Chapter 2
From 1D convolutional codes to 2D convolutional codes of rate 1/n*

Paulo Almeida, Diego Napp, and Raquel Pinto

Abstract In this paper we introduce a new type of superregular matrices that give rise to novel constructions of two-dimensional (2D) convolutional codes with finite support. These codes are of rate $1/n$ and degree δ with $n \geq \delta + 1$ and achieve the maximum possible distance among all 2D convolutional codes with finite support with the same parameters.

Key words 1D and 2D convolutional codes, MDS codes, superregular matrix

Contents

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1 Introduction

When considering data recorded in two dimensions, like pictures and video, two-dimensional (2D) convolutional codes [3–5, 7, 8] seem to be a better framework to encode such data than one-dimensional (1D) codes, since it takes advantage of the interdependence of the data in more than one direction. In [3] the distance properties of 2D convolutional codes of rate $1/n$ and degree δ were studied, and constructions of such codes with maximum possible distance (MDS) were given for $n \geq \frac{(\delta+1)(\delta+2)}{2}$. In this paper we relax this restriction and present 2D MDS convolutional codes of rate $1/n$ with $n \geq \delta + 1$. The idea is to consider 1D convolutional codes obtained as the projection of the 2D code on the vertical lines. For that we need to introduce a new type of matrices of a particular structure and show that they are superregular.

2 1D and 2D convolutional codes

In this section we give some basic results on 1D and 2D convolutional codes that will be useful throughout the paper.

Let $\mathbb{F}$ be a finite field and $\mathbb{F}[z_2]$ the ring of polynomials in one indeterminate with coefficients in $\mathbb{F}$. A 1**D (finite support) convolutional code** $\mathscr{C}$ of rate k/n is an $\mathbb{F}[z_2]$-submodule of $\mathbb{F}[z_2]^n$, where k is the rank of $\mathscr{C}$ (see [6]). A full column rank matrix $\widehat{G}(z_2) \in \mathbb{F}[z_2]^{n\times k}$ whose columns constitute a basis for $\mathscr{C}$ is called an **encoder** of $\mathscr{C}$. So,

$$\begin{aligned}\mathscr{C} &= \mathrm{im}_{\mathbb{F}[z_2]}\, \widehat{G}(z_2) \\ &= \left\{ \widehat{v}(z_2) \in \mathbb{F}[z_2]^n \mid \widehat{v}(z_2) = \widehat{G}(z_2)\widehat{u}(z_2) \text{ with } \widehat{u}(z_2) \in \mathbb{F}[z_2]^k \right\}.\end{aligned}$$

The **weight** of a word $\widehat{v}(z_2) = \sum_{i\geq 0} v(i) z_2^i \in \mathbb{F}[z_2]^n$ is given by

$$\mathrm{wt}(\widehat{v}(z_2)) = \sum_{i\in\mathbb{N}} \mathrm{wt}(v(i)),$$

where the weight of a constant vector $v(i)$ is the number of nonzero entries of $v(i)$ and the **distance** of a 1D convolutional code $\mathscr{C}$ is defined as

$$\mathrm{dist}(\mathscr{C}) = \min \left\{ \mathrm{wt}(\widehat{v}(z_2)) \mid \widehat{v}(z_2) \in \mathscr{C}, \text{ with } \widehat{v}(z_2) \neq 0 \right\}.$$

If $\mathscr{C}$ is a 1D convolutional code of rate $1/n$, then all its encoders differ by a nonzero constant. The degree of $\mathscr{C}$ is defined as the column degree of any encoder of $\mathscr{C}$. The next result follows immediately.

Corollary 1. *[6] Let $\mathscr{C}$ be a 1D convolutional code of rate $1/n$ with degree ν. Then*

$$\mathrm{dist}(\mathscr{C}) \leq n(\nu + 1).$$

A 1D convolutional code of rate $1/n$ with degree ν and distance $n(\nu+1)$ is said to be Maximum Distance Separable (MDS). In [3] constructions of such codes were given for $n \geq \nu+1$ as stated in the next theorem.

Theorem 2. *Let $\nu, n \in \mathbb{N}$ with $n \geq \nu+1$ and $\mathscr{G} = [G_0\ G_1\ \cdots G_\nu]$, with $G_i \in \mathbb{F}^n$, $i = 0,1,\ldots,\nu$, be a matrix such that all its minors of any order are different from zero. Then $\mathscr{C} = \mathrm{Im}_{\mathbb{F}[z_2]} \sum_{i=0}^{\nu} G_i z_2^i$ is an MDS 1D convolutional code of rate $1/n$ and degree ν.*

We are going to consider now convolutional codes whose codewords belong to $\mathbb{F}[z_1,z_2]^n$, where $\mathbb{F}[z_1,z_2]$ is the ring of polynomials in two indeterminates with coefficients in $\mathbb{F}$. Such codes are called 2D (finite support) convolutional codes. More precisely, a **2D (finite support) convolutional code** $\mathscr{C}$ of rate k/n is a free $\mathbb{F}[z_1,z_2]$-submodule of $\mathbb{F}[z_1,z_2]^n$ of rank k (see [7, 8]). An encoder of $\mathscr{C}$ is a full column rank matrix $\widehat{G}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ whose columns constitute a basis for $\mathscr{C}$. Therefore,

$$\begin{aligned}\mathscr{C} &= \mathrm{im}_{\mathbb{F}[z_1,z_2]} \widehat{G}(z_1,z_2)\\ &= \left\{\widehat{v}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^n \mid \widehat{v}(z_1,z_2) = \widehat{G}(z_1,z_2)\widehat{u}(z_1,z_2) \text{ with } \widehat{u}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^k\right\}.\end{aligned}$$

The **weight** of a word $\widehat{v}(z_1,z_2) = \sum_{(i,j)\in\mathbb{N}^2} v(i,j) z_1^i z_2^j \in \mathbb{F}[z_1,z_2]^n$ is defined in a similar way to the 1D case as

$$\mathrm{wt}(\widehat{v}(z_1,z_2)) = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{wt}(v(i,j)),$$

and the **distance** of $\mathscr{C}$ as

$$\mathrm{dist}(\mathscr{C}) = \min\left\{\mathrm{wt}(\widehat{v}(z_1,z_2)) \mid \widehat{v}(z_1,z_2) \in \mathscr{C}, \text{ with } \widehat{v}(z_1,z_2) \neq 0\right\}.$$

In this paper, we restrict to 2D convolutional codes of rate 1/n. Given an encoder

$$\widehat{G}(z_1,z_2) = \sum_{(i,j)\in\mathbb{N}^2} G(i,j) z_1^i z_2^j \in \mathbb{F}[z_1,z_2]^n$$

of a 2D convolutional code $\mathscr{C}$ of rate 1/n, we define the **degree** of $\widehat{G}(z_1,z_2)$ as $\delta = \max\{i+j \mid G(i,j) \neq 0\}$. Since two encoders of $\mathscr{C}$ differ by a nonzero constant, all encoders of $\mathscr{C}$ have the same degree and we define the **degree** of $\mathscr{C}$ as the degree of any of its encoders.

If $\mathscr{C}$ is a 2D convolutional code of rate 1/n and degree δ, then

$$\mathrm{dist}(\mathscr{C}) \leq \frac{(\delta+1)(\delta+2)}{2} n. \tag{1}$$

Such bound is called the 2D generalized Singleton bound and if the distance of $\mathscr{C}$ equals such bound, $\mathscr{C}$ is said to be MDS (see [3]).

3 Superregular matrices

In [1] a new type of superregular matrices was introduced. The superregular matrices we are going to construct in this work have similar entries and, therefore, some properties are the same, even if the structure of these new matrices is different. Before we develop our new construction, we will recall some definitions pertinent to this type of superregular matrices.

Let $A = [\mu_{i\ell}]$ be a square matrix of order m over $\mathbb{F}$ and S_m the symmetric group of order m. The determinant of A is given by

$$|A| = \sum_{\sigma \in S_m} (-1)^{\mathrm{sgn}(\sigma)} \mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}.$$

Whenever we use the word *term*, we will be considering one product of the form $\mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}$, with $\sigma \in S_m$, and the word *component* will be reserved to refer to each of the $\mu_{i\sigma(i)}$, with $1 \leq i \leq m$ in a term. Denote $\mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}$ by μ_σ.

A *trivial term* of the determinant is a term μ_σ, with at least one component $\mu_{i\sigma(i)}$ equal to zero. If A is a square submatrix of a matrix B with entries in $\mathbb{F}$, and all the terms of the determinant of A are trivial, we say that $|A|$ is a **trivial minor** of B (if $B = A$ we simply say that $|A|$ is a trivial minor). We say that B is **superregular** if all its nontrivial minors are different from zero.

The next theorem states that square matrices over $\mathbb{F}$ of a certain form are superregular.

Theorem 3. *Let α be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and $A = [\mu_{i\ell}]$ be a square matrix over $\mathbb{F}$ of order m, with the following properties*

1. *if $\mu_{i\ell} \neq 0$ then $\mu_{i\ell} = \alpha^{\beta_{i\ell}}$ for a positive integer $\beta_{i\ell}$;*
2. *if $\hat{\sigma} \in S_m$ is the permutation defined by $\hat{\sigma}(i) = m - i + 1$, then $\mu_{\hat{\sigma}}$ is a nontrivial term of $|A|$;*
3. *if $\ell \geq m - i + 1$, $\ell < \ell'$, $\mu_{i\ell} \neq 0$ and $\mu_{i\ell'} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i\ell'}$;*
4. *if $\ell \geq m - i + 1$, $i < i'$, $\mu_{i\ell} \neq 0$ and $\mu_{i'\ell} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i'\ell}$.*

Suppose $|A|$ is a nontrivial minor and N is greater than any exponent of α appearing as a nontrivial term of $|A|$. Then $|A| \neq 0$.

Proof. Let $\sigma \in S_m$ such that μ_σ is a nontrivial term of $|A|$. By property 1, we have $\mu_\sigma = \alpha^{\beta_\sigma}$, for a positive integer β_σ.

Let $T_m = \{\sigma \in S_m \mid \sigma \neq \hat{\sigma}$ and μ_σ is a nontrivial term of $|A|\}$. If $T_m = \emptyset$ then $|A| = \mu_{\hat{\sigma}} = \alpha^{\beta_{\hat{\sigma}}} \neq 0$.

If $T_m \neq \emptyset$, let $\sigma \in T_m$. We are going to prove that if $\sigma \neq \hat{\sigma}$ then $\beta_{\hat{\sigma}} < \beta_\sigma$. Since μ_σ in a nontrivial term of $|A|$, for any $1 \leq i \leq m$, there exists $\ell \geq i$ such that $\sigma(\ell) \geq m - i + 1$. Now, For any $1 \leq \ell \leq m$ define

$$S_\ell = \{i \mid i \leq \ell \text{ and } \sigma(\ell) \geq m - i + 1\}.$$

Notice that $\bigcup_{1\leq j\leq m} S_\ell = \{1,2,\ldots,m\}$ and, since $\sigma \neq \hat{\sigma}$, exists at least one ℓ_0, such that $1 \leq \ell_0 \leq m$ and $S_{\ell_0} = \emptyset$. By properties 3 and 4, we have that $\sum_{i\in S_\ell} \beta_{i\, m-i+1} \leq \beta_{\ell\ \sigma(\ell)}$. Therefore $\beta_{\hat{\sigma}} = \sum_{i=1}^{m} \beta_{i\, m-i+1} \leq \sum_{\substack{\ell=1\\ S_\ell\neq 0}}^{m} \beta_{\ell\ \sigma(\ell)} < \sum_{\ell=1}^{m} \beta_{\ell\ \sigma(\ell)}$. So $|A| = \alpha^{\beta_{\hat{\sigma}}} + \sum_{h=\beta_{\hat{\sigma}}+1}^{N-1} \varepsilon_h \alpha^h$, where $\varepsilon_h \in \mathbb{F}_p$. Hence $|A| \neq 0$. □

Let us now construct specific types of superregular matrices, which will be useful in the next section. Let p be a prime number, N a positive integer and α be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$. For $0 \leq a \leq \delta$ and $0 \leq b \leq \delta - a$ define the $n \times 1$ matrix $G(a,b)$ as

$$G(a,b) = \left[\alpha^{2^{(\delta-a-b)n(\delta+1)+a}} \quad \alpha^{2^{((\delta-a-b)n+1)(\delta+1)+a}} \quad \ldots \quad \alpha^{2^{((\delta-a-b+1)n-1)(\delta+1)+a}}\right]^T \tag{2}$$

For example, if $\delta = 2$

$$G(0,2) = \begin{bmatrix} \alpha^{2^0} \\ \alpha^{2^3} \\ \vdots \\ \alpha^{2^{3(n-1)}} \end{bmatrix} \qquad G(0,0) = \begin{bmatrix} \alpha^{2^{6n}} \\ \alpha^{2^{3(2n+1)}} \\ \vdots \\ \alpha^{2^{3(3n-1)}} \end{bmatrix} \qquad G(1,1) = \begin{bmatrix} \alpha^{2^1} \\ \alpha^{2^4} \\ \vdots \\ \alpha^{2^{3(n-1)+1}} \end{bmatrix}$$

The following technical lemmas will be useful in the next section.

Lemma 4. *Let $\delta \geq 0$, $0 \leq j \leq \delta$ and $n \geq \delta - j + 1$. Then for $N \geq 2^{9n-2}$, the matrices*

$$\mathscr{G}_j = [G(j,0)\ \ G(j,1)\cdots G(j,\delta-j)] \in \mathbb{F}_{p^N}^{n\times(\delta-j+1)}$$

have all its minors of any dimension, different from zero.

Note that all elements of $\mathscr{G}_j$ are different from zero, which means that all its minors are nontrivial. Moreover, up to column permutations, the minors of $\mathscr{G}_j$ satisfy Theorem 3.

Lemma 5. *Let $N \geq 2^{9n-1}$ and α a primitve element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$. Let $n \geq 3$ and $G(a,b) \in \mathbb{F}^n$, with $0 \leq a \leq 2$ and $0 \leq b \leq 2-a$, be defined as in (2). Then the following matrices are superregular:*

$$A_1 = \begin{bmatrix} G(0,2) & G(0,1) & G(0,0) \\ G(0,1) & G(0,0) & 0 \\ G(0,0) & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ G(0,2) & G(0,1) & G(0,0) \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & G(0,2) \\ 0 & 0 & G(0,2) & G(0,1) \\ G(0,2) & G(0,1) & G(0,0) & 0 \\ G(0,1) & G(0,0) & 0 & 0 \\ G(0,0) & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ 0 & G(0,1) & G(0,0) \\ G(0,2) & G(0,0) & 0 \\ G(0,1) & 0 & 0 \\ G(0,0) & 0 & 0 \end{bmatrix},$$

$$\textit{and } A_5 = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ G(0,2) & G(0,1) & G(0,0) \\ G(0,1) & G(0,0) & 0 \\ G(0,0) & 0 & 0 \end{bmatrix}.$$

4 Constructions of MDS 2D convolutional codes of rate 1/n and degree $\delta \leq 2$ for $n \geq \delta + 1$

In this section we will consider 2D convolutional codes of rate $1/n$ and degree δ and we will give constructions of MDS codes for $\delta \leq 2$. Let $\widehat{G}(z_1,z_2) = \sum_{0\leq i+j\leq\delta} G(i,j)z_1^i z_2^j$ be an encoder of $\mathscr{C}$, with $G(i,j) \in \mathbb{F}_{p^N}$ defined as in (2). We can write

$$\widehat{G}(z_1,z_2) = \sum_{j=0}^{\delta} G_j(z_2)z_1^j, \tag{3}$$

where $G_j(z_2) = \sum_{i=0}^{\delta-j} G(j,i)z_2^i \in \mathbb{F}[z_2]^n$, $j = 0,1,\dots,\delta$. Analogously, given $\widehat{u}(z_1,z_2) \in \mathbb{F}[z_1,z_2]$ and $\widehat{v}(z_1,z_2) = \widehat{G}z_1,z_2)\widehat{u}(z_1,z_2)$, we can write them in the same way, i.e.,

$$\widehat{u}(z_1,z_2) = \sum_{j=0}^{\ell} \widehat{u}_j(z_2)z_1^j \quad \text{and} \quad \widehat{v}(z_1,z_2) = \sum_{j=0}^{\delta+\ell} \widehat{v}_j(z_2)z_1^j, \tag{4}$$

with $\ell \in \mathbb{N}$, where $\widehat{u}_j(z_2) = \sum_{i\geq 0} u(j,i)z_2^i \in \mathbb{F}[z_2]$, for any $j = 0,\dots,\ell$ and

$$\widehat{v}_s(z_2) = \sum_{i=0}^{\delta} G_i(z_2)\widehat{u}_{s-i}(z_2), \text{ for any } s = 0,\dots,\delta+\ell$$

(we consider $\widehat{u}_a(z_2) = 0$ if $a < 0$ or if $a > \ell$). Note that $\widehat{v}_s(z_2)$ are codewords of 1D convolutional codes.

We conjecture that for $n \geq \delta + 1$ and N sufficiently large, the 2D convolutional code $\mathscr{C} = \mathrm{Im}_{\mathbb{F}[z_1,z_2]}\widehat{G}(z_1,z_2)$ is MDS. Next theorem considers such codes for $\delta \leq 2$.

Theorem 6. *Let $N \geq 2^{9n-1}$, $\delta \leq 2$, $n \geq \delta + 1$ and $\widehat{G}(z_1,z_2)$ as defined in (3). Then $\mathscr{C} = \mathrm{Im}_{\mathbb{F}[z_1,z_2]}\widehat{G}(z_1,z_2)$ is a 2D MDS convolutional code of rate $1/n$ and degree δ.*

Proof. It is obvious that $\mathscr{C}$ has rate $1/n$ and degree δ. Let us consider first $\delta = 2$. To prove that $\mathscr{C}$ is MDS we have to show that all nonzero codewords of $\mathscr{C}$,

$\widehat{v}(z_1,z_2)$, have weight greater or equal than $6n$. Let $\widehat{v}(z_1,z_2)$ be a nonzero codeword of $\mathscr{C}$ and $\widehat{u}(z_1,z_2) \in \mathbb{F}_{p^N}[z_1,z_2]$ such that $\widehat{v}(z_1,z_2) = \widehat{G}(z_1,z_2)\widehat{u}(z_1,z_2)$ and let us represent both vectors as in (4). It is obvious that $\widehat{u}(z_1,z_2) \neq 0$ and in order to calculate the weight of $\widehat{v}(z_1,z_2)$ we can assume without loss of generality that $\widehat{u}_0(z_2) \neq 0$. Thus $\widehat{v}_0(z_2) = G_0(z_2)\widehat{u}_0(z_2)$, $\widehat{v}_1(z_2) = [G_0(z_2) \;\; G_1(z_2)] \begin{bmatrix} \widehat{u}_1(z_2) \\ \widehat{u}_0(z_2) \end{bmatrix}$ and $\widehat{v}_2(z_2) = [G_0(z_2) \;\; G_1(z_2) \;\; G_2(z_2)] \begin{bmatrix} \widehat{u}_2(z_2) \\ \widehat{u}_1(z_2) \\ \widehat{u}_0(z_2) \end{bmatrix}$, are all nonzero vectors, i.e., the weight any of these vectors is at least one. By definition, $\widehat{u}(z_1,z_2) = \widehat{u}_0(z_2) + \widehat{u}_1(z_2)z_1 + \cdots + \widehat{u}_\ell(z_2)z_1^\ell$, with $\widehat{u}_\ell(z_2) \neq 0$ for some $\ell \in \mathbb{N}$. Then, since $\widehat{v}_{2+\ell}(z_2) = G(2,0)\widehat{u}_\ell(z_2)$ it follows that $\mathrm{wt}(\widehat{v}_{2+\ell}(z_2)) = n\,\mathrm{wt}(\widehat{u}_\ell(z_2)) \geq n$. Since $\widehat{v}_0(z_2) = G_0(z_2)\widehat{u}_0(z_2)$ then, by Lemma 4, $\mathrm{wt}(\widehat{v}_0(z_2)) \geq 3n$. Now, if $\widehat{u}_1(z_2) = 0$ we have $\widehat{v}_1(z_2) = G_1(z_2)\widehat{u}_0(z_2)$ and again by Lemma 4, $\mathrm{wt}(\widehat{v}_1(z_2)) \geq 2n$. Hence

$$\mathrm{wt}(\widehat{v}(z_1,z_2)) \geq \mathrm{wt}(\widehat{v}_0(z_2) + \widehat{v}_1(z_2)z_1 + \widehat{v}_{2+\ell}(z_2)z_1^{2+\ell}) \geq 6n.$$

Next, we consider $\widehat{u}_1(z_2) \neq 0$. Suppose $\mathrm{wt}(\widehat{u}_0(z_2)) \geq 4$ and let min Supp $(\widehat{u}_0(z_2)) = i_1$ and max Supp $(\widehat{u}_0(z_2)) = i_2$. Since $i_2 \geq i_1 + 3$, we have that

$$\begin{bmatrix} v(0,i_1+2) \\ v(0,i_1+1) \\ v(0,i_1) \end{bmatrix} = \begin{bmatrix} G(0,2) & G(0,1) & G(0,0) \\ G(0,1) & G(0,0) & 0 \\ G(0,0) & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0,i_1) \\ u(0,i_1+1) \\ u(0,i_1+2) \end{bmatrix}$$

and

$$\begin{bmatrix} v(0,i_2+2) \\ v(0,i_2+1) \\ v(0,i_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ G(0,2) & G(0,1) & G(0,0) \end{bmatrix} \begin{bmatrix} u(0,i_2-2) \\ u(0,i_2-1) \\ u(0,i_2) \end{bmatrix}.$$

Since the matrices A_1 and A_2 in Lemma 5 are superregular, we obtain, for $s \in \{i_1, i_2\}$, $\mathrm{wt}(v(0,s)z_2^s + v(0,s+1)z_2^{s+1} + v(0,s+2)z_2^{s+2}) \geq 3n-2$. Then $\mathrm{wt}(\widehat{v}_0(z_2)) \geq 6n-4$. Therefore, $\mathrm{wt}(\widehat{v}(z_1,z_2)) \geq \mathrm{wt}(\widehat{v}_0(z_2) + \widehat{v}_1(z_2)z_1 + \widehat{v}_{2+\ell}(z_2)z_1^{2+\ell}) \geq 6n-4+1+n \geq 6n$, since $n \geq 3$.

Assume now that $\mathrm{wt}(\widehat{u}_0(z_2)) = 3$. If $\mathrm{Supp}(\widehat{u}_0(z_2)) = \{i, i+1, i+2\}$, for some $i \in \mathbb{N}$, then $\widehat{v}_0(z_2) = v(0,i)z_2^i + v(0,i+1)z_2^{i+1} + v(0,i+2)z_2^{i+2} + v(0,i+3)z_2^{i+3} + v(0,i+4)z_2^{i+4}$, where

$$\begin{bmatrix} v(0,i+4) \\ v(0,i+3) \\ v(0,i+2) \\ v(0,i+1) \\ v(0,i) \end{bmatrix} = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ G(0,2) & G(0,1) & G(0,0) \\ G(0,1) & G(0,0) & 0 \\ G(0,0) & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0,i) \\ u(0,i+1) \\ u(0,i+2) \end{bmatrix},$$

and, since A_6 is superregular, by Lemma 5, $\mathrm{wt}(\widehat{v}_0(z_2)) \geq 5n-2$. We now need $\mathrm{wt}(\widehat{u}_1(z_2))$. Let $j = \min \mathrm{Supp}(\widehat{u}_1(z_2))$. If $j < i$, then $v(1,j) = G(0,0)u(1,j)$, if $j > i$ then $v(1,i) = G(1,0)u(0,i)$ and if $j = i$, then $v(1,i) = G(1,0)u(0,i)G(0,0)u(1,i)$, so, in any case, we get $\text{wt}(\widehat{v}(z_1,z_2)) = \text{wt}(\widehat{v}_0(z_2) + \widehat{v}_1(z_2)z_1 + \widehat{v}_{2+\ell}(z_2)z_1^{2+\ell}) \geq 5n-2+n-1+n = 7n-3 \geq 6n$, since $n \geq 3$.

Suppose now that $\text{Supp}(\widehat{u}_0(z_2)) = \{i_1,i_2,i_3\}$ with $i_1 < i_2 < i_3$ and $i_2 - i_1 > 1$ or $i_3 - i_2 > 1$. In this case, we will always obtain $\text{wt}(\widehat{v}_0(z_2)) \geq 6n-2$. For example, If $i_2 - i_1 = 2$ and $i_3 - i_2 = 1$, we have that

$$\begin{bmatrix} v(0,i_1+5) \\ v(0,i_1+4) \\ v(0,i_1+3) \\ v(0,i_1+2) \\ v(0,i_1+1) \\ v(0,i_1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & G(0,2) \\ 0 & G(0,2) & G(0,1) \\ 0 & G(0,1) & G(0,0) \\ G(0,2) & G(0,0) & 0 \\ G(0,1) & 0 & 0 \\ G(0,0) & 0 & 0 \end{bmatrix} \begin{bmatrix} u(0,i_1) \\ u(0,i_2) \\ u(0,i_3) \end{bmatrix},$$

so $\text{wt}(\widehat{v}_0(z_2)) \geq \text{wt}\left(v(0,i_1)z_2^{i_1} + v(0,i_1+1)z_2^{i_1+1} + \cdots + v(0,i_1+5)z_2^{i_1+5}\right) \geq 6n-2$. Thus $\text{wt}(\widehat{v}(z_1,z_2)) = \text{wt}(\widehat{v}_0(z_2) + \widehat{v}_{2+\ell}(z_2)z_1^{2+\ell}) \geq 6n-2+n \geq 6n$, since $n \geq 3$.

Suppose now $\text{wt}(\widehat{u}_0(z_2)) = 2$ and $\text{Supp}(\widehat{u}_0(z_2)) = \{i,j\}$ with $i < j$, with similar arguments as before we get $\text{wt}(\widehat{v}_0(z_2)) \geq 4n-1$. The worst case is when $j = i+1$ where

$$\begin{bmatrix} v(0,i_1+3) \\ v(0,i_1+2) \\ v(0,i_1+1) \\ v(0,i_1) \end{bmatrix} = \begin{bmatrix} 0 & G(0,2) \\ G(0,2) & G(0,1) \\ G(0,1) & G(0,0) \\ G(0,0) & 0 \end{bmatrix} \begin{bmatrix} u(0,i) \\ u(0,i+1) \end{bmatrix},$$

which implies $\text{wt}(\widehat{v}_0(z_2)) \geq 4n-1$. We also have $\text{wt}(\widehat{v}_1(z_2)) \geq 2n-2$ always. Thus $\text{wt}(\widehat{v}(z_1,z_2)) = \text{wt}(\widehat{v}_0(z_2) + \widehat{v}_1(z_2)z_1 + \widehat{v}_{2+\ell}(z_2)z_1^{2+\ell}) \geq 4n-1+2n-2+n = 7n-3 \geq 6n$, since $n \geq 3$.

Finally, assume that $\text{wt}(\widehat{u}_0(z_2)) = 1$. Here, we obtain $\text{wt}(\widehat{v}_0(z_2)) \geq 3n$ and $\text{wt}(\widehat{v}_1(z_2)) \geq 3n-1$. Hence, $\text{wt}(\widehat{v}(z_1,z_2)) \geq 6n$, for $n \geq 3$.

For $\delta \leq 1$ the theorem follows immediately. □

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