ a.e. in Ω, $u_0|_{\partial\Omega} = 0$.

Nonlinear regularity (see [12]) implies $u_0 \in C_+ \backslash \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}_1$(iv). Then

$$-\Delta_p u_0(z) + \xi_\rho u_0(z)^{p-1} = f\bigl(z, u_0(z)\bigr) + \xi_\rho u_0(z)^{p-1} \geq 0 \quad \text{a.e. in } \Omega,$$

hence

$$\Delta_p u_0(z) \leq \xi_\rho u_0(z)^{p-1} \quad \text{a.e. in } \Omega,$$

therefore

$$u_0 \in \operatorname{int} C_+ \quad \bigl(\text{see Vasquez [21]}\bigr).$$

Similarly, using hypothesis $\mathbf{H}_1$(ii), we can find $\underline{v} \in \mathit{int}\, C_+$ such that

$$-\Delta_p \underline{v}(z) = f\big(z, \underline{v}(z)\big) \quad \text{a.e. in } \Omega.$$

Then truncating $f(z,\cdot)$ at $\{\underline{v}(z), 0\}$ and reasoning as above, we generate a second constant sign smooth solution $v_0 \in -\mathit{int}\, C_+$. $\square$

To produce a third nontrivial smooth solution with sign information, we need to strengthen the hypotheses on $f(z,\cdot)$ near zero. More precisely, the new hypotheses on the reaction $f(z,x)$ are the following:

$\mathbf{H}_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ a.e. in Ω and hypotheses $\mathbf{H}_2$(i), (ii), (iv) are the same as the corresponding hypotheses $\mathbf{H}_1$(i), (ii), (iv) and
(iii) there exists $\beta_1 > \beta_0 > \widehat{\lambda}_2$ such that

$$\beta_0 \le \liminf_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \limsup_{x \to 0} \frac{f(z,x)}{|x|^{p-2}x} \le \beta_1 \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark. Hypothesis $\mathbf{H}_2$(iii) restricts the growth of $f(z,\cdot)$ to be $(p-1)$-linear near zero.

With this stronger condition on $f(z,\cdot)$ near zero, we show that problem (1.1) admits extremal constant sign solutions, i.e., there is a smallest nontrivial positive solution $u_* \in \mathit{int}\, C_+$ and a biggest nontrivial negative solution $v_* \in -\mathit{int}\, C_+$.

Proposition 4. *If hypotheses $\mathbf{H}_2$ hold, then problem (1.1) admits extremal constant sign solutions $u_* \in \mathit{int}\, C_+$ and $v_* \in -\mathit{int}\, C_+$.*

Proof. Let S_+ be the set of nontrivial positive solutions of (1.1) in the order interval $[0, \bar{u}]$. From Proposition 4 and its proof, we know that $S_+ \neq \varnothing$ and $S_+ \subseteq \mathit{int}\, C_+$.

Let $C \subseteq S_+$ be a chain (i.e. a totally ordered subset of S_+). From Dunford and Schwartz [9, p. 336], we know that we can find $\{u_n\}_{n\ge1} \subseteq C$ such that

$$\inf C = \inf_{n\ge1} u_n.$$

We have

$$A(u_n) = N_f(u_n) \quad \text{for all } n \ge 1, \tag{3.9}$$

hence $\{u_n\}_{n\ge1} \subseteq W_0^{1,p}(\Omega)$ is bounded (recall $u_n \le \bar{u}$ for all $n \ge 1$ and see $\mathbf{H}_2$(i)).

So, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^p(\Omega). \tag{3.10}$$

On (3.9) we act with $u_n - u$, pass to the limit as $n \to \infty$ and use (3.10). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

therefore

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad (\text{see Proposition 2}). \tag{3.11}$$

Passing to the limit as $n \to \infty$ in (3.9) and using (3.11), we obtain $A(u) = N_f(u)$, hence

$$u \in C_+ \quad \text{is a solution of (1.1).}$$

We show that $u \neq 0$. Arguing by contradiction, suppose that $u = 0$. Then $u_n \to 0$ in $W_0^{1,p}(\Omega)$. Let

$$y_n = \frac{u_n}{\|u_n\|}, \quad n \geq 1.$$

Since $\|y_n\| = 1$ for all $n \geq 1$, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \quad \text{in } L^p(\Omega) \quad \text{as } n \to \infty.$$

Hypotheses $\mathbf{H}_2$(i), (iii) imply that

$$\left|f(z,x)\right| \leq c_2|x|^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \rho := \|\bar{u}\|_\infty.$$

So, it follows that $\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ is bounded (recall $u_n \in [0, \bar{u}]$ for all $n \geq 1$). Therefore we may assume that

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} h \quad \text{in } L^{p'}(\Omega). \tag{3.12}$$

As in Aizicovici, Papageorgiou and Staicu [1] (see the proof of Proposition 31), we show that

$$h = \eta y^{p-1} \quad \text{with } \beta_0 \leq \eta(z) \leq \beta_1 \text{ a.e. in } \Omega. \tag{3.13}$$

From (3.9) we have

$$A(y_n) = \frac{N_f(u_n)}{\|u_n\|^{p-1}}, \quad n \geq 1 \tag{3.14}$$

hence

$$\left\langle A(y_n), y_n - y\right\rangle = \int\limits_\Omega \frac{f(z,u_n)}{\|u_n\|^{p-1}}(y_n - y)dz,$$

therefore

$$\lim_{n\to\infty} \left\langle A(y_n), y_n - y\right\rangle = 0$$

and by Proposition 2 we conclude

$$y_n \to y \quad \text{in } W_0^{1,p}(\Omega), \quad \text{hence} \quad \|y\| = 1, \quad y \geq 0. \tag{3.15}$$

Then passing to the limit as $n \to \infty$ in (3.14) and using (3.12), (3.13) and (3.15), we obtain

$$A(y) = \eta y^{p-1},$$

hence

$$-\Delta_p y(z) = \eta(z)y(z)^{p-1} \quad \text{a.e. in } \Omega, \qquad y|_{\partial\Omega} = 0.$$

By virtue of Proposition 2 and (3.13), we have

$$\widehat{\lambda}_1(\eta) \le \widehat{\lambda}_1(\beta_0) < \widehat{\lambda}_1(\widehat{\lambda}_1) = 1$$

and so, from (3.15), we have a contradiction (recall that only the principal eigenfunctions are of constant sign). Therefore $u \neq 0$ and so $u \in S_+$, $u = \inf C$.

Since C is an arbitrary chain, by the Kuratowski–Zorn lemma, we can find $u_* \in S_+ \subseteq \operatorname{int} C_+$ a minimal element. From Filippakis, Kristaly and Papageorgiou [10, Lemma 4.3], we have that S_+ is downward directed (i.e. if $u, y \in S_+$, then there exists $v \in S_+$ such that $v \le u$, $v \le y$). So, it follows that $u_* \in \operatorname{int} C_+$ is the smallest nontrivial positive solution of (1.1).

Similarly, let S_- be the set of nontrivial negative solutions of (1.1) in the order interval $[\underline{v}, 0]$. Reasoning as above and using the fact that S_- is upward directed (i.e. if $v, y \in S_-$, then we can find $u \in S_-$ such that $v \le u$, $y \le u$) we can produce $v_* \in -\operatorname{int} C_+$ the biggest nontrivial negative solution of (1.1). □

Using these two extremal constant sign solutions, we can produce a third nontrivial smooth solution of (1.1) which is nodal (sign changing).

Proposition 5. *If hypotheses* $\mathbf{H}_2$ *hold, then problem* (1.1) *admits a nodal solution* $y_0 \in C_0^1(\overline{\Omega})$ *such that* $v_*(z) \le y_0(z) \le u_*(z)$ *for all* $z \in \overline{\Omega}$.

Proof. Let $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ be the two extremal nontrivial constant sign solutions of (1.1) produced in Proposition 4. Using them, we introduce the following truncation of the reaction $f(z, \cdot)$:

$$g(z,x) = \begin{cases} f(z, v_*(z)) & \text{if } x < v_*(z), \\ f(z,x) & \text{if } v_*(z) \le x \le u_*(z), \\ f(z, u_*(z)) & \text{if } x > u_*(z). \end{cases} \tag{3.16}$$

This is a Carathéodory function. Let $G(z,x) = \int_0^x g(z,s)ds$ and consider the C^1-functional $\widehat{\varphi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G\big(z, u(z)\big)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Also let $g_\pm(z,x) = g(z, \pm x^\pm)$, $G_\pm(z,x) = \int_0^x g_\pm(z,s)ds$ and consider the C^1-functionals $\widehat{\varphi}_\pm : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\pm(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega G_\pm\big(z, u(z)\big)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

As in the proof of Proposition 3, we show that

$$K_{\widehat{\varphi}} \subseteq [v_*, u_*], \qquad K_{\widehat{\varphi}_+} \subseteq [0, u_*] \quad \text{and} \quad K_{\widehat{\varphi}_-} \subseteq [v_*, 0].$$

The extremality of the solutions $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ implies that

$$K_{\widehat{\varphi}} \subseteq [v_*, u_*], \qquad K_{\widehat{\varphi}_+} \subseteq \{0, u_*\} \quad \text{and} \quad K_{\widehat{\varphi}_-} \subseteq \{v_*, 0\}. \tag{3.17}$$

Claim. u_* *and* v_* *are local minimizers of the functional* $\widehat{\varphi}$.

Clearly $\widehat{\varphi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(\widetilde{u}) = \inf\{\widehat{\varphi}_+(u) : u \in W_0^{1,p}(\Omega)\} = \widehat{m}_+^*.$$

As in the proof of Proposition 3, using hypothesis $\mathbf{H}_2$(iii), we show that $\widehat{\varphi}_+(\widetilde{u}) = \widehat{m}_+^* < 0 = \widehat{\varphi}_+(0)$, i.e. $\widetilde{u} \neq 0$, hence $\widetilde{u} = u_*$ (see (3.17)).

Since $u_* \in int\, C_+$ and $\widehat{\varphi}|_{C^+} = \widehat{\varphi}_+|_{C^+}$, it follows that u_* is a local $C_0^1(\overline{\Omega})$-minimizer of $\widehat{\varphi}$. Hence, by virtue of Theorem 2 of Garcia Azorero, Manfredi and Peral Alonso [11], we have that u_* is a local $W_0^{1,p}(\Omega)$-minimizer of $\widehat{\varphi}$. Similarly for $v_* \in -\, int\, C_+$. This proves the Claim.

Without any loss of generality, we may assume that $\widehat{\varphi}(v_*) \le \widehat{\varphi}(u_*)$. Because of the Claim and reasoning as in Aizicovici, Papageorgiou and Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}(v_*) \le \widehat{\varphi}(u_*) < \inf\{\widehat{\varphi}(u) : \|u - u_*\| = \rho\} = \widehat{\eta}_\rho. \tag{3.18}$$

Since $\widehat{\varphi}$ is coercive (see (3.16)) it satisfies the PS-condition. This fact and (3.18) permit the use of Theorem 1 (the *mountain pass theorem*). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\eta}_\rho \le \widehat{\varphi}(y_0) = \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} \widehat{\varphi}(\gamma(t)) \tag{3.19}$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = v_*,\ \gamma(1) = u_*\}$ and

$$\widehat{\varphi}'(y_0) = 0. \tag{3.20}$$

From (3.19) we have that $y_0 \notin \{v_*, u_*\}$, while from (3.20), we have that $y_0 \in C_0^1(\overline{\Omega})$ and $y_0 \in [v_*, u_*]$ (see (3.17)), hence y_0 is a solution of (1.1) (see (3.16)).

We need to show that $y_0 \neq 0$. According to the minimax characterization of $\widehat{\varphi}(y_0)$ in (3.19), to show the nontriviality of y_0, it suffices to produce $\gamma_* \in \Gamma$ such that

$$\widehat{\varphi}|_{\gamma_*} < 0.$$

By virtue of hypothesis $\mathbf{H}_2$(iii), we can find $\beta_2 > \widehat{\lambda}_2$ and $\delta > 0$ such that

$$\frac{\beta_2}{p}|x|^p \le F(z,x) \quad \text{for a.a. } z \in \Omega,\ \text{all } |x| \le \delta. \tag{3.21}$$

Recall that

$$\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}, \qquad M = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p}.$$

Also we set $M_c = M \cap C_0^1(\overline{\Omega})$. We endow M with the relative $W_0^{1,p}(\Omega)$-topology and M_c with the relative $C_0^1(\overline{\Omega})$-topology. Evidently M_c is dense in M.

Recall that

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1,\ \widehat{\gamma}(1) = \widehat{u}_1\}$$

and define

$$\widehat{\Gamma}_c = \{\widehat{\gamma} \in C([-1,1], M_c) : \widehat{\gamma}(-1) = -\widehat{u}_1,\ \widehat{\gamma}(1) = \widehat{u}_1\}.$$

Clearly $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$. So, by virtue of Proposition 1, we can find $\widehat{\gamma} \in \widehat{\Gamma}_c$ such that

$$\max_{-1\le t\le 1}\big\|D\widehat{\gamma}(t)\big\|_p^p \le \beta_3 \quad \text{with } \beta_3 \in (\widehat{\lambda}_2, \beta_2) \quad \big(\text{see (3.21)}\big). \tag{3.22}$$

Because $\widehat{\gamma} \in \widehat{\Gamma}_c$, $u_* \in int\, C_+$ and $v_* \in -\, int\, C_+$, we can find $\theta \in (0,1)$ small such that

$$\theta\widehat{\gamma}(t) \in [v_*, u_*] \quad \text{and} \quad \theta\big|\widehat{\gamma}(t)(z)\big| \le \delta \quad \text{for all } z \in \overline{\Omega},\ \text{all } t \in [-1,1]. \tag{3.23}$$

Then, for all $t \in [-1,1]$, we have

$$\begin{aligned}
\widehat{\varphi}\big(\theta\widehat{\gamma}(t)\big) &= \frac{\theta^p}{p}\big\|D\widehat{\gamma}(t)\big\|_p^p - \int_\Omega F\big(z, \theta\widehat{\gamma}(t)(z)\big)\,dz \\
&\le \frac{\theta^p}{p}(\beta_3 - \beta_2) \quad \big(\text{see (3.21), (3.22) and (3.23)}\big) \\
&< 0 \quad \big(\text{recall } \beta_3 \in (0, \beta_2)\big).
\end{aligned} \tag{3.24}$$

So, if we set $\gamma_0 = \theta\widehat{\gamma}$, then from (3.24) it follows that

$$\widehat{\varphi}|_{\gamma_0} < 0. \tag{3.25}$$

Next we produce a continuous path in $W_0^{1,p}(\Omega)$ which connects $\theta\widehat{u}_1$ and u_* and along which $\widehat{\varphi}$ is negative. To this end, let

$$a = \widehat{\varphi}_+(y_0) = \widehat{m}_+^* < 0 = \widehat{\varphi}_+(0).$$

Applying Theorem 2 (the *second deformation theorem*), we can find a deformation

$$\begin{gathered}
h : [0,1] \times \big(\widehat{\varphi}_+^0 \backslash K_{\widehat{\varphi}_+}^0\big) \to \widehat{\varphi}_+^0 \quad \text{such that} \quad h(t,\cdot)|_{K_{\widehat{\varphi}_+}^0} = id|_{K_{\widehat{\varphi}_+}^0} \quad \forall t \in [0,1], \\
h\big(1, \widehat{\varphi}_+^0 \backslash K_{\widehat{\varphi}_+}^0\big) \subseteq \widehat{\varphi}_+^a
\end{gathered} \tag{3.26}$$

and

$$\varphi\big(h(t,u)\big) \le \varphi\big(h(s,u)\big) \quad \text{for all } s,t \in [0,1],\ s \le t,\ \text{all } u \in \widehat{\varphi}_+^0 \backslash K_{\widehat{\varphi}_+}^0. \tag{3.27}$$

Note that $\widehat{\varphi}_+^a = \{u_*\}$ (see (3.17) and recall $a < 0$). We set

$$\gamma_+(t) = h(t, \theta\widehat{u}_1)^+ \quad \text{for all } t \in [0,1].$$

This is a continuous path in $W_0^{1,p}(\Omega)$ and

$$\gamma_+(0) = h(0, \theta\widehat{u}_1)^+ = \theta\widehat{u}_1, \qquad \gamma_+(1) = h(1, \theta\widehat{u}_1)^+ = u_*$$

(see (3.26)). Moreover, we have

$$\widehat{\varphi}_+(\theta\widehat{u}_1) = \widehat{\varphi}(\theta\widehat{u}_1) = \widehat{\varphi}\big(\gamma_+(0)\big) < 0 \quad \big(\text{see (3.25)}\big),$$

hence

$$\widehat{\varphi}_+|_{\gamma_+} < 0 \quad \big(\text{see (3.27)}\big).$$

Since $\gamma_+(t) \geq 0$ for all $t \in [0,1]$, we also have

$$\widehat{\varphi}|_{\gamma_+} < 0. \tag{3.28}$$

In a similar fashion, we produce another continuous path γ_- in $W_0^{1,p}(\Omega)$ which connects $-\theta\widehat{u}_1$ and v_* such that

$$\widehat{\varphi}|_{\gamma_-} < 0. \tag{3.29}$$

We concatenate γ_-, γ_0 and γ_+ and produce a path $\gamma_* \in \Gamma$ such that

$$\widehat{\varphi}|_{\gamma_*} < 0$$

(see (3.25), (3.28) and (3.29)), which implies that $y_0 \neq 0$ and $y_0 \in C_0^1(\overline{\Omega})$ is a nodal smooth solution of (1.1). $\square$

So, we can state the following multiplicity theorem for problem (1.1).

Theorem 3. *If hypotheses $\mathbf{H}_2$ hold, then problem (1.1) has at least three nontrivial smooth solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in C_0^1(\overline{\Omega})$ nodal. Moreover, problem (1.1) has extremal constant sign smooth solutions.*

As an application of this theorem, we consider the following parametric Dirichlet problem

$$-\Delta_p u(z) = \lambda u^+(z)^{p-1} - \mu u^-(z)^{p-1} - g\big(z, u(z)\big) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{3.30}$$

On the perturbation $g(z,x)$, we impose the following hypotheses:

$\mathbf{H}_3$: $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, such that
- (i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that $|g(z,x)| \leq a_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leq \rho$;
- (ii) $\lim_{x\to\pm\infty} \frac{g(z,x)}{|x|^{p-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$;
- (iii) $\lim_{x\to 0} \frac{g(z,x)}{|x|^{p-2}x} = 0$ uniformly for a.a. $z \in \Omega$.

Recall that C_1 denotes the first Fucik curve of $-\Delta_p^D$. As a direct consequence of Theorem 3, we have the following result.

Corollary 1. *If hypotheses $\mathbf{H}_3$ hold and (λ,μ) is above C_1, then problem (3.30) has at least three nontrivial smooth solutions $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ and $y_0 \in C_0^1(\overline{\Omega})$ nodal.*

Remarks. If $\lambda = \mu > \widehat{\lambda}_2$, Corollary 1 improves the multiplicity theorem of Papageorgiou–Papageorgiou [18]. In [18] no sign information is provided for the third solution. Also, in [18] it is assumed that $f(z,x) \geq 0$ for a.a. $z \in \Omega$, all $x \in R$ (sign condition) and $f(z,\cdot)$ has subcritical growth both from above and below. Therefore the function

$$g(z,x) = g(x) = |x|^{r-2}x - |x|^{s-2}x \quad \text{for all } x \in \mathbb{R} \text{ with } p < s < r < \infty$$

satisfies hypotheses $\mathbf{H}_3$ but not those of [18]. When $p = 2$ (semilinear case), Corollary 1 generalizes significantly the works of [3,4,19,20], where $g(z,x) = g(x)$ with $g \in C^1(\mathbb{R})$ (see [3,4,19]) or g Lipschitz (see [20]) and no sign information is given for the third solution.

In fact, for the semilinear problem (i.e. $p = 2$), by strengthening the regularity of $f(z,\cdot)$, we can have a stronger multiplicity theorem, producing four nontrivial smooth solutions, two of constant sign and two nodal. This is done using a combination of variational methods and Morse theory.

4. Semilinear problem

In this section we deal with the semilinear version (i.e. $p = 2$) of problem (1.1). So, the boundary value problem under consideration is the following:

$$-\Delta u(z) = f\big(z, u(z)\big) \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{4.1}$$

The hypotheses on the reaction $f(z,x)$ are the following:

H$_4$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function, such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \le a(z) + c|x|^{r-2}$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $c > 0$, $2 \le r \le 2^*$;

(ii) $\lim_{x\to\pm\infty} \frac{f(z,x)}{x} = -\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists an integer $m \ge 1$ such that:

$$f'_x(z,0) \in [\widehat{\lambda}_m, \widehat{\lambda}_{m+1}] \quad \text{for a.a. } z \in \Omega,$$
$$f'_x(z,\cdot) \ne \widehat{\lambda}_m, \qquad f'_x(z,\cdot) \ne \widehat{\lambda}_{m+1}, \quad \text{and}$$
$$f'_x(z,0) = \lim_{x\to 0} \frac{f(z,x)}{x} \quad \text{uniformly for a.a. } z \in \Omega$$

(recall that $\{\widehat{\lambda}_m\}_{m\ge1}$ are all distinct eigenvalues of $-\Delta^D$).

Remark. Evidently, the hypotheses on the reaction $f(z,\cdot)$ are stronger. Now, we require that $f(z,\cdot)$ is C^1, it cannot have arbitrary growth from below but instead it is subcritical and at zero in contrast to **H**$_2$(iii), we do not allow for an asymmetric behavior as we approach zero from the left and the right respectively. These stronger conditions lead to the existence of a second nodal solution.

Theorem 4. *If hypotheses* **H**$_4$ *hold, then problem* (4.1) *has at least four nontrivial smooth solutions* $u_0 \in \mathit{int}\, C_+$, $v_0 \in -\mathit{int}\, C_+$ *and* $y_0, \widehat{y} \in C^1_0(\overline{\Omega})$ *nodal.*

Proof. From Theorem 3, we already have three nontrivial smooth solutions $u_0 \in \mathit{int}\, C_+$, $v_0 \in -\mathit{int}\, C_+$ and $y_0 \in C^1_0(\overline{\Omega})$ nodal. We may assume that u_0, v_0 are the extremal constant sign solutions. Recall that $y_0 \in [v_0, u_0]$ (see the proof of Proposition 6).

Let $\rho = \max\{\|v_0\|_\infty, \|u_0\|_\infty\}$. Since $f(z,\cdot) \in C^1(\mathbb{R})$ and using **H**$_4$(i) via the mean value theorem, we see that we can find $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, $x \to f(z,x) + \widehat{\xi}_\rho x$ is increasing on $[-\rho, \rho]$. Then

$$\begin{aligned} &-\Delta(u_0 - y_0)(z) + \widehat{\xi}_\rho(u_0 - y_0)(z) \\ &\quad = f\big(z, u_0(z)\big) + \widehat{\xi}_\rho u_0(z) - f\big(z, y_0(z)\big) + \widehat{\xi}_\rho y_0(z) \ge 0 \quad \text{a.e. in } \Omega, \end{aligned}$$

hence

$$-\Delta(u_0 - y_0)(z) \le \widehat{\xi}_\rho(u_0 - y_0)(z) \quad \text{a.e. in } \Omega,$$

therefore

$$u_0 - y_0 \in \mathit{int}\, C_+ \quad \big(\text{see Vasquez [21]}\big).$$

In a similar fashion, we show that $y_0 - v_0 \in int\, C_+$. Therefore

$$y_0 \in int_{C_0^1(\overline{\Omega})} [v_0, u_0]. \tag{4.2}$$

From the proof of Proposition 5 (see the Claim), we show that u_0, v_0 are both local minimizers of the functional $\widehat{\varphi}$ defined there by truncating $f(z, \cdot)$ at $\{v_0(z), u_0(z)\}$. So, we have

$$C_k(\widehat{\varphi}, u_0) = C_k(\widehat{\varphi}, v_0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.3}$$

Let $\varphi : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional for problem (4.1) defined by

$$\varphi(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega F\big(z, u(z)\big)dz \quad \text{for all } u \in H_0^1(\Omega).$$

Hypotheses $\mathbf{H}_4$ imply that $\varphi \in C^2(H_0^1(\Omega))$. Hypothesis $\mathbf{H}_4$(iii) and the unique continuation property of the eigenspaces imply that $u = 0$ is a nondegenerate critical point of φ with Morse index

$$d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i)$$

($E(\widehat{\lambda}_i)$ being the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_i$, $i \geq 1$). Hence

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0 \quad \big(\text{see (2.4)}\big). \tag{4.4}$$

Note that $\varphi|_{[v_0,u_0]} = \widehat{\varphi}|_{[v_0,u_0]}$ and $v_0 \in -int\, C_+$, $u_0 \in int\, C_+$. Hence

$$C_k(\varphi|_{C_0^1(\overline{\Omega})}, 0) = C_k(\widehat{\varphi}|_{C_0^1(\overline{\Omega})}, 0) \quad \text{for all } k \geq 0. \tag{4.5}$$

From Palais [17] (see also Chang [7, p. 14]), we know that

$$C_k(\varphi|_{C_0^1(\overline{\Omega})}, 0) = C_k(\varphi, 0) \quad \text{and} \quad C_k(\widehat{\varphi}|_{C_0^1(\overline{\Omega})}, 0) = C_k(\widehat{\varphi}, 0) \quad \text{for all } k \geq 0. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we infer that

$$C_k(\widehat{\varphi}, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.7}$$

We have

$$\big\langle \varphi''(y_0)u, v\big\rangle = \int_\Omega (Du, Dv)_{\mathbb{R}^N}dz - \int_\Omega f_x'(z, y_0)uvdz \quad \text{for all } u, v \in H_0^1(\Omega).$$

Let $\sigma(\varphi''(y_0))$ be the spectrum of $\varphi''(y_0)$ and assume that $\sigma(\varphi''(y_0)) \subseteq [0, +\infty)$. Then

$$\|Du\|_2^2 \geq \int_\Omega f_x'(z, y_0)u^2dz \quad \text{for all } u, v \in H_0^1(\Omega). \tag{4.8}$$

If $u \in \ker \varphi''(y_0)$, then

$$-\Delta u(z) = f_x'(z, y_0)u(z) \quad \text{a.e. in } \Omega, \qquad u|_{\partial\Omega} = 0. \tag{4.9}$$

From (4.8) we see that $\widehat{\lambda}_+(f'_x(\cdot, y_0(\cdot))) \geq 1$. Hence (4.9) implies that

$$\dim \ker \varphi''(y_0) \leq 1. \tag{4.10}$$

By virtue of (4.2) we have

$$C_k(\varphi|_{C_0^1(\overline{\Omega})}, y_0) = C_k(\widehat{\varphi}|_{C_0^1(\overline{\Omega})}, y_0) \quad \text{for all } k \geq 0,$$

hence

$$C_k(\varphi, y_0) = C_k(\widehat{\varphi}, y_0) \quad \text{for all } k \geq 0 \quad \big(\text{see [7,17]}\big). \tag{4.11}$$

But we know that y_0 is a critical point of mountain pass type for the functional $\widehat{\varphi}$ (see the proof of Proposition 5). Hence

$$C_1(\widehat{\varphi}, y_0) \neq 0 \quad \big(\text{see Chang [7] and Mawhin and Willem [16]}\big)$$

hence

$$C_1(\varphi, y_0) \neq 0 \quad \big(\text{see (4.11)}\big). \tag{4.12}$$

From (4.10), (4.12) and Proposition 2.5 of Bartsch [5] and from Mawhin and Willem [16], we have

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0$$

hence

$$C_k(\widehat{\varphi}, y_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0 \quad \big(\text{see (4.11)}\big). \tag{4.13}$$

Recall that $\widehat{\varphi}$ is coercive (see (3.16)). Therefore

$$C_k(\widehat{\varphi}, \infty) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \geq 0. \tag{4.14}$$

Suppose $K_{\widehat{\varphi}} = \{0, u_0, v_0, y_0\}$. From (4.3), (4.7), (4.13), (4.14) and the Morse relation with $t = -1$ (see (2.3)), we have

$$2(-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0,$$

hence

$$(-1)^{d_m} = 0, \quad \text{a contradiction.}$$

Therefore we can find $\widehat{y} \in K_{\widehat{\varphi}}$, $\widehat{y} \notin \{0, u_0, v_0, y_0\}$. Hence $\widehat{y} \in C_0^1(\overline{\Omega})$ solves (4.1) and it is nodal (see (3.17)). $\square$

5. Problems concave near the origin

In Theorems 3 and 4, the reaction $f(z,\cdot)$ is restricted to be $(p-1)$-linear near zero. This precludes nonlinearities concave near the origin. This raises the question of whether we can have a multiplicity theorem with sign information for the solutions, in the presence of concave nonlinearities. Recently, Hu and Papageorgiou [13] studied a parametric class of such problems and proved certain bifurcation type results.

We introduce the following hypotheses on the reaction $f(z,x)$:

H$_5$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, such that:

(i) for every $\rho>0$, there exists $a_\rho\in L^\infty(\Omega)_+$ such that $|f(z,x)|\le a_\rho(z)$ for a.a. $z\in\Omega$, all $|x|\le\rho$;

(ii) $\lim_{x\to\pm\infty}\frac{f(z,x)}{|x|^{p-2}x}=-\infty$ uniformly for a.a. $z\in\Omega$;

(iii) there exist $r\in(p,p^*)$ and constants $\widehat{c}_1,\widehat{c}_2>0$, $\widehat{c}_1>\widehat{\lambda}_2$ such that

$$f(z,x)x\ge\widehat{c}_1|x|^p-\widehat{c}_2|x|^r\quad\text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R}.$$

Remark. Now the growth of $f(z,\cdot)$ is subcritical both from above and below.

Example. A simple function satisfying hypotheses **H**$_5$ is the following

$$f(x)=|x|^{q-2}x-\widehat{c}|x|^{r-2}x\quad\text{for all } x\in\mathbb{R},\ \text{with } 1<q\le p<r<\infty,\ \widehat{c}>\widehat{\lambda}_2.$$

We start by considering the following auxiliary Dirichlet problem:

$$-\Delta_p u(z)=\widehat{c}_1\big|u(z)\big|^{p-2}u(z)-\widehat{c}_2\big|u(z)\big|^{r-2}u(z)\quad\text{in }\Omega,\qquad u|_{\partial\Omega}=0. \tag{5.1}$$

Proposition 6. *Problem (5.1) has a unique nontrivial positive solution $\underline{u}\in \operatorname{int}C_+$ and a unique nontrivial negative solution $\bar{v}=-\underline{u}\in-\operatorname{int}C_+$.*

Proof. Let $\psi_+:W_0^{1,p}(\Omega)\to\mathbb{R}$ be the C^1-functional defined by

$$\psi_+(u)=\frac{1}{p}\|Du\|_p^p-\frac{\widehat{c}_1}{p}\big\|u^+\big\|_p^p+\frac{\widehat{c}_2}{p}\big\|u^+\big\|_r^r\quad\text{for all } u\in W_0^{1,p}(\Omega).$$

Then

$$\psi_+(u)\ge\frac{1}{p}\|Du\|_p^p-\widehat{c}_3\big\|u^+\big\|_r^p+\frac{\widehat{c}_2}{r}\big\|u^+\big\|_r^r\quad\text{for some } \widehat{c}_3>0\quad(\text{recall } r>p). \tag{5.2}$$

Because $r>p$, from (5.2) we infer that ψ_+ is coercive. Also, since $r<p^*$, by the Sobolev embedding theorem, we have that ψ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\underline{u}\in W_0^{1,p}(\Omega)$ such that

$$\psi_+(\underline{u})=\inf\big\{\psi_+(u):u\in W_0^{1,p}(\Omega)\big\}=m_+. \tag{5.3}$$

Since $\widehat{c}_2>\widehat{\lambda}_2$, for $t\in(0,1)$ small, we have $\psi(t\widehat{u}_1)<0$, hence

$$\psi_+(\underline{u})=m_+<0=\psi_+(0),\quad\text{i.e.}\quad\underline{u}\ne0.$$

From (5.3), we have $\psi_+'(\underline{u})=0$, hence

$$A(\underline{u})=\widehat{c}_1\big(\underline{u}^+\big)^{p-1}-\widehat{c}_2\big(\underline{u}^+\big)^{r-1}. \tag{5.4}$$

On (5.4) we act with $-\underline{u}^- \in W_0^{1,p}(\Omega)$ and obtain $\underline{u} \geq 0$, $\underline{u} \neq 0$. Hence

$$A(\underline{u}) = \widehat{c}_1 \underline{u}^{p-1} - \widehat{c}_2 \underline{u}^{r-1},$$

and we obtain that

$$\underline{u} \in C_+ \backslash \{0\} \quad \text{solves problem (5.1)}.$$

If $\rho = \|\underline{u}\|_\infty$, then $\Delta_p \underline{u}(z) \leq \widehat{c}_2 \rho^{r-p} \underline{u}(z)^{p-1}$ a.e. in Ω, hence

$$\underline{u} \in \mathit{int}\, C_+ \quad \big(\text{see Vasquez [21]}\big).$$

Now, we show the uniqueness of $\underline{u} \in \mathit{int}\, C_+$. To this end, let u, v be two nontrivial positive solutions of (5.1). From the above argument, we have that $u, v \in \mathit{int}\, C_+$. Let

$$R(u,v)(z) = \big\|Du(z)\big\|^p - \big\|Dv(z)\big\|^p \left(Dv(z), D\left(\frac{u(z)^p}{v(z)^{p-1}}\right)\right)_{\mathbb{R}^N}.$$

From the generalized Picone's identity of Allegretto and Huang [2], we have

$$R(u,v)(z) \geq 0 \quad \text{a.e. in } \Omega.$$

Note that

$$\begin{aligned}
&\int\limits_\Omega \big(\widehat{c}_1 - \widehat{c}_2 u^{r-p}\big)\big(u^p - v^p\big)dz \\
&\quad = \int\limits_\Omega \left(\frac{\widehat{c}_1 u^{p-1} - \widehat{c}_2 u^{r-1}}{u^{p-1}}\right)\big(u^p - v^p\big)dz \\
&\quad = \int\limits_\Omega -\Delta_p u\left(u - \frac{v^p}{u^{p-1}}\right)dz \quad \big(\text{see (5.1)}\big) \\
&\quad = \int\limits_\Omega \|Du\|^{p-2}\left(Du, Du - D\left(\frac{v^p}{u^{p-1}}\right)\right)_{\mathbb{R}^N} dz \\
&\qquad \big(\text{by the nonlinear Green's identity see [12, p. 210]}\big) \\
&\quad = \|Du\|_p^p - \|Dv\|_p^p + \int\limits_\Omega R(v,u)dz.
\end{aligned} \tag{5.5}$$

Reversing the roles of u and v in the above argument, we obtain

$$\int\limits_\Omega \big(\widehat{c}_1 - \widehat{c}_2 v^{r-p}\big)\big(v^p - u^p\big)dz = \|Dv\|_p^p - \|Du\|_p^p + \int\limits_\Omega R(u,v)dz. \tag{5.6}$$

Adding (5.5) and (5.6), we obtain

$$0 \geq \int\limits_\Omega \widehat{c}_2\big(v^{r-p} - u^{r-p}\big)\big(u^p - v^p\big)dz = \int\limits_\Omega \big(R(v,u) + R(u,v)\big)dz \geq 0,$$

hence

$$u = v \quad \big(\text{since } x \to \widehat{c}_2 x^{r-p} \text{ is strictly increasing in } (0,\infty)\big).$$

This proves the uniqueness of $\underline{u} \in int\, C_+$.

By the oddness of (5.1), $\bar{v} = -\underline{u} \in -\, int\, C_+$ is the unique nontrivial negative solution of (5.1). $\square$

Let S_+ (resp. S_-) be the set of nontrivial positive (resp. negative) solutions of (1.1). From Proposition 3, we show that $S_+, S_- \neq \varnothing$ and $S_+ \subseteq int\, C_+$, $S_- \subseteq -\, int\, C_+$.

Note that hypothesis $\mathbf{H}_1$(iii) allows the presence of concave terms.

Proposition 7. *If hypotheses $\mathbf{H}_5$ hold and $\widetilde{u} \in S_+$ (resp. $\widetilde{v} \in S_-$), then $\underline{u} \le \widetilde{u}$ (resp. $\widetilde{v} \le \bar{v} = -\underline{u}$).*

Proof. We introduce the following Carathéodory function

$$\gamma_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ \widehat{c}_1 x^{p-1} - \widehat{c}_2 x^{r-1} & \text{if } 0 \le x \le \widetilde{u}(z) \\ \widehat{c}_1 \widetilde{u}(z)^{p-1} - \widehat{c}_2 \widetilde{u}(z)^{r-1} & \text{if } x > \widetilde{u}(z). \end{cases} \tag{5.7}$$

Let $\Gamma_+(z,x) = \int_0^x \gamma_+(z,s)ds$ and consider the C^1-functional $\sigma_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_+(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \Gamma_+\big(z,u(z)\big)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Evidently σ_+ is coercive (see (5.7)) and it is sequentially weakly lower semicontinuous (recall that $r < p^*$). So, by the Weierstrass theorem, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\sigma_+(\widehat{u}) = \inf\big\{\sigma_+(u) : u \in W_0^{1,p}(\Omega)\big\}. \tag{5.8}$$

As before, since $\widehat{c}_1 > \widehat{\lambda}_2$, we have $\sigma_+(\widehat{u}) < 0 = \sigma_+(0)$, hence $\widehat{u} \neq 0$. From (5.8), we have $\sigma_+'(\widehat{u}) = 0$, hence

$$A(\widehat{u}) = N_{\gamma_+}(\widehat{u}). \tag{5.9}$$

Acting on (5.9) with $-\widehat{u}^- \in W_0^{1,p}(\Omega)$ and with $(\widehat{u} - \widetilde{u})^+ \in W_0^{1,p}(\Omega)$, we show that $\widehat{u} \in [0, \widetilde{u}]$, $\widehat{u} \neq 0$ (see the proof of Proposition 3). Hence from (5.9) and (5.7), we have

$$A(\widehat{u}) = \widehat{c}_1 \widehat{u}^{p-1} - \widehat{c}_2 \widehat{u}^{r-1},$$

hence

$$\widehat{u} \text{ is a nontrivial solution of (5.1).}$$

Then $\widehat{u} = \underline{u} \in int\, C_+$ (see Proposition 6), and we conclude that

$$\underline{u} \le \widetilde{u}.$$

Similarly, we show that $\widetilde{v} \le \bar{v} = -\underline{u} \in -\, int\, C_+$. $\square$

Using this proposition, we can show the existence of extremal constant sign solutions for problem (1.1). This is done as in the proof of Proposition 4, using the Kuratowski–Zorn lemma. In this case, the nontriviality of the limit function u (see (3.10)) follows from the fact that $u \ge \underline{u}$ (see Proposition 7).

Similarly for the extremal nontrivial negative solution. So, we can state the following proposition.

Proposition 8. *If hypotheses* $\mathbf{H}_5$ *hold, then problem* (1.1) *admits extremal constant sign solutions* $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$.

Having these extremal constant sign solutions and reasoning as in the proof of Proposition 5, we have the following multiplicity theorem.

Theorem 5. *If hypotheses* $\mathbf{H}_5$ *hold, then problem* (1.1) *admits at least three nontrivial smooth solutions* $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$ *and* $y_0 \in C_0^1(\overline{\Omega})$ *nodal.*

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