Chapter

2D Elastostatic Problems in Parabolic Coordinates

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Abstract

In the present chapter, the boundary value problems are considered in a parabolic coordinate system. In terms of parabolic coordinates, the equilibrium equation system and Hooke's law are written, and analytical (exact) solutions of 2D problems of elasticity are constructed in the homogeneous isotropic body bounded by coordinate lines of the parabolic coordinate system. Analytical solutions are obtained using the method of separation of variables. The solution is constructed using its general representation by two harmonic functions. Using the MATLAB software, numerical results and constructed graphs of the some boundary value problems are obtained.

Keywords: parabolic coordinates, separation of variables, elasticity, boundary, value problem, harmonic function

1. Introduction

In order to solve boundary value and boundary-contact problems in the areas with curvilinear border, it is purposeful to examine such problems in the relevant curvilinear coordinate system. Namely, the problems for the regions bounded by a circle or its parts are considered in the polar coordinate system [1–4], while the problems for the regions bounded by an ellipse or its parts or hyperbola are considered in the elliptic coordinate system [5–13], and the problems for the regions with parabolic boundaries are considered in the parabolic coordinate system [14–16]. The problems for the regions bounded by the circles with different centers and radiuses are considered in the bipolar coordinate system [17–19]. For that purpose, first the governing differential equations are expressed in terms of the relevant curvilinear coordinates. Then a number of important problems involving the relevant curvilinear coordinates are solved.

The chapter consists of five paragraphs.

Many problems are very easily cast in terms of parabolic coordinates. To this end, first the governing differential equations discussed in present chapter are expressed in terms of parabolic coordinates; then two concrete (test) problems involving parabolic coordinates are solved.

The second section, following the Introduction, gives the equilibrium equations and Hooke's law written down in the parabolic coordinate system and the setting of boundary value problems in the parabolic coordinate system.

Section 3 considers the method used to solve internal and external boundary value problems of elasticity for a homogeneous isotropic body bounded by parabolic curves.

Section 4 solves the concrete problems, gains the numerical results, and constructs the relevant graphs.

Section 5 is a conclusion.

2. Problems statement

2.1 Equilibrium equations and Hooke's law in parabolic coordinates

It is known that elastic equilibrium of an isotropic homogeneous elastic body free of volume forces is described by the following differential equation [20]:

$$(\lambda + 2\mu)\mathrm{grad}\,\mathrm{div}\,\vec{U} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\vec{U} = 0 \tag{1}$$

where $\lambda = E\nu/[(1+\nu)(1-2\nu)]$, $\mu = E/[2(1-\nu)]$ are elastic Lamé constants; ν is the Poisson's ratio; E is the modulus of elasticity; and $\vec{U}$ is a displacement vector.

By projecting Eq. (1) onto the tangent lines of the curves of the parabolic coordinate system (see Appendix A), we obtain the system of equilibrium equations in the parabolic coordinates.

In the parabolic coordinate system, the equilibrium equations with respect to the function D, K, u, v and Hooke's law can be written as [20–22]:

$$\begin{aligned}
&(\mathrm{a})\, D_{,\xi} - K_{,\eta} = 0, \qquad (\mathrm{c})\, \overline{u}_{,\xi} + \overline{\mathrm{v}}_{,\eta} = (\kappa - 2)/(\kappa\mu)\cdot h_0^2 D,\\
&(\mathrm{b})\, D_{,\eta} + K_{,\xi} = 0, \qquad (\mathrm{d})\, \overline{\mathrm{v}}_{,\xi} - \overline{u}_{,\eta} = 1/\mu \cdot h_0^2 K.
\end{aligned} \tag{2}$$

$$\begin{aligned}
\sigma_{\eta\eta} &= h_0^{-1}\left[\lambda\overline{u}_{,\xi} + (\lambda+2\mu)\overline{\mathrm{v}}_{,\eta} + \left[(\lambda+\mu) - \mu h_0^{-2}\right](\xi\overline{u} + \eta\overline{\mathrm{v}})\right],\\
\sigma_{\xi\xi} &= h_0^{-1}\left[(\lambda+2\mu)\overline{u}_{,\xi} + \lambda\overline{\mathrm{v}}_{,\eta} + \left[(\lambda+\mu) + \mu h_0^{-2}\right](\xi\overline{u} + \eta\overline{\mathrm{v}})\right],\\
\tau_{\xi\eta} &= \mu h_0^{-1}\left[\left(\mathrm{v}_{,\xi} + u_{,\eta}\right) - h_0^{-2}(\xi\overline{\mathrm{v}} + \eta\overline{u})\right],
\end{aligned} \tag{3}$$

where $\kappa = 4(1-\nu)$, $\overline{u} = hu/c^2$, $\overline{\mathrm{v}} = h\mathrm{v}/c^2$, $h_0 = \sqrt{\xi^2 + \eta^2}$, $h = h_\xi = h_\eta = c\sqrt{\xi^2+\eta^2}$ are Lamé coefficients (see Appendix A), u, v are the components of the displacement vector $\vec{U}$ along the tangents of η, ξ curved lines, and c is the scale factor (see Appendix A). And in the present paper, we take $c = 1$, $(\kappa-2)/(\kappa\mu)\cdot D$ is the divergence of the displacement vector, K/μ is the rotor component of the displacement vector; $\sigma_{\xi\xi}$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta} = \tau_{\eta\xi}$ are normal and tangential stresses; and sub-indexes $()_{,\xi}$ and $()_{,\eta}$ denotes partial derivatives with relevant coordinates, for example, $K_{,\xi} = \frac{\partial K}{\partial \xi}$.

2.2 Boundary conditions

In the parabolic system of coordinates ξ, η $(-\infty < \xi < \infty,\ 0 \le \eta < \infty)$, exact solutions of two-dimensional static boundary value problems of elasticity are constructed for homogeneous isotropic bodies occupying domains bounded by coordinate lines of the parabolic coordinate system (see Appendix A).

The elastic body occupies the following domain (see **Figures 1** and **2**):

$$(\mathrm{a})\ D_1 = \{0 < \xi < \xi_1,\ 0 < \eta < \eta_1\},\ (\mathrm{b})\ D = \{-\xi_1 < \xi < \xi_1,\ 0 < \eta < \eta_1\}, \tag{4}$$

$$(\mathrm{a})\ \Omega_1 = \{0 < \xi < \xi_1,\ \eta_1 < \eta < \infty\},\ (\mathrm{b})\ \Omega = \{-\xi_1 \le \xi < \xi_1,\ \eta_1 \le \eta < \infty\}. \tag{5}$$

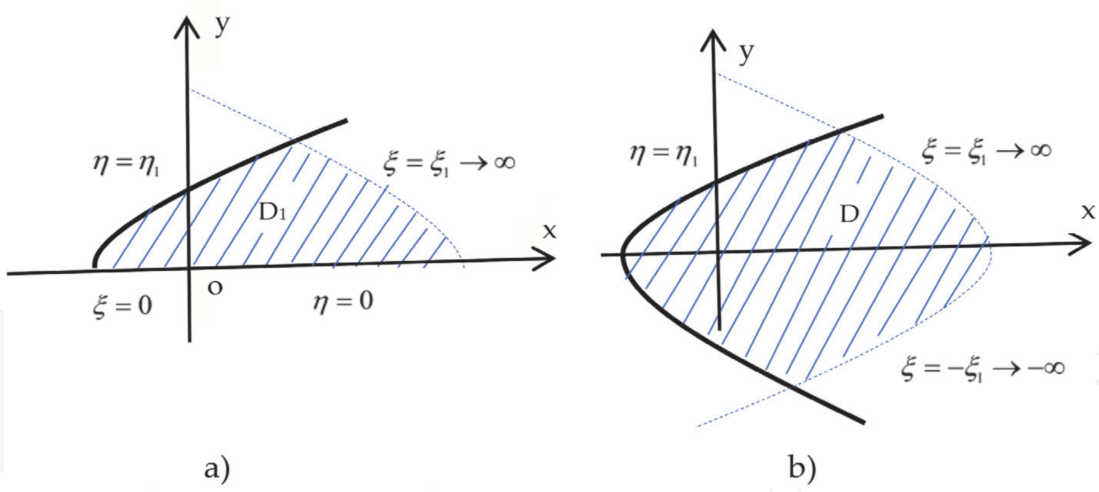

Figure 1.
(a) $D_1 = \{0 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ domain bounded by parabolic curve $\eta = \eta_1$ and line y = 0 and (b) $D = \{-\xi_1 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ domain bounded by parabola $\eta = \eta_1$.

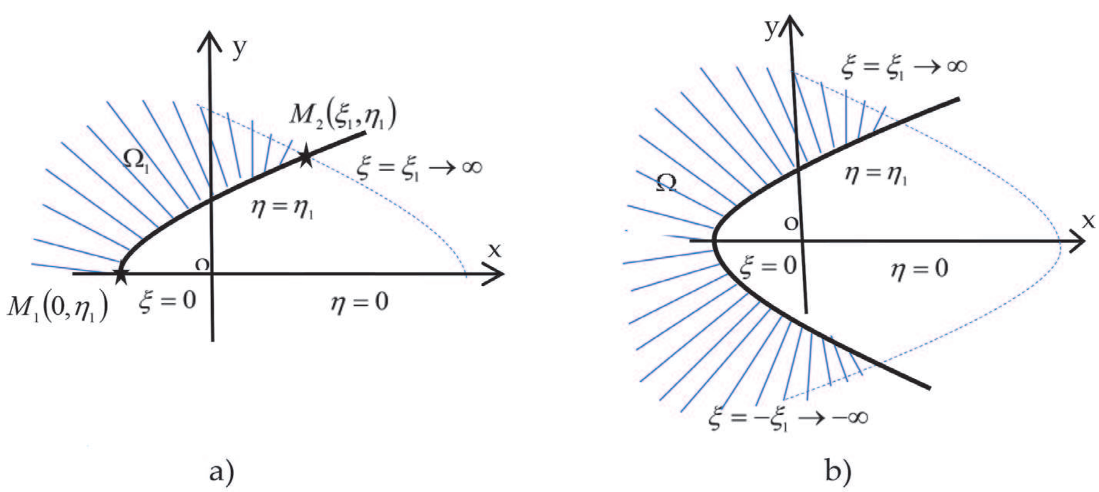

Figure 2.
Infinite domain (a) $\Omega_1 = \{0 < \xi < \xi_1,\ \eta_1 < \eta < \infty\}$ bounded by parabolic curve $\eta = \eta_1$ and line y = 0 and (b) $\Omega = \{-\xi_1 < \xi < \xi_1,\ \eta_1 < \eta < \infty\}$ bounded by parabola $\eta = \eta_1$.

Boundary conditions that appear in the chapter have the following form:

$$\text{for}\quad \xi = \xi_1: \quad (\text{a})\, \sigma_{\xi\xi} = F^1{}_{(i)}(\eta), \quad \tau_{\xi\eta} = F_2^{(i)}(\eta) \quad \text{or} \quad (\text{b})\, u = G^1{}_{(i)}(\eta), \quad \text{v} = G_2^{(i)}(\eta), \tag{6}$$

$$\text{for}\quad \eta = \eta_1: \quad (\text{a})\, \sigma_{\eta\eta} = Q^1{}_{(i)}(\xi), \ \tau_{\xi\eta} = Q_2^{(i)}(\xi) \quad \text{or} \quad (\text{b})\, u = H^1{}_{(i)}(\xi), \quad \text{v} = H_2^{(i)}(\xi), \tag{7}$$

$$\text{for}\quad \xi = 0: \quad (\text{a})\, \text{v} = 0, \quad \sigma_{\xi\xi} = 0 \quad \text{or} \quad (\text{b})\, u = 0, \quad \tau_{\xi\eta} = 0, \tag{8}$$

$$\text{for}\, \eta = 0: \quad (\text{a})\, u = 0, \quad \sigma_{\eta\eta} = 0, \quad \text{or} \quad (\text{b})\, \text{v} = 0, \quad \tau_{\xi\eta} = 0, \tag{9}$$

$$\text{for}\ \xi_1 \to \pm\infty: \quad \sigma_{\eta\eta} \to 0, \quad \tau_{\xi\eta} \to 0, \quad u \to 0, \quad \text{v} \to 0. \tag{10}$$

$$\text{for}\ \eta \to \infty: \quad \sigma_{\eta\eta} \to 0, \quad \tau_{\xi\eta} \to 0, \quad u \to 0, \quad \text{v} \to 0, \tag{10a}$$

where F_i, Q_i $(i = 1, 2)$ with the first derivative and G_i, H_i with the first and second derivatives can be decomposed into the trigonometric absolute and uniform convergent Fourier series.

Boundary conditions on the linear parts $\xi = 0$ and $\eta = 0$ of the consideration area enable us to continue the solutions continuously (symmetrically or anti-symmetrically) in the domain, that is, the mirror reflection of the consideration area in a relationship $y = 0$ line (see **Figures 1b** and **2b**).

3. Solution of stated boundary value problems

In this section we will be considered internal and external problems for a homogeneous isotropic body bounded by parabolic curves.

3.1 Interior boundary value problems

Let us find the solution of problems (2), (3), (4a) (see **Figure 1a**), and (7)–(10) in class $C^2(D)$ (for D area shown in **Figure 1b**). The solution is presented by two harmonious φ_1 and φ_2 functions (see Appendix B). From formulas (B11)–(B13), after inserting $\alpha = \eta_1$ and making simple transformations, we will obtain:

$$\begin{aligned}
\overline{u} &= -\left[\eta\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right) + (\kappa-1)\varphi\right]\xi + \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right) - (\kappa-1)\varphi_2\right]\eta, \\
\overline{\mathrm{v}} &= \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right) + (\kappa-1)\varphi_1\right]\eta + \left[\eta\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right) - (\kappa-1)\varphi_2\right]\xi;
\end{aligned} \tag{11}$$

$$D = \frac{\kappa\mu}{h_0^2}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\eta - \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\xi\right], \quad K = \frac{\kappa\mu}{h_0^2}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\xi + \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\eta\right],$$

where

$$\frac{1}{h^2}\left(\varphi_{i,\xi\xi} + \varphi_{i,\eta\eta}\right) = 0, \quad i = 1, 2. \tag{12}$$

The stress tensor components can be written as

$$\begin{aligned}
\frac{h_0^2}{2\mu}\sigma_{\eta\eta} &= -\left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta}\right) - \frac{\kappa}{2}\varphi_{1,\eta} - \frac{\kappa-2}{2}\varphi_{2,\xi}\right]\eta \\
&\quad + \left[\eta\left(\varphi_{1,\xi\eta} - \varphi_{2,\eta\eta}\right) + \frac{\kappa-2}{2}\varphi_{1,\xi} - \frac{\kappa}{2}\varphi_{2,\eta}\right]\xi \\
&\quad - \frac{\eta_1^2 - \eta}{\xi^2 + \eta^2}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\eta - \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\xi\right], \\
\frac{h_0^2}{2\mu}\tau_{\xi\eta} &= \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi}\right) + \frac{\kappa-2}{2}\varphi_{1,\xi} - \frac{\kappa}{2}\varphi_{2,\eta}\right]\eta \\
&\quad + \left[\eta\left(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta}\right) - \frac{\kappa}{2}\varphi_{1,\eta} - \frac{\kappa-2}{2}\varphi_{2,\xi}\right]\xi \\
&\quad - \frac{\eta_1^2 - \eta}{\xi^2 + \eta^2}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\xi + \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\eta\right], \\
\frac{h_0^2}{2\mu}\sigma_{\xi\xi} &= \left[\frac{\eta_1^2}{\eta}\left(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta}\right) - \frac{\kappa-4}{2}\varphi_{1,\eta} - \frac{\kappa+2}{2}\varphi_{2,\xi}\right]\eta \\
&\quad - \left[\eta\left(\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi}\right) + \frac{\kappa+2}{2}\varphi_{1,\xi} - \frac{\kappa-4}{2}\varphi_{2,\eta}\right]\xi \\
&\quad + \frac{\eta_1^2 - \eta}{\xi^2 + \eta^2}\left[\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right)\eta - \left(\varphi_{1,\xi} + \varphi_{2,\eta}\right)\xi\right].
\end{aligned} \tag{13}$$

From (12) by the separation of variables method, we obtain (see Appendix A)

$$\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \quad i = 1, 2, \tag{14}$$

where

$$\varphi_{1n} == A_{1n} \cosh(n\eta) \cos(n\xi), \qquad \varphi_{2n} == A_{2n} \sinh(n\eta) \sin(n\xi)$$

or

$$\varphi_{1n} == A_{1n} \sinh(n\eta) \sin(n\xi), \qquad \varphi_{2n} == A_{2n} \cosh(n\eta) \cos(n\xi).$$

For $n = 0$: $\varphi_{10} = A_{10} + a_{02}\xi + +a_{03}\eta + a_{04}\xi\eta$, $\varphi_{20} = A_{20} + b_{02}\xi + b_{03}\eta + b_{04}\xi\eta$, where $A_{10}, a_{02}, \dots, b_{04}$ are constant coefficients. When $n = 0$ and $0 < \xi < \xi_1$, then the terms ξ, η and $\xi\eta$ will not be contained in φ_{10} and φ_{20}. If the foregoing solutions are presented in expressions of φ_{10} and φ_{20}, then it would be impossible on $\xi = \xi_1$ to satisfy the boundary conditions, and grad $\varphi_{i0} = \left(\varphi_{i0,\xi} + \varphi_{i0,\eta}\right)/h$ $(i = 1, 2)$ will not be bounded in the point $M(0, 0)$.

Provision. We are introducing the following assumptions:

1. ξ_1 is a sufficiently great positive number (see Appendix C).
2. The boundary conditions given on $\eta = \eta_1$, i.e., stresses or displacements equal zero at interval $\tilde{\xi}_1 < \xi < \xi_1$.
3. When stresses are given on $\eta = \eta_1$, the main vector and main moment equal zero.

It is clear that

$$D = \kappa\left(\sigma_{\xi\xi} + \sigma_{\eta\eta}\right)/4, \sigma_{\xi\xi} = 4D/\kappa - \sigma_{\eta\eta}.$$

By ultimately opening expressions $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ (in details), we can demonstrate that at point $M(0, 0)$, $\sigma_{\eta\eta}$ and $\tau_{\xi\eta}$ (and naturally, $\sigma_{\xi\xi}$, too) are determined, i.e., they are finite.

When at $\eta = \eta_1$ $\overline{u}$ and $\overline{\mathrm{v}}$ are given, then it is expedient to take instead of them as their equivalent the following expressions:

$$\begin{aligned} \frac{1}{h_0^2}\left(\overline{u} \cdot \eta_1 + \overline{\mathrm{v}} \cdot \xi\right) &= \eta_1\left(\varphi_{1,\xi} + \varphi_{2,\eta}\right) - (\kappa - 1)\varphi_2, \\ \frac{1}{h_0^2}\left(\overline{u} \cdot \xi - \overline{\mathrm{v}} \cdot \eta_1\right) &= \eta_1\left(\varphi_{1,\eta} - \varphi_{2,\xi}\right) + (\kappa - 1)\varphi_1, \end{aligned} \tag{15}$$

and if at $\eta = \eta_1$ $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$ are given, then instead of them we have to take their equivalent following expressions:

$$\begin{aligned} \frac{1}{2\mu}\left(\sigma_{\eta\eta} \cdot \eta_1 - \sigma_{\xi\eta} \cdot \xi\right) &= -\eta_1\left(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta}\right) - \frac{\kappa}{2}\varphi_{1,\eta} - \frac{\kappa - 2}{2}\varphi_{2,\xi}, \\ \frac{1}{2\mu}\left(\sigma_{\eta\eta} \cdot \xi + \sigma_{\xi\eta} \cdot \eta_1\right) &= \eta_1\left(\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi}\right) + \frac{\kappa - 2}{2}\varphi_{1,\xi} - \frac{\kappa}{2}\varphi_{2,\eta}. \end{aligned} \tag{16}$$

Considering the homogeneous boundary conditions of the concrete problem, we will insert φ_1 and φ_2 functions selected from the (14) in the right sides of (15) or

(16), and we will expand the left sides in the Fourier series. In both sides expressions which are with identical combinations of trigonometric functions will equate to each other and will receive the infinite system of linear algebraic equations to unknown coefficients A_{1n} and A_{2n} of harmonic functions, with its main matrix having a block-diagonal form. The dimension of each block is 2×2, and determinant is not equal to zero, but in infinite the determinant of block strives to the finite number different to zero.

It is very easy to establish the convergence of (11) and (13) functional series on the area $\overline{D} = \{-\xi_1 \le \xi \le \xi_1,\ 0 \le \eta \le \eta_1\}$ by construction of the corresponding uniform convergent numerical majorizing series. So we have the following:

Proposal 1. The functional series corresponding to (11) and (13) are absolute and uniform by convergent series on the area $\overline{D} = \{-\xi_1 \le \xi \le \xi_1,\ 0 \le \eta \le \eta_1\}$.

3.2 Exterior boundary value problems

We have to find the solution of problems (2), (3), (5a) (see **Figure 2a**), (7), (8), (10), and (10′), which belongs to the class $C^2(\Omega)$ (see region Ω on **Figure 2b**). The solution is constructed using its general representation by harmonic functions φ_1, φ_2 (see Appendix B). From formulas (B11)–(B13), following inserting $\alpha = \eta_1$ and simple transformations, we obtain the following expressions:

$$\overline{u} = -\left[(\varphi_{1,\xi} + \varphi_{2,\eta})\eta_1 + (\varphi_{1,\eta} - \varphi_{2,\xi})\xi\right](\eta - \eta_1) - \left[(\kappa - 1)\varphi_1 + \varphi_{3,\eta}\right]\xi - \left[(\kappa - 1)\varphi_2 - \varphi_{3,\xi}\right]\eta,$$

$$\overline{\mathrm{v}} = \left[(\varphi_{1,\xi} + \varphi_{2,\eta})\xi - (\varphi_{1,\eta} - \varphi_{2,\xi})\eta_1\right](\eta - \eta_1) + \left[(\kappa - 1)\varphi_1 + \varphi_{3,\eta}\right]\eta - \left[(\kappa - 1)\varphi_2 - \varphi_{3,\xi}\right]\xi, \tag{17}$$

$$D = \frac{\kappa\mu}{h_0^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\eta - (\varphi_{1,\xi} + \varphi_{2,\eta})\xi\right],\ K = \frac{\kappa\mu}{h_0^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\xi + (\varphi_{1,\xi} + \varphi_{2,\eta})\eta\right],$$

where

$$\frac{1}{h^2}\left(\varphi_{i,\xi\xi} + \varphi_{i,\eta\eta}\right) = 0, \quad i = 1, 2, 3. \tag{18}$$

The stress tensor components can be written as:

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\eta\eta} = {} & \left[(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta})\eta_1 + (\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi})\xi\right](\eta - \eta_0) \\ & + \left(\frac{\kappa}{2}\varphi_{1,\eta} + \frac{\kappa - 2}{2}\varphi_{2,\xi} - \varphi_{3,\xi\xi}\right)\eta \\ & + \left(\frac{\kappa - 2}{2}\varphi_{1,\xi} - \frac{\kappa}{2}\varphi_{2,\eta} + \varphi_{3,\xi\eta}\right)\xi \\ & + \frac{\eta^2 - \eta_1^2}{\xi^2 + \eta^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\eta - (\varphi_{1,\xi} + \varphi_{2,\eta})\xi\right],\end{aligned}$$

$$\begin{aligned}\frac{h_0^2}{2\mu}\tau_{\xi\eta} = {} & \left[(\varphi_{1,\xi\xi} + \varphi_{2,\xi\eta})\xi - (\varphi_{1,\xi\eta} - \varphi_{2,\xi\xi})\eta_{\cdot 1}\right](\eta - \eta_1) - \left(\frac{\kappa}{2}\varphi_{1,\eta} + \frac{\kappa - 2}{2}\varphi_{2,\xi} - \varphi_{3,\xi\xi}\right)\xi \\ & + \left(\frac{\kappa - 2}{2}\varphi_{1,\xi} - \frac{\kappa}{2}\varphi_{2,\eta} + \varphi_{3,\xi\eta}\right)\eta + \frac{\eta^2 - \eta_1^2}{\xi^2 + \eta^2}\left[(\varphi_{1,\eta} - \varphi_{2,\xi})\xi - (\varphi_{1,\xi} + \varphi_{2,\eta})\eta\right],\end{aligned} \tag{19}$$

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\xi\xi} = &-\left[\left(\varphi_{1,\xi\xi}+\varphi_{2,\xi\eta}\right)\eta_1+\left(\varphi_{1,\xi\eta}-\varphi_{2,\xi\xi}\right)\xi\right](\eta-\eta_0)\\ &-\left(\frac{\kappa-4}{2}\varphi_{1,\eta}+\frac{\kappa+2}{2}\varphi_{2,\xi}-\varphi_{3,\xi\xi}\right)\eta-\left(\frac{\kappa+2}{2}\varphi_{1,\xi}-\frac{\kappa-2}{2}\varphi_{2,\eta}+\varphi_{3,\xi\eta}\right)\xi\\ &-\frac{\eta^2-\eta_1^2}{\xi^2+\eta^2}\left[\left(\varphi_{1,\eta}-\varphi_{2,\xi}\right)\eta-\left(\varphi_{1,\xi}+\varphi_{2,\eta}\right)\xi\right]\end{aligned}$$

If $\overline{u}$ and $\overline{v}$ are given for $\eta=\eta_1$, then we take $\varphi_3=0$, and when $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$ is given for $\eta=\eta_1$, then $\varphi_3=\frac{\kappa-2}{2}\int\varphi_2 d\xi$.

From (18), by the separation of variables method, we obtain

$$\varphi_i=\sum_{n=1}^{\infty}\varphi_{in},\quad i=1,2,3, \tag{20}$$

where

$$\varphi_{1n}=B_{1n}e^{-n\eta}\sin(n\xi),\qquad \varphi_{2n}=B_{2n}e^{-n\eta}\cos(n\xi),\qquad \varphi_{3n}=\frac{\kappa-2}{2n}B_{2n}e^{-n\eta}\sin(n\xi)$$

or

$$\varphi_{1n}=B_{1n}e^{-n\eta}\cos(n\xi),\qquad \varphi_{2n}=B_{2n}e^{-n\eta}\sin(n\xi),\qquad \varphi_{3n}=-\frac{\kappa-2}{2n}B_{2n}e^{-n\eta}\cos(n\xi).$$

When $n=0$, then $\varphi_{10}=A_{10}+a_{02}\xi+a_{03}\eta+a_{04}\xi\eta$, $\varphi_{20}=A_{20}+b_{02}\xi+b_{03}\eta+b_{04}\xi\eta$, where $A_{10}, a_{02}, \dots, b_{04}$ are constants. From limited of functions φ_{i0} $(i=1,2)$ in $\eta\to\infty$ and satisfying boundary condition for $\xi=\xi_1$, it implies that $a_{02}=0$, $b_{02}=0$, $a_{03}=0$, $b_{03}=0$, $a_{04}=0$, $b_{04}=0$. Therefore, $\varphi_{10}=0$, $\varphi_{20}=A_{20}$ or $\varphi_{10}=A_{10}$, $\varphi_{20}=0$.

Provision. As in the previous subsection we make the following assumptions:

- ξ_1 is a sufficiently large positive number (see Appendix C).
- At $\eta=\eta_1$ given boundary conditions, i.e., displacements or stresses on interval $\tilde{\xi}_1<\xi<\xi_1$, will equal zero.
- When stresses are given on $\eta=\eta_1$, the main vector and main moment will equal zero.

When $\overline{u}$ and $\overline{v}$ are given at $\eta=\eta_1$, then instead of them, it is expedient to take the following expressions as their equivalent:

$$\frac{1}{h_0^2(\kappa-1)}\left(\overline{u}\xi-\overline{v}\eta_1\right)=\varphi_1,\qquad -\frac{1}{h_0^2(\kappa-1)}\left(\overline{u}\eta_1+\overline{v}\xi\right)=\varphi_2, \tag{21}$$

and if at $\eta=\eta_1$ $\frac{h_0^2}{2\mu}\sigma_{\eta\eta}$ and $\frac{h_0^2}{2\mu}\sigma_{\xi\eta}$ are given, then instead of them we have to take the following expressions as their equivalent:

$$\begin{aligned}&\frac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\eta_1-\sigma_{\xi\eta}\cdot\xi\right)=\frac{\kappa}{2}\varphi_{1,\eta},\\ &\frac{1}{2\mu}\left(\sigma_{\eta\eta}\cdot\xi+\sigma_{\xi\eta}\cdot\eta_1\right)=\frac{\kappa-2}{2}\varphi_{1,\xi}-\varphi_{2,\eta}.\end{aligned} \tag{22}$$

Just like that in the previous subsection, considering the homogeneous boundary conditions of the concrete problem, we will insert φ_1 and φ_2 functions selected from (20) in Eq. (21) or (22), and we will expand the left sides in the Fourier series. Both sides of the expressions, which show the identical combinations of trigonometric functions, will equate to each other and will receive the infinite system of linear algebraic equations to unknown coefficients A_{1n} and A_{2n} of harmonic functions, with its main matrix having a block-diagonal form. The dimension of each block is 2×2, and the determinant does not equate to zero, but in the infinity, the determinant of block tends to the finite number different from zero.

As in the previous subsection, we received the following:

Proposition 2. The functional series corresponding to (17) and (19) are absolute and a uniformly convergent series on region $\overline{\Omega} = \{-\xi_1 \le \xi \le \xi_1,\ \eta_1 \le \eta < \infty\}$.

4. Test problems

In this section we will be obtained numerical results of internal and external problems for a homogeneous isotropic body bounded by parabolic curves when normal stress distribution is applied to the parabolic border.

4.1 Internal problem

We will set and solve the concrete internal boundary value problem in stresses. Let us find the solution of equilibrium equation system (2) of the homogeneous isotropic body in the area $\Omega_1 = \{0 < \xi < \xi_1,\ 0 < \eta < \eta_1\}$ (see **Figure 1a**), which satisfies boundary conditions (7a), (8a), (9a), and (10).

From (14), (8a), and (9a)

$$\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \quad i = 1, 2, \tag{23}$$

where $\varphi_{1n} == A_{1n} \sinh(n\eta)\sin(n\xi)$, $\varphi_{2n} == A_{2n}\cosh(n\eta)\cos(n\xi)$.

By inserting (23) in (11) and (13), we will receive the following expressions for the displacements:

$$\begin{aligned}\overline{u} &= \sum_{n=1}^{\infty}\{-[n\eta\xi\cosh(n\eta)(A_{1n}+A_{2n}) + (\kappa-1)\xi\sinh(n\eta)A_{1n}]\sin(n\xi) \\ &\quad + \left[n\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n}) - (\kappa-1)\eta\cosh(n\eta)A_{2n}\right]\cos(n\xi)\}, \\ \overline{\mathrm{v}} &= \sum_{n=1}^{\infty}\{\left[n\eta_1^2\cosh(n\eta)(A_{1n}+A_{2n}) + (\kappa-1)\eta\sinh(n\eta)A_{1n}\right]\sin(n\xi) \\ &\quad + [n\eta\xi\sinh(n\eta)(A_{1n}+A_{2n}) - (\kappa-1)\xi\cosh(n\eta)A_{2n}]\cos(n\xi)\},\end{aligned} \tag{24}$$

but for the stresses the following:

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\eta\eta} = \sum_{n=1}^{\infty}\Bigg\{&\left[n^2\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n}) + n\eta\cosh(n\eta)\left(\frac{\kappa}{2}A_{1n} - \frac{\kappa-2}{2}A_{2n}\right)\right]\sin(n\xi) \\ &+ \left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n}) + n\xi\sinh(n\eta)\left(\frac{\kappa-2}{2}A_{1n} - \frac{\kappa}{2}A_{2n}\right)\right]\cos(n\xi) \\ &- \frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi) - n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)]\Bigg\},\end{aligned}$$

$$\frac{h_0^2}{2\mu}\tau_{\xi\eta}=\sum_{n=1}^{\infty}\left\{\left[n^2\eta_1^2\cosh(n\eta)(A_{1n}+A_{2n})+n\eta\sinh(n\eta)\left(\frac{\kappa-2}{2}A_{1n}-\frac{\kappa}{2}A_{2n}\right)\right]\cos(n\xi)\right.$$
$$-\left[n^2\eta\xi\sinh(n\eta)(A_{1n}+A_{2n})+n\xi\cosh(n\eta)\left(\frac{\kappa}{2}A_{1n}-\frac{\kappa-2}{2}A_{2n}\right)\right]\sin(n\xi)$$
$$\left.-\frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\xi\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)+n\eta\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\right\}, \tag{25}$$

$$\frac{h_0^2}{2\mu}\sigma_{\xi\xi}=\sum_{n=1}^{\infty}\left\{-\left[n^2\eta_1^2\sinh(n\eta)(A_{1n}+A_{2n})+n\eta\cosh(n\eta)\left(\frac{\kappa-4}{2}A_{1n}-\frac{\kappa+2}{2}A_{2n}\right)\right]\sin(n\xi)\right.$$
$$-\left[n^2\eta\xi\cosh(n\eta)(A_{1n}+A_{2n})+n\xi\sinh(n\eta)\left(\frac{\kappa+2}{2}A_{1n}-\frac{\kappa-4}{2}A_{2n}\right)\right]\cos(n\xi)$$
$$\left.+\frac{\eta_1^2-\eta^2}{\xi^2+\eta^2}\left[n\eta\cosh(n\eta)(A_{1n}+A_{2n})\sin(n\xi)-n\xi\sinh(n\eta)(A_{1n}+A_{2n})\cos(n\xi)\right]\right\}.$$

We have to solve problem (2), (7a), (8a), and (9a) when $Q_1(\xi)=P$ and $Q_2(\xi)=0$, i.e., at $\eta=\eta_1$ boundary the normal load $\frac{1}{2\mu}\sigma_{\eta\eta}=\frac{P}{h_0^2}$ is given, but tangent stress is equal to zero. From (16), and (23), we obtain the following equations:

$$\sum_{n=1}^{\infty}\left[n^2\eta_1\sinh(n\eta_1)(A_{1n}+A_{2n})-n\cosh(n\eta_1)\left(\frac{\kappa}{2}A_{1n}-\frac{\kappa-2}{2}A_{2n}\right)\right]\sin(n\xi)=\frac{P\eta_1}{\xi^2+\eta_1^2},$$
$$\sum_{n=1}^{\infty}\left[n^2\eta_1\cosh(n\eta_1)(A_{1n}+A_{2n})+n\sinh(n\eta_1)\left(\frac{\kappa-2}{2}A_{1n}-\frac{\kappa}{2}A_{2n}\right)\right]\cos(n\xi)=\frac{P\xi}{\xi^2+\eta_1^2}.$$

From here an infinite system of the linear algebraic equations with unknown A_{1n} and A_{2n} coefficients is obtained:

$$\begin{gathered}\left[\left(n^2\eta_1\sinh(n\eta_1)-n\frac{\kappa}{2}\cosh(n\eta_1)\right)A_{1n}\right.\\ \left.+\left(n^2\eta_1\sinh(n\eta_1)+n\frac{\kappa-2}{2}\cosh(n\eta_1)\right)A_{2n}\right]=\tilde{F}_{1n},\\ \left[\left(n^2\eta_1\cosh(n\eta_1)+n\frac{\kappa-2}{2}\sinh(n\eta_1)\right)A_{1n}\right.\\ \left.+\left(n^2\eta_1\cosh(n\eta_1)-n\frac{\kappa}{2}\sinh(n\eta_1)\right)A_{2n}\right]=\tilde{F}_{2n},\quad n=1,2,\dots\end{gathered} \tag{26}$$

where $\tilde{F}_{1n}$ and $\tilde{F}_{2n}$ are the coefficients of expansion into the Fourier series $f_1(\xi)=\sum_{n=1}^{\infty}\tilde{F}_{1n}\sin(n\xi)$ and $f_2(\xi)=\sum_{n=1}^{\infty}\tilde{F}_{2n}\cos(n\xi)$, respectively, $f_1(\xi)=\frac{P\eta_1}{\xi^2+\eta_1^2}$ and $f_2(\xi)=\frac{P\xi}{\xi^2+\eta_1^2}$ functions.

As seen, the main matrix of system (26) has a block-diagonal form, dimension of each block is 2×2. Thus, two equations with two A_{1n} and A_{2n} unknown values will be solved. After solving this system, we find A_{1n} and A_{2n} coefficients, and in putting them into formulas (24) and (25), we get displacements and stresses at any points of the body.

Numerical values of displacements and stresses are obtained at the points of the finite size region bounded by curved lines $\eta=\eta_1$ and $\xi=\xi_1$ (see **Figure 1a**), and relevant 3D graphs are drafted. The numerical results are obtained for the following data: $\nu = 0.3$, $E = 2 \times 10^6 \text{kg/cm}^2$, $P = -10\,\text{kg/cm}^2$, $0.1 \le \eta_1 \le 3$, $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, and $\xi_1 = 6 * \pi$. Numerical calculations and the visual presentation are made by MATLAB software.

Figures 3 and **4** show the distribution of stresses and displacements in the region bounded by curved lines $\eta = \eta_1$ and $\xi = \xi_{1k} := \xi_1$ (see **Figure 1a**), when (7a), (8a), and (9a) boundary conditions are valid and normal stress is applied to the parabolic boundary. Following conditions (8a) and (9a), at points of the linear parts $\xi = 0$ and $\eta = 0$ of consideration area $\sigma_{\xi\xi}(0, \eta)$, $\sigma_{\eta\eta}(\xi, 0)$ stresses and $u(\xi, 0)$, $\mathrm{v}(0, \eta)$ displacements equal zero which is seen in **Figures 3** and **4**.

4.2 External problem

We will set and solve the concrete external boundary value problem in stresses. Let us find the solution of equilibrium equation system (2) of the homogeneous

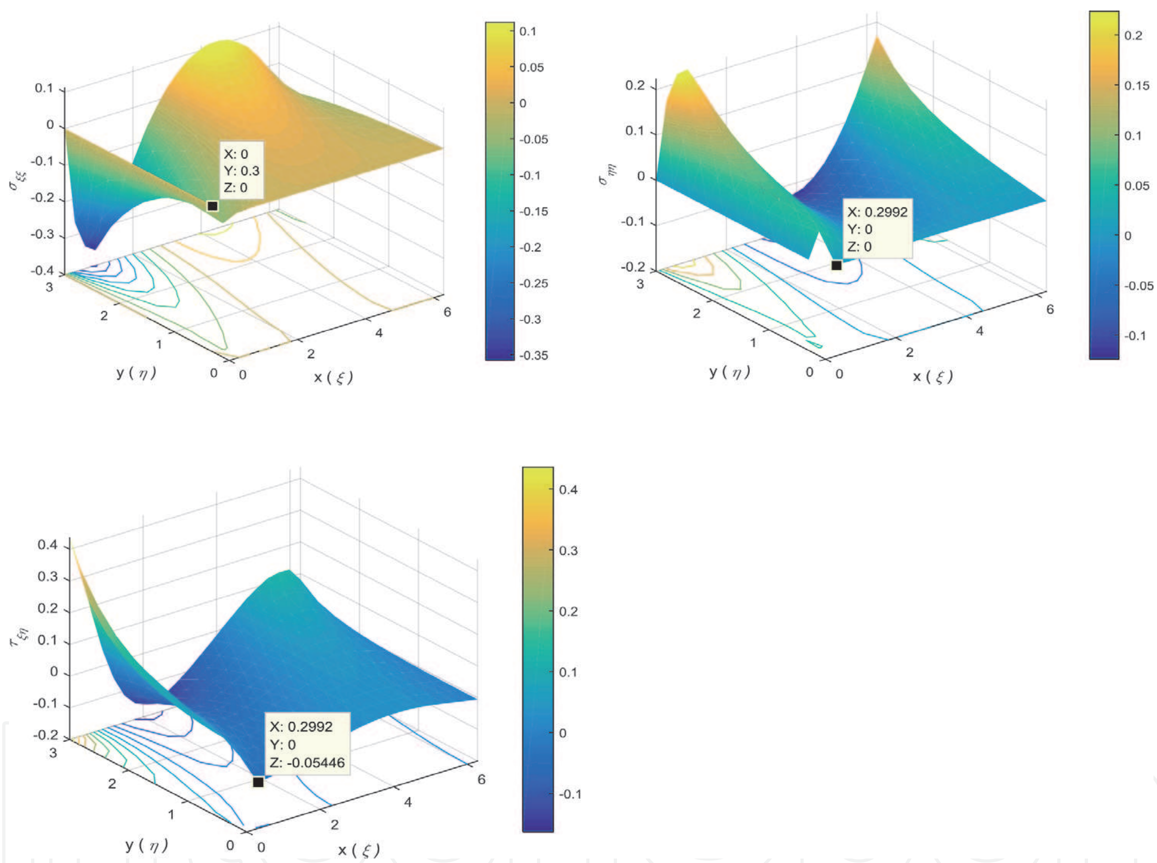

Figure 3.
Distribution of stresses in the region bounded by curved lines $\eta = \eta_1$ and $\xi = \xi_1$.

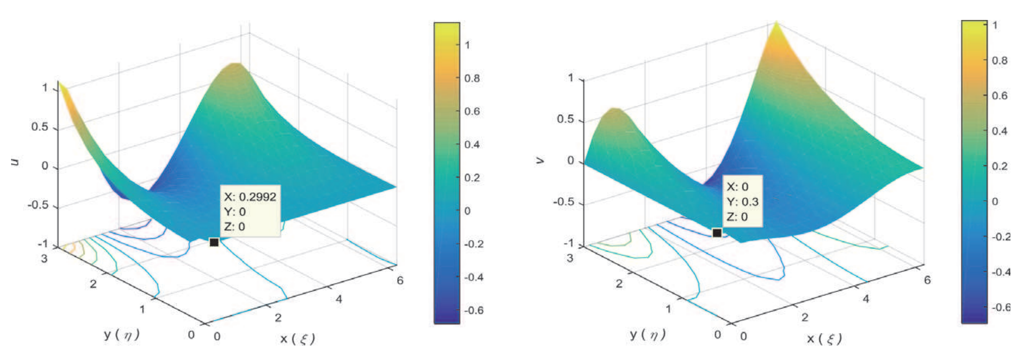

Figure 4.
Distribution of displacements in the region bounded by curved lines $\eta = \eta_1$ and $\xi = \xi_1$.

isotropic body in the region $\Omega_1 = \{0 < \xi < \xi_1, \eta_1 < \eta < \infty\}$, which satisfies the following boundary conditions: (7a), (8a), (10), and (10′).

From (20) and (8a)

$$\varphi_i = \sum_{n=1}^{\infty} \varphi_{in}, \quad i = 1, 2, 3, \tag{27}$$

where $\varphi_{1n} = B_{1n}e^{-n\eta}\sin(n\xi)$, $\varphi_{2n} = B_{2n}e^{-n\eta}\cos(n\xi)$, $\varphi_{3n} = \frac{\kappa-2}{2n}B_{2n}e^{-n\eta}\sin(n\xi)$.

By inserting (27) in (17) and (19), we will obtain the following expressions for displacements:

$$\begin{aligned}\overline{u} &= \sum_{n=0}^{\infty}\{-ne^{-n\eta}[(B_{1n}-B_{2n})\eta_1\cos(n\xi)+(B_{1n}-B_{2n})\xi\sin(n\xi)](\eta-\eta_1)\\ &\quad -e^{-n\eta}[(\kappa-1)B_{1n}-(\kappa-2)B_{2n}]\xi\sin(n\xi)-\frac{\kappa}{2}e^{-n\eta}B_{2n}\eta\cos(n\xi)\Big\},\\ \overline{v} &= \sum_{n=1}^{\infty}\{ne^{-n\eta}[(B_{1n}-B_{2n})\xi\cos(n\xi)+(B_{1n}-B_{2n})\eta_1\sin(n\xi)](\eta-\eta_1)\\ &\quad +e^{-n\eta}[(\kappa-1)B_{1n}-(\kappa-2)B_{2n}]\eta\sin(n\xi)-\frac{\kappa}{2}e^{-n\eta}B_{2n}\xi\cos(n\xi)\Big\},\end{aligned} \tag{28}$$

and for the stresses, we obtain the following formula:

$$\begin{aligned}\frac{h_0^2}{2\mu}\sigma_{\eta\eta} &= \sum_{n=1}^{\infty}\{-n^2e^{-n\eta}[(B_{1n}-B_{2n})\eta_1\sin(n\xi)+(B_{1n}-B_{2n})\xi\cos(n\xi)](\eta-\eta_1)\\ &\quad -ne^{-n\eta}\left[\frac{\kappa}{2}B_{1n}\eta\sin(n\xi)-\left(\frac{\kappa-2}{2}B_{1n}+B_{2n}\right)\xi\cos(n\xi)\right]\\ &\quad -\frac{\eta^2-\eta_1^2}{\xi^2+\eta^2}ne^{-n\eta}(B_{1n}-B_{2n})[\eta\sin(n\xi)+\xi\cos(n\xi)]\Big\},\\ \frac{h_0^2}{2\mu}\tau_{\xi\eta} &= \sum_{n=1}^{\infty}\{-n^2e^{-n\eta}[(B_{1n}-B_{2n})\xi\sin(n\xi)-(B_{1n}-B_{2n})\eta_1\cos(n\xi)](\eta-\eta_1)\\ &\quad -ne^{-n\eta}\left[\frac{\kappa}{2}B_{1n}\xi\sin(n\xi)-\left(\frac{\kappa-2}{2}B_{1n}+B_{2n}\right)\eta\cos(n\xi)\right],\\ &\quad -\frac{\eta^2-\eta_1^2}{\xi^2+\eta^2}ne^{-n\eta}(B_{1n}-B_{2n})[\xi\sin(n\xi)+\eta\cos(n\xi)]\Big\},\\ \frac{h_0^2}{2\mu}\sigma_{\xi\xi} &= \sum_{n=1}^{\infty}\{n^2e^{-n\eta}[(B_{1n}-B_{2n})\eta_1\sin(n\xi)+(B_{1n}-B_{2n})\xi\cos(n\xi)](\eta-\eta_1)\\ &\quad +ne^{-n\eta}\left[\left(\frac{\kappa-4}{2}B_{1n}+2B_{2n}\right)\eta\sin(n\xi)+\frac{\kappa+2}{2}B_{1n}\xi\cos(n\xi)\right]\\ &\quad +\frac{\eta^2-\eta_1^2}{\xi^2+\eta^2}ne^{-n\eta}(B_{1n}-B_{2n})[\eta\sin(n\xi)+\xi\cos(n\xi)]\Big\}.\end{aligned} \tag{29}$$

Next, we will obtain the numerical results of the following example.

We have to solve problem (2), (7a), and (8a), when $Q_1(\xi) = P$ and $Q_2(\xi) = 0$, i.e., at $\eta = \eta_1$ boundary the normal load $\frac{1}{2\mu}\sigma_{\eta\eta} = \frac{P}{h_0^2}$ is given, but tangent stress is equal to zero. From (22) and (27), we obtain the following equations:

$$\sum_{n=1}^{\infty} ne^{-n\eta_1}\frac{\kappa}{2}B_{1n}\sin(n\xi) = -\frac{P\eta_1}{\xi^2+\eta_1^2}, \quad \sum_{n=1}^{\infty} ne^{-n\eta_1}\left(\frac{\kappa-2}{2}B_{1n}+B_{2n}\right)\cos(n\xi) = \frac{P\xi}{\xi^2+\eta_1^2}.$$

Consequently, we obtain the infinite system of the linear algebraic equations with unknown B_{1n} and B_{2n} coefficients:

$$\sum_{n=1}^{\infty} ne^{-n\eta_1}\tfrac{\kappa}{2}B_{1n}\sin(n\xi) = -\sum_{n=1}^{\infty}\tilde{P}_{1n}\sin(n\xi), \quad \sum_{n=1}^{\infty} ne^{-n\eta_1}\left(\tfrac{\kappa-2}{2}B_{1n}+B_{2n}\right)\cos(n\xi) = \sum_{n=1}^{\infty}\tilde{P}_{2n}\cos(n\xi),$$

i.e.,

$$ne^{-n\eta_1}\frac{\kappa}{2}B_{1n} = -\tilde{P}_{1n}, \quad ne^{-n\eta_1}\left(\frac{\kappa-2}{2}B_{1n}+B_{2n}\right) = \tilde{P}_{2n}, \quad n = 1, 2, \ldots . \tag{30}$$

Hence,

$$B_{1n} = -\frac{2}{\kappa n}e^{n\eta_1}\tilde{P}_{1n}, \quad B_{2n} = \frac{e^{n\eta_1}}{n}\left(\tilde{P}_{2n}+\frac{\kappa-2}{\kappa}\tilde{P}_{1n}\right),$$

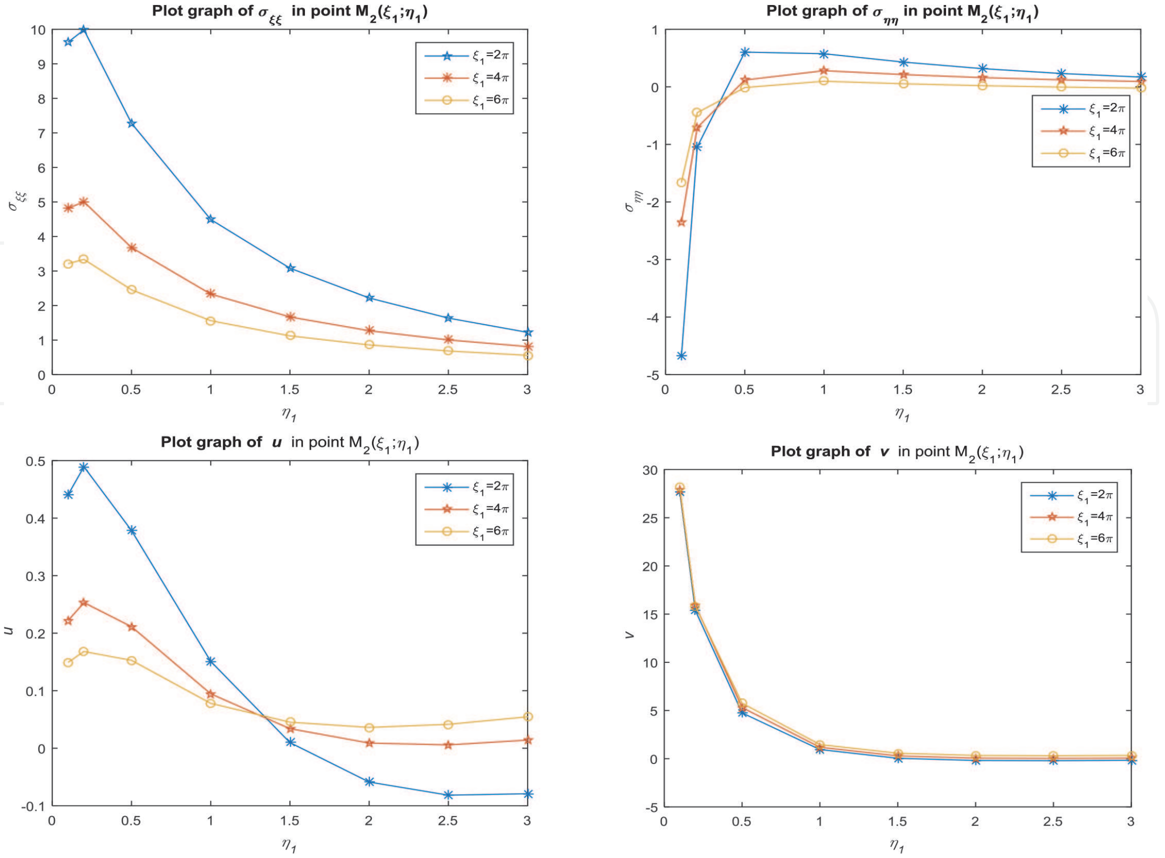

Figure 5.
Stresses and displacements at points $M_2(\xi_1, \eta_1)$ for $\xi_1 = 2*\pi$, $\xi_1 = 4*\pi$, and $\xi_1 = 6*\pi$, when $0.01 \le \eta_1 \le 3$.

where $\tilde{P}_{1n}$ and $\tilde{P}_{2n}$ are the coefficients of expansion into the Fourier series of functions $f_1(\xi) = -\frac{P\eta_1}{\xi^2+\eta_1^2}$ and $f_2(\xi) = \frac{P\xi}{\xi^2+\eta_1^2}$, respectively ($f_1(\xi)$, according to sinuses, and $f_2(\xi)$, according to cosines).

As it can be seen, the main matrix of system (30) has a block-diagonal form, and the dimension of each block is 2×2. Thus, two equations with two B_{1n} and B_{2n} unknown values will be solved. After solving this system, we find the values of B_{1n} and B_{2n} coefficients and put them into formulas (28) and (29) to get displacements and stresses at any points of the body.

Numerical results are obtained for some characteristic points of the body, in particular, $M_1(0, \eta_1)$, $M_2(\xi_1, \eta_1)$ points (see. **Figure 2a**), for the following data: $\nu = 0.3$, $E = 2 * 10^6 \text{kg/cm}^2$, $P = -10\,\text{kg/cm}^2$, $0.01 \le \eta_1 \le 3$, $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, and $\xi_1 = 6 * \pi$.

The above-presented graphs (see **Figures 5** and **6**) show how displacements and stresses change at some characteristic points of body, namely, at points $M_1^{(j)}\left(0, \eta_1^{(j)}\right)$ and $M_2^{(j)}\left(\xi_1, \eta_1^{(j)}\right)$ $(j = 1, 2, \ldots, 8)$, when $0.01 \le \eta_1 \le 3$ (see **Figure 7**).

From the presented results, we obtain the following:

- At points $M_1^{(j)}\left(0, \eta_1^{(j)}\right)$, $\max|u^t| < \max|u^n|$, $\quad v^t = v^n = 0$.

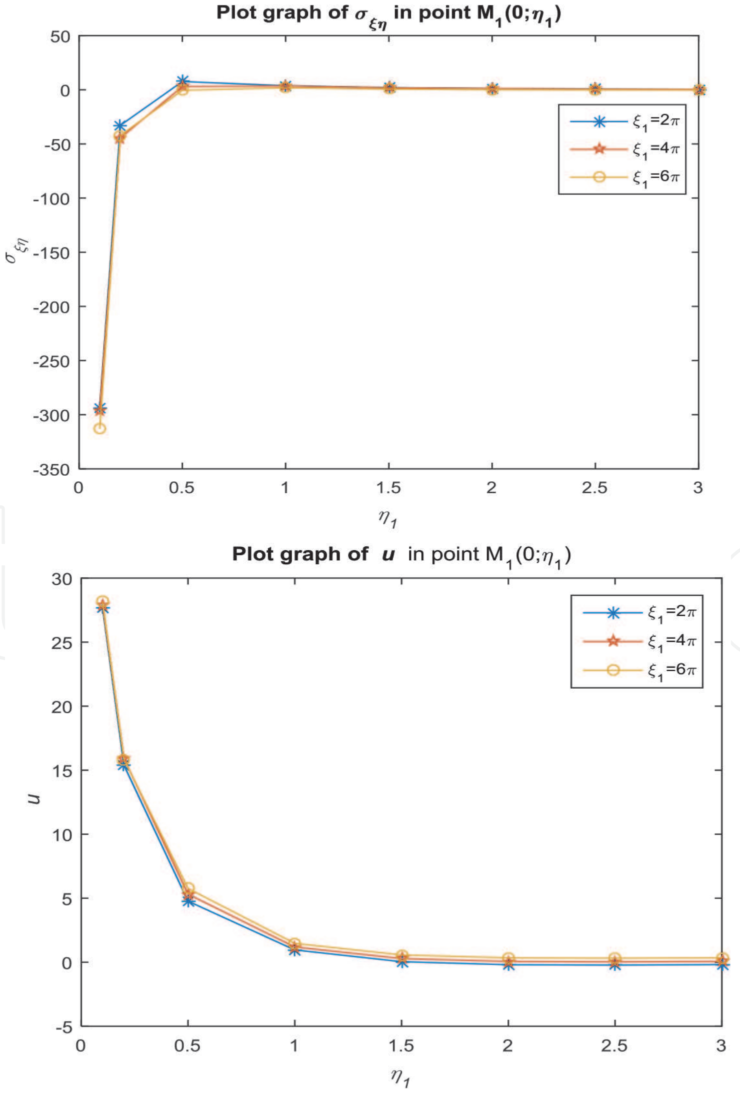

Figure 6.
Tangential stress and normal displacements at points $M_1(0, \eta_1)$ for $\xi_1 = 2 * \pi$, $\xi_1 = 4 * \pi$, and $\xi_1 = 6 * \pi$, when $0.01 \le \eta_1 \le 3$.

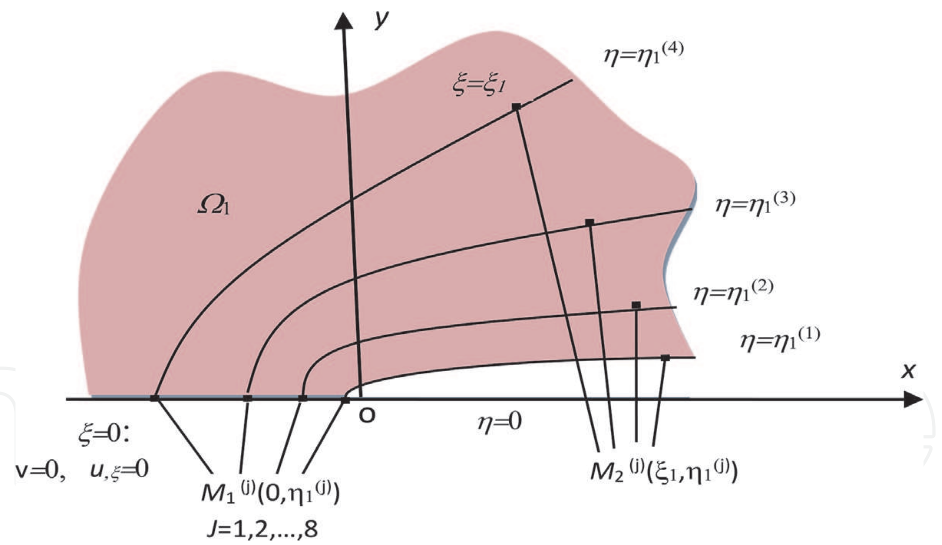

Figure 7.
Infinite region bounded by parabola marked with points, when obtaining the above-presented numerical results.

- At points $M_2^{(j)}\left(\xi_1, \eta_1^{(j)}\right)$, $\max\left|\sigma_{\xi\xi}^t\right| > \max\left|\sigma_{\xi\xi}^n\right|$, $\max|u^t| > \max|u^n|$, $\max|\mathrm{v}^t| < \max|\mathrm{v}^n|$.

- When $\xi_1 \to \infty$, then displacements and stresses tend to zero, that is, the boundary conditions (10) are satisfied.

- When $\eta_1 \to \infty$, then displacements and stresses tend to zero, that is, the boundary conditions ($10'$) are satisfied.

- When $\eta_1 \to 0$ (in this case there is a crack), then (a) at points $M_1^{(j)}\left(0, \eta_1^{(j)}\right)$ tangential stresses and normal displacements tend to ∞, but other components equal to zero. It can be seen from the boundary conditions (8a) (b) at points $M_2^{(j)}\left(\xi_1, \eta_1^{(j)}\right)$ that all components of the displacements and stresses tend to ∞.

Here superscript t and n denote the tangential and normal displacement or the stress, respectively.

5. Conclusion

The main results of this chapter can be formulated as follows:

- The equilibrium equations and Hooke's law are written in terms of parabolic coordinates.

- The solution of the equilibrium equations is obtained by the method of separation of variables. The solution is constructed using its general representation by harmonic functions.

- In parabolic coordinates, analytical solutions of 2D static boundary value problems for the elasticity are constructed for homogeneous isotropic finite and infinite bodies occupying domains bounded by coordinate lines of parabolic coordinate system.

- Two concrete internal and external boundary value problems in stresses are set and solved.

The bodies bounded by the parabola are common in practice, for example, in building, mechanical engineering, biology, medicine, etc., the study of the deformed state of such bodies is topical, and consequently, in my opinion, setting the problems considered in the chapter and the method of their solution is interesting in a practical view.

Notations

x, y	Cartesian coordinates
ξ, η	parabolic coordinates
E and v	modulus of elasticity and Poisson's ratio
λ, μ	elastic Lamé constants
$\vec{U}(u, \mathrm{v})$	displacement vector
$\sigma_{\xi\xi}, \sigma_{\eta\eta}, \tau_{\xi\eta} = \tau_{\eta\xi}$	normal and tangential stresses

Appendix

A. Some basic formulas in parabolic coordinates

In orthogonal parabolic coordinate system $\xi, \eta(-\infty < \xi < \infty,\ 0 \leq \eta < \infty$, see **Figure A1**) [23, 24]; we have

$$h_\xi = h_\eta = h = c\sqrt{\xi^2 + \eta^2}, \quad x = c\left(\xi^2 - \eta^2\right)/2, \quad y = c\xi\eta,$$

where h_ξ, h_η are Lame's coefficients of the system of parabolic coordinates, c is a scale coefficient, x, y are the Cartesian coordinates.

The coordinate axes are parabolas

$$y^2 = -2c\xi_0^2\left(x - c\xi_0^2/2\right), \quad \xi_0 = \text{const}, \quad y^2 = -2c\eta_0^2\left(x + c\eta_0^2/2\right), \quad \eta_0 = \text{const}.$$

Laplace's equation $\Delta f = 0$, where $f = f(\xi, \eta)$, in the parabolic coordinates has the form

$$\left(f_{,\xi\xi} + f_{,\eta\eta}\right)/c^2\left(\xi^2 + \eta^2\right) = 0.$$

We have to find solution of the equation in following form

$$f = X(\xi) \cdot E(\eta),$$

and then by separation of variables, we will receive

$$\frac{1}{c^2\left(\xi^2 + \eta^2\right)}\left[\frac{X''}{X} + \frac{E'}{E}\right] = 0.$$

From here

$$X'' + mX = 0, \qquad E'' - mE = 0,$$

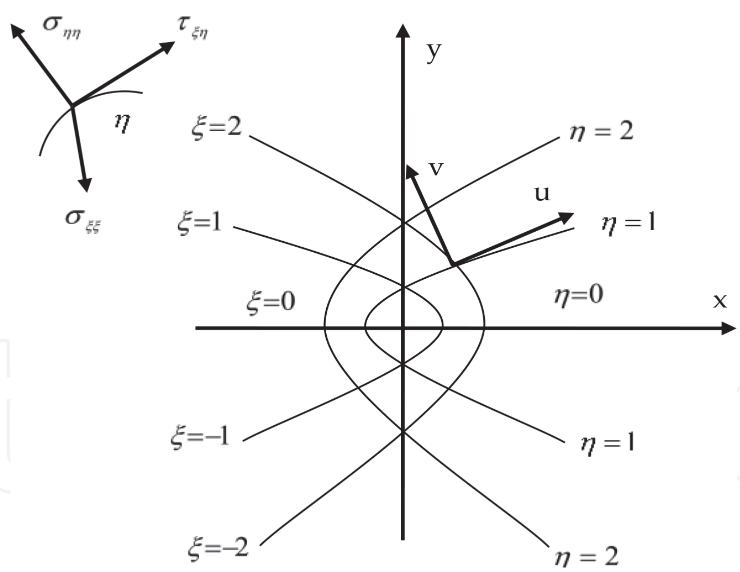

Figure A1.
Parabolic coordinate system.

where m is any constant, their solutions are [25]

$$X = C_1 \cos(m\xi) + C_2 \sin(m\xi), \quad E = C_3 e^{m\eta} + C_4 e^{-m\eta} \\ = C_3^* \cosh(m\eta) + C_4^* \sinh(m\eta).$$

So

$$f(\xi,\eta) = (C_3 e^{m\eta} + C_4 e^{-m\eta})(C_1 \cos(m\xi) + C_2 \sin(m\xi))$$

or

$$f(\xi,\eta) = \left(C_3^* \cosh(m\eta) + C_4^* \sinh(m\eta)\right)(C_1 \cos(m\xi) + C_2 \sin(m\xi)),$$

B. Solution of system of partial differential equations

We solve the system of partial differential equations (2).
We have introduced φ_1 harmonic function, and if we take

$$\text{(a)} \quad D = \frac{\kappa\mu}{h_0^2}\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \qquad \text{(b)}\, K = \frac{\kappa\mu}{h_0^2}\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \tag{B1}$$

then Eqs. (2a) and (2b) will be satisfied identically, while Eqs. (2c) and (2d) will receive the following form:

$$\text{(a)} \quad \overline{u}_{,\xi} + \overline{\mathrm{v}}_{,\eta} = (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \quad \text{(b)}\, \overline{\mathrm{v}}_{,\xi} - \overline{u}_{,\eta} = \kappa\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \tag{B2}$$

$$\text{(a)} \quad \overline{u}_{,\xi} + \overline{\mathrm{v}}_{,\eta} = (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), \quad \text{(b)}\, \left(\overline{\mathrm{v}} - \kappa\varphi_1\eta\right)_{,\xi} = \left(\overline{u} + \kappa\varphi_1\xi\right)_{,\eta}. \tag{B3}$$

From equation (B3b) imply that exists such type harmonic function φ, for which fulfill the following

$$\overline{u} = \varphi_{,\xi} - \kappa\varphi_1\xi, \qquad \overline{\mathrm{v}} = \varphi_{,\eta} + \kappa\varphi_1\eta. \tag{B4}$$

Considering (B4), from Equation (B3a), the following will be obtained:

$$h^2\Delta\varphi = \varphi_{,\xi\xi} + \varphi_{,\eta\eta} = \kappa\varphi_1 + \kappa\varphi_{1,\xi}\xi - \kappa\varphi_1 - \kappa\varphi_{1,\eta}\eta + (\kappa - 2)\left(\varphi_{1,\eta}\eta - \varphi_{1\xi}\xi\right) \\ = 2\left(\varphi_{1,\xi}\xi - \varphi_{1,\eta}\eta\right). \tag{B5}$$

General solution of the system (B2) can be written in the form $\overline{u} = \psi_1$, $\overline{v} = \psi_2$, where

$$\psi_{1,\xi} + \psi_{2,\eta} = 0, \qquad \psi_{2,\xi} - \psi_{1,\eta} = 0.$$

The full solution of equation system (B2) is written in the following form:

$$\overline{u} = \varphi_{,\xi} - \kappa\varphi_1\xi + \psi_1, \qquad \overline{v} = \varphi_{,\eta} + \kappa\varphi_1\eta + \psi_2, \tag{B6}$$

where φ is the partial solution of the (B5).
If we take $\kappa = \text{const}$, then

$$\varphi = \frac{\xi^2 - \eta^2}{2}\varphi_1,$$

and (B6) formula will receive the following form:

$$\overline{u} = \frac{\xi^2 - \eta^2}{2}\varphi_{1,\xi} - (\kappa - 1)\varphi_1\xi + \psi_1, \qquad \overline{v} = \frac{\xi^2 - \eta^2}{2}\varphi_{1,\eta} + (\kappa - 1)\varphi_1\eta + \psi_2.$$

From here

$$\overline{u} = \left(\frac{\xi^2 - \eta^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) - \xi\eta\varphi_{1,\eta} - (\kappa - 1)\varphi_1\xi + \psi_1,$$
$$\overline{v} = \left(\frac{\xi^2 - \eta^2}{2}\varphi_{1,\eta} - \xi\eta\varphi_{1,\xi}\right) + \xi\eta\varphi_{1,\xi} + (\kappa - 1)\varphi_1\eta + \psi_2.$$

Without losing the generality, the expression in brackets can be taken as zero, because we already have in $\overline{u}$ and $\overline{v}$ of the solutions Laplacian (we mean ψ_1 and ψ_2). Therefore, the solutions of system (2) are given in the following form:

$$\begin{aligned} &(\text{a})\ \ h_0^2 D = \kappa\mu\left(\varphi_{1,\eta}\eta - \varphi_{1,\xi}\xi\right), &&(\text{b})\, h_0^2 K = \kappa\mu\left(\varphi_{1,\eta}\xi + \varphi_{1,\xi}\eta\right), \\ &(\text{c})\, \overline{u} = -\xi\eta\varphi_{1,\eta} - (\kappa - 1)\varphi_1\xi + \psi_1, &&(\text{d})\, \overline{v} = \xi\eta\varphi_{1,\xi} + (\kappa - 1)\varphi_1\eta + \psi_2. \end{aligned} \tag{B7}$$

Now we have to write down three versions of ψ_1 and ψ_2 function representation. In the first version

$$\psi_1 = \overline{\varphi}_{1,\eta} + \tilde{\varphi}_{1,\eta} + \varphi_{2,\eta}, \qquad \psi_2 = \overline{\varphi}_{1,\xi} + \tilde{\varphi}_{1,\xi} + \varphi_{2,\xi}, \tag{B8}$$

$\overline{\varphi}_1, \tilde{\varphi}_1, \varphi_2$ are harmonic functions; in addition, $\overline{\varphi}_1, \tilde{\varphi}_1$ are selected so that at $\eta = \alpha$, where $\alpha = \eta_1$ or $\alpha = \eta_2$, the following equations will be satisfied:

$$-\xi\eta\varphi_{1,\eta} - (\kappa - 1)\varphi_1\xi + \overline{\varphi}_{1,\eta} + \tilde{\varphi}_{1,\eta} = 0, \qquad \xi\eta\varphi_{1,\xi} + (\kappa - 1)\varphi_1\xi + \overline{\varphi}_{1,\xi} + \tilde{\varphi}_{1,\xi} = 0,$$

In the second version

$$\begin{aligned} \psi_1 &= -\alpha\left(\frac{\xi^2 - (\eta - \alpha)^2}{2}\varphi_{1,\xi} + \xi\eta\varphi_{1,\eta}\right) + \frac{\xi^2 - \eta^2}{2}\varphi_{2,\xi} + \xi\eta\varphi_{2,\eta}, \\ \psi_2 &= \alpha\left(\xi\eta\varphi_{1,\xi} - \frac{\xi^2 - (\eta - \alpha)^2}{2}\varphi_{1,\eta}\right) + \frac{\xi^2 - \eta^2}{2}\varphi_{2,\eta} - \xi\eta\varphi_{2,\xi}, \end{aligned} \tag{B9}$$

where φ_2 is the harmonic function.
In the third version

$$\begin{aligned} \psi_1 &= -\alpha^2\left(\frac{\xi^2-\eta^2}{2}\varphi_{1,\xi}+\xi\eta\varphi_{1,\eta}\right)+\frac{\xi^2-\eta^2}{2}\varphi_{2,\xi}+\xi\eta\varphi_{2,\eta}, \\ \psi_2 &= \alpha^2\left(\xi\eta\varphi_{1,\xi}-\frac{\xi^2-\eta^2}{2}\varphi_{1,\eta}\right)+\frac{\xi^2-\eta^2}{2}\varphi_{2,\eta}-\xi\eta\varphi_{2,\xi}. \end{aligned} \tag{B10}$$

Inserting (B8) in (B7c and d), we will get

$$\begin{aligned} &(\mathrm{a})\ \overline{u} = -\xi\eta\varphi_{1,\eta}-(\kappa-1)\varphi_1\xi+\overline{\varphi}_{1,\eta}+\tilde{\varphi}_{1,\eta}+\varphi_{2,\eta}, \\ &(\mathrm{b})\ \overline{\mathrm{v}} = \xi\eta\varphi\xi+(\kappa-1)\varphi_1\xi+\overline{\varphi}_{1,\xi}+\tilde{\varphi}_{1,\xi}+\varphi_{2,\xi}. \end{aligned} \tag{B11}$$

Inserting (B9) in (B7c and d), we will have

$$\begin{aligned} &(\mathrm{a})\ \overline{u} = -\alpha\left(\frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\xi}+\xi\eta\varphi_{1,\eta}\right)-\xi\eta\varphi_{1,\eta}-(\kappa-1)\varphi_1\xi+\frac{\xi^2-\eta^2}{2}\varphi_{2,\xi}+\xi\eta\varphi_{2,\eta}, \\ &(\mathrm{b})\ \overline{\mathrm{v}} = \alpha\left(\xi\eta\varphi_{1,\xi}-\frac{\xi^2-(\eta-\alpha)^2}{2}\varphi_{1,\eta}\right)+\xi\eta\varphi_{1,\xi}+(\kappa-1)\varphi_1\eta+\frac{\xi^2-\eta^2}{2}\varphi_{2,\eta}-\xi\eta\varphi_{2,\xi}. \end{aligned} \tag{B12}$$

Inserting (B10) in (B7c and d), we will get

$$\begin{aligned} &(\mathrm{a})\ \overline{u} = -\alpha^2\left(\frac{\xi^2-\eta^2}{2}\varphi_{1,\xi}+\xi\eta\varphi_{1,\eta}\right)-\xi\eta\varphi_{1,\eta}-(\kappa-1)\varphi_1\xi+\frac{\xi^2-\eta^2}{2}\varphi_{2,\xi}+\xi\eta\varphi_{2,\eta}, \\ &(\mathrm{b})\ \overline{\mathrm{v}} = \alpha^2\left(\xi\eta\varphi_{1,\xi}-\frac{\xi^2-\eta^2}{2}\varphi_{1,\eta}\right)+\xi\eta\varphi_{1,\xi}+(\kappa-1)\varphi_1\eta+\frac{\xi^2-\eta^2}{2}\varphi_{2,\eta}-\xi\eta\varphi_{2,\xi}. \end{aligned} \tag{B13}$$

C. Finding of ξ_1

After the boundary value problem with relevant boundary conditions on $\xi=\xi_1=\xi_{11}$ is solved, the following condition is examined: $F_{11}/F_{10}<\varepsilon$.

ε is a sufficiently small positive number given in advance ($\varepsilon = 0,001-0,0001$).

$$F_{11}=\left[\int_0^{\eta_1}\left(|\sigma_{\xi\xi}|+|\sigma_{\eta\eta}|+|\tau_{\xi\eta}|\right)hd\eta\right]_{\xi=\xi_1},\quad F_{10}=\left[\int_0^{\eta_1}\left(|\sigma_{\xi\xi}|+|\sigma_{\eta\eta}|+|\tau_{\xi\eta}|\right)hd\eta\right]_{\xi=g\tilde{\xi}_1}.$$

g number will be selected so that on boundary $\eta=\eta_1$, point $M(g\tilde{\xi}_1,\eta_1)$ should correspond to the highest value of expression $\left[\sigma_{\eta\eta}(g\tilde{\xi}_1,\eta_1)\right]^2+\left[\tau_{\xi\eta}(g\tilde{\xi}_1,\eta_1)\right]^2$ (when stresses are given) or to the highest value of expression $\left[\overline{u}(g\tilde{\xi}_1,\eta_1)\right]^2+\left[\overline{\mathrm{v}}(g\tilde{\xi}_1,\eta_1)\right]^2$ (when displacements are given).

If condition $F_{11}/F_{10}<\varepsilon$ is not valid for$\xi_1=\xi_{11}$, the same problem will be solved at the beginning, but $\xi_1=\xi_{12}$ will be used instead of $\xi_1=\xi_{11}$. In addition, $\xi_{12}>\xi_{11}$. Then, if condition $F_{12}/F_{10}<\varepsilon$ is not still valid, we will continue with the boundary problem, where $\xi_1=\xi_{13}$; besides, $\xi_{13}>\xi_{12}>\xi_{11}$, and we will examine condition $F_{13}/F_{10}<\varepsilon$. The process will be over at the kth stage, if condition $F_{1k}/F_{10}<\varepsilon$ is valid.

Finding such $\xi_1 = \xi_{1k}$***, for which*** $F_{1k}/F_{10} < \varepsilon$***.***

Distance l between surfaces $\xi = \xi_1$ and $\xi = \tilde{\xi}_1$, which gives the guarantee for condition $F_{1k}/F_{10} < \varepsilon$ to be valid in the parabolic coordinate system, will be taken along the axis of the parabola , and the following expression will be obtained:

$$\xi_1 = \sqrt{l/c + \tilde{\xi}_1^2}.$$

By relying on the known solutions of the relevant plain problems of elasticity, it is purposeful to admit that $l/c = 4, 5, 6, \ldots,$ which allows finding ξ_1 from the relevant equation. Let us note that when $l/c = 4$, we will denote value ξ_1 by ξ_{11}, when $l/c = 5$; by ξ_{12}, when $l/c = 6$; by ξ_{13}, etc. If after selecting $\xi_1 = \xi_{1k}$, inequality $F_{1k}/F_{10} < \varepsilon$ is valid; in order to check the righteousness of the selection, it is necessary to once again make sure that, together with condition $F_{1k}/F_{10} < \varepsilon$, condition $\varepsilon > F_{1k}/F_{10} > F_{1k+1}/F_{10} > F_{1k+2}/F_{10} > \ldots$ is valid, too.

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