EMBEDDINGS OF BESOV SPACES ON FRACTAL h-SETS

ANTÓNIO M. CAETANO[1] AND DOROTHEE D. HAROSKE[2*]

ABSTRACT. Let Γ be a fractal h-set and $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ be a trace space of Besov type defined on Γ. While we dealt in [9] with growth envelopes of such spaces mainly and investigated the existence of traces in detail in [12], we now study continuous embeddings between different spaces of that type on Γ. We obtain necessary and sufficient conditions for such an embedding to hold, and can prove in some cases complete characterisations. It also includes the situation when the target space is of type $L_r(\Gamma)$ and, as a by-product, under mild assumptions on the h-set Γ we obtain the exact conditions on σ, p and q for which the trace space $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ exists. We can also refine some embedding results for spaces of generalised smoothness on $\mathbb{R}^n$.

INTRODUCTION

We study in this paper continuous embeddings of trace spaces on a fractal h-set Γ. It is the natural continuation and so far final step of our project to characterise traces taken on some h-set Γ, which led to the papers [9, 12] already. Questions of that type are of particular interest in view of boundary value problems of elliptic operators, where the solutions belong to some appropriate Besov (or Sobolev) space. One standard method is to start with assertions about traces on hyperplanes and then to transfer these results to bounded domains with sufficiently smooth boundary afterwards. Further studies may concern compactness or regularity results, leading to the investigation of spectral properties. However, when it comes to irregular (or fractal) boundaries, one has to circumvent a lot of difficulties following that way, such that another method turned out to be more appropriate. This approach was proposed by Edmunds and Triebel in connection with smooth boundaries first in [21] and then extended to fractal d-sets in [22,23,47]. Later the setting of d-sets was extended to (d, Ψ)-sets by Moura [40] and finally to the more general h-sets by Bricchi [6].
The idea is rather simple to describe, but the details are much more complicated: at first one determines the trace spaces of certain Besov (or Sobolev) spaces as precisely as possible, studies (compact) embeddings of such spaces into appropriate target spaces together with their entropy and approximation numbers afterwards, and finally applies Carl's or Weyl's inequalities to link eigenvalues and entropy or approximation numbers. If one is in the lucky situation that, on one hand, one has atomic or wavelet decomposition results for the corresponding spaces, and, on the other hand, the irregularity of the fractal can be characterised by its local

---

behaviour (within 'small' cubes or balls), then there is some chance to shift all the arguments to appropriate sequence spaces which are usually easier to handle. This explains our preference for fractal h-sets and Besov spaces. But still the problem is not so simple and little is known so far.

Dealing with spaces on h-sets we refer to [14,15,34,36], and, probably closest to our approach here, [49, Ch. 7]. There it turns out that one first needs a sound knowledge about the existence and quality of the corresponding trace spaces. Returning to the first results in that respect in [6], see also [5,7,8], we found that the approach can (and should) be extended for later applications. More precisely, for a positive continuous and non-decreasing function $h : (0,1] \to \mathbb{R}$ (a gauge function) with $\lim_{r \to 0} h(r) = 0$, a non-empty compact set $\Gamma \subset \mathbb{R}^n$ is called h-set if there exists a finite Radon measure μ in $\mathbb{R}^n$ with supp $\mu = \Gamma$ and

$$\mu(B(\gamma, r)) \sim h(r), \qquad r \in (0,1], \ \gamma \in \Gamma,$$

see also [43, Ch. 2] and [37, p. 60]. In the special case $h(r) = r^d$, $0 < d < n$, Γ is called d-set (in the sense of [47, Def. 3.1], see also [31,37] – be aware that this is different from the sense used in [24]). Recall that some self-similar fractals are outstanding examples of d-sets; for instance, the usual (middle-third) Cantor set in $\mathbb{R}^1$ is a d-set for $d = \ln 2 / \ln 3$, and the Koch curve in $\mathbb{R}^2$ is a d-set for $d = \ln 4 / \ln 3$.

The trace is defined by completion of pointwise traces of $\varphi \in \mathcal{S}(\mathbb{R}^n)$, assuming that for $0 < p < \infty$ we have in addition $\|\varphi|_{\Gamma} |L_p(\Gamma)\| \lesssim \|\varphi|B_{p,q}^t(\mathbb{R}^n)\|$ for suitable parameters $t \in \mathbb{R}$ and $0 < q < \infty$. In case of a compact d-set Γ, $0 < d < n$, this results in

$$\operatorname{tr}_{\Gamma} B_{p,q}^{\frac{n-d}{p}}(\mathbb{R}^n) = L_p(\Gamma) \quad \text{if} \quad 0 < q \le \min(p,1) \tag{0.1}$$

and, for $s > \frac{n-d}{p}$,

$$\operatorname{tr}_{\Gamma} B_{p,q}^s(\mathbb{R}^n) = \mathbb{B}_{p,q}^{s - \frac{n-d}{p}}(\Gamma),$$

see [47] with some later additions in [48,49]. Here $B_{p,q}^s(\mathbb{R}^n)$ are the usual Besov spaces defined on $\mathbb{R}^n$. In the classical case $d = n - 1$, $0 < p < \infty$, $0 < q \le \min(p,1)$, (0.1) reproduces – up to our compactness requirement – the well-known trace result $\operatorname{tr}_{\mathbb{R}^{n-1}} B_{p,q}^{\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^{n-1})$.

In case of h-sets Γ one needs to consider Besov spaces of generalised smoothness $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ which naturally extend $B_{p,q}^s(\mathbb{R}^n)$: instead of the smoothness parameter $s \in \mathbb{R}$ one now admits sequences $\boldsymbol{\sigma} = (\sigma_j)_{j \in \mathbb{N}_0}$ of positive numbers which satisfy $\sigma_j \sim \sigma_{j+1}$, $j \in \mathbb{N}_0$. Such spaces are special cases of the scale $B_{p,q}^{\boldsymbol{\sigma}, N}(\mathbb{R}^n)$ which is well studied by different approaches, namely the interpolation one [18,38] and the theory developed independently by Gol'dman and Kalyabin in the late 70's and early 80's of the last century, see the survey [33] and the appendix [35] which cover the extensive (Russian) literature at that time. We shall rely on the Fourier-analytical approach as presented in [25]. It turns out that the classical smoothness $s \in \mathbb{R}$ has to be replaced by certain regularity indices $\bar{\mathfrak{s}}(\boldsymbol{\sigma})$, $\underline{\mathfrak{s}}(\boldsymbol{\sigma})$ of $\boldsymbol{\sigma}$. In case of $\boldsymbol{\sigma} = (2^{js})_j$ the spaces $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ and $B_{p,q}^s(\mathbb{R}^n)$ coincide and $\bar{\mathfrak{s}}(\boldsymbol{\sigma}) = \underline{\mathfrak{s}}(\boldsymbol{\sigma}) = s$.

Dealing with traces on h-sets Γ in a similar way as for d-sets, one obtains

$$\operatorname{tr}_{\Gamma} B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n) = \mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma),$$

where the sequence $\boldsymbol{\tau}$ (representing smoothness) depends on $\boldsymbol{\sigma}$, h (representing the geometry of Γ) and the underlying $\mathbb{R}^n$; in particular, with $\boldsymbol{h} := (h(2^{-j}))_j$, $\boldsymbol{h_p} = \left(h(2^{-j})^{\frac{1}{p}} 2^{j\frac{n}{p}}\right)_j$, the counterpart of (0.1) reads as

$$\mathrm{tr}_\Gamma B^{\boldsymbol{h_p}}_{p,q}(\mathbb{R}^n) = L_p(\Gamma), \qquad 0 < q \leq \min(p,1).$$

These results were already obtained in [8] under some additional restrictions. In [9] we studied sufficient conditions for the existence of such traces again (in the course of dealing with growth envelopes, characterising some singularity behaviour) and returned in [12] to the subject to obtain 'necessary' conditions, or, more precisely, conditions for the non-existence of traces. This problem is closely connected with the so-called *dichotomy*; we refer to [12] for further details.

In the present paper we study embeddings of type

$$\mathrm{id}_\Gamma : \mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma) \to \mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma) \tag{0.2}$$

in some detail. But before we concentrate on these trace spaces we first inspect and slightly improve the related embedding results for spaces on $\mathbb{R}^n$. In Theorem 3.1 we can show that for two admissible sequences $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2, \leq \infty$, and q^* given by $\frac{1}{q^*} = \max\left(\frac{1}{q_2} - \frac{1}{q_1}, 0\right)$,

$$\mathrm{id}_{\mathbb{R}^n} : B^{\boldsymbol{\sigma}}_{p_1,q_1}(\mathbb{R}^n) \to B^{\boldsymbol{\tau}}_{p_2,q_2}(\mathbb{R}^n)$$

exists and is bounded if, and only if, $p_1 \leq p_2$ and $\left(\sigma_j^{-1}\tau_j 2^{jn(\frac{1}{p_1} - \frac{1}{p_2})}\right)_j \in \ell_{q^*}$. Next we study embeddings similar to (0.2), but where the target space is a Lebesgue space $L_r(\Gamma)$. The corresponding results are of the following type: Assume that Γ is an h-set satisfying some additional conditions, such that the corresponding trace spaces exist, $\boldsymbol{\sigma}$ is an admissible sequence, $0 < p \leq r < \infty$ and $0 < q \leq \min(r,1)$. Then $\mathrm{id}_\Gamma : \mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \to L_r(\Gamma)$ exists and is bounded if, and only if, $\left(\sigma_j^{-1}h(2^{-j})^{\frac{1}{r} - \frac{1}{p}}\right)_j \in \ell_\infty$, cf. Proposition 3.12. There are further similar characterisations referring to different parameter settings. When $r = p$, the outcome reads as $\mathrm{id}_\Gamma : \mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \to L_p(\Gamma)$ exists and is bounded if, and only if,

$$\boldsymbol{\sigma}^{-1} \in \begin{cases} \ell_{q'}, & \text{if} \quad 1 \leq p < \infty \quad \text{or} \quad 0 < q \leq p < 1, \\ \ell_{v_p}, & \text{if} \quad 0 < p < 1 \quad \text{and} \quad 0 < p < q < \infty, \end{cases} \tag{0.3}$$

where the number q' is given by $\frac{1}{q'} = \max\left(1 - \frac{1}{q}, 0\right)$ and v_p is given by $\frac{1}{v_p} = \frac{1}{p} - \frac{1}{q}$; cf. Corollary 3.17. Actually, such a Corollary shows that, under mild assumptions on the h-set Γ, (0.3) gives the exact conditions on σ, p and q for which the trace spaces $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ exist.

Our other main results can be found in Theorems 3.23 and 3.28, where we prove necessary and sufficient conditions for the embedding (0.2) to hold. Apart from some more technical assumptions, ensuring, in particular, the existence of the related trace spaces, the main requirement for the continuity of (0.2) reads

as

$$\left(\sigma_j^{-1}\tau_j h(2^{-j})^{-\max\left(\frac{1}{p_1}-\frac{1}{p_2},0\right)}\right)_j \in \ell_{q^*}. \tag{0.4}$$

This condition clearly reflects the expected interplay between smoothness and regularity parameters of the spaces, as well as the underlying geometry of the h-set Γ where the traces are taken. Moreover, we can show that an assumption like (0.4) is also necessary for the embedding (0.2), at least when $p_1 \le p_2$.

   The paper is organised as follows. In Section 1 we collect some fundamentals about h-sets and admissible sequences. In Section 2 we start by recalling the definition of Besov spaces of generalised smoothness and their wavelet and atomic characterisations. Then we describe the corresponding trace spaces in some detail and present the results on growth envelopes that we shall use afterwards. Our main embedding results are contained in Section 3, first concerning spaces of generalised smoothness on $\mathbb{R}^n$, then for spaces defined on Γ. Throughout the paper we add remarks, discussions and examples to illustrate the (sometimes technically involved) arguments and results.

1. Preliminaries

1.1. **General notation.** As usual, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We use the equivalence '$\sim$' in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_k \le b_k \le c_2\, a_k \quad \text{or} \quad c_1\, \varphi(x) \le \psi(x) \le c_2\, \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and φ, ψ are non-negative functions. Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous.
All unimportant positive constants will be denoted by c, occasionally with additional subscripts within the same formula. If not otherwise indicated, log is always taken with respect to base 2. For some $\varkappa \in \mathbb{R}$ let

$$\varkappa_+ = \max(\varkappa, 0) \quad \text{and} \quad \lfloor \varkappa \rfloor = \max\{k \in \mathbb{Z} : k \le \varkappa\}.$$

Moreover, for $0 < r \le \infty$ the number r' is given by $\frac{1}{r'} := \left(1 - \frac{1}{r}\right)_+$.
   For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ be the closed ball

$$B(x, r) = \{y \in \mathbb{R}^n : |y - x| \le r\}.$$

   Let $\mathbb{Z}^n$ stand for the lattice of all points in $\mathbb{R}^n$ with integer-valued components, Q_{jm} denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m = (2^{-j}m_1, \ldots, 2^{-j}m_n)$, and with side length 2^{-j}, where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$, then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. For $0 < p < \infty$, $j \in \mathbb{N}_0$, and

$m \in \mathbb{Z}^n$ we denote by $\chi_{j,m}^{(p)}$ the p-normalised characteristic function of the cube $Q_{j,m}$,

$$\chi_{j,m}^{(p)}(x) = 2^{\frac{jn}{p}} \chi_{j,m}(x) = \begin{cases} 2^{\frac{jn}{p}} & \text{for} \quad x \in Q_{j,m}, \\ 0 & \text{for} \quad x \notin Q_{j,m}, \end{cases}$$

hence $\|\chi_{j,m}^{(p)} | L_p\| = 1$.

Let $C(\mathbb{R}^n)$ be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual.

1.2. h-**sets** Γ. A central concept for us is the theory of so-called h-sets and corresponding measures. We refer to [43] for a comprehensive treatment of this concept.

Definition 1.1.
(i) Let $\mathbb{H}$ denote the class of all positive continuous and non-decreasing functions $h : (0, 1] \to \mathbb{R}$ *(gauge functions)* with $\lim_{r \to 0} h(r) = 0$.
(ii) Let $h \in \mathbb{H}$. A non-empty compact set $\Gamma \subset \mathbb{R}^n$ is called h-*set* if there exists a finite Radon measure μ with

$$\text{supp } \mu = \Gamma, \tag{1.1}$$

$$\mu(B(\gamma, r)) \sim h(r), \qquad r \in (0, 1], \ \gamma \in \Gamma. \tag{1.2}$$

If for a given $h \in \mathbb{H}$ there exists an h-set $\Gamma \subset \mathbb{R}^n$, we call h a *measure function (in $\mathbb{R}^n$)* and any related measure μ with (1.1) and (1.2) will be called h-*measure (related to Γ)*.

Certainly one of the most prominent examples of these sets are the famous d-sets, see also Example 1.6 below, but it is also well-known that in many cases more general approaches are necessary, cf. [37, p. 60]. Here we essentially follow the presentation in [5–8], see also [37] for basic notions and concepts. For convenience we quote some results on h-sets and give examples afterwards; we refer to the literature for a more detailed account on geometric properties of h-sets.

In view of (ii) the question arises which $h \in \mathbb{H}$ are measure functions. The complete characterisation is given in [7].

Proposition 1.2. *Let $h \in \mathbb{H}$. There is a compact set Γ and a Radon measure μ with (1.1) and (1.2) if, and only if, there are constants $0 < c_1 \le c_2 < \infty$ and a function $\tilde{h} \in \mathbb{H}$ such that*

$$c_1 \tilde{h}(t) \le h(t) \le c_2 \tilde{h}(t), \quad t \in (0, 1],$$

and

$$\tilde{h}(2^{-j}) \le 2^{kn} \tilde{h}(2^{-k-j}), \qquad \text{for all} \quad j, k \in \mathbb{N}_0.$$

Remark 1.3. As a consequence of the above proposition we have that for a measure function $h \in \mathbb{H}$ there is some $c > 0$ such that for all $j, k \in \mathbb{N}_0$,

$$\frac{h(2^{-k-j})}{h(2^{-j})} \ge c \, 2^{-kn}. \tag{1.3}$$

Note that every h-set Γ satisfies the *doubling condition*, i.e., there is some $c > 0$ such that

$$\mu(B(\gamma, 2r)) \leq c \, \mu(B(\gamma, r)), \qquad r \in (0, 1], \ \gamma \in \Gamma. \tag{1.4}$$

Obviously one can regard (1.3) as a refined version of (1.4) for the function h, in which the dimension n of the underlying space $\mathbb{R}^n$ is taken into account (as expected).

Proposition 1.4. *Let Γ be an h-set in $\mathbb{R}^n$. All h-measures μ related to Γ are equivalent to $\mathcal{H}^h|\Gamma$, where the latter stands for the restriction to Γ of the generalised Hausdorff measure with respect to the gauge function h.*

Remark 1.5. A proof of this result is given in [6, Thm. 1.7.6]. Concerning the theory of generalised Hausdorff measures $\mathcal{H}^h$ we refer to [43, Ch. 2] and [37, p. 60]; in particular, if $h(r) = r^d$, then $\mathcal{H}^h$ coincides with the usual d-dimensional Hausdorff measure.

Example 1.6. We restrict ourselves to a few examples only and refer to [8, Ex. 3.8] for further results. All functions are defined for $r \in (0, \varepsilon)$, suitably extended on $(0, 1]$ afterwards.

Let Ψ be a continuous *admissible* function or a continuous *slowly varying* function, respectively. An *admissible* function Ψ in the sense of [22], [40] is a positive monotone function on $(0, 1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}$. A positive and measurable function Ψ defined on the interval $(0, 1]$ is said to be *slowly varying* (in Karamata's sense) if

$$\lim_{t \to 0} \frac{\Psi(st)}{\Psi(t)} = 1, \quad s \in (0, 1].$$

For such functions it is known, for instance, that for each $\varepsilon > 0$ there is a decreasing function ϕ and an increasing function φ with $t^{-\varepsilon} \Psi(t) \sim \phi(t)$, and $t^{\varepsilon} \Psi(t) \sim \varphi(t)$; we refer to the monograph [3] for details and further properties; see also [50, Ch. V], [20], and [41, 42]. In particular,

$$\Psi_b(x) = (1 + |\log x|)^b, \quad x \in (0, 1], \quad b \in \mathbb{R}, \tag{1.5}$$

may be considered as a prototype both for an admissible function and a slowly varying function.

Let $0 < d < n$. Then

$$h(r) = r^d \, \Psi(r), \quad r \in (0, 1], \tag{1.6}$$

is a typical example of $h \in \mathbb{H}$. The limiting cases $d = 0$ and $d = n$ can be included, assuming additional properties of Ψ in view of (1.3) and $h(r) \to 0$ for $r \to 0$, e.g.

$$h(r) = (1 + |\log r|)^b, \quad b < 0, \quad r \in (0, 1], \tag{1.7}$$

referring to (1.5). Such functions h given by (1.6) are related to so-called (d, Ψ)-sets studied in [22], [40], whereas the special setting $\Psi \equiv 1$ leads to

$$h(r) = r^d, \quad r \in (0, 1], \quad 0 < d < n, \tag{1.8}$$

connected with the famous d-sets. Apart from (1.6) also functions of type $h(r) = \exp(b|\log r|^{\varkappa})$, $b < 0$, $0 < \varkappa < 1$, are admitted, for example.

We shall need another feature of some h-sets, namely the so-called 'porosity' condition, see also [37, p. 156] and [48, Sects. 9.16-9.19].

Definition 1.7. A Borel set $\Gamma \neq \emptyset$ satisfies the *porosity condition* if there exists a number $0 < \eta < 1$ such that for any ball $B(\gamma, r)$ centred at $\gamma \in \Gamma$ and with radius $0 < r \leq 1$ there is a ball $B(x, \eta r)$ centred at some $x \in \mathbb{R}^n$ satisfying

$$B(\gamma, r) \supset B(x, \eta r), \quad B(x, \eta r) \cap \overline{\Gamma} = \emptyset. \tag{1.9}$$

Replacing, if necessary, η by $\frac{\eta}{2}$, we can complement (1.9) by

$$\text{dist}\left(B(x, \eta r), \overline{\Gamma}\right) \geq \eta r, \quad 0 < r \leq 1.$$

This definition coincides with [47, Def. 18.10]. There is a complete characterisation of measure functions h such that the corresponding h-sets Γ satisfy the porosity condition; this can be found in [48, Prop. 9.18]. We recall it for convenience.

Proposition 1.8. *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Then Γ satisfies the porosity condition if, and only if, there exist constants $c > 0$ and $\varepsilon > 0$ such that*

$$\frac{h\left(2^{-j-k}\right)}{h\left(2^{-j}\right)} \geq c \, 2^{-(n-\varepsilon)k}, \quad j, k \in \mathbb{N}_0. \tag{1.10}$$

Note that an h-set Γ satisfying the porosity condition has Lebesgue measure $|\Gamma| = 0$, but the converse is not true. This can be seen from (1.10) and the result [49, Prop. 1.153],

$$|\Gamma| = 0 \quad \text{if, and only if,} \quad \lim_{r \to 0} \frac{r^n}{h(r)} = 0$$

for all h-sets Γ.

Remark 1.9. In view of our above examples and (1.10) it is obvious that h from (1.6) and (1.8) with $d = n$ does not satisfy the porosity condition, unlike in case of $d < n$.

Remark 1.10. Later it will turn out that some additional property of h is desirable; it is in some sense 'converse' to the porosity criterion (1.10): Triebel calls the measure μ corresponding to h and Γ with $|\Gamma| = 0$ *strongly isotropic* if $h \in \mathbb{H}$ is strictly increasing (with $h(1) = 1$) and there exists some $k \in \mathbb{N}$ such that, for all $j \in \mathbb{N}_0$,

$$2h\left(2^{-j-k}\right) \leq h\left(2^{-j}\right). \tag{1.11}$$

It is known that μ is strongly isotropic if, and only if,

$$\sum_{j=l}^{\infty} h\left(2^{-j}\right) \sim h\left(2^{-l}\right) \tag{1.12}$$

for all $l \in \mathbb{N}_0$, and this is equivalent to

$$\sum_{j=0}^{m} h\left(2^{-j}\right)^{-1} \sim h\left(2^{-m}\right)^{-1} \tag{1.13}$$

for $m \in \mathbb{N}_0$, see [49, Def. 7.18, Prop. 7.20]. Plainly, $h(r) = r^d$, $0 < d < n$, satisfies this condition, unlike, e.g., $h(r) = (1 + |\log r|)^b$, $b < 0$, from (1.7).

1.3. **Admissible sequences and regularity indices.** We collect some basic concepts and notions for later use when we introduce function spaces of generalised smoothness and their trace spaces on an h-set Γ.

Definition 1.11. A sequence $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ of positive numbers is called *admissible* if there are two positive constants d_0, d_1 such that

$$d_0\, \sigma_j \;\leq\; \sigma_{j+1} \;\leq\; d_1\, \sigma_j, \quad j \in \mathbb{N}_0. \tag{1.14}$$

Remark 1.12. If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are admissible sequences, then $\boldsymbol{\sigma\tau} := (\sigma_j\tau_j)_j$ and $\boldsymbol{\sigma}^r := (\sigma_j^r)_j$, $r \in \mathbb{R}$, are admissible, too. For convenience, let us further introduce the notation

$$(a) := (2^{ja})_{j\in\mathbb{N}_0} \quad \text{where} \quad a \in \mathbb{R}.$$

Obviously, for $a, b \in \mathbb{R}$, $r > 0$, and $\boldsymbol{\sigma}$ admissible, $(a)(b) = (a+b)$, $\left(\frac{a}{r}\right) = (a)^{1/r}$, and $(a)\boldsymbol{\sigma} = (2^{ja}\sigma_j)_{j\in\mathbb{N}_0}$.

Example 1.13. We restrict ourselves to the sequence $\boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j$, $s \in \mathbb{R}$, Ψ an admissible function in the sense of Example 1.6 above. This includes, in particular, $\boldsymbol{\sigma} = (s)$, $s \in \mathbb{R}$. We refer to [25] for a more general approach and further examples.

We introduce some 'regularity' indices for $\boldsymbol{\sigma}$.

Definition 1.14. Let $\boldsymbol{\sigma}$ be an admissible sequence, and

$$\underline{\mathfrak{s}}(\boldsymbol{\sigma}) := \liminf_{j\to\infty}\ \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right) \tag{1.15}$$

and

$$\overline{\mathfrak{s}}(\boldsymbol{\sigma}) := \limsup_{j\to\infty}\ \log\left(\frac{\sigma_{j+1}}{\sigma_j}\right). \tag{1.16}$$

Remark 1.15. These indices were introduced and used in [6]. For admissible sequences $\boldsymbol{\sigma}$ according to (1.14) we have $\log d_0 \leq \underline{\mathfrak{s}}(\boldsymbol{\sigma}) \leq \overline{\mathfrak{s}}(\boldsymbol{\sigma}) \leq \log d_1$. One easily verifies that

$$\overline{\mathfrak{s}}(\boldsymbol{\sigma}) = \underline{\mathfrak{s}}(\boldsymbol{\sigma}) = s \quad \text{in case of} \quad \boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j \tag{1.17}$$

for all admissible functions Ψ and $s \in \mathbb{R}$. In contrast to this one can find examples in [25], due to Kalyabin, showing that an admissible sequence has not necessarily a fixed main order. Moreover, it is known that for any $0 < a \leq b < \infty$ there is an admissible sequence $\boldsymbol{\sigma}$ with $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = a$ and $\overline{\mathfrak{s}}(\boldsymbol{\sigma}) = b$, that is, with prescribed upper and lower indices.

It is more or less obvious from the definitions (1.15), (1.16) that for admissible sequences $\boldsymbol{\sigma}, \boldsymbol{\tau}$ one obtains $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = -\overline{\mathfrak{s}}(\boldsymbol{\sigma}^{-1})$, $\underline{\mathfrak{s}}(\boldsymbol{\sigma}^r) = r\underline{\mathfrak{s}}(\boldsymbol{\sigma})$, $r \geq 0$, and $\overline{\mathfrak{s}}(\boldsymbol{\sigma\tau}) \leq \overline{\mathfrak{s}}(\boldsymbol{\sigma}) + \overline{\mathfrak{s}}(\boldsymbol{\tau})$, $\underline{\mathfrak{s}}(\boldsymbol{\sigma\tau}) \geq \underline{\mathfrak{s}}(\boldsymbol{\sigma}) + \underline{\mathfrak{s}}(\boldsymbol{\tau})$. In particular, for $\boldsymbol{\sigma} = (a)$, $a \in \mathbb{R}$, this can be sharpened by $\overline{\mathfrak{s}}(\boldsymbol{\tau}(a)) = a + \overline{\mathfrak{s}}(\boldsymbol{\tau})$, $\underline{\mathfrak{s}}(\boldsymbol{\tau}(a)) = a + \underline{\mathfrak{s}}(\boldsymbol{\tau})$. Observe that, given $\varepsilon > 0$, there are two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1\, 2^{(\underline{\mathfrak{s}}(\boldsymbol{\sigma})-\varepsilon)j} \leq \sigma_j \leq c_2\, 2^{(\overline{\mathfrak{s}}(\boldsymbol{\sigma})+\varepsilon)j}, \qquad j \in \mathbb{N}_0. \tag{1.18}$$

Plainly this implies that whenever $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$, then $\boldsymbol{\sigma}^{-1}$ belongs to any space ℓ_u, $0 < u \leq \infty$, whereas $\bar{\mathfrak{s}}(\boldsymbol{\sigma}) < 0$ leads to $\boldsymbol{\sigma}^{-1} \notin \ell_\infty$.

Remark 1.16. Note that in some later papers, cf. [5], instead of (1.15) and (1.16) the so-called *upper* and *lower Boyd indices of* $\boldsymbol{\sigma}$ are considered, given by

$$\alpha_{\boldsymbol{\sigma}} = \lim_{j \to \infty} \frac{1}{j} \, \log \left(\sup_{k \in \mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) = \inf_{j \in \mathbb{N}} \frac{1}{j} \, \log \left(\sup_{k \in \mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right)$$

and

$$\beta_{\boldsymbol{\sigma}} = \lim_{j \to \infty} \frac{1}{j} \, \log \left(\inf_{k \in \mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right) = \sup_{j \in \mathbb{N}} \frac{1}{j} \, \log \left(\inf_{k \in \mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \right)$$

respectively. In general we have

$$\underline{\mathfrak{s}}(\boldsymbol{\sigma}) \leq \beta_{\boldsymbol{\sigma}} \leq \alpha_{\boldsymbol{\sigma}} \leq \bar{\mathfrak{s}}(\boldsymbol{\sigma}),$$

but one can construct admissible sequences with $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) < \beta_{\boldsymbol{\sigma}}$ and $\alpha_{\boldsymbol{\sigma}} < \bar{\mathfrak{s}}(\boldsymbol{\sigma})$.

Remark 1.17. We briefly mention some convenient feature of admissible sequences that will be used later several times. Assume that $\boldsymbol{\gamma}$ is some admissible sequence with the property that for some (fixed) $\iota_0 \in \mathbb{N}$ and some $u \in (0, \infty]$ the special subsequence $(\gamma_{k\iota_0})_k \subset \boldsymbol{\gamma}$ satisfies $(\gamma_{k\iota_0})_k \in \ell_u$. This already implies $\boldsymbol{\gamma} \in \ell_u$, since, for arbitrary $m \in \mathbb{N}$,

$$\sum_{j=1}^{m\iota_0} \gamma_j^u = \sum_{k=1}^{m} \gamma_{k\iota_0}^u + \sum_{l=1}^{\iota_0-1} \sum_{k=0}^{m-1} \gamma_{l+k\iota_0}^u \sim \sum_{k=1}^{m} \gamma_{k\iota_0}^u$$

with constants depending on $\boldsymbol{\gamma}$, u and ι_0, but not on $m \in \mathbb{N}$. Thus $\boldsymbol{\gamma} \in \ell_u$ if, and only if, $(\gamma_{k\iota_0})_k \in \ell_u$. The modifications for $u = \infty$ are obvious.

We shall need the following short lemma several times below and insert it here. It mainly relies on an inequality of Landau, see [26, Thm. 161, p. 120], see also [13, Prop. 4.1].

Lemma 1.18. *Let* $\boldsymbol{\alpha} = (\alpha_k)_{k \in \mathbb{N}_0}$, $\boldsymbol{\beta} = (\beta_k)_{k \in \mathbb{N}_0}$ *be sequences of non-negative numbers, let* $0 < q_1, q_2 \leq \infty$, *and* q^* *be given by*

$$\frac{1}{q^*} := \left(\frac{1}{q_2} - \frac{1}{q_1} \right)_+. \tag{1.19}$$

Then

$$\boldsymbol{\alpha} \in \ell_{q^*} \quad \text{if, and only if,} \quad \boldsymbol{\alpha}\boldsymbol{\beta} \in \ell_{q_2} \quad \text{whenever} \quad \boldsymbol{\beta} \in \ell_{q_1}.$$

Proof. The 'only if'-part is a consequence of Hölder's inequality (when $q^* < \infty$) or the monotonicity of the ℓ_u-scale (when $q^* = \infty$). Moreover, the 'if'-part in case of $q^* < \infty$ is covered by the above-mentioned result of Landau, see [26, Thm. 161, p. 120]. So we are left to show the necessity of $\boldsymbol{\alpha} \in \ell_\infty$ when assuming that $\boldsymbol{\alpha}\boldsymbol{\beta} \in \ell_{q_2}$ whenever $\boldsymbol{\beta} \in \ell_{q_1}$, where in particular we have now that $q_1 \leq q_2$. This is surely well-known, but for the sake of completeness we insert a short argument here. The case $q_1 = \infty$ being obvious, we need to deal with the case $q_1 < \infty$ merely.

Assume, to the contrary, that $\boldsymbol{\alpha}$ is unbounded. Then we can find a subsequence

$(\alpha_{k_j})_j \subset \boldsymbol{\alpha}$ which is strictly increasing and satisfies $\alpha_{k_j} \xrightarrow[j\to\infty]{} \infty$. We refine this subsequence further by the following procedure: let $\alpha_{t_1} = \alpha_{k_1}$ and choose consecutively the index t_{j+1}, $j \in \mathbb{N}$, such that

$$\frac{\alpha_{t_{j+1}-1}}{\alpha_{t_j}} < 2 \quad \text{and} \quad \frac{\alpha_{t_{j+1}}}{\alpha_{t_j}} \geq 2, \quad j \in \mathbb{N},$$

that is, t_{j+1} is the smallest index larger than t_j such that $\alpha_{t_{j+1}} \geq 2\alpha_{t_j}$. Now consider the sequence $\boldsymbol{\beta} = (\beta_k)_k$ given by

$$\beta_k = \begin{cases} j^{\frac{1}{q_1}} \alpha_{t_j}^{-1}, & k = t_j, \ j \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|\boldsymbol{\beta}|\ell_{q_1}\| = \Big(\sum_{j=1}^{\infty} j \alpha_{t_j}^{-q_1} \Big)^{\frac{1}{q_1}} = \Big(\sum_{j=1}^{\infty} j \Big(\frac{\alpha_{t_j}}{\alpha_{t_{j-1}}} \cdots \frac{\alpha_{t_2}}{\alpha_{t_1}} \Big)^{-q_1} \alpha_{t_1}^{-q_1} \Big)^{\frac{1}{q_1}}$$

$$\leq \alpha_{k_1}^{-1} \Big(\sum_{j=1}^{\infty} j 2^{-(j-1)q_1} \Big)^{\frac{1}{q_1}} < \infty,$$

whereas

$$\|\boldsymbol{\alpha}\boldsymbol{\beta}|\ell_{q_2}\| = \Big(\sum_{j=1}^{\infty} \alpha_{t_j}^{q_2} j^{\frac{q_2}{q_1}} \alpha_{t_j}^{-q_2} \Big)^{\frac{1}{q_2}} = \Big(\sum_{j=1}^{\infty} j^{\frac{q_2}{q_1}} \Big)^{\frac{1}{q_2}} = \infty$$

(modification if $q_2 = \infty$), which disproves $\boldsymbol{\alpha}\boldsymbol{\beta} \in \ell_{q_2}$. $\square$

We return to the notion of strongly isotropic measures given in Remark 1.10 and will from now on use the following abbreviations introduced in [8]: Let Γ be some h-set, $h \in \mathbb{H}$ a measure function. Then we denote

$$\boldsymbol{h} := (h_j)_{j\in\mathbb{N}_0} \quad \text{with} \quad h_j := h(2^{-j}), \quad j \in \mathbb{N}_0, \tag{1.20}$$

for the sequence connected with $h \in \mathbb{H}$.

Remark 1.19. It is easy to verify that the hypothesis $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ implies that there exists $\kappa_0 > 1$ such that $h_{j+1} \leq \kappa_0^{-1} h_j$ for $j \geq j_0$ and some appropriately chosen $j_0 \in \mathbb{N}_0$. Therefore $\boldsymbol{h} \in \ell_1$ and we can also say that the measure μ corresponding to h is strongly isotropic (at least if we relax, for convenience, the assumptions $h(1) = 1$ and $|\Gamma| = 0$ and also admit that (1.11) holds for all $j \in \mathbb{N}$ with $j \geq j_0$ for some fixed starting term $j_0 \in \mathbb{N}$ only), cf. Remark 1.10. More precisely, for arbitrary admissible sequences $\boldsymbol{\sigma}$ the assumption $\bar{\mathfrak{s}}(\boldsymbol{\sigma}) < 0$ leads to $\boldsymbol{\sigma} \in \ell_1$ and also to the equivalences (1.12), (1.13) with $\boldsymbol{h}$ replaced by $\boldsymbol{\sigma}$.

2. BESOV SPACES

The main object of the paper are embeddings of Besov spaces defined (as trace spaces) on some h-set Γ. We approach this concept now.

2.1. **Besov spaces of generalised smoothness.** First we want to introduce function spaces of generalised smoothness and need to recall some notation. By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\widehat{\varphi}(\xi) \equiv (\mathcal{F}\varphi)(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{2.1}$$

denotes the Fourier transform of φ. As usual, $\mathcal{F}^{-1}\varphi$ or $\varphi^\vee$ stands for the inverse Fourier transform, given by the right-hand side of (2.1) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad \text{supp } \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\},$$

and for each $j \in \mathbb{N}$ let

$$\varphi_j(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad x \in \mathbb{R}^n.$$

Then the sequence $(\varphi_j)_{j=0}^\infty$ forms a smooth dyadic resolution of unity.

Definition 2.1. Let $\boldsymbol{\sigma}$ be an admissible sequence, $0 < p, q \leq \infty$, and $(\varphi_j)_{j=0}^\infty$ a smooth dyadic resolution of unity as described above. Then

$$B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \; : \left(\sum_{j=0}^\infty \sigma_j^q \left\| \mathcal{F}^{-1}\varphi_j \mathcal{F}f | L_p(\mathbb{R}^n) \right\|^q \right)^{1/q} < \infty \right\}$$

(with the usual modification if $q = \infty$).

Remark 2.2. These spaces are quasi-Banach spaces, independent of the chosen resolution of unity, and $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ when $p < \infty$ and $q < \infty$. Taking $\boldsymbol{\sigma} = (s)$, $s \in \mathbb{R}$, we obtain the classical Besov spaces $B_{p,q}^s(\mathbb{R}^n)$, whereas $\boldsymbol{\sigma} = (2^{js}\Psi(2^{-j}))_j$, $s \in \mathbb{R}$, Ψ an admissible function, leads to spaces $B_{p,q}^{(s,\Psi)}(\mathbb{R}^n)$, studied in [39, 40] in detail. Moreover, the above spaces $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ are special cases of the more general approach investigated in [25]. For the theory of spaces $B_{p,q}^s(\mathbb{R}^n)$ we refer to the series of monographs [45–49].

For later use, we briefly describe the wavelet and atomic characterisation of Besov spaces with generalised smoothness obtained in [1] and [25], respectively. Let $\widetilde{\phi}$ be a scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\widetilde{\psi}$ be an associated compactly supported wavelet. Then the tensor-product ansatz yields a scaling function ϕ and associated wavelets $\psi_1, \ldots, \psi_{2^n-1}$, all defined now on $\mathbb{R}^n$. We suppose $\widetilde{\phi} \in C^{N_1}(\mathbb{R})$ and supp $\widetilde{\phi}$, supp $\widetilde{\psi} \subset [-N_2, N_2]$ for certain natural numbers N_1 and N_2. This implies

$$\phi, \psi_l \in C^{N_1}(\mathbb{R}^n) \quad \text{and} \quad \text{supp } \phi, \text{supp } \psi_l \subset [-N_3, N_3]^n, \tag{2.2}$$

for $l = 1, \ldots, 2^n - 1$. We use the standard abbreviations

$$\phi_{j,m}(x) = \phi(2^j x - m) \quad \text{and} \quad \psi_{j,m}^l(x) = \psi_l(2^j x - m).$$

To formulate the result we introduce some suitable sequence spaces. For $0 < p < \infty$, $0 < q \leq \infty$, and $\boldsymbol{\sigma}$ an admissible sequence, let

$$b_{p,q}^{\boldsymbol{\sigma}} := \left\{ \lambda = \{\lambda_{j,m}\}_{j,m} : \ \lambda_{j,m} \in \mathbb{C}, \right.$$

$$\left. \| \lambda \,|b_{p,q}^{\boldsymbol{\sigma}}\| = \left\| \left\{ \sigma_j 2^{-j\frac{n}{p}} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j,m} \, \chi_{j,m}^{(p)} | L_p(\mathbb{R}^n) \right\| \right\}_{j \in \mathbb{N}_0} |\ell_q \right\| < \infty \right\}.$$

In view of the definition of $Q_{j,m}$ and $\chi_{j,m}^{(p)}$ one can easily verify that

$$\| \lambda | b_{p,q}^{\boldsymbol{\sigma}} \| \sim \left\| \left\{ \sigma_j 2^{-j\frac{n}{p}} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{j,m}|^p \Big)^{1/p} \right\}_{j \in \mathbb{N}_0} |\ell_q \right\|.$$

Theorem 2.3. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\boldsymbol{\sigma}$ be an admissible sequence. Then there exists a number $N_0 = N_0(\boldsymbol{\sigma}, p, n)$ and for any $N_1 > N_0$ a scaling function ϕ and wavelets ψ^l, $l = 1, \ldots, 2^n - 1$, as above satisfying* (2.2), *such that the following holds: A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ if, and only if, it can be represented by*

$$f = \sum_{m \in \mathbb{Z}^n} \mu_m \phi_{0,m} + \sum_{l=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^l \psi_{j,m}^l \quad \text{with} \quad \mu \in \ell_p \quad \text{and} \quad \lambda^l \in b_{p,q}^{\boldsymbol{\sigma}},$$

$l = 1, \ldots, 2^n - 1$ (unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$). Moreover, the coefficients can be uniquely determined by

$$\mu_m = 2^{jn} \langle f, \phi_{0,m} \rangle \quad \text{and} \quad \lambda_{j,m}^l = 2^{jn} \langle f, \psi_{j,m}^l \rangle, \quad m \in \mathbb{Z}^n, \ j \in \mathbb{N}_0, \ l = 1, \ldots, 2^n - 1,$$

and

$$\| f \,|B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)\|^{\star} = \left\| \left\{ \langle f, \phi_{0,m} \rangle \right\}_{m \in \mathbb{Z}^n} |\ell_p \right\| + \sum_{l=1}^{2^n-1} \left\| \left\{ \langle f, \psi_{j,m}^l \rangle \right\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n} |b_{p,q}^{\widetilde{\boldsymbol{\sigma}}} \right\|$$

is an equivalent (quasi-) norm in $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$, where $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(n)$.

Remark 2.4. It follows from Theorem 2.3 that, under the conditions given, the mapping

$$T : f \ \mapsto \ \left(\left\{ \langle f, \phi_{0,m} \rangle \right\}_{m \in \mathbb{Z}^n}, \left\{ \langle f, \psi_{j,m}^l \rangle \right\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n, l = 1, \ldots, 2^n - 1} \right)$$

is an isomorphism of $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ onto $\ell_p \oplus \big(\oplus_{l=1}^{2^n-1} b_{p,q}^{\widetilde{\boldsymbol{\sigma}}} \big)$, $\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(n)$, cf. [1, Cor. 17]. Moreover, if $q < \infty$ and N_1 is chosen large enough, then the wavelet system forms an unconditional Schauder basis in $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$, [1, Cor. 16].

Definition 2.5. Let $K \in \mathbb{N}_0$ and $c > 1$.

(i) A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called *an 1_K-atom* if

$$\text{supp } a \subset c\, Q_{0m} \quad \text{for some} \quad m \in \mathbb{Z}^n,$$

and

$$|\mathrm{D}^\alpha a(x)| \leq 1 \quad \text{for} \quad |\alpha| \leq K.$$

(ii) Let $L + 1 \in \mathbb{N}_0$, and $\boldsymbol{\sigma}$ admissible. A K times differentiable complex-valued function $a(x)$ in $\mathbb{R}^n$ (continuous if $K = 0$) is called *an* $(\boldsymbol{\sigma}, p)_{K,L}$-*atom* if, for some $j \in \mathbb{N}_0$,

$$\operatorname{supp} a \subset c\, Q_{jm} \quad \text{for some} \quad m \in \mathbb{Z}^n,$$

$$|\mathrm{D}^\alpha a(x)| \leq \sigma_j^{-1}\, 2^{j(\frac{n}{p} + |\alpha|)} \quad \text{for} \quad |\alpha| \leq K, \quad x \in \mathbb{R}^n,$$

and

$$\int\limits_{\mathbb{R}^n} x^\beta a(x)\mathrm{d}x = 0 \quad \text{if} \quad |\beta| \leq L.$$

We adopt the usual convention to denote atoms located at Q_{jm} as above by $a_{j,m}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$. Let $b_{p,q} = b_{p,q}^{(n/p)}$, $0 < p < \infty$, $0 < q \leq \infty$. The atomic decomposition theorem for $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ reads as follows, see [25, Thm. 4.4.3, Rem. 4.4.8].

Proposition 2.6. *Let* $\boldsymbol{\sigma}$ *be admissible,* $0 < p, q \leq \infty$, $c > 1$, $K \in \mathbb{N}_0$ *and* $L + 1 \in \mathbb{N}_0$ *with*

$$K > \bar{\mathfrak{s}}\,(\boldsymbol{\sigma}) \qquad and \qquad L > -1 + n\left(\frac{1}{\min(1, p)} - 1\right) - \underline{\mathfrak{s}}\,(\boldsymbol{\sigma})$$

be fixed. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}\, a_{j,m}(x), \quad \text{convergence being in } \mathcal{S}'(\mathbb{R}^n), \qquad (2.3)$$

where $\lambda \in b_{p,q}$ *and* $a_{j,m}(x)$ *are* 1_K-*atoms* $(j = 0)$ *or* $(\boldsymbol{\sigma}, p)_{K,L}$-*atoms* $(j \in \mathbb{N})$ *according to Definition 2.5. Furthermore*

$$\inf \|\lambda \,|\, b_{p,q}\|,$$

where the infimum is taken over all admissible representations (2.3), *is an equivalent quasi-norm in* $B_{p,q}^{\boldsymbol{\sigma}}(\mathbb{R}^n)$.

2.2. **Trace spaces.** Let Γ be some h-set, $h \in \mathbb{H}$, and recall the notation (1.20) for the sequence connected with $h \in \mathbb{H}$. Given $0 < p \leq \infty$, and similarly as in $L_p(\mathbb{R}^n)$, but now with respect to the measure $\mu \sim \mathcal{H}^h|\Gamma$ related to the h-set Γ, $L_p(\Gamma) = L_p(\Gamma, \mu)$ stands for the quasi-Banach space of p-integrable (measurable, essentially bounded if $p = \infty$) functions on Γ with respect to the measure μ, quasi-normed in the obvious way. Suppose there exists some $c > 0$ such that, for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\|\varphi|_\Gamma \,|\, L_p(\Gamma)\right\| \leq c\, \left\|\varphi|B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)\right\|, \qquad (2.4)$$

where the trace on Γ is taken pointwise. By the density of $\mathcal{S}(\mathbb{R}^n)$ in $B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)$ for $p, q < \infty$ and the completeness of $L_p(\Gamma)$ one can thus define for $f \in B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)$ in those cases its trace $\operatorname{tr}_\Gamma f = f|_\Gamma$ on Γ by completion of pointwise restrictions.

Remark 2.7. In this way, $\operatorname{tr}_\Gamma : B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n) \to L_p(\Gamma)$ will be uniquely determined by the fact that it is linear, continuous and coincides with the pointwise restriction when restricted to $\mathcal{S}(\mathbb{R}^n)$. The definition of $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ as the trace space of

$B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$ means to take $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) = \mathrm{tr}_\Gamma B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)$ for $\boldsymbol{\tau} = \boldsymbol{\sigma h}^{1/p}(n)^{1/p}$ for which the above approach works.

The starting point for us was the trace result of Bricchi [8, Thm. 5.9],

$$\mathrm{tr}_\Gamma B_{p,q}^{\boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) = L_p(\Gamma), \tag{2.5}$$

where $0 < p < \infty$, $0 < q \le \min(p,1)$, and Γ is an h-set satisfying the porosity condition (1.9) (this condition not being required in the proof of the embedding $\hookrightarrow$) . Based on this result Bricchi proposes in [6, Def. 3.3.5] the approach to define spaces $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ as trace spaces, conjugating general embedding results for Besov spaces on $\mathbb{R}^n$ (as seen for example in [10, Thm. 3.7] or [6, Prop. 2.2.16]) with the fact that (2.4) above holds for $\boldsymbol{\tau} = \boldsymbol{h}^{1/p}(n)^{1/p}$, $0 < p < \infty$, $0 < q \le \min(p,1)$ (cf. [6, Thm. 3.3.1(i)]. Then we have, following [6, Def. 3.3.5], that for $0 < p,q < \infty$ and $\boldsymbol{\sigma}$ admissible with $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$, Besov spaces $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ on Γ are defined as

$$\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) := \mathrm{tr}_\Gamma B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n), \tag{2.6}$$

more precisely,

$$\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) := \left\{ f \in L_p(\Gamma) : \exists\, g \in B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n), \mathrm{tr}_\Gamma g = f \right\}, \tag{2.7}$$

equipped with the quasi-norm

$$\left\| f | \mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \right\| = \inf \left\{ \left\| g | B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \right\| : \mathrm{tr}_\Gamma g = f, g \in B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \right\}. \tag{2.8}$$

This was extended in the following way in [9, Def. 3.7]:

Definition 2.8. Let $0 < p,q < \infty$, $\boldsymbol{\sigma}$ be an admissible sequence, and Γ be an h-set. Assume that

(i) in case of $p \ge 1$ or $q \le p < 1$,

$$\boldsymbol{\sigma}^{-1} \in \ell_{q'},$$

(ii) in case of $0 < p < 1$ and $p < q$,

$$\boldsymbol{\sigma}^{-1} \boldsymbol{h}^{\frac{1}{r}-\frac{1}{p}} \in \ell_{v_r} \quad \text{for some} \quad r \in [p, \min(q,1)] \quad \text{and} \quad \frac{1}{v_r} = \frac{1}{r} - \frac{1}{q},$$

is satisfied. Then (it makes sense to) define $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ as in (2.6), (2.7), that is, as the trace space of $B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$, again with the quasi-norm given by (2.8).

Remark 2.9. Note that the – in (2.6) implicitly given – correspondence of smoothness (1) (that is, 0, in classical notation) on Γ and smoothness $\boldsymbol{h}^{1/p}(n)^{1/p}$ on $\mathbb{R}^n$ is in good agreement with (2.5).

Remark 2.10. For simplicity we restrict ourselves to the case $p < \infty$ and $q < \infty$ in the above definition and also in our results below, though both in [6] and in [9] the definition of Besov spaces on Γ also covers the cases when p or q can be ∞. Then the above approach has to be modified appropriately. Moreover, we refer to [9] (and also [12]) for some argument to what extent our definition extends the former approach by Bricchi.

Remark 2.11. For $h(r) = r^d$, $0 < d < n$, Γ a d-set, and $\boldsymbol{\sigma} = (s)$, $s \in \mathbb{R}$, the above definition (2.6) can be rewritten as

$$\mathbb{B}_{p,q}^s(\Gamma) = \operatorname{tr}_\Gamma B_{p,q}^{s+\frac{n-d}{p}}(\mathbb{R}^n),$$

assuming $0 < p, q < \infty$, $s > 0$, and

$$\mathbb{B}_{p,q}^0(\Gamma) = \operatorname{tr}_\Gamma B_{p,q}^{\frac{n-d}{p}}(\mathbb{R}^n) = L_p(\Gamma),$$

if $0 < p < \infty$, $0 < q \le \min(p,1)$. This coincides with [47, Def. 20.2]. There is a parallel result for (d, Ψ)-sets Γ studied by Moura in [39, 40], which yields for $0 < p, q < \infty$, $s > 0$, and admissible Ψ,

$$\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma) = \operatorname{tr}_\Gamma B_{p,q}^{(s+\frac{n-d}{p}, \Psi^{1+\frac{1}{p}})}(\mathbb{R}^n),$$

see [40, Def. 2.2.7].

In Section 3.2 we briefly return to our above approach (2.4) and ask what happens if we replace the target space L_p by some possibly larger or smaller space L_r.

2.3. **Growth envelopes.** We want to apply some result on growth envelopes obtained in [9] later and thus briefly recall this concept and the result for spaces $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$. Therefore we concentrate on this specific setting mainly and refer for further details to [28] as well as [9].

Let Γ be an h-set according to Definition 1.1 with the corresponding Radon measure $\mu \sim \mathcal{H}^h|\Gamma$ and f be a $\mu-$measurable function on Γ, finite $\mu-$a.e. Its decreasing (i.e., non-increasing) rearrangement $f^{*,\mu}$ is the function $f^{*,\mu}$ defined on $[0, \infty)$ by

$$f^{*,\mu}(t) = \inf \left\{ s \ge 0 : \mu\left(\{\gamma \in \Gamma : |f(\gamma)| > s\}\right) \le t \right\} \quad, \quad t \ge 0 \,.$$

We put $\inf \emptyset = \infty$, as usual. Note that $f^{*,\mu}$ is non-negative, decreasing and right-continuous on $[0, \infty)$, and f and $f^{*,\mu}$ are equi-measurable. Furthermore, $f^{*,\mu}(0) = \|f|L_\infty(\Gamma)\|$, and $f^{*,\mu}(t) = 0$ for $t > \mu(\Gamma)$.

Remark 2.12. There is plenty of literature on the topic of non-increasing rearrangements in general measure spaces; we refer to [2, Ch. 2, Prop. 1.7] and [19, Ch. 2, §2], for instance. We decided to indicate the measure μ here since when dealing with functions on $\mathbb{R}^n$ and their trace $\operatorname{tr}_\Gamma f = f|_\Gamma$ on a set $\Gamma \subset \mathbb{R}^n$, typically with $|\Gamma| = 0$, then it naturally matters which measure ν (on $\mathbb{R}^n$ or Γ) is taken for the decreasing rearrangements $f^{*,\nu}$. But we want to avoid any further discussion here and stick to the above situation.

Definition 2.13. Let $[\Omega, \nu]$ be some measure space and X a quasi-normed space of ν-measurable functions on Ω, finite a.e. in Ω. A non-negative function $\mathcal{E}_{\mathsf{G}}^X$ defined on some interval $(0, \varepsilon]$, $\varepsilon \in (0, 1)$, is called the (*local*) *growth envelope function* of X if

$$\mathcal{E}_{\mathsf{G}}^X(t) \sim \sup_{\|f|X\| \le 1} f^{*,\nu}(t) \,, \quad t \in (0, \varepsilon].$$

Remark 2.14. For a more general approach, appropriate interpretations as well as a detailed account on basic properties of $\mathcal{E}_G^X$ we refer to [27] and [28]. Note that there exist function spaces X which do not possess a growth envelope function in the sense that $\mathcal{E}_G^X$ is not finite for any $t > 0$. It is well-known that $X \hookrightarrow L_\infty$ if, and only if, $\mathcal{E}_G^X$ is bounded. Let us finally mention the convenient 'monotonicity' feature:

$$X_1 \hookrightarrow X_2 \quad \text{implies} \quad \mathcal{E}_G^{X_1}(t) \leq c\, \mathcal{E}_G^{X_2}(t) \quad \text{for some } c > 0 \text{ and all } t \in (0, \varepsilon), \tag{2.9}$$

where $\varepsilon = \varepsilon(X_1, X_2) > 0$ has to be chosen sufficiently small.

We refine the characterisation of X provided by the growth envelope function and introduce some 'characteristic' index u_G^X, which gives a finer measure of the (local) integrability of functions belonging to X. By μ_G we mean the Borel measure associated with the non-decreasing function $-\log \mathcal{E}_G^X$, where the growth envelope function $\mathcal{E}_G^X$ of X is assumed to be positive, non-increasing and continuous on some interval $(0, \varepsilon]$, with $\varepsilon > 0$, and $X \not\hookrightarrow L_\infty$. This approach essentially coincides with the one presented by Triebel in [48, Sect. 12.1] and [48, Sect. 12.8], see also [28] and [9] for further details.

Definition 2.15. Let $[\Omega, \nu]$ be some measure space and $X \not\hookrightarrow L_\infty$ some quasi-normed space of ν-measurable functions, finite a.e. on Ω. Let $\mathcal{E}_G^X$ be a positive, non-increasing and continuous growth envelope function defined on $(0, \varepsilon]$ for some sufficiently small $\varepsilon > 0$. The index u_G^X, $0 < u_G^X \leq \infty$, is defined as the infimum of all numbers v, $0 < v \leq \infty$, such that

$$\left(\int\limits_0^\varepsilon \left[\frac{f^*(t)}{\mathcal{E}_G^X(t)} \right]^v \mu_G(\mathrm{d}t) \right)^{1/v} \leq c\, \|f|X\| \tag{2.10}$$

(meaning $\sup_{0 < t < \varepsilon} \frac{f^*(t)}{\mathcal{E}_G^X(t)} \leq c\, \|f|X\|$ when $v = \infty$) holds for some $c > 0$ and all $f \in X$. Then

$$\mathfrak{E}_G(X) = \left(\mathcal{E}_G^X(\cdot), u_G^X \right)$$

is called the *(local) growth envelope* for the function space X.

Remark 2.16. Since (2.10) holds with $v = \infty$ in any case and the corresponding expressions on the left-hand side are, up to multiplicative (positive) constants, non-increasing in v by [48, Prop. 12.2], it is reasonable to ask for the *smallest* parameter v satisfying (2.10). Note that parallel to (2.9) there is a similar 'monotonicity' feature for corresponding indices that can be found in [28].

Example 2.17. Let $\Gamma \subset \mathbb{R}^n$ be an h-set in the sense of Definition 1.1 and $0 < p < \infty$. Then

$$\mathfrak{E}_G(L_p(\Gamma)) = \left(t^{-\frac{1}{p}}, p \right).$$

This result can be found in [9, Prop. 2.11] (as a special case of a corresponding result for the more general Lorentz spaces $L_{p,q}(\Gamma)$). In case of d-sets, $h(r) = r^d$, $0 < d < n$, the result remains unchanged and coincides with the usual $\mathbb{R}^n$-situation as well as more abstract settings, cf. [28, Sect. 4].

Before we can present our results for spaces $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ we need some preparation. We shall use a convenient notation introduced in [16] and adapted in [10]. A function $\Lambda : (0,\infty) \to (0,\infty)$ is called *admissible* if it is continuous and satisfies for any $b > 0$ that $\Lambda(y) \sim \Lambda(by)$, $y > 0$, with equivalence constants independent of y. Let $\boldsymbol{\tau}$ be an admissible sequence according to Definition 1.11, and $\boldsymbol{N} = (N_j)_j$ a sequence of positive numbers such that for some $\lambda_0 > 1$ it holds $\lambda_0 N_j \leq N_{j+1}$, $j \in \mathbb{N}_0$. In [10, Ex. 2.3] a construction of an admissible function $\Lambda = \Lambda_{\boldsymbol{\tau},\boldsymbol{N}}$ is given, that satisfies

$$\Lambda_{\boldsymbol{\tau},\boldsymbol{N}}(y) \ \sim \ \tau_j, \quad y \in [N_j, N_{j+1}], \quad j \in \mathbb{N}_0. \tag{2.11}$$

Let $\boldsymbol{\sigma}$ be an admissible sequence and, for $h \in \mathbb{H}$, let $N_j = h_j^{-1}$, $j \in \mathbb{N}_0$, satisfy $\lambda_0 N_j \leq N_{j+1}$, $j \in \mathbb{N}_0$, for some $\lambda_0 > 1$. We denote $\Sigma_{\boldsymbol{\sigma},h} := \Lambda_{\boldsymbol{\sigma},h^{-1}}$. It satisfies, in particular,

$$\Sigma_{\boldsymbol{\sigma},h}(y) \ \sim \ \sigma_j, \quad y \in \left[h_j^{-1}, h_{j+1}^{-1}\right], \quad j \in \mathbb{N}_0. \tag{2.12}$$

Let $\ J_0 \in \mathbb{N}$ be chosen such that $h_{J_0} < 1$. For $0 < r, u \leq \infty$ and $\Sigma_{\boldsymbol{\sigma},h}$ we introduce the function $\Psi_{r,u} : (0, h_{J_0}] \to \mathbb{R}$ by

$$\Psi_{r,u}(t) := \left(\int_t^1 y^{-\frac{u}{r}} \, \Sigma_{\boldsymbol{\sigma},h}(y^{-1})^{-u} \, \frac{\mathrm{d}y}{y} \right)^{1/u}, \tag{2.13}$$

with the usual modification for $u = \infty$,

$$\Psi_{r,\infty}(t) := \sup_{t \leq y \leq 1} \ y^{-\frac{1}{r}} \, \Sigma_{\boldsymbol{\sigma},h}(y^{-1})^{-1}. \tag{2.14}$$

In particular, $\Psi_{r,u}$ is positive, monotonically decreasing and continuous, being also differentiable when $u \neq \infty$ — cf. [10, Lemma 2.5]. The main result on growth envelopes in Besov spaces $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ was obtained in [9].

Theorem 2.18. *Let $0 < p, q < \infty$, $\boldsymbol{\sigma}$ be admissible, Γ be an h-set satisfying the porosity condition. Assume*

$$- n \leq \underline{\mathfrak{s}}\,(\boldsymbol{h}) \leq \overline{\mathfrak{s}}\,(\boldsymbol{h}) < 0, \tag{2.15}$$

$$\boldsymbol{\sigma}^{-1} \boldsymbol{h}^{-\frac{1}{p}} \notin \ell_{q'} \tag{2.16}$$

and

$$\underline{\mathfrak{s}}\,(\boldsymbol{\sigma}) > -\underline{\mathfrak{s}}\,(\boldsymbol{h}) \left(\frac{1}{p} - 1 \right)_+ . \tag{2.17}$$

Let $\Psi_{r,u}$ be given by (2.13). *Then*

$$\mathfrak{E}_{\mathsf{G}} \mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) = (\Psi_{p,q'}, q). \tag{2.18}$$

Remark 2.19. Observe that the hypotheses that Γ satisfies the porosity condition, $\underline{\mathfrak{s}}\,(\boldsymbol{h}) \geq -n$ and (2.17) are not needed for the proof of the optimality of the exponent q, as long as we assume that $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ exists as the trace of $B_{p,q}^{\boldsymbol{\sigma} \boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$. Moreover, (2.16) ensures that $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ is not contained in $L_\infty(\Gamma)$; see also (3.4) below.

Remark 2.20. In some applications it will be enough to use a sharp estimate from below for the growth envelope function, in which case we can also dispense with most of the hypotheses considered in the Theorem above, as follows from [9, Prop. 4.4]: up to a multiplicative constant factor, $\Psi_{p,q'}$ from (2.18) is a lower bound for the local growth envelope function of $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ near 0 as long as this space is assumed to exist as the trace of $B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$ and the h-set Γ satisfies the condition $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$.

Example 2.21. If Γ is a d-set, $h(r) = r^d$, $0 < d < n$, and $0 < p, q < \infty$, $d\left(\frac{1}{p} - 1\right)_+ < s < \frac{d}{p}$, then Theorem 2.18 reads as

$$\mathfrak{E}_{\mathsf{G}}\mathbb{B}_{p,q}^s(\Gamma) = \left(t^{\frac{s}{d} - \frac{1}{p}}, q\right). \tag{2.19}$$

In the limiting case $s = \frac{d}{p}$, now with $1 < q < \infty$, we obtain

$$\mathfrak{E}_{\mathsf{G}}\mathbb{B}_{p,q}^s(\Gamma) = \left(|\log t|^{1/q'}, q\right).$$

Remark 2.22. Note that in this last example in case (2.19) the growth envelope function $\mathcal{E}_{\mathsf{G}}^X(t) \sim t^{\frac{s}{d} - \frac{1}{p}}$ does not distinguish between spaces $\mathbb{B}_{p_1,q_1}^{s_1}(\Gamma)$ and $\mathbb{B}_{p_2,q_2}^{s_2}(\Gamma)$ whenever $s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}$, not even in case of $s_1 = s_2$, $p_1 = p_2$, but $q_1 \neq q_2$. Therefore the introduction of the additional fine index is justified.

Remark 2.23. We would like to emphasise the resemblance of the above results in Example 2.21 with those obtained for spaces $B_{p,q}^s(\mathbb{R}^n)$, see [48, Thms. 13.2, 15.2]: apparently one just had to replace n by d to get the right expressions in the context of d-sets. But, as already remarked in [9], this obvious similarity has indeed some geometrical meaning, as it is now easy to see that, for all possible relations between parameters given above, and for small positive t,

$$\mathcal{E}_{\mathsf{G}}^{\mathbb{B}_{p,q}^s(\Gamma)}\left(t^d\right) \sim \mathcal{E}_{\mathsf{G}}^{B_{p,q}^{s+\frac{n-d}{p}}(\mathbb{R}^n)}\left(t^n\right), \tag{2.20}$$

with coincidence of the corresponding indices. In particular, the argument on the left-hand side of (2.20) represents the geometry of the d-set Γ on which the trace space is defined, whereas the argument on the right-hand side corresponds to the geometry of the underlying $\mathbb{R}^n$ for the space $B_{p,q}^{s+(n-d)/p}(\mathbb{R}^n)$ from which the trace (on Γ) is taken.

Example 2.24. For the convenience of the reader we briefly return to the study of (d, Ψ)-sets Γ and corresponding trace spaces $\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma)$, recall Remark 2.11. In this case the assumptions in Theorem 2.18 are satisfied whenever $0 < d < n$, $0 < p, q < \infty$, $s > d(\frac{1}{p} - 1)_+$, Ψ is admissible and

$$\left(2^{j(\frac{d}{p} - s)}\Psi(2^{-j})^{-\frac{1}{p} - 1}\right)_j \notin \ell_{q'},$$

that is, when

$$\begin{cases} s < \frac{d}{p} & \text{or} \\ s = \frac{d}{p} & \text{and } \left(\Psi(2^{-j})^{-\frac{1}{p} - 1}\right)_j \notin \ell_{q'}. \end{cases}$$

This simply follows from the setting $h_j \sim 2^{-jd}\Psi(2^{-j})$ and $\sigma_j = 2^{js}\Psi(2^{-j})$, $j \in \mathbb{N}$, in this special case of (d,Ψ)-sets and spaces of type $B_{p,q}^{(s,\Psi)}$. This implies $\Sigma_{\sigma,h}\left(2^{jd}\Psi(2^{-j})^{-1}\right) \sim 2^{js}\Psi(2^{-j})$, in view of (2.12), and straightforward calculations taking advantage of the admissibility of Ψ — cf. also Remark 2.25 below — lead to

$$\mathfrak{E}_{\mathsf{G}}\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma) = \left(t^{\frac{s}{d}-\frac{1}{p}}\Psi(t)^{-1-\frac{s}{d}}, q\right) \tag{2.21}$$

if $s < \frac{d}{p}$, and to

$$\mathfrak{E}_{\mathsf{G}}\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma) = \left(\left(\int_{t^{1/d}}^{1} \Psi(y)^{-q'\left(\frac{1}{p}+1\right)}\frac{\mathrm{d}y}{y}\right)^{1/q'}, q\right)$$

if $s = \frac{d}{p}$ and $\left(\Psi(2^{-j})^{-\frac{1}{p}-1}\right)_j \notin \ell_{q'}$. In case of $0 < q \le 1$, that is, $q' = \infty$, this last expression has to be understood as

$$\mathfrak{E}_{\mathsf{G}}\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma) = \left(\sup_{t^{1/d} \le y \le 1} \Psi(y)^{-\left(\frac{1}{p}+1\right)}, q\right), \tag{2.22}$$

assuming now that $\left(\Psi(2^{-j})^{-\frac{1}{p}-1}\right)_j \notin \ell_\infty$ and $s = \frac{d}{p}$. Actually, with this assumption, and recalling the definition of Ψ, the growth envelope function in (2.22) can simply be written as $\Psi(t)^{-\left(\frac{1}{p}+1\right)}$.

Remark 2.25. In the spirit of Remark 2.23 we would like to compare the outcome in Example 2.24 with its counterpart for growth envelopes of spaces $B_{p,q}^{(s,\Psi)}(\mathbb{R}^n)$ obtained in [16, 17]. In the sub-critical case $n(\frac{1}{p}-1)_+ < s < \frac{n}{p}$ that result reads as

$$\mathfrak{E}_{\mathsf{G}}\left(B_{p,q}^{(s,\Psi)}(\mathbb{R}^n)\right) = \left(t^{\frac{s}{n}-\frac{1}{p}}\Psi(t)^{-1}, q\right),$$

whereas in the limiting case $s = \frac{n}{p}$ with the additional assumption $(\Psi(2^{-j})^{-1})_j \notin \ell_{q'}$ it is

$$\mathfrak{E}_{\mathsf{G}}\left(B_{p,q}^{(s,\Psi)}(\mathbb{R}^n)\right) = \left(\tilde{\Phi}_{p,q'}(t), q\right) \qquad \text{with} \quad \tilde{\Phi}_{p,q'}(t) = \left(\int_{t^{1/n}}^{1} \Psi(y)^{-q'}\frac{\mathrm{d}y}{y}\right)^{1/q'}$$

(modified similarly as in (2.22) when $q' = \infty$). We claim that

$$\mathcal{E}_{\mathsf{G}}^{\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma)}\left(t^d\Psi(t)\right) \sim \mathcal{E}_{\mathsf{G}}^{B_{p,q}^{(s+\frac{n-d}{p},\Psi^{1+\frac{1}{p}})}(\mathbb{R}^n)}\left(t^n\right), \tag{2.23}$$

parallel to (2.20) in case of d-sets (i.e., with $\Psi \equiv 1$).

We first deal with the sub-limiting case. Plainly, (2.21) leads to

$$\mathcal{E}_{\mathsf{G}}^{\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma)}\left(t^d\Psi(t)\right) \sim \left(t^d\Psi(t)\right)^{\frac{s}{d}-\frac{1}{p}} \Psi\left(t^d\Psi(t)\right)^{-\frac{s}{d}-1} \sim t^{s-\frac{d}{p}}\Psi(t)^{-\frac{1}{p}-1}$$

$$\sim \left(t^n\right)^{\frac{1}{n}(s+\frac{n-d}{p})-\frac{1}{p}} \Psi\left(t^n\right)^{-\frac{1}{p}-1} \sim \mathcal{E}_{\mathsf{G}}^{B_{p,q}^{(s+\frac{n-d}{p},\Psi^{1+\frac{1}{p}})}(\mathbb{R}^n)}\left(t^n\right),$$

where we used twice the admissibility of Ψ, which implies $\Psi(t) \sim \Psi(t^a\Phi(t))$, $a > 0$, Φ admissible, for small $t > 0$. As to the critical case, the correspondence

between the additional assumptions on Ψ when $s = \frac{d}{p}$ (on the side of the trace spaces on Γ) and when $s = \frac{n}{p}$ (on the side of the corresponding spaces on $\mathbb{R}^n$) are obvious, whereas (2.23) follows again from the admissibility of Ψ.

3. EMBEDDINGS

3.1. **Sharp embeddings for spaces on $\mathbb{R}^n$.** First we study criteria for embeddings of spaces of type $B_{p,q}^{\sigma}(\mathbb{R}^n)$ and slightly extend corresponding results in [10, Thm. 3.7] with fore-runners in [40] and [8, Prop. 4.12].

Theorem 3.1. *Let σ and τ be two admissible sequences, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$, and q^* be given by (1.19).*
(i) *The embedding*
$$\mathrm{id} : B_{p_1,q_1}^{\sigma}(\mathbb{R}^n) \longrightarrow B_{p_2,q_2}^{\tau}(\mathbb{R}^n) \tag{3.1}$$
   exists and is bounded if, and only if,
$$p_1 \leq p_2 \qquad and \qquad \sigma^{-1}\tau(n)^{\frac{1}{p_1}-\frac{1}{p_2}} \in \ell_{q^*} . \tag{3.2}$$
(ii) *The embedding* (3.1) *is never compact.*

Proof. We begin with (i). The sufficiency of (3.2) for the existence and boundedness of the embedding (3.1) is covered by [10, Thm. 3.7] already. So it remains to show the necessity. Here we use the wavelet characterisation in Theorem 2.3, see also Remark 2.4, and assume to choose the wavelet system in such a way that it is applicable for both spaces simultaneously. Hence the existence and continuity of the embedding (3.1) implies that $\ell_{p_1} \hookrightarrow \ell_{p_2}$ and

$$b_{p_1,q_1}^{\widetilde{\sigma}} \hookrightarrow b_{p_2,q_2}^{\widetilde{\tau}}. \tag{3.3}$$

Obviously, the first embedding immediately yields $p_1 \leq p_2$. As for (3.3), note first that $\widetilde{\sigma}^{-1}\widetilde{\tau}(n)^{\frac{1}{p_1}-\frac{1}{p_2}} = \sigma^{-1}\tau(n)^{\frac{1}{p_1}-\frac{1}{p_2}}$. We consider the sequence $\lambda = \{\lambda_{j,m}\}_{j \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, defined by

$$\lambda_{j,m} = \begin{cases} \sigma_j^{-1} 2^{jn(\frac{1}{p_1}-\frac{1}{2})} \beta_j, & m = 0, \ j \in \mathbb{N}_0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\beta \in \ell_{q_1}$ is arbitrary. Then by straightforward calculation,

$$\|\lambda|b_{p_1,q_1}^{\widetilde{\sigma}}\| = \|\beta|\ell_{q_1}\| \qquad and \qquad \|\lambda|b_{p_2,q_2}^{\widetilde{\tau}}\| = \left\|\tau\sigma^{-1}(n)^{\frac{1}{p_1}-\frac{1}{p_2}}\beta|\ell_{q_2}\right\|.$$

Application of Lemma 1.18 with $\alpha = \tau\sigma^{-1}(n)^{\frac{1}{p_1}-\frac{1}{p_2}}$ leads to the desired result (3.2).

We deal with (ii) and benefit from the fact that embeddings between classical spaces of type $B_{p,q}^s(\mathbb{R}^n)$ are never compact independently of the choice of the parameters. We proceed by contradiction. Assume that (3.1) was compact. Choose $s > \overline{\mathfrak{s}}(\sigma)$ and $t < \underline{\mathfrak{s}}(\tau)$, such that $B_{p_1,q_1}^s(\mathbb{R}^n) \hookrightarrow B_{p_1,q_1}^{\sigma}(\mathbb{R}^n)$ and $B_{p_2,q_2}^{\tau}(\mathbb{R}^n) \hookrightarrow B_{p_2,q_2}^t(\mathbb{R}^n)$ in view of (i) and Remark 1.15, in particular (1.18). Hence we obtain the compactness of $B_{p_1,q_1}^s(\mathbb{R}^n) \hookrightarrow B_{p_2,q_2}^t(\mathbb{R}^n)$ which can never happen. So we have proved (ii). □

Remark 3.2. Note that there is an extension in [29] to spaces of type $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$.

Example 3.3. We return to the setting of (d, Ψ)-sets, $0 < d < n$, Ψ admissible, as described in Example 1.6, in particular in (1.6), and the corresponding spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$ discussed in Remark 2.2, with $\sigma = (2^{js}\Psi(2^{-j}))_j$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, studied in [39, 40] in detail. Let $s_i \in \mathbb{R}$, Ψ_i admissible, $0 < p_i < \infty$, $0 < q_i \leq \infty$, $i = 1, 2$. Using the well-known abbreviation for the difference of the corresponding differential smoothnesses,

$$\delta = s_1 - \frac{n}{p_1} - \left(s_2 - \frac{n}{p_2} \right),$$

Theorem 3.1(i) reads as

$$B^{(s_1,\Psi_1)}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{(s_2,\Psi_2)}_{p_2,q_2}(\mathbb{R}^n)$$

if, and only if,

$$p_1 \leq p_2 \qquad \text{and} \qquad \left(2^{-j\delta} \frac{\Psi_2(2^{-j})}{\Psi_1(2^{-j})} \right)_j \in \ell_{q*}.$$

This coincides (in the limiting case $\delta = 0$) with our earlier result [11, Prop. 4.3] which already refined an outcome of Moura in [40, Prop. 1.1.13].

For matter of comparison we recall the result for target spaces L_p, see [10, Cor. 3.18] and the slightly weaker version [8, Prop. 4.13]. Here we rely on the recently obtained characterisation (in the more general setting of spaces $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$) in [13, Thm. 4.3, Cor. 4.6(i)].

Proposition 3.4. *Let* $0 < p < \infty$, $0 < q \leq \infty$, σ *be admissible. Then*

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{\max(p,1)}(\mathbb{R}^n)$$

if, and only if,

$$\begin{cases} \sigma^{-1}(n)^{\left(\frac{1}{p}-1\right)} \in \ell_{q'}, & \text{if} \quad 0 < p \leq 1, \ 0 < q \leq \infty, \\ \sigma^{-1} \in \ell_{\infty}, & \text{if} \quad 1 < p < \infty, \ 0 < q \leq \min(p,2), \\ \sigma^{-1} \in \ell_{\frac{pq}{q-p}}, & \text{if} \quad 1 < p \leq 2, \ \min(p,2) < q \leq \infty, \\ \sigma^{-1} \in \ell_{\frac{2q}{q-2}}, & \text{if} \quad 2 < p < \infty, \ \min(p,2) < q \leq \infty. \end{cases}$$

We come to the target space $L_{\infty}(\mathbb{R}^n)$ now.

Proposition 3.5. *Let* σ *be an admissible sequence,* $0 < p, q \leq \infty$. *Then*

$$B^{\sigma}_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \text{if, and only if,} \quad \sigma^{-1}(n)^{\frac{1}{p}} \in \ell_{q'},$$

where $C(\mathbb{R}^n)$ *can be replaced by* $L_{\infty}(\mathbb{R}^n)$.

This characterisation was proved in [10, Cors. 3.10, 4.9] in full generality; it has a fore-runner in [32], restricted to $1 < p, q < \infty$. In case of spaces $B^{(s,\Psi)}_{p,q}(\mathbb{R}^n)$, that is, when $\sigma = (2^{js}\Psi(2^{-j}))_j$, $s \in \mathbb{R}$, the result was already obtained in [16, Prop. 3.11].

Remark 3.6. The case $p = \infty$ missing in Proposition 3.4 can be read from Proposition 3.5 with $p = \infty$.

3.2. **Sharp embeddings for spaces on Γ.** We now study embedding assertions of type

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \hookrightarrow L_r(\Gamma) \qquad \text{and} \qquad \mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma) \hookrightarrow \mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma) \ .$$

In contrast to previous considerations in [6], see also [4], we are especially interested in 'sharp' embeddings, that is, to give necessary and sufficient conditions for the existence of such embeddings, whereas in the above-mentioned papers the focus was rather on the compactness of related embeddings.

First we collect what is already known. Note that whenever $0 < p, q < \infty$, $\boldsymbol{\sigma}$ is an admissible sequence, and Γ is an h-set, then

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \hookrightarrow L_{\max(p,1)}(\Gamma) \qquad \text{if} \qquad \boldsymbol{\sigma}^{-1}\boldsymbol{h}^{-\left(\frac{1}{p}-1\right)_+} \in \ell_{q'} \ ,$$

see [9, Prop. 3.9]. We can sharpen this assertion below in Corollary 3.13. Moreover, in [9, Prop. 3.11, Cor. 4.5] there is also a complete characterisation of embeddings in $L_\infty(\Gamma)$:

Corollary 3.7. *Let $0 < p, q < \infty$ and $\boldsymbol{\sigma}$ be an admissible sequence. Let Γ be an h-set with $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ and assume there exists the space $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ as the trace of $B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n)$ in $L_p(\Gamma)$. Then*

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \hookrightarrow L_\infty(\Gamma) \quad \textit{if, and only if,} \quad \boldsymbol{\sigma}^{-1}\boldsymbol{h}^{-\frac{1}{p}} \in \ell_{q'}, \qquad (3.4)$$

where $L_\infty(\Gamma)$ can be replaced by $C(\Gamma)$.

Remark 3.8. We would like to mention that for the 'if'-part in (3.4) we do not need to assume a priori the existence of $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ (this comes as a consequence then). For that part we also do not need the assumption $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$.

Example 3.9. For d-sets Γ and the special choice $\boldsymbol{\sigma} = (s)$ we obtain $\mathbb{B}^s_{p,q}(\Gamma) \hookrightarrow C(\Gamma)$ if, and only if, $s > \frac{d}{p}$, or (in the limiting case) $s = \frac{d}{p}$ and $0 < q \leq 1$; this is in perfect coincidence with the results in [47, Thm. 20.6, Sect. 21.1], and compares well with the classical situation on $\mathbb{R}^n$, see [44, Thm. 3.3.1(ii)].

Now we return to the problem of 'sharp' embeddings into spaces L_r with $r < \infty$. First we deal with the question what happens if we replace the target space L_p in (2.4) by some possibly larger or smaller space L_r. More precisely, we study the boundedness of the trace operator

$$\mathrm{tr}_\Gamma : B^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n) \to L_r(\Gamma), \quad f \mapsto \mathrm{tr}_\Gamma f, \qquad (3.5)$$

defined first pointwise for smooth functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and then extended by completion in such spaces where $\mathcal{S}(\mathbb{R}^n)$ is dense. Therefore we require $\max(p, q) < \infty$ again. We recall that this trace is well-defined if we assume (2.4) to hold (with $\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}$ and L_p replaced by L_r). It will turn out that, in all cases we shall study, (3.5) exists and is continuous if, and only if, the trace space $\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma)$ exists and

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \hookrightarrow L_r(\Gamma) \qquad (3.6)$$

holds. We begin with the case $r \le p$.

Proposition 3.10. *Let Γ be an h-set satisfying the porosity condition, $\boldsymbol{\sigma}$ be an admissible sequence and either*

(a): $1 \le p < \infty,\ 0 < q < \infty,\ 0 < r \le p$ *or*
(b): $0 < q, r \le p < 1$.

Then

$$\operatorname{tr}_\Gamma : B_{p,q}^{\boldsymbol{\sigma} \boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \longrightarrow L_r(\Gamma) \tag{3.7}$$

exists and is bounded if, and only if,

$$\boldsymbol{\sigma}^{-1} \in \ell_{q'}.$$

Proof. Note that the sufficiency is already covered by our considerations in Section 2.2, together with the embeddings $L_p(\Gamma) \hookrightarrow L_r(\Gamma)$ for $0 < r \le p$ and the compacity of Γ. It remains to show the necessity. But this comes as an easy consequence of the density results which we have obtained in [12, Rem. 3.20,Thm. 3.18], namely that when $\boldsymbol{\sigma}^{-1} \notin \ell_{q'}$, then $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B_{p,q}^{\boldsymbol{\sigma} \boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$. For the convenience of the reader, we essentially repeat the argument presented in the beginning of Section 3.2 in [12], which shows that, in the presence of the just mentioned density, the trace (3.7) cannot exist:

Assume that $\mathcal{D}(\mathbb{R}^n \setminus \Gamma)$ is dense in $B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ on a neighbourhood of Γ. Clearly, $\varphi \in \mathcal{S}(\mathbb{R}^n) \cap B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n)$ and therefore there exists a sequence $(\psi_k)_k \subset \mathcal{D}(\mathbb{R}^n \setminus \Gamma) \subset \mathcal{S}(\mathbb{R}^n)$ with

$$\left\| \varphi - \psi_k | B_{p,q}^{\boldsymbol{\tau}}(\mathbb{R}^n) \right\| \xrightarrow[k \to \infty]{} 0.$$

If the trace $\operatorname{tr}_\Gamma : B_{p,q}^{\boldsymbol{\tau}} \longrightarrow L_r(\Gamma)$ were to exist, this would imply

$$0 = \psi_k|_\Gamma = \operatorname{tr}_\Gamma \psi_k \xrightarrow[k \to \infty]{} \operatorname{tr}_\Gamma \varphi = \varphi|_\Gamma = 1 \quad \text{in } L_r(\Gamma),$$

which is a contradiction. $\qquad\square$

Remark 3.11. In view of Definition 2.8(i) the above result states that, under the given conditions, there is a trace in $L_p(\Gamma)$ if, and only if, there is a trace in $L_r(\Gamma)$. Note that Triebel proved in [49, Cor. 7.21] a related result in case of $1 < p < \infty$, $1 \le q < \infty$, $1 \le r \le p$, and $\boldsymbol{\sigma}(n)^{\frac{1}{p}} \boldsymbol{h}^{\frac{1}{p}} = (s)$ with $s > 0$: also in that case the trace in $L_r(\Gamma)$ exists if, and only if, $\boldsymbol{\sigma}^{-1} \in \ell_{q'}$ (the further restrictions on the parameters are partly caused by the used argument, in particular, duality). Moreover, when $q > 1$, then he can even show compactness of the trace operator in that setting.

We return to (3.6) now in case of $r \ge p$.

Proposition 3.12. *Let Γ be an h-set such that $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ and $\boldsymbol{\sigma}$ be an admissible sequence.*
(i) *Let $0 < p \le r < \infty$ and $0 < q \le \min(r, 1)$. Then*

$$\operatorname{tr}_\Gamma : B_{p,q}^{\boldsymbol{\sigma} \boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \longrightarrow L_r(\Gamma)$$

exists and is bounded if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{\frac{1}{r}-\frac{1}{p}} \in \ell_\infty.$$

(ii) *Let* $0 < q < \infty$ *and* $0 < p \le r \le \min(q,1)$ *and denote* $\frac{1}{v_r} := \frac{1}{r} - \frac{1}{q}$. *Then*

$$\mathrm{tr}_\Gamma : B_{p,q}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \longrightarrow L_r(\Gamma)$$

exists and is bounded if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{\frac{1}{r}-\frac{1}{p}} \in \ell_{v_r}.$$

Proof. Step 1. We show the sufficiency of the conditions by applying Theorem 3.1 to get

$$B_{p,q}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n) \hookrightarrow B_{r,\min(r,1)}^{\boldsymbol{h}^{1/r}(n)^{1/r}}(\mathbb{R}^n),$$

so that, with the help of (2.5), we see that the required traces exist.

Step 2. We come to the necessity of the condition in (i). Assuming the existence of the trace in assertion (i), since $p \le r$ and Γ is compact we have that $L_r(\Gamma) \hookrightarrow L_p(\Gamma)$ and therefore $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ exists as the trace of $B_{p,q}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$ and, moreover,

$$\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_r(\Gamma). \tag{3.8}$$

Then Remark 2.20 can be applied to get that

$$\mathcal{E}_\mathsf{G}^{\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)}(h_j) \gtrsim \sup_{k=0,\dots,j} \sigma_k^{-1} h_k^{-\frac{1}{p}} \ge \sigma_j^{-1} h_j^{-\frac{1}{p}} \quad \text{for } j \text{ large enough.} \tag{3.9}$$

On the other hand, from (3.8) and (2.9) we can write

$$\mathcal{E}_\mathsf{G}^{\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)}(t) \lesssim \mathcal{E}_\mathsf{G}^{L_r(\Gamma)}(t), \quad t > 0.$$

From this, (3.9) and Example 2.17 it follows that

$$\sigma_j^{-1} h_j^{-\frac{1}{p}} \lesssim h_j^{-\frac{1}{r}} \quad \text{for large enough } j \in \mathbb{N},$$

that is, $\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{\frac{1}{r}-\frac{1}{p}} \in \ell_\infty$.

Step 3. It remains to deal with the necessity of the condition in (ii). Let $L \in \mathbb{N}$ be such that $L > -1 + n(\frac{1}{p}-1)_+ - \mathfrak{s}(\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p})$ and consider a C^∞-function ϕ on $\mathbb{R}^n$ such that

- there exist $C_1, C_2, C_3 > 0$ with $C_1 < C_3 < 2C_1$, $\phi(x) \ge C_2$ for $|x|_\infty \le C_1$ and $\phi(x) = 0$ for $|x|_\infty \ge C_3$,
- $\int_{\mathbb{R}^n} x^\beta \phi(x)\mathrm{d}x = 0$ whenever $\beta = (\beta_j)_{j=1}^n \in \mathbb{N}_0^n$ with $|\beta|_1 \le L$.

Such a function exists (cf. [10, Lemma 4.6]). Moreover,

$$\sigma_j^{-1} h_j^{-\frac{1}{p}} \phi(2^j(\cdot - \gamma_0)), \quad j \in \mathbb{N}, \quad \gamma_0 \text{ fixed in } \Gamma,$$

are (up to multiplicative constants) $(\boldsymbol{\sigma}\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}, p)_{K,L}$-atoms located at $Q_{j,m(j)}$, where K is a fixed natural number satisfying $K > \mathfrak{s}(\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p})$ and $m(j) \in \mathbb{N}_0^n$ is one of the possible n-tuples satisfying $|\gamma_0 - 2^{-j}m(j)|_\infty \le 2^{-j-1}$.

Let $\boldsymbol{b} := (b_j)_{j \in \mathbb{N}} \in \ell_q$ be any sequence of non-negative numbers. For any given $T \in \mathbb{N}$ consider the sequence $\boldsymbol{b^T}$ formed by the terms of $\boldsymbol{b}$ until the order T and by null terms afterwards and also consider

$$g^{\boldsymbol{b^T}}(x) := \sum_{j=1}^{T} b_j \sigma_{j\iota_0}^{-1} h_{j\iota_0}^{-\frac{1}{p}} \phi(2^{j\iota_0}(x - \gamma_0)), \quad x \in \mathbb{R}^n, \tag{3.10}$$

for the γ_0 fixed in Γ considered above and with $\iota_0 \in \mathbb{N}$ at our disposal.

From the atomic representation Proposition 2.6 applied to $B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ and the above considerations, there exists $c' > 0$ independent of $\boldsymbol{b}$ and T such that

$$\|g^{\boldsymbol{b^T}}|B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)\| \le c' \|\boldsymbol{b^T}|\ell_q\| \le c' \|\boldsymbol{b}|\ell_q\| < \infty. \tag{3.11}$$

From our existence hypothesis and the fact that $g^{\boldsymbol{b^T}} \in C_0^{\infty}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, there also exists $c'' > 0$ independent of $\boldsymbol{b^T}$ and T such that

$$\|g^{\boldsymbol{b^T}}|_{\Gamma}|L_r(\Gamma)\| \le c'' \|g^{\boldsymbol{b^T}}|B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)\|$$

and therefore

$$\|g^{\boldsymbol{b^T}}|_{\Gamma}|L_r(\Gamma)\| \le c'' c' \|\boldsymbol{b}|\ell_q\| < \infty. \tag{3.12}$$

For each $m \in \mathbb{N}$ define

$$P_m := \{x \in \mathbb{R}^n : C_3 2^{-(m+1)\iota_0} < |x - \gamma_0|_{\infty} \le C_1 2^{-m\iota_0}\}.$$

Since $C_1 < C_3 < 2C_1 \le 2^{\iota_0} C_1$, then $C_1 2^{-(m+1)\iota_0} < C_3 2^{-(m+1)\iota_0} < C_1 2^{-m\iota_0}$ and therefore $P_m \ne \emptyset$ and $P_{m+1} \cap P_m = \emptyset$. Observe that if $x \in P_m$, for some given $m \in \mathbb{N}$, then

$$\phi(2^j(x - \gamma_0)) \ge C_2 \quad \text{if } 1 \le j \le m\iota_0 \quad \text{and}$$
$$\phi(2^j(x - \gamma_0)) = 0 \quad \text{if } j \ge (m + 1)\iota_0.$$

We can then write that

$$\begin{aligned}
\|g^{\boldsymbol{b^T}}|_{\Gamma}|L_r(\Gamma)\| &\ge \left(\sum_{m=1}^{T} \int_{\Gamma \cap P_m} \left(\sum_{j=1}^{m} b_j \sigma_{j\iota_0}^{-1} h_{j\iota_0}^{-\frac{1}{p}} C_2\right)^r d\mu(\gamma)\right)^{\frac{1}{r}} \\
&\ge C_2 \left(\sum_{m=1}^{T} b_m^r \sigma_{m\iota_0}^{-r} h_{m\iota_0}^{-\frac{r}{p}} \mu(\Gamma \cap P_m)\right)^{\frac{1}{r}} \tag{3.13}
\end{aligned}$$

where, using the notation $Q_t(\gamma) := \{x \in \mathbb{R}^n : |x - \gamma|_{\infty} \le t\}$,

$$\mu(\Gamma \cap P_m) = \mu(Q_{C_1 2^{-m\iota_0}}(\gamma_0)) - \mu(Q_{C_3 2^{-(m+1)\iota_0}}(\gamma_0)).$$

Now we recall that Γ is an h-set and therefore, besides the use of balls in Definition 1.1(ii), we can also say that, given any $t_0 > 0$, there exist constants $a_0, a_1 > 0$ such that, for any $\gamma \in \Gamma$ and $t \in (0, t_0]$, $a_0 h(t) \le \mu(Q_t(\gamma)) \le a_1 h(t)$. By using Remark 1.3, the fact that h is non-decreasing and the hypothesis $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$, it is then possible to choose $\iota_0 \in \mathbb{N}$ large enough in order that

$$\begin{aligned}
\mu(\Gamma \cap P_m) &\ge a_0 h(C_1 2^{-m\iota_0}) - a_1 h(C_3 2^{-(m+1)\iota_0}) \\
&\ge a_2(\iota_0) h_{m\iota_0},
\end{aligned}$$

where $a_2(\iota_0)$ can be chosen positive and independent of m. Using this in (3.13) and conjugating with (3.12) we get that

$$(b_m \sigma_{m\iota_0}^{-1} h_{m\iota_0}^{\frac{1}{r}-\frac{1}{p}})_{m\in\mathbb{N}} \in \ell_r.$$

Let $\boldsymbol{\beta} = \boldsymbol{b}$, $\boldsymbol{\alpha} = (\sigma_{m\iota_0}^{-1} h_{m\iota_0}^{\frac{1}{r}-\frac{1}{p}})_{m\in\mathbb{N}}$. We apply Lemma 1.18 with $q_1 = q$ and $q_2 = r$ and obtain that the subsequence $(\sigma_{m\iota_0}^{-1} h_{m\iota_0}^{\frac{1}{r}-\frac{1}{p}})_m \in \ell_{v_r}$. The extension to $\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{\frac{1}{r}-\frac{1}{p}} \in \ell_{v_r}$ is a consequence of Remark 1.17. $\qquad\square$

Corollary 3.13. *Let $0 < p, q < \infty$, $\boldsymbol{\sigma}$ be an admissible sequence and Γ be an h-set satisfying the porosity condition and such that $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$. The following two assertions are equivalent:*

(i): $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ *exists as the trace space of $B_{p,q}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ in $L_p(\Gamma)$ and*

$$\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_{\max(p,1)}(\Gamma);$$

(ii): $\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{-\left(\frac{1}{p}-1\right)_+} \in \ell_{q'}$.

Proof. (ii) $\Rightarrow$ (i): This was already proved in [9, Prop. 3.9], even without special restrictions on the h-set.

(i) $\Rightarrow$ (ii): Consider first the case $1 \leq p < \infty$. Since Γ is compact and $p \geq 1$, the hypothesis implies that $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_1(\Gamma)$ and the result follows by applying Proposition 3.10 with $r = 1$.

Consider now the case $0 < p < 1$. Then the result follows from Proposition 3.12 with $r = 1$, using its part (i) in the subcase $0 < q \leq 1$ and its part (ii) in the subcase $1 < q < \infty$. $\qquad\square$

Remark 3.14. If we compare the above corollary with its $\mathbb{R}^n$-counterpart Proposition 3.4, the condition in case of $0 < p \leq 1$ is quite similar, unlike in the case $1 < p < \infty$; this might be surprising at first glance, but the reason is presumably hidden in our assumption or conclusion, respectively, that the trace spaces should exist.

Remark 3.15. Note that in the criterion (3.4) for $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow C(\Gamma)$ only the smoothness parameter $\boldsymbol{\sigma}$ and the underlying fractal geometry $\boldsymbol{h}$ are involved (and not the underlying $\mathbb{R}^n$-space) – as it should be; the same applies to Corollary 3.13. For similar discussions concerning the independence of the dimension of the underlying space we refer to [47, Sects. 18.9, 20.3].

Example 3.16. If Γ is a d-set, $h(r) = r^d$, $0 < d < n$, and $\boldsymbol{\sigma} = (s)$ with $s > 0$, then Corollary 3.13 reads as $\mathbb{B}_{p,q}^s(\Gamma) \hookrightarrow L_{\max(p,1)}(\Gamma)$ if, and only if, $s > d(\frac{1}{p}-1)_+$, $0 < q < \infty$, or $s = d(\frac{1}{p}-1)_+$ when $0 < q \leq 1$. This obviously goes well with the classical situation in $\mathbb{R}^n$, cf. [44, Thm. 3.3.2] for a complete characterisation, and [47, Sect. 20.3] for the case of d-sets.

Corollary 3.17. *Let $0 < p, q < \infty$, $\boldsymbol{\sigma}$ be an admissible sequence and Γ be an h-set satisfying the porosity condition and such that $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$. Then $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma)$ exists*

as the trace space of $B_{p,q}^{\boldsymbol{\sigma h}^{1/p}(n)^{1/p}}(\mathbb{R}^n)$ *in* $L_p(\Gamma)$ *if, and only if,*

$$\boldsymbol{\sigma}^{-1} \in \begin{cases} \ell_{q'}, & \text{if} \quad 1 \le p < \infty \quad \text{or} \quad 0 < q \le p < 1, \\ \ell_{v_p}, & \text{if} \quad 0 < p < 1 \quad \text{and} \quad 0 < p < q < \infty, \end{cases}$$

where $\frac{1}{v_p} := \frac{1}{p} - \frac{1}{q}$.

Proof. The sufficiency comes from Definition 2.8 (which is backed up by the discussion in [9, Sect. 3.1]), even without special restrictions on the h-set.

As to the necessity, for the first line of the brace it comes from Proposition 3.10 with $r = p$ (and the assumption $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ is not needed here), while for the second line it comes from part (ii) of Proposition 3.12 with $r = p$ (and the porosity condition is not needed here). $\qquad\square$

Remark 3.18. Recall that $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ implies that μ is strongly isotropic in the sense of Remark 1.19 and that $\boldsymbol{h} \in \ell_1$. In particular, this implies (cf. [9, Sect. 2.1]) that if $\boldsymbol{\sigma}^{-1} \notin \ell_{v_p}$ (with $0 < p < 1$ and $0 < p < q < \infty$), then $\boldsymbol{\sigma}^{-1}\boldsymbol{h}^{\frac{1}{r} - \frac{1}{p}} \notin \ell_{v_r}$ for all $r \in [p, \min(q, 1)]$ — which had to be, otherwise the above result would contradict Definition 2.8(ii).

Remark 3.19. In [12, Conj. 3.21] we conjectured that, instead of $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$, an extra assumption like $\lim_{j\to\infty} h_j \sigma_j^{v_p} = 0$ would give the equivalence above in the case $0 < p < 1$ and $0 < p < q < \infty$ together with a so-called dichotomy result. We have nothing new to add here as far as the dichotomy is concerned, but we would like to draw the attention of the reader to [12, Rem. 3.22], which shows that a result like the one above for $0 < p < 1$ and $0 < p < q < \infty$ cannot hold without extra assumptions. In particular, even if not knowing whether $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$ can be improved or not as an extra requirement, at least we know that something extra must be required in the mentioned case of the result above.

Remark 3.20. Bricchi obtained in [6, Thm. 4.3.2] that for $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$, $1 \le r \le p < \infty$, $0 < q < \infty$, and with some additional assumptions on Γ and h, the embedding $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_r(\Gamma)$ is compact and he estimated the corresponding entropy numbers by

$$e_{\lfloor h_j^{-1} \rfloor}\left(\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_r(\Gamma)\right) \sim \sigma_j^{-1}, \quad j \in \mathbb{N}. \tag{3.14}$$

As already pointed out in Remark 1.15, $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) > 0$ implies $\boldsymbol{\sigma}^{-1} \in \ell_v$ for arbitrary v, but we conjecture that the embedding $\mathbb{B}_{p,q}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow L_r(\Gamma)$ remains compact for $\underline{\mathfrak{s}}(\boldsymbol{\sigma}) = 0$ as long as $\boldsymbol{\sigma}^{-1} \in \ell_{q'}$ and $q > 1$, that is, in particular, with $\sigma_j^{-1} \to 0$ for $j \to \infty$, in good agreement with (3.14).

Remark 3.21. We return to Remark 3.11 and the special setting Triebel studied in [49, Cor. 7.21]. For $1 < p = q < \infty$, and $\boldsymbol{\sigma}(n)^{\frac{1}{p}}\boldsymbol{h}^{\frac{1}{p}} = (s)$ with $0 < s \le \frac{n}{p}$, the trace operator

$$\text{tr}_\Gamma : B_{p,p}^s(\mathbb{R}^n) \to L_p(\Gamma)$$

is not only compact, but one can also determine the asymptotic behaviour of its approximation numbers: Under the additional assumption that μ is strongly

isotropic in the sense of Remark 1.10 and satisfies

$$\sum_{j \geq J} 2^{-jp'(s-\frac{n}{p})} h_j^{\frac{p'}{p}} \sim 2^{-Jp'(s-\frac{n}{p})} h_J^{\frac{p'}{p}}, \quad J \in \mathbb{N},$$

then, denoting the inverse function of $\boldsymbol{h}$ by H, Triebel proved in [49, Thm. 7.22] that

$$a_k \left(\text{tr}_\Gamma : B^s_{p,p}(\mathbb{R}^n) \to L_p(\Gamma) \right) \sim k^{-\frac{1}{p}} H(k^{-1})^{s-\frac{n}{p}}, \quad k \in \mathbb{N}.$$

Following the proof in [49, Thm. 7.22] and our different notation this can be rewritten as

$$a_{\lfloor h_j^{-1} \rfloor} \left(\text{tr}_\Gamma : B^{\boldsymbol{\sigma} \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,p}(\mathbb{R}^n) \to L_p(\Gamma) \right) \sim \sigma_j^{-1}, \quad j \in \mathbb{N},$$

assuming that $1 < p < \infty$, $\boldsymbol{\sigma}(n)^{\frac{1}{p}} \boldsymbol{h}^{\frac{1}{p}} = (s)$ with $0 < s \leq \frac{n}{p}$, and $\sum_{j \geq J} \sigma_j^{-p'} \sim \sigma_J^{-p'}$, $J \in \mathbb{N}$. Though in a slightly different setting, the similarity to (3.14) is obvious.

Example 3.22. In case of $0 < p \leq r \leq 1$, $0 < q < \infty$, and $\bar{\mathfrak{s}}(\boldsymbol{h}) < 0$, we obtain from Proposition 3.12 that

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p,q}(\Gamma) \hookrightarrow L_r(\Gamma)$$

if, and only if,

$$\boldsymbol{\sigma}^{-1} \boldsymbol{h}^{\frac{1}{r} - \frac{1}{p}} \in \ell_{v_r^+} \quad \text{with} \quad \frac{1}{v_r^+} = \left(\frac{1}{r} - \frac{1}{q} \right)_+ .$$

In particular, when Γ is a d-set with $0 < d < n$, and $\boldsymbol{\sigma} = (s)$, then this result reads as

$$\mathbb{B}^s_{p,q}(\Gamma) \hookrightarrow L_r(\Gamma)$$

if, and only if, $s - \frac{d}{p} \geq -\frac{d}{r}$ if $q \leq r$, or $s - \frac{d}{p} > -\frac{d}{r}$ if $q > r$. In other words, we have for the limiting case $s - \frac{d}{p} = -\frac{d}{r}$ that

$$\mathbb{B}^{d(\frac{1}{p} - \frac{1}{r})}_{p,q}(\Gamma) \hookrightarrow L_r(\Gamma) \quad \text{if, and only if,} \quad q \leq r.$$

This behaviour is well-known from the classical $\mathbb{R}^n$-situation, see [44, Rem. 3.3.5] or [48, Thm. 11.4] for the case $r \geq 1$, and [30, Thm. 1.15] otherwise. There it is proved by some interpolation argument involving Lorentz spaces. Note that one has to be quite careful with the definition of Besov spaces when $0 < p < 1$ and $0 < s \leq n(\frac{1}{p} - 1)$, that is, whether the corresponding spaces are defined as spaces of tempered distributions or as subspaces of L_p.

We finally come to embeddings between two such Besov spaces on some h-set Γ. We begin with sufficient conditions for such an embedding.

Theorem 3.23. *Let Γ be an h-set, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be admissible sequences. Let q^* be given by (1.19). If $\underline{\mathfrak{s}}(\boldsymbol{\tau}) > 0$ and*

$$\boldsymbol{\sigma}^{-1} \boldsymbol{\tau} \boldsymbol{h}^{-\left(\frac{1}{p_1} - \frac{1}{p_2} \right)_+} \in \ell_{q^*}, \tag{3.15}$$

then there exist the trace spaces $\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma)$ and $\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)$ and

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma) \hookrightarrow \mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma) . \tag{3.16}$$

Proof. Step 1. We consider the case $p_1 \leq p_2$. Since in that case (3.15) ensures that

$$\left[\boldsymbol{\sigma}\boldsymbol{h}^{\frac{1}{p_1}}(n)^{\frac{1}{p_1}}\right]^{-1}\left[\boldsymbol{\tau}\boldsymbol{h}^{\frac{1}{p_2}}(n)^{\frac{1}{p_2}}\right](n)^{\frac{1}{p_1}-\frac{1}{p_2}} = \boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{h}^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)_+} \in \ell_{q^*} \ ,$$

then, by Theorem 3.1, the embedding

$$\mathrm{id}_1 : B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n) \to B_{p_2,q_2}^{\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}(n)^{1/p_2}}(\mathbb{R}^n) \tag{3.17}$$

exists and is bounded.

On the other hand, following our discussion in Section 2.2, the hypothesis $\underline{\mathfrak{s}}(\boldsymbol{\tau}) > 0$ guarantees the existence of the trace space $\mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$, that is, the existence and boundedness of the trace operator

$$\mathrm{tr}_\Gamma : B_{p_2,q_2}^{\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}(n)^{1/p_2}}(\mathbb{R}^n) \to L_{p_2}(\Gamma)$$

such that $\mathrm{tr}_\Gamma \varphi = \varphi|_\Gamma$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $p_1 \leq p_2$ and Γ is compact, we also have that the embedding

$$\mathrm{id}_2 : L_{p_2}(\Gamma) \to L_{p_1}(\Gamma)$$

exists and is bounded. Therefore

$$\mathrm{id}_2 \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1 : B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n) \to L_{p_1}(\Gamma)$$

is a bounded linear operator such that, given any $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$(\mathrm{id}_2 \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1)\,\varphi = \varphi|_\Gamma \ .$$

By Remark 2.7 it must be the trace operator of $B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n)$ in $L_{p_1}(\Gamma)$, hence there exists also

$$\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma) = (\mathrm{id}_2 \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1)\,B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n).$$

Moreover, given $f \in \mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma)$, there is some $g \in B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n)$ such that $f = (\mathrm{id}_2 \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1)\,g$. But (3.17) implies also $g \in B_{p_2,q_2}^{\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}(n)^{1/p_2}}(\mathbb{R}^n)$, thus also $f \in \mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$, and

$$\begin{aligned}
\left\|f|\mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)\right\| &= \inf_{\mathrm{tr}_\Gamma g = f} \left\|g|B_{p_2,q_2}^{\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}(n)^{1/p_2}}(\mathbb{R}^n)\right\| \\
&\leq c \inf_{(\mathrm{id}_2 \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_1)g = f} \left\|g|B_{p_1,q_1}^{\boldsymbol{\sigma}\boldsymbol{h}^{1/p_1}(n)^{1/p_1}}(\mathbb{R}^n)\right\| \\
&= c \left\|f|\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma)\right\| .
\end{aligned}$$

Step 2. We deal with the case $p_1 > p_2$. Applying the preceding step to $p_1 = p_2$, we have

$$\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow \mathbb{B}_{p_1,q_2}^{\boldsymbol{\tau}}(\Gamma)$$

since (3.15) reads then as $\boldsymbol{\sigma}^{-1}\boldsymbol{\tau} \in \ell_{q^*}$. This also covers the existence of the trace spaces. The hypothesis $\underline{\mathfrak{s}}(\boldsymbol{\tau}) > 0$ guarantees, as before, the existence of the trace space $\mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$, and from the assumption $p_1 > p_2$ it follows that

$$\mathbb{B}_{p_1,q_2}^{\boldsymbol{\tau}}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma).$$

This can be proved by using atomic representations for the corresponding spaces on $\mathbb{R}^n$, thus extending a related result in [6, Step 2 of the proof of Thm. 4.3.2]. Putting the two previous embeddings together we arrive at $\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$, as desired. $\square$

Remark 3.24. From the above proof we see that the hypothesis $\underline{\mathfrak{s}}(\boldsymbol{\tau}) > 0$ was only used to guarantee the existence of some trace spaces. We can dispense with it if we, instead, assume that $\mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$ exists, and, in case $p_1 > p_2$, that also $\mathbb{B}_{p_1,q_2}^{\boldsymbol{\tau}}(\Gamma)$ exists.

Remark 3.25. Our result above extends a corresponding one by Bricchi in [6, Thm. 4.3.1] as far as the existence of trace spaces is concerned, whereas Bricchi obtained also compactness assertions there.

Example 3.26. Let $0 < d < n$, Γ be a d-set, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$, $s, t > 0$. Then

$$\mathbb{B}_{p_1,q_1}^{s}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{t}(\Gamma) \quad \text{if} \quad \begin{cases} s - t \geq d\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right)_{+} & \text{if } q_1 \leq q_2, \\ s - t > d\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right)_{+} & \text{if } q_1 > q_2 \,. \end{cases}$$

This is an immediate consequence of Theorem 3.23. We thus recover (part of) the result of Triebel in [47, Thm. 20.6] where he also dealt with the asymptotic behaviour of the entropy numbers of the compact embedding $\mathbb{B}_{p_1,q_1}^{s}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{t}(\Gamma)$ when $s - t > d\left(\dfrac{1}{p_1} - \dfrac{1}{p_2}\right)_{+}$ (including further limiting cases of $t = 0$, $p_i, q_i = \infty$).

Example 3.27. Similarly, if we consider (d, Ψ)-sets Γ, $0 < d < n$, Ψ an admissible function, and corresponding trace spaces $\mathbb{B}_{p,q}^{(s,\Psi)}(\Gamma)$, recall Remark 2.11, then we can compare our result with a corresponding one of Moura in [40, Thm. 3.3.2] where the focus was again on compactness assertions rather than on (limiting) continuous embeddings. Explicating Theorem 3.23 in that setting we obtain for $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$, $s, t > 0$, that

$$\mathbb{B}_{p_1,q_1}^{(s,\Psi)}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{(t,\Psi)}(\Gamma) \quad \text{if} \quad \left(2^{-j(s-t-d(\frac{1}{p_1}-\frac{1}{p_2})_+)}\Psi(2^{-j})^{-(\frac{1}{p_1}-\frac{1}{p_2})_+}\right)_j \in \ell_{q^*}.$$

We study the necessity of the condition (3.15) for the embedding (3.16) now.

Theorem 3.28. *Let Γ be an h-set satisfying the porosity condition with* (2.15), $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$, *$\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be admissible sequences with*

$$\bar{\mathfrak{s}}\left(\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}\right) < 0 \tag{3.18}$$

and

$$\underline{\mathfrak{s}}\left(\boldsymbol{\tau}\right) > -\underline{\mathfrak{s}}\left(\boldsymbol{h}\right)\left(\frac{1}{p_2} - 1\right)_{+}. \tag{3.19}$$

Let q^* be given by (1.19). *If the trace spaces $\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma)$ and $\mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma)$ exist and*

$$\mathbb{B}_{p_1,q_1}^{\boldsymbol{\sigma}}(\Gamma) \hookrightarrow \mathbb{B}_{p_2,q_2}^{\boldsymbol{\tau}}(\Gamma), \tag{3.20}$$

then

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{h}^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \in \ell_{q^*}\,. \tag{3.21}$$

Proof. Step 1. We would like to start by remarking that the assumption that $\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)$ exists is not really necessary, as it comes as a consequence of (2.15) and (3.19), due to Corollary 3.17 and (the end of) Remark 1.15. The latter also guarantees, together with (3.18), that $\boldsymbol{\tau}^{-1}\boldsymbol{h}^{-1/p_2} \notin \ell_\infty$, therefore

$$\boldsymbol{\tau}^{-1}\boldsymbol{h}^{-1/p_2} \notin \ell_{q_2'}. \tag{3.22}$$

Hence, by Theorem 2.18, $\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)$ possesses a non-trivial growth envelope. We return to the example $g^{\boldsymbol{b}^T}$ given by (3.10),

$$g^{\boldsymbol{b}^T}(x) = \sum_{r=1}^{T} b_r \sigma^{-1}_{r\iota_0} h^{-\frac{1}{p_1}}_{r\iota_0} \phi(2^{r\iota_0}(x-\gamma_0)), \quad x \in \mathbb{R}^n,$$

where now the sequence $\boldsymbol{b} = (b_r)^\infty_{r=1}$ of non-negative numbers belongs to ℓ_{q_1}. According to (2.8) and (3.11) we have

$$\|g^{\boldsymbol{b}^T} \mid \mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma)\| \leq c_1^{-1} \|\boldsymbol{b} \mid \ell_{q_1}\|$$

(recall that we assume that $\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma)$ exists, see also the discussion in Section 2.2). On the other hand, similarly as in Step 3 of the proof of Proposition 3.12 (or directly from [9, Lemma 3.3]), if $\iota_0 \in \mathbb{N}$ is chosen large enough we have

$$(g^{\boldsymbol{b}^T})^{*,\mu}(c_2 h_{k\iota_0}) \geq c' \sum_{r=1}^{k} b_r \sigma^{-1}_{r\iota_0} h^{-\frac{1}{p_1}}_{r\iota_0}, \quad k \in \mathbb{N}, \tag{3.23}$$

with $c_2 \in (0,1]$. We apply Theorem 2.18 to $\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)$ and can thus estimate, also due to our hypothesis (3.20),

$$\begin{aligned}
\|\boldsymbol{b}|\ell_{q_1}\| &\geq c_1 \left\|g^{\boldsymbol{b}^T}|\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma)\right\| \\
&\geq c_3 \left\|g^{\boldsymbol{b}^T}|\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)\right\| \\
&\geq c_4 \left(\int_0^\varepsilon \left[\frac{(g^{\boldsymbol{b}^T})^{*,\mu}(t)}{\Psi^{\boldsymbol{\tau}}_{p_2,q_2'}(t)}\right]^{q_2} \mu_{\Psi^{\boldsymbol{\tau}}}(\mathrm{d}t)\right)^{1/q_2}
\end{aligned} \tag{3.24}$$

where $\Psi^{\boldsymbol{\tau}}_{p_2,q_2'}$ is given by (2.13) or (2.14) with $r = p_2$, $u = q_2'$ and $\boldsymbol{\sigma}$ replaced by $\boldsymbol{\tau}$, and $\mu_{\Psi^{\boldsymbol{\tau}}}$ stands for the corresponding Borel measure μ_{G} as explained before Definition 2.15.

Step 2. We first deal with the case $q_2 > 1$. We proceed from (3.24), where we can write, equivalently, the last expression as

$$\left(\int_0^\varepsilon \left[\frac{(g^{\boldsymbol{b}^T})^{*,\mu}(t)}{\Psi^{\boldsymbol{\tau}}_{p_2,q_2'}(t)^{q_2'}}\right]^{q_2} t^{-\frac{q_2'}{p_2}-1}\Lambda_{\boldsymbol{\tau}}(t^{-\frac{1}{n}})^{-q_2'}\mathrm{d}t\right)^{1/q_2}, \tag{3.25}$$

where $\Lambda_{\boldsymbol{\tau}}$ is the same as in (2.11) with $\boldsymbol{N} = \boldsymbol{h}^{-1}$ (see also [10, Rem. 4.15]). Using (3.23), the admissibility of the sequences $\boldsymbol{\tau}$ and $\boldsymbol{h}$, and discretising (2.13) (see

also [10, Lemma 2.6, Prop. 2.7]), (3.25) is bounded from below by

$$\Big(\sum_{k=k_0}^{\infty} \Big(\sum_{r=1}^{k} b_r \sigma_{r\iota_0}^{-1} h_{r\iota_0}^{-\frac{1}{p_1}} \Big)^{q_2} \Big(\sum_{r=1}^{k\iota_0} h_r^{-\frac{q_2'}{p_2}} \tau_r^{-q_2'} \Big)^{-q_2} h_{k\iota_0}^{-\frac{q_2'}{p_2}} \tau_{k\iota_0}^{-q_2'} \Big)^{\frac{1}{q_2}} \tag{3.26}$$

up to a constant factor, where $k_0 \in \mathbb{N}$ is chosen sufficiently large (depending on $\boldsymbol{h}$, ι_0 and ε). Observe now that (3.18) implies that there exists $k_1 > 1$ and $j_0 \in \mathbb{N}$ such that

$$\tau_{j+1}^{-1} h_{j+1}^{-\frac{1}{p_2}} \geq k_1 \, \tau_j^{-1} h_j^{-\frac{1}{p_2}}, \quad j \geq j_0, \tag{3.27}$$

from which follows, using also the admissibility of $\boldsymbol{\tau}$ and $\boldsymbol{h}$, that

$$\Big(\sum_{r=1}^{k\iota_0} h_r^{-\frac{q_2'}{p_2}} \tau_r^{-q_2'} \Big)^{-1} \sim h_{k\iota_0}^{\frac{q_2'}{p_2}} \tau_{k\iota_0}^{q_2'} \tag{3.28}$$

(cf. also Remark 1.19). Therefore, putting together (3.24)-(3.26), we can write

$$\|\boldsymbol{b}|\ell_{q_1}\| \geq c_5 \Big(\sum_{k=k_0}^{\infty} b_k^{q_2} \sigma_{k\iota_0}^{-q_2} h_{k\iota_0}^{-\frac{q_2}{p_1} + \frac{q_2'}{p_2}(q_2-1)} \tau_{k\iota_0}^{q_2'(q_2-1)} \Big)^{\frac{1}{q_2}}$$

$$= c_5 \Big(\sum_{k=k_0}^{\infty} b_k^{q_2} \sigma_{k\iota_0}^{-q_2} h_{k\iota_0}^{q_2(\frac{1}{p_2} - \frac{1}{p_1})} \tau_{k\iota_0}^{q_2} \Big)^{\frac{1}{q_2}}.$$

Let $\boldsymbol{\beta} = \boldsymbol{b}$, $\boldsymbol{\alpha} = \Big(\sigma_{k\iota_0}^{-1} h_{k\iota_0}^{\frac{1}{p_2} - \frac{1}{p_1}} \tau_{k\iota_0} \Big)_k$. Then Lemma 1.18 yields

$$\Big(\sigma_{k\iota_0}^{-1} h_{k\iota_0}^{\frac{1}{p_2} - \frac{1}{p_1}} \tau_{k\iota_0} \Big)_k \in \ell_{q^*}$$

and the result (3.21) follows by Remark 1.17.

Step 3. We turn to the case $0 < q_2 \leq 1$. Then $\Psi_{p_2, q_2'}^{\boldsymbol{\tau}} = \Psi_{p_2, \infty}^{\boldsymbol{\tau}}$ and a discretisation argument (see e.g. [10, Lemma 2.6, Prop. 2.7, Rem. 2.8]) gives us, with a small enough positive constant $C \leq c_2$,

$$\Psi_{p_2, \infty}^{\boldsymbol{\tau}}(Ch_k) \sim \sup_{1 \leq i \leq k} \tau_i^{-1} h_i^{-\frac{1}{p_2}}, \quad k \in \mathbb{N}.$$

On the other hand, similarly to (3.27) and (3.28), assumption (3.18) and the admissibility of $\boldsymbol{\tau}$ and $\boldsymbol{h}$ implies that

$$\sup_{1 \leq i \leq k} \tau_i^{-1} h_i^{-\frac{1}{p_2}} \sim \tau_k^{-1} h_k^{-\frac{1}{p_2}}, \quad k \in \mathbb{N}.$$

Denote by C_1 and C_2 positive constants such that

$$C_1 \tau_k^{-1} h_k^{-\frac{1}{p_2}} \leq \Psi_{p_2, \infty}^{\boldsymbol{\tau}}(Ch_k) \leq C_2 \tau_k^{-1} h_k^{-\frac{1}{p_2}}, \quad k \in \mathbb{N}. \tag{3.29}$$

Now we should be careful that ι_0 has been chosen large enough in such a way that also

$$\frac{\tau_{(k+1)\iota_0}^{-1} h_{(k+1)\iota_0}^{-\frac{1}{p_2}}}{\tau_{k\iota_0}^{-1} h_{k\iota_0}^{-\frac{1}{p_2}}} \geq 2\frac{C_2}{C_1}, \quad k \in \mathbb{N}. \tag{3.30}$$

This is clearly possible, due again to (3.18). We now discretise the right-hand side of (3.24) so that, up to a constant factor, it is bounded from below by

$$\Big(\sum_{k=k_0}^{\infty} \Big(\sum_{r=1}^{k} b_r \sigma_{r\iota_0}^{-1} h_{r\iota_0}^{-\frac{1}{p_1}}\Big)^{q_2} \tau_{k\iota_0}^{q_2} h_{k\iota_0}^{\frac{q_2}{p_2}} \mu_{\Psi^{\tau}} \big(\big[Ch_{(k+1)\iota_0}, Ch_{k\iota_0}\big]\big)\Big)^{\frac{1}{q_2}}. \tag{3.31}$$

Observing that

$$\mu_{\Psi^{\tau}} \big(\big[Ch_{(k+1)\iota_0}, Ch_{k\iota_0}\big]\big) = -\log \Psi_{p_2,\infty}^{\tau}(Ch_{k\iota_0}) + \log \Psi_{p_2,\infty}^{\tau}(Ch_{(k+1)\iota_0})$$

$$= \log \frac{\Psi_{p_2,\infty}^{\tau}(Ch_{(k+1)\iota_0})}{\Psi_{p_2,\infty}^{\tau}(Ch_{k\iota_0})}$$

$$\geq \log \frac{C_1 \tau_{(k+1)\iota_0}^{-1} h_{(k+1)\iota_0}^{-\frac{1}{p_2}}}{C_2 \tau_{k\iota_0}^{-1} h_{k\iota_0}^{-\frac{1}{p_2}}} \geq 1,$$

where we used first (3.29) and afterwards (3.30), we can further estimate (3.31) from below by

$$\Big(\sum_{k=k_0}^{\infty} b_k^{q_2} \sigma_{k\iota_0}^{-q_2} h_{k\iota_0}^{-\frac{q_2}{p_1}} \tau_{k\iota_0}^{q_2} h_{k\iota_0}^{\frac{q_2}{p_2}}\Big)^{\frac{1}{q_2}},$$

so that, returning to (3.24) we get

$$\|\boldsymbol{b}|\ell_{q_1}\| \geq c_5 \Big(\sum_{k=k_0}^{\infty} b_k^{q_2} \sigma_{k\iota_0}^{-q_2} h_{k\iota_0}^{q_2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)} \tau_{k\iota_0}^{q_2}\Big)^{\frac{1}{q_2}}.$$

We are now in the same situation as at the end of Step 2, so the last part of the proof follows as there. $\square$

Remark 3.29. Analysing the proof we see that in the case $q_2 > 1$ we could have managed with weaker hypotheses, namely replacing $\bar{\mathfrak{s}}\left(\boldsymbol{\tau}\boldsymbol{h}^{1/p_2}\right) < 0$, cf. (3.18), by $\boldsymbol{\tau}^{-1}\boldsymbol{h}^{-1/p_2} \notin \ell_{q_2'}$, cf. (3.22), and

$$\sum_{r=1}^{k\iota_0} h_r^{-\frac{q_2'}{p_2}} \tau_r^{-q_2'} \sim h_{k\iota_0}^{-\frac{q_2'}{p_2}} \tau_{k\iota_0}^{-q_2'}, \quad k \in \mathbb{N}_0,$$

cf. (3.28). In the argument presented above these two assertions are consequences of our assumption (3.18), but what is really needed in Step 2 are (3.22) and (3.28) only. Note that (3.28) is connected with requirements of strong isotropicity for the measure μ corresponding to h and Γ, as already pointed out in Remark 1.19.

In view of the fact that (3.19) and (2.15) immediately imply $\underline{\mathfrak{s}}(\boldsymbol{\tau}) > 0$, and (3.15) and (3.21) coincide for $p_1 \leq p_2$, we can combine Theorems 3.23 and 3.28 in that case.

Corollary 3.30. *Let Γ be an h-set satisfying the porosity condition with (2.15), $0 < p_1 \leq p_2 < \infty$, $0 < q_1, q_2 < \infty$, and q^* be given by (1.19). Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be admissible sequences with (3.18), (3.19). Then the trace spaces $\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma)$ and $\mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma)$ exist and*

$$\mathbb{B}^{\boldsymbol{\sigma}}_{p_1,q_1}(\Gamma) \hookrightarrow \mathbb{B}^{\boldsymbol{\tau}}_{p_2,q_2}(\Gamma),$$

if, and only if,

$$\boldsymbol{\sigma}^{-1}\boldsymbol{\tau}\boldsymbol{h}^{-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \in \ell_{q^*} . \tag{3.32}$$

Remark 3.31. Unlike in case of $p_1 \leq p_2$, there is an obvious gap for $p_1 > p_2$ between the sufficient condition (3.15) in Theorem 3.23 and the necessary one (3.21) in Theorem 3.28 in view of the embedding (3.16). We are not yet able to close it. But in view of similar observations in the classical case (embeddings of Besov spaces on a bounded domain Ω), or special cases studied before, we claim that (3.15) is the right one, not (3.21). In other words, we conjecture that Corollary 3.30 remains true irrespective of the relation between p_1 and p_2 if we replace (3.32) by (3.15).

We end this paper with explicating Corollary 3.30 in case of d- and (d, Ψ)-sets, recall also Remarks 1.9 and 2.11.

Example 3.32. Let $0 < d < n$, Γ be a d-set, $0 < p_1 \leq p_2 < \infty$, $0 < q_1, q_2 < \infty$, $s > 0$, $d\left(\frac{1}{p_2} - 1\right)_+ < t < \frac{d}{p_2}$. Then

$$\mathbb{B}^{s}_{p_1,q_1}(\Gamma) \hookrightarrow \mathbb{B}^{t}_{p_2,q_2}(\Gamma) \quad \text{if, and only if,} \quad \begin{cases} s - t \geq d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) & \text{if } q_1 \leq q_2, \\ s - t > d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) & \text{if } q_1 > q_2 . \end{cases}$$

Similarly, if we consider (d, Ψ)-sets Γ, $0 < d < n$, Ψ an admissible function, and corresponding trace spaces $\mathbb{B}^{(s,\Psi)}_{p,q}(\Gamma)$, recall Remark 2.11, we get for $0 < p_1 \leq p_2 < \infty$, $0 < q_1, q_2 < \infty$, $s > 0$, $d\left(\frac{1}{p_2} - 1\right)_+ < t < \frac{d}{p_2}$, that

$$\mathbb{B}^{(s,\Psi)}_{p_1,q_1}(\Gamma) \hookrightarrow \mathbb{B}^{(t,\Psi)}_{p_2,q_2}(\Gamma)$$

if, and only if,

$$\left(2^{-j(s-t-d(\frac{1}{p_1}-\frac{1}{p_2}))}\Psi(2^{-j})^{-(\frac{1}{p_1}-\frac{1}{p_2})}\right)_j \in \ell_{q^*}.$$

Here we have also used (1.17) again.

References

[1] A. Almeida, *Wavelet bases in generalized Besov spaces*, J. Math. Anal. Appl. **304** (2005), no. 1, 198–211.

[2] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Boston, 1988.

[3] N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge Univ. Press, Cambridge, 1987.

[4] M. Bricchi, *Compact embeddings between Besov spaces defined on h-sets*, Funct. Approx. Comment. Math. **30** (2002), no. 1, 7–36.

[5] _____, *Existence and properties of h-sets*, Georgian Math. J. **9** (2002), no. 1, 13–32.

[6] _____, *Tailored function spaces and related h-sets*, Ph.D. thesis, Friedrich-Schiller-Universität Jena, Germany, 2002.

[7] _____, *Complements and results on h-sets*, Function Spaces, Differential Operators and Nonlinear Analysis - The Hans Triebel Anniversary Volume (D.D. Haroske, Th. Runst, and H.J. Schmeißer, eds.), Birkhäuser, Basel, 2003, pp. 219–230.

[8] _____, *Tailored Besov spaces and h-sets*, Math. Nachr. **263-264** (2004), no. 1, 36–52.

[9] A.M. Caetano, *Growth envelopes of Besov spaces on fractal h-sets*, Math. Nachr. **286** (2013), no. 5-6, 550–568.

[10] A.M. Caetano and W. Farkas, *Local growth envelopes of Besov spaces of generalized smoothness*, Z. Anal. Anwendungen **25** (2006), 265–298.

[11] A.M. Caetano and D.D. Haroske, *Continuity envelopes of spaces of generalised smoothness: a limiting case; embeddings and approximation numbers*, J. Funct. Spaces Appl. **3** (2005), no. 1, 33–71.

[12] _____, *Traces for Besov spaces on fractal h-sets and dichotomy results*, submitted, 2014.

[13] A.M. Caetano and H.-G. Leopold, *On generalized Besov and Triebel-Lizorkin spaces of regular distributions*, J. Funct. Anal. **264** (2013), 2676–2703.

[14] A.M. Caetano and S. Lopes, *The fractal Dirichlet Laplacian*, Rev. Mat. Complut. **24** (2011), no. 1, 189–209.

[15] _____, *Spectral theory for the fractal Laplacian in the context of h-sets*, Math. Nachr. **284** (2011), no. 1, 5–38.

[16] A.M. Caetano and S.D. Moura, *Local growth envelopes of spaces of generalized smoothness: the critical case*, Math. Inequal. Appl. **7** (2004), no. 4, 573–606.

[17] _____, *Local growth envelopes of spaces of generalized smoothness: the subcritical case*, Math. Nachr. **273** (2004), 43–57.

[18] F. Cobos and D.L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, Function spaces and applications (M. Cwikel and J. Peetre, eds.), Lecture Notes in Mathematics, vol. 1302, Proc. US-Swed. Seminar held in Lund, June, 1986, Springer, 1988, pp. 158–170.

[19] R. DeVore and G.G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften, vol. 303, Springer, Berlin, 1993.

[20] D.E. Edmunds, R. Kerman, and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000), 307–355.

[21] D.E. Edmunds and H. Triebel, *Function spaces, entropy numbers, differential operators*, Cambridge Univ. Press, Cambridge, 1996.

[22] _____, *Spectral theory for isotropic fractal drums*, C. R. Acad. Sci. Paris **326** (1998), no. 11, 1269–1274.

[23] _____, *Eigenfrequencies of isotropic fractal drums*, Oper. Theory Adv. Appl. **110** (1999), 81–102.

[24] K.J. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press, Cambridge, 1985.

[25] W. Farkas and H.-G. Leopold, *Characterisation of function spaces of generalised smoothness*, Ann. Mat. Pura Appl. **185** (2006), no. 1, $1 - 62$.

[26] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge Univ. Press, Cambridge, 1952.

[27] D.D. Haroske, *Limiting embeddings, entropy numbers and envelopes in function spaces*, Habilitationsschrift, Friedrich-Schiller-Universität Jena, Germany, 2002.

[28] _____, *Envelopes and sharp embeddings of function spaces*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 437, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[29] D.D. Haroske and S.D. Moura, *Continuity envelopes and sharp embeddings in spaces of generalized smoothness*, J. Funct. Anal. **254** (2008), no. 6, 1487–1521.

[30] D.D. Haroske and C. Schneider, *Besov spaces with positive smoothness on $\mathbb{R}^n$, embeddings and growth envelopes*, J. Approx. Theory **161** (2009), no. 2, 723–747.

[31] A. Jonsson and H. Wallin, *Function spaces on subsets of $\mathbb{R}^n$*, Math. Rep. Ser. 2, No.1, xiv + 221 p., 1984.

[32] G.A. Kalyabin, *Criteria for the multiplicativity of spaces of Besov-Lizorkin- Triebel type and their imbedding into C*, Math. Notes **30** (1982), 750–755, Transl. from Mat. Zametki 30, 517-526 (1981).

[33] G.A. Kalyabin and P.I. Lizorkin, *Spaces of functions of generalized smoothness*, Math. Nachr. **133** (1987), 7–32.

[34] V. Knopova and M. Zähle, *Spaces of generalized smoothness on h-sets and related Dirichlet forms*, Studia Math. **174** (2006), no. 3, 277–308.

[35] P.I. Lizorkin, *Spaces of generalized smoothness*, Mir, Moscow (1986), 381–415, Appendix to Russian ed. of [45]; Russian.

[36] S. Lopes, *Besov spaces and the Laplacian on fractal h-sets*, Ph.D. thesis, Universidade de Aveiro, Portugal, 2009.

[37] P. Mattila, *Geometry of sets and measures in euclidean spaces*, Cambridge Univ. Press, Cambridge, 1995.

[38] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, Interpolation spaces and allied topics in analysis (M. Cwikel and J. Peetre, eds.), Lecture Notes in Mathematics, vol. 1070, Proc. Conf., Lund/Swed. 1983, Springer, 1984, pp. 183–201.

[39] S.D. Moura, *Function spaces of generalised smoothness*, Dissertationes Math. **398** (2001), 88 pp.

[40] _____, *Function spaces of generalised smoothness, entropy numbers, applications*, Ph.D. thesis, Universidade de Coimbra, Portugal, 2002.

[41] J.S. Neves, *Lorentz-Karamata spaces, Bessel and Riesz potentials and embeddings*, Dissertationes Math. **405** (2002), 46 pp.

[42] _____, *Spaces of Bessel-potential type and embeddings: the super-limiting case*, Math. Nachr. **265** (2004), 68–86.

[43] C.A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, London, 1970.

[44] W. Sickel and H. Triebel, *Hölder inequalities and sharp embeddings in function spaces of $B_{p,q}^s$ and $F_{p,q}^s$ type*, Z. Anal. Anwendungen **14** (1995), 105–140.

[45] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983, Reprint (Modern Birkhäuser Classics) 2010.

[46] _____, *Theory of function spaces II*, Birkhäuser, Basel, 1992, Reprint (Modern Birkhäuser Classics) 2010.

[47] _____, *Fractals and spectra*, Birkhäuser, Basel, 1997, Reprint (Modern Birkhäuser Classics) 2011.

[48] _____, *The structure of functions*, Birkhäuser, Basel, 2001.

[49] _____, *Theory of function spaces III*, Birkhäuser, Basel, 2006.

[50] A. Zygmund, *Trigonometric series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1977.

[1] Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

E-mail address: acaetano@ua.pt

[2] INSTITUTE OF MATHEMATICS, FRIEDRICH-SCHILLER-UNIVERSITY JENA, 07737 JENA, GERMANY

E-mail address: dorothee.haroske@uni-jena.de