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Chapter

Approximate Solutions of Some Boundary Value Problems by Using Operational Matrices of Bernstein Polynomials

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Abstract

In this chapter, we develop an efficient numerical scheme for the solution of boundary value problems of fractional order differential equations as well as their coupled systems by using Bernstein polynomials. On using the mentioned polynomial, we construct operational matrices for both fractional order derivatives and integrations. Also we construct a new matrix for the boundary condition. Based on the suggested method, we convert the considered problem to algebraic equation, which can be easily solved by using Matlab. In the last section, numerical examples are provided to illustrate our main results.

Keywords: Bernstein polynomials, coupled systems, fractional order differential equations, operational matrices of integration, approximate solutions
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1. Introduction

Generalization of classical calculus is known as fractional calculus, which is one of the fastest growing area of research, especially the theory of fractional order differential equations because this area has wide range of applications in real-life problems. Differential equations of fractional order provide an excellent tool for the description of many physical biological phenomena. The said equations play important roles for the description of hereditary characteristics of various materials and genetical problems in biological models as compared with integer order differential equations in the form of mathematical models. Nowadays, most of its applications are found in bio-medical engineering as well as in other scientific and engineering disciplines such as mechanics, chemistry, viscoelasticity, control theory, signal and image processing phenomenon, economics, optimization theory, etc.; for details, we refer the reader to study [8, 9, 11–13, 23–26] and the references there in. Due to these important applications of fractional order differential equations, mathematicians are taking interest in the study of these equations because their models are more realistic and practical. In the last decade, many researchers have studied the existence and uniqueness of solutions to boundary value problems and their coupled systems for fractional order differential equations (see [1–7, 10]). Hence the area devoted to existence theory has been very well explored. However, every fractional differential equation cannot be solved for its analytical solutions easily due to the complex nature of fractional derivative; so, in such a situation, approximate solutions to such a problem is most efficient and helpful. Recently, many methods such as finite difference method, Fourier series method, Adomian decomposition method (ADM), inverse Laplace technique (ILT), variational iteration methods (VIM), fractional transform method (FTM), differential transform method (DTM), homotopy analysis method (HAM), method of radial base function (MRBM), wavelet techniques (WT), spectral methods and many more (for more details, see [15–22, 26–37, 39, 40]) have been developed for obtaining numerical solutions of such differential equations. These methods have their own merits and demerits. Some of them provide a very good approximation. However, in the last few years, some operational matrices were constructed to achieve good approximation as in [14]. After this, a variety of operational matrices were developed for different wavelet methods. This method uses operational matrices, where every operation, for example differentiation and integration, involved in these equations is performed with the help of a matrix. A large variety of operational matrices are available in the literature for different orthogonal polynomials like Legendre, Laguerre, Jacobi and Bernstein polynomials [40–47]. Motivated by the above applications and uses of fractional differential equations, in this chapter, we developed a numerical scheme based on operational matrices via Bernstein polynomials. Our proof is more generalized and there is no need to convert the Bernstein polynomial function vector to another basis like block pulse function or Legendre polynomials. To the best of our knowledge, the method we consider provides a very good approximation to the solution. By the use of these operational matrices, we apply our scheme to a single fractional order differential equation with given boundary conditions as

$$\begin{cases} D^{\alpha}y(t) + AD^{\mu}y(t) + By(t) = f(t), \quad 1 < \alpha \le 2, \quad 0 < \mu \le 1, \\ y(0) = a, \quad y(1) = b, \end{cases} \tag{1}$$

where $f(t)$ is the source term, A, B are any real numbers; then we extend our method to solve a boundary value problem of coupled system of fractional order differential equations of the form

$$\begin{cases} D^{\alpha}x(t) + A_1D^{\mu_1}x(t) + B_1D^{\nu_1}y(t) + C_1x(t) + D_1y(t) = f(t), \quad 1 < \alpha \le 2, \quad 0 < \mu_1, \nu_1 \le 1, \\ D^{\beta}y(t) + A_2D^{\mu_2}x(t) + B_2D^{\nu_2}y(t) + C_2x(t) + D_2y(t) = g(t), \quad 1 < \beta \le 2, \quad 0 < \mu_2, \nu_2 \le 1, \\ y(0) = a, \ y(1) = b, \ y(0) = c, \ y(1) = d, \end{cases} \tag{2}$$

where $f(t), g(t)$ are source terms of the system, $A_i, B_i, C_i, D_i (i = 1, 2)$ are any real constants. Also we compare our approximations to exact values and approximations of other methods like Jacobi polynomial approximations and Haar wavelets methods to evaluate the efficiency of the proposed method. We also provide some examples for the illustration of our main results.

This chapter is designed in five sections. In the first section of the chapter, we have cited some basic works related to the numerical and analytical solutions of differential equations of arbitrary order by various methods. The necessary definitions and results related to fractional calculus and Bernstein polynomials along with the construction of some operational matrices are given in Section 2. In Section 3, we have discussed the main theory for the numerical procedure. Section 4 contains some interesting practical examples and their images. Section 5 describes the conclusion of the chapter.

2. Basic definitions and results

In this section, we recall some fundamental definitions and results from the literature, which can be found in [1–7].

Definition 2.1. *The fractional integral of order* $\gamma \in \mathbb{R}_+$ *of a function* $y \in L^1([0,1],\mathbb{R})$ *is defined as*

$$I^{\gamma}_{0+}y(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-\tau)^{\gamma-1}y(\tau)d\tau.$$

Definition 2.2. *The Caputo fractional order derivative of a function y on the interval* $[0,1]$ *is defined by*

$$D^{\gamma}_{0+}y(t) = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\tau)^{n-\gamma-1}y^{(n)}(\tau)d\tau,$$

where $n = [\gamma]+1$ and $[\gamma]$ represents the integer part of γ.

Lemma 2.1. *The fractional differential equation of order* $\gamma > 0$

$$D^{\gamma}y(t) = 0, n-1 < \gamma \leq n,$$

has a unique solution of the form $y(t) = d_0 + d_1 t + d_2 t^2 + \ldots + d_{n-1}t^{n-1}$, where $d_k \in R$ and $k = 0,1,2,3,\ldots,n-1$..

Lemma 2.2. *The following result holds for fractional differential equations*

$$I^{\gamma}D^{\gamma}y(t) = y(t) + d_0 + c_1 t + d_2 t^2 + \ldots + d_{n-1}t^{n-1},$$

for arbitrary $d_k \in R$, $k = 0,1,2,\ldots,n-1$.

Hence it follows that

$$D^{\gamma}t^k = \frac{\Gamma(k+1)}{\Gamma(k-\gamma+1)}t^{k-\gamma}, \quad I^{\gamma}t^k = \frac{\Gamma(k+1)}{\Gamma(k+\gamma+1)}t^{k+\gamma} \text{ and } D^{\gamma}[constant] = 0.$$

2.1 The Bernstein polynomials

The Bernstein polynomials $B_{i,m}(t)$ on $[0,1]$ can be defined as

$$B_{i,m}(t) = \binom{m}{i} t^i (1-t)^{m-i}, \text{for } i = 0,1,2\ldots m,$$

where $\binom{m}{i} = \frac{m!}{(m-i)!i!}$, which on further simplification can be written in the most simplified form as

$$B_{i,m}(t) = \sum_{k=0}^{m-i} \Theta_{(i,k,m)} t^{k+i}, \quad i = 0,1,2\ldots m, \tag{3}$$

where

$$\Theta_{(i,k,m)} = (-1)^k \binom{m}{i}\binom{m-i}{k}.$$

Note that the sum of the Bernstein polynomials converges to 1.

Lemma 2.3. ***Convergence Analysis:*** *Assume that the function* $g \in C^{m+1}[0,1]$ *that is* $m+1$ *times continuously differentiable function and let* $X = \langle B_{0,m}, B_{1,m}, \dots, B_{m,m}\rangle$. *If* $C^T\Psi(x)$ *is the best approximation of* g *out of* X*, then the error bound is presented as*

$$\|g - C^T\Psi\|_2 \le \frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{\Gamma(m+2)\sqrt{2m+3}},$$

where $M = \max_{x\in[0,1]}|g^{(m+1)}(x)|, \ \ S = \max\{1-x_0, x_0\}$.

Proof. In view of Taylor polynomials, we have

$$F(x) = g(x_0) + (x-x_0)g^{(1)}(x_0) + \frac{(x-x_0)^2}{\Gamma 3}g^{(2)} + \dots + \frac{(x-x_0)^m}{\Gamma(m+1)}g^{(m)},$$

from which we know that

$$|g - F(x)| = |g^{(m+1)}(\eta)|\frac{(x-x_0)^{m+1}}{\Gamma(m+2)}, \ \text{ there exist } \ \eta \in (0,1).$$

Due the best approximation $C^T\Psi(x)$ of g, we have

$$\begin{aligned}
\|g - C^T\Psi(x)\|_2^2 &\le \|g - F\|_2^2 \\
&= \int_0^1 (g(x) - F(x))^2 dx \\
&= \int_0^1 \left[|g^{(m+1)}(\eta)|\frac{(x-x_0)^{m+1}}{\Gamma(m+2)}\right]^2 dx \\
&\le \frac{M^2}{\Gamma^2(m+2)}\int_0^1 (x-x_0)^{2m+2}dx \\
&\le \frac{2M^2S^{2m+3}}{\Gamma^2(m+2)(2m+3)}.
\end{aligned}$$

Hence we have

$$\|g - C^T\Psi(x)\|_2 \le \frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{\Gamma(m+2)\sqrt{(2m+3)}}. \quad \square$$

Let $H = L^2[0,1]$ be a Hilbert space, then the inner product can be defined as

$$<f, g> = \int_0^1 f(x).g(x)dx$$

and

$$Y = span\{B_{0,m}, B_{1,m}, \dots B_{m,m}\}$$

is a finite dimensional and closed subspace. So if $f \in H$ is an arbitrary element then its best approximation is unique in Y. This terminology can be achieved by using $y_0 \in Y$ and for all $y \in Y$, we have $\|f - y_0\| \le \|f - y\|$. Thus any function can be approximated in terms of Bernstein polynomials as

$$f(t) = \sum_{i=0}^{m} c_i B_{(i,m)}, \tag{4}$$

where coefficient c_i can easily be calculated by multiplying (4) by $B_{(j,m)}(t)$, $j = 0, 1, 2, \dots m$ and integrating over $[0, 1]$ by using inner product and $d_i = \int_0^1 B_{(i,m)}(t) f(t) dt$, $\theta_{(i,j)} = \int_0^1 B_{(i,m)}(t) B_{(j,m)}(t) dt$, $i, j = 0, 1, 2. \dots m$, we have

$$\int_0^1 f(t) B_{(j,m)}(t) dt = \int_0^1 \sum_{i=0}^{m} c_i B_{(i,m)}(t).B_{(j,m)}(t) dt, \quad j = 0, 1, 2 \dots m,$$

which implies that $\int_0^1 f(t) B_{(j,m)}(t) dt = \sum_{i=0}^{m} c_i \int_0^1 B_{(i,m)}(t).B_{(j,m)}(t) dt, \quad j = 0, 1, 2 \dots m$

which implies that $$[d_0 \ \ d_1 \ \ \dots. \ \ d_m] = [c_0 \ \ c_1. \dots c_m] \begin{bmatrix} \theta_{(0,0)} & \theta_{(0,1)} & \cdots & \theta_{(0,r)} & \cdots & \theta_{(0,m)} \\ \theta_{(1,0)} & \theta_{(1,1)} & \cdots & \theta_{(1,r)} & \cdots & \theta_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{(r,0)} & \theta_{(r,1)} & \cdots & \theta_{(r,r)} & \cdots & \theta_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{(m,0)} & \theta_{(m,1)} & \cdots & \theta_{(m,r)} & \cdots & \theta_{(m,m)} \end{bmatrix}. \tag{5}$$

Let $X_M = [d_0 \ \ d_1 \ \ \dots. \ \ d_m]$, $C_M = [c_0 \ \ c_1. \dots c_m]$, where $M = m + 1$ where M is the scale level and $\Phi_{M \times M} = \begin{bmatrix} \theta_{(0,0)} & \theta_{(0,1)} & \cdots & \theta_{(0,r)} & \cdots & \theta_{(0,m)} \\ \theta_{(1,0)} & \theta_{(1,1)} & \cdots & \theta_{(1,r)} & \cdots & \theta_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{(r,0)} & \theta_{(r,1)} & \cdots & \theta_{(r,r)} & \cdots & \theta_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{(m,0)} & \theta_{(m,1)} & \cdots & \theta_{(m,r)} & \cdots & \theta_{(m,m)} \end{bmatrix}$, so

$$X_M = C_M \Phi_{M \times M} \Rightarrow C_M = X_M \Phi_{M \times M}^{-1}. \tag{6}$$

where $\Phi_{m \times M}$ is called the dual matrix of the Bernstein polynomials. After calculating $c_{i,}$ (4) can be written as

$$f(t) = C_M B_M^T(t), \quad C_M \text{ is coefficient matrix}$$

where

$$B_M(t) = \left[B_{(0,m)}, B_{(1,m)}, \dots. B_{(m,m)}\right]. \tag{7}$$

Lemma 2.4. *Let $B_M^T(t)$ be the function vector defined in* (3), *then the fractional order integration of $B_M^T(t)$ is given by*

$$I^{\alpha}B_M^T(t) = P_{M\times M}^{\alpha}B_M^T(t), \tag{8}$$

where $P_{M\times M}^{\alpha}$ is the fractional integration's operational matrix defined as

$$P_{M\times M}^{\alpha} = \hat{P}_{M\times M}^{\alpha}\Phi_{M\times M}^{-1}$$

and $\Phi_{M\times M}^{-1}$ *is given in* (3) *and* $P_{M\times M}^{\alpha}$ *is given by*

$$\hat{P}_{M\times M}^{\alpha} = \begin{bmatrix} \Psi_{(0,0)} & \Psi_{(0,1)} & \cdots & \Psi_{(0,r)} & \cdots & \Psi_{(0,m)} \\ \Psi_{(1,0)} & \Psi_{(1,1)} & \cdots & \Psi_{(1,r)} & \cdots & \Psi_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{(r,0)} & \Psi_{(r,1)} & \cdots & \Psi_{(r,r)} & \cdots & \Psi_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{(m,0)} & \Psi_{(m,1)} & \cdots & \Psi_{(m,r)} & \cdots & \Psi_{(m,m)} \end{bmatrix}, \tag{9}$$

where

$$\Psi_{i,j} = \sum_{k=0}^{m-i}\sum_{l=0}^{m-j}\theta_{(i,k,m)}\theta_{(j,l,m)}\frac{\Gamma(k+i+1)}{(i+j+k+l+\alpha+1)\Gamma(k+i+\alpha+1)}. \tag{10}$$

Proof. Consider

$$B_{i,m}(t) = \sum_{k=0}^{m-i}\theta_{(i,k,m)}t^{k+i} \tag{11}$$

taking fractional integration of both sides, we have

$$I^{\alpha}B_{i,m}(t) = \sum_{k=0}^{m-i}\theta_{(i,k,m)}I^{\alpha}t^{k+i} = \sum_{k=0}^{m-i}\theta_{(i,k,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}t^{k+i+\alpha}. \tag{12}$$

Now to approximate right-hand sides of above

$$\sum_{k=0}^{m-i}\theta_{(i,k,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}t^{k+i+\alpha} = C_M^{(i)}B_M^T(t) \tag{13}$$

where $C_M^{(i)}$ can be approximated by using (3) as

$$C_M^{(i)} = X_M^{(i)}\Phi_{M\times M}^{-1}, \tag{14}$$

where entries of the vector $X_M^{(i)}$ can be calculated in generalized form as

$$X_M^{(j)} = \int_0^1\sum_{k=0}^{m-i}\theta_{(i,k,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}t^{k+i+\alpha}B_{j,m}(t)dt, \; j = 0,1,2.\ldots m$$

$$\Rightarrow X_M^{(j)} = \int_0^1\sum_{k=0}^{m-i}\theta_{(i,k,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}\sum_{l=0}^{m-j}\theta_{(j,l,m)}t^{k+i+\alpha}.t^{l+j}dt, \; j = 0,1,2.\ldots m$$

$$= \sum_{k=0}^{m-i}\theta_{(i,k,m)}\sum_{l=0}^{m-j}\theta_{(j,l,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}\frac{1}{(k+l+j+i+\alpha+1)}, \; j = 0,1,2,\ldots m \tag{15}$$

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evaluating this result for i = 0,1,2....m, we have

$$\begin{bmatrix} I^{\alpha}B_{0,m}(t) \\ I^{\alpha}B_{1,m}(t) \\ \vdots \\ \vdots \\ \vdots \\ I^{\alpha}B_{m,m}(t) \end{bmatrix} = \begin{bmatrix} X_M^{(0)}\Phi_{M\times M}^{-1}B_M^T(t) \\ X_M^{(1)}\Phi_{M\times M}^{-1}B_M^T(t) \\ \vdots \\ \vdots \\ \vdots \\ X_M^{(m)}\Phi_{M\times M}^{-1}B_M^T(t)) \end{bmatrix} \tag{16}$$

further writing

$$\Psi_{i,j} = \sum_{k=0}^{m-i}\sum_{l=0}^{m-j}\theta_{(i,k,m)}.\theta_{(j,l,m)}\frac{\Gamma(k+i+\alpha)}{\Gamma(k+i+\alpha+1)}\frac{1}{(k+l+j+i+\alpha+1)}$$

we get

$$I^{\alpha}B_M^T(t) = \hat{P}_{M\times M}^{\alpha}.\Phi_{M\times M}^{-1}.B_M^T(t). \tag{17}$$

Let us represent

$$\hat{P}_{M\times M}^{\alpha}.\Phi_{M\times M}^{-1} = P_{M\times M}^{\alpha}$$

thus

$$I^{\alpha}B_M^T(t) = P_{M\times M}^{\alpha}.B_M^T(t). \tag{18}$$

□

Lemma 2.5. *Let $B_M^T(t)$ be the function vector as defined in (3), then fractional order derivative is defined as*

$$D^{\alpha}B_M^T(t) = G_{M\times M}^{\alpha}.B_M^T(t) \tag{19}$$

where $G_{M\times M}^{\alpha}$ is the operational matrix of fractional order derivative given by

$$G_{M\times M}^{\alpha} = \hat{G}_{M\times M}^{\alpha}\Phi_{M\times M}^{-1}, \tag{20}$$

where $\Phi_{M\times M}$ is the dual matrix given in (3) and

$$\hat{G}_{M\times M}^{\alpha} = \begin{bmatrix} \Psi_{(0,0)} & \Psi_{(0,1)} & \cdots & \Psi_{(0,r)} & \cdots & \Psi_{(0,m)} \\ \Psi_{(1,0)} & \Psi_{(1,1)} & \cdots & \Psi_{(1,r)} & \cdots & \Psi_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{(r,0)} & \Psi_{(r,1)} & \cdots & \Psi_{(r,r)} & \cdots & \Psi_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Psi_{(m,0)} & \Psi_{(m,1)} & \cdots & \Psi_{(m,r)} & \cdots & \Psi_{(m,m)} \end{bmatrix}, \tag{21}$$

where $\Psi_{(i,j)}$ is defined for two different cases as

Case I: $(i < [\alpha])$

$$\Psi_{i,j} = \sum_{k=[\alpha]}^{m-i} \sum_{l=0}^{m-j} \theta_{(i,k,m)}.\theta_{(j,l,m)} \frac{\Gamma(k+i-\alpha)}{\Gamma(k+i-\alpha+1)} \frac{1}{(k+l+j+i-\alpha+1)} \tag{22}$$

Case II: $(i \geq [\alpha])$

$$\Psi_{i,j} = \sum_{k=0}^{m-i} \sum_{l=0}^{m-j} \theta_{(i,k,m)}.\theta_{(j,l,m)} \frac{\Gamma(k+i-\alpha)}{\Gamma(k+i-\alpha+1)} \frac{1}{(k+l+j+i-\alpha+1)}. \tag{23}$$

Proof. Consider the general element as

$$D^\alpha B_{i,m}(t) = D^\alpha \left(\sum_{k=0}^{m-i} \theta_{(i,k,m)}.t^{k+i} \right) = \sum_{k=0}^{m-i} \theta_{(i,k,m)} D^\alpha t^{k+i}. \tag{24}$$

It is to be noted in the polynomial function $B_{i,m}$ the power of the variable 't' is an ascending order and the lowest power is 'i' therefore the first $[\alpha - 1]$ terms becomes zero when we take derivative of order α.

Case I: $(i < [\alpha])$By the use of definition of fractional derivative

$$D^\alpha B_{i,m}(t) = \sum_{k=[\alpha]}^{m-i} \theta_{(i,k,m)} \frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)} t^{k+i-\alpha}. \tag{25}$$

Now approximating RHS of (25) as

$$\sum_{k=[\alpha]}^{m-i} \theta_{(i,k,m)} \frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)} t^{k+i-\alpha} = C_M^{(i)} B_M^T(t) \tag{26}$$

further implies that

$$X_M^{(j)} = \int_0^1 \sum_{k=[\alpha]}^{m-i} \theta_{(i,k,m)} \frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)} t^{k+i-\alpha} B_{j,m}(t)dt, \;\; j = 0,1,2. \dots m$$

$$\Rightarrow X_M^{(j)} = \sum_{k=[\alpha]}^{m-i} \theta_{(i,k,m)} \sum_{l=0}^{m-j} \theta_{(j,l,m)} \frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)(k+i+l+j-\alpha+1)}, \;\; j = 0,1,2. \dots m \tag{27}$$

Case II: $(i \geq [\alpha])$ if $i \leq [\alpha]$ then

$$X_M^{(j)} = \sum_{k=0}^{m-i} \theta_{(i,k,m)} \sum_{l=0}^{m-j} \theta_{(j,l,m)} \frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)(k+i+l+j-\alpha+1)}, \;\; j = 0,1,2. \dots m. \tag{28}$$

After careful simplification, we get

$$\begin{bmatrix} D^{\alpha}B_{0,m}(t) \\ D^{\alpha}B_{1,m}(t) \\ \vdots \\ \vdots \\ \vdots \\ D^{\alpha}B_{m,m}(t) \end{bmatrix} = \begin{bmatrix} X_M^{(0)}\Phi_{M\times M}^{-1}B_M^T(t) \\ X_M^{(1)}\Phi_{M\times M}^{-1}B_M^T(t) \\ \vdots \\ \vdots \\ \vdots \\ X_M^{(m)}\Phi_{M\times M}^{-1}B_M^T(t)) \end{bmatrix}. \tag{29}$$

On further simplification, we have

$$\Psi_{i,j} = \sum_{k=[\alpha]}^{m-i}\sum_{l=0}^{m-j}\theta_{(i,k,m)}.\theta_{(j,l,m)}\frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)}\frac{1}{(k+l+j+i-\alpha+1)} \quad (i<[\alpha])$$

$$\Psi_{i,j} = \sum_{k=0}^{m-i}\sum_{l=0}^{m-j}\theta_{(i,k,m)}.\theta_{(j,l,m)}\frac{\Gamma(k+i+1)}{\Gamma(k+i-\alpha+1)}\frac{1}{(k+l+j+i-\alpha+1)}$$

we get $D^{\alpha}B_M^T(t) = \hat{G}_{M\times M}^{\alpha}\Phi_{M\times M}^{-1}.B_M^T(t)$.

(30)

Let

$$\hat{G}_{M\times M}^{\alpha}\Phi_{M\times M}^{-1} = G_{M\times M}^{\alpha}$$

so

$$D^{\alpha}B_M^T(t) = G_{M\times M}^{\alpha}B_M^T(t)$$

which is the desired result. □

Lemma 2.6. *An operational matrix corresponding to the boundary condition by taking $B_M^T(t)$ is function vector and K is coefficient vector by taking the approximation*

$$u(t) = K\hat{B}(t)$$

is given by

$$Q_{M\times M}^{\alpha,\phi} = \begin{bmatrix} \Omega_{(0,0)} & \Omega_{(0,1)} & \cdots & \Omega_{(0,r)} & \cdots & \Omega_{(0,m)} \\ \Omega_{(1,0)} & \Omega_{(1,1)} & \cdots & \Omega_{(1,r)} & \cdots & \Omega_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{(r,0)} & \Omega_{(r,1)} & \cdots & \Omega_{(r,r)} & \cdots & \Omega_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{(m,0)} & \Omega_{(m,1)} & \cdots & \Omega_{(m,r)} & \cdots & \Omega_{(m,m)} \end{bmatrix}, \tag{31}$$

where

$$\Omega_{i,j} = \int_0^1 \Delta_{i,m}\phi(t)B_j(t)dt, \quad i,j = 0,1,2....m.$$

Proof. Let us take $u(t) = K\hat{B}(t)$, then

$$ {}_0I_1^{\alpha} K\hat{B}(t) = K\,{}_0I_1^{\alpha}\hat{B}(t) = K\begin{bmatrix} {}_0I_1^{\alpha}B_0(t) \\ {}_0I_1^{\alpha}B_1(t) \\ \vdots \\ \vdots \\ \vdots \\ {}_0I_1^{\alpha}B_m(t) \end{bmatrix}. $$

Let us evaluate the general terms

$$ \begin{aligned} {}_0I_1^{\alpha}B_i(t)dt &= \frac{1}{\Gamma\alpha}\int_0^1 (1-s)^{\alpha-1}B_{i,m}(s)ds \\ &= \frac{1}{\Gamma\alpha}\sum_{k=0}^{m-i}\Theta_{i,k,m}\int_0^1 (1-s)^{\alpha-1}s^{k+i}ds. \end{aligned} \tag{32} $$

Since by

$$ L\left(\int_0^1 (1-s)^{\alpha-1}s^{k+i}ds\right) = \frac{\Gamma\alpha\Gamma(k+i+1)}{\tau^{k+\alpha+i}} $$

taking inverse Laplace of both sides, we get

$$ \int_0^1 (1-s)^{\alpha-1}.s^{k+i}ds = L^{-1}\left[\frac{\Gamma\alpha.\Gamma(k+i+1)}{\tau^{k+\alpha+i}}\right] = \frac{\Gamma\alpha.\Gamma(k+i+1)}{\Gamma(k+i+\alpha+1)} $$

now Eq. (32) implies that

$$ {}_0I_1^{\alpha}B_i(t)dt = \sum_{k=0}^{m-i}\Theta_{i,k,m}\frac{\Gamma(k+i+1)}{\Gamma(k+i+\alpha+1)} = \Delta_{i,m}. \tag{33} $$

Now using the approximation $\Delta_{i,m}\phi(t) = \sum_{i=0}^{m}\hat{c}_iB_i(t) = C_M^iB_M^T$, and using Eq. (3) we get $C_M^i = K_M^i\Phi_{M\times M}^{-1}B_M^T$ and using $c_j = \int_0^1\phi(t)B_j(t)dt$,

$$ \begin{aligned} \phi(t)KI^{\alpha}\hat{B}(t) &= K\begin{bmatrix} \Delta_{0,m}\phi(t) \\ \Delta_{1,m}\phi(t) \\ \vdots \\ \vdots \\ \vdots \\ \Delta_{m,m}\phi(t) \end{bmatrix} = K\begin{bmatrix} C_M^0\Phi_{M\times M}^{-1}B_M^T(t) \\ C_M^1\Phi_{M\times M}^{-1}B_M^T(t) \\ \vdots \\ \vdots \\ \vdots \\ C_M^m\Phi_{M\times M}^{-1}B_M^T(t) \end{bmatrix} \\ &= K\begin{bmatrix} c_0^0 & c_1^0 & \cdots & c_r^0 & \cdots & c_m^0 \\ c_0^1 & c_1^1 & \cdots & c_r^1 & \cdots & c_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_0^r & c_1^r & \cdots & c_r^r & \cdots & c_m^r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_0^m & c_1^m & \cdots & c_r^m & \cdots & c_m^m \end{bmatrix}\begin{bmatrix} \Phi_{M\times M}^{-1}B_M^T(t) \\ \Phi_{M\times M}^{-1}B_M^T(t) \\ \vdots \\ \vdots \\ \vdots \\ \Phi_{M\times M}^{-1}B_M^T(t) \end{bmatrix}. \end{aligned} \tag{34} $$

On further simplification, we get

$$\phi(t)KI^{\alpha}\hat{B}(t) = K\begin{bmatrix} \Omega_{(0,0)} & \Omega_{(0,1)} & \cdots & \Omega_{(0,r)} & \cdots & \Omega_{(0,m)} \\ \Omega_{(1,0)} & \Omega_{(1,1)} & \cdots & \Omega_{(1,r)} & \cdots & \Omega_{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{(r,0)} & \Omega_{(r,1)} & \cdots & \Omega_{(r,r)} & \cdots & \Omega_{(r,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{(m,0)} & \Omega_{(m,1)} & \cdots & \Omega_{(m,r)} & \cdots & \Omega_{(m,m)} \end{bmatrix}\begin{bmatrix} B_0(t) \\ B_1(t) \\ \vdots \\ \vdots \\ \vdots \\ B_m(t) \end{bmatrix}. \tag{35}$$

So

$$\phi(t)_0I_1^{\alpha}u(t) = KQ_{M\times M}^{\alpha,\phi}B_M^T(t),$$

and

$$\Omega_{i,j} = \int_0^1 \Delta_{i,m}\phi(t)B_j(t)dt, \quad i,j = 0,1,2.\ldots m. \tag{36}$$

which is the required result. □

3. Applications of operational matrices

In this section, we are going to approximate a boundary value problem of fractional order differential equation as well as a coupled system of fractional order boundary value problem. The application of obtained operational matrices is shown in the following procedure.

3.1 Fractional differential equations

Consider the following problem in generalized form of fractional order differential equation

$$\begin{aligned} &D^{\alpha}y(t) + AD^{\mu}y(t) + By(t) = f(t),\ \ 1<\alpha\le 2,\ \ 0<\mu\le 1, \\ &\text{subject to the boundary conditions } \ y(0) = a, \quad y(1) = b, \end{aligned} \tag{37}$$

where $f(t)$ is a source term; A, B are any real constants and $y(t)$ is an unknown solution which we want to determine. To obtain a numerical solution of the above problem in terms of Bernstein polynomials, we proceed as

$$\text{Let } \ D^{\alpha}y(t) = K_M B_M^T(t). \tag{38}$$

Applying fractional integral of order α we have

$$y(t) = K_M P_{M\times M}^{\alpha} B_M^T(t) - c_0 + c_1 t$$

using boundary conditions, we have

$$c_0 = a, \quad c_1 = b - a - K_M P_{M\times M}^{\alpha} B_M^T(t)\big|_{t=1}.$$

Using the approximation and Lemma 2.2

$$a + t(b - a) = F_M^{(1)} B_M^T(t), \quad tP^{\alpha}_{M\times M} B_M^T(t)\Big|_{t=1} = Q^{\alpha,\phi}_{M\times M} B_M^T(t).$$

Hence

$$y(t) = K_M P^{\alpha}_{M\times M} B_M^T(t) + a + t(b-a) - tK_M P^{\alpha}_{M\times M} B_M^T(t)\big|_{t=1},$$

$$\text{which gives } y(t) = K_M P^{\alpha}_{M\times M} B_M^T(t) + F_M^{(1)} B_M^T(t) - Q^{\alpha,\phi}_{M\times M} B_M^T(t) \qquad (39)$$

$$= K_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) B_M^T(t) + F_M^{(1)} B_M^T(t).$$

Now

$$\begin{aligned} D^{\mu} y(t) &= D^{\mu}\left[K_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) B_M^T(t) + F_M^{(1)} B_M^T(t)\right] \\ &= K_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu}_{M\times M} B_M^T(t) + F_M^{(1)} G^{\mu}_{M\times M} B_M^T(t) \end{aligned} \qquad (40)$$

and

$$f(t) = F_M^{(2)} B_M^T(t). \qquad (41)$$

Putting Eqs. (38)–(41) in Eq. (37), we get

$$\begin{aligned} &K_M B_M^T(t) + AK_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu}_{M\times M} B_M^T(t) + AF_M^{(1)} G^{\mu}_{M\times M} B_M^T(t) \\ &\quad + BK_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) B_M^T(t) + BF_M^{(1)} B_M^T(t) = F_M^{(2)} B_M^T(t). \end{aligned} \qquad (42)$$

In simple form, we can write (42) as

$$\begin{aligned} &K_M B_M^T(t) + AK_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu}_{M\times M} B_M^T(t) + AF_M^{(1)} G^{\mu}_{M\times M} B_M^T(t) \\ &\quad + BK_M\left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) B_M^T(t) + BF_M^{(1)} B_M^T(t) - F_M^{(2)} B_M^T(t) = 0 \qquad (43) \\ &K_M + K_M\left(A\hat{P}^{\alpha}_{M\times M} G^{\mu}_{M\times M} + B\hat{P}^{\alpha}_{M\times M}\right) + AF_M^{(1)} G^{\mu}_{M\times M} + BF_M^{(1)} - F_M^{(2)}, \end{aligned}$$

where

$$\hat{P}^{\alpha}_{M\times M} = P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}.$$

Eq. (43) is an algebraic equation of Lyapunov type, which can be easily solved for the unknown coefficient vector K_M. When we find K_M, then putting this in Eq. (39), we get the required approximate solution of the problem.

3.2 Coupled system of boundary value problem of fractional order differential equations

Consider a coupled system of a fractional order boundary value problem

$$\begin{aligned} &D^{\alpha}x(t) + A_1 D^{\mu_1}x(t) + B_1 D^{\nu_1}y(t) + C_1 x(t) + D_1 y(t) = f(t), \quad 1 < \alpha \le 2,\ 0 < \mu_1, \nu_1 \le 1, \\ &D^{\beta}y(t) + A_2 D^{\mu_2}x(t) + B_2 D^{\nu_2}y(t) + C_2 x(t) + D_2 y(t) = g(t), \quad 1 < \beta \le 2,\ 0 < \mu_2, \nu_2 \le 1, \end{aligned} \qquad (44)$$

subject to the boundary conditions

$$x(0) = a, \quad x(1) = b \quad y(0) = c, \quad y(1) = d, \tag{45}$$

where $A_i, B_i, C_i, D_i (i = 1, 2)$ are any real constants, $f(t), g(t)$ are given source terms. We approximate the solution of the above system in terms of Bernstein polynomials such as

$$D^{\alpha}x(t) = K_M B_M^T(t), \quad D^{\beta}y(t) = L_M B_M^T(t)$$
$$x(t) = K_M P^{\alpha}_{M\times M} B_M^T(t) + c_0 + c_1 t, \quad y(t) = L_M \left(P^{\beta}_{M\times M} B_M^T(t) + d_0 + d_1 t \right.$$

applying boundary conditions, we have

$$x(t) = K_M (P^{\alpha}_{M\times M} B_M^T(t) + a + t(b-a) - tK_M P^{\alpha}_{M\times M} B_M^T(t)\Big|_{t=1},$$
$$y(t) = K_M (P^{\beta}_{M\times M} B_M^T(t) + c + t(d-c) - tK_M P^{\beta}_{M\times M} B_M^T(t)\Big|_{t=1}.$$

let us approximate

$$a + t(b-a) = F^1_M B_M^T(t), c + t(d-c) = F^2_M B_M^T(t)$$
$$tP^{\alpha}_{M\times M} B_M^T(t)\Big|_{t=1} = Q^{\alpha,\phi}_{M\times M} B_M^T(t), tP^{\beta}_{M\times M} B_M^T(t)\Big|_{t=1} = Q^{\beta,\phi}_{M\times M} B_M^T(t)$$

then

$$x(t) = K_M P^{\alpha}_{M\times M} B_M^T(t) + F_M^{(1)} B_M^T(t) - K_M Q^{\alpha,\phi}_{M\times M} B_M^T(t)$$
$$y(t) = L_M P^{\beta}_{M\times M} B_M^T(t) + F_M^{(2)} B_M^T(t) - L_M Q^{\beta,\phi}_{M\times M} B_M^T(t)$$
$$D^{\mu_1}x(t) = \left[K_M P^{\alpha}_{M\times M} B_M^T(t) + F_M^{(1)} B_M^T(t) - K_M Q^{\alpha,\phi}_{M\times M} B_M^T(t) \right]$$
$$= K_M \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M} \right) G^{\mu_1}_{M\times M} + F_M^{(1)} G^{\mu_1}_{M\times M} B_M^T(t)$$
$$D^{\nu_1}y(t) = D^{\nu_1}\left[L_M P^{\beta}_{M\times M} B_M^T(t) + F_M^{(2)} B_M^T(t) - L_M Q^{\beta,\phi}_{M\times M} B_M^T(t) \right]$$
$$= L_M \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M} \right) G^{\nu_1}_{M\times M} + F_M^{(2)} G^{\nu_1}_{M\times M} B_M^T(t)$$
$$D^{\mu_2}x(t) = D^{\mu_2}\left[K_M P^{\alpha}_{M\times M} B_M^T(t) + F_M^{(1)} B_M^T(t) - K_M Q^{\alpha,\phi}_{M\times M} B_M^T(t) \right]$$
$$= K_M \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M} \right) G^{\mu_2}_{M\times M} + F_M^{(1)} G^{\mu_2}_{M\times M} B_M^T(t)$$

and

$$D^{\nu_2}y(t) = D^{\nu_2}\left[K_M P^{\beta}_{M\times M} B_M^T(t) + F_M^{(2)} B_M^T(t) - K_M Q^{\beta,\phi}_{M\times M} B_M^T(t) \right]$$
$$= L_M \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M} \right) G^{\nu_2}_{M\times M} + F_M^{(2)} G^{\nu_2}_{M\times M} B_M^T(t)$$
$$f(t) = F^{(3)} B_M^T(t), \quad g(t) = F^{(4)} B_M^T(t).$$

Thus system (44) implies that

$$
\begin{aligned}
&K_M B_M^T(t) + A_1 K_M \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu_1}_{M\times M} + A_1 F_M^{(1)} G^{\mu_1}_{M\times M} B_M^T(t) \\
&\quad + B_1 L_M \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) G^{\nu_1}_{M\times M} + B_1 F_M^{(2)} G^{\nu_1}_{M\times M} B_M^T(t) + C_1 K_M P^{\alpha}_{M\times M} B_M^T(t) \\
&\quad + C_1 F_M^{(1)} B_M^T(t) - C_1 K_M Q^{\alpha,\phi}_{M\times M} B_M^T(t) + D_1 L_M P^{\beta}_{M\times M} B_M^T(t) + D_1 F_M^{(2)} B_M^T(t) \\
&\quad - D_1 L_M Q^{\beta,\phi}_{M\times M} B_M^T(t) = F^{(3)} B_M^T(t) \\
&L_M B_M^T(t) + A_2 K_M \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu_2}_{M\times M} + A_2 F_M^{(1)} G^{\mu_2}_{M\times M} B_M^T(t) \\
&\quad + B_2 L_M \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) G^{\nu_2}_{M\times M} + B_2 F_M^{(2)} G^{\nu_2}_{M\times M} B_M^T(t) + C_2 K_M P^{\alpha}_{M\times M} B_M^T(t) \\
&\quad + C_2 F_M^{(1)} B_M^T(t) - C_2 K_M Q^{\alpha,\phi}_{M\times M} B_M^T(t) + D_2 L_M P^{\beta}_{M\times M} B_M^T(t) + D_2 F_M^{(2)} B_M^T(t) \\
&\quad - D_2 L_M Q^{\beta,\phi}_{M\times M} B_M^T(t) = F^{(4)} B_M^T(t).
\end{aligned}
\tag{46}
$$

Rearranging the terms in the above system and using the following notation for simplicity in Eq. (46)

$$
\begin{aligned}
\hat{Q}^{\alpha}_{M\times M} &= A_1 \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu_1}_{M\times M} + C_1 \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) \\
\hat{Q}^{\beta}_{M\times M} &= B_1 \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) G^{\nu_1}_{M\times M} + D_1 \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) \\
\hat{R}^{\alpha}_{M\times M} &= A_2 \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) G^{\mu_2}_{M\times M} + C_2 \left(P^{\alpha}_{M\times M} - Q^{\alpha,\phi}_{M\times M}\right) \\
\hat{R}^{\beta}_{M\times M} &= B_2 \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) G^{\nu_2}_{M\times M} + D_2 \left(P^{\beta}_{M\times M} - Q^{\beta,\phi}_{M\times M}\right) \\
F_M &= A_1 F_M^{(1)} G^{\mu_1}_{M\times M} + B_1 F_M^{(2)} G^{\nu_1}_{M\times M} + C_1 F_M^{(1)} + F_M^{(2)} - D_1 F_M^{(3)} \\
G_M &= A_2 F_M^{(1)} G^{\mu_2}_{M\times M} + B_2 F_M^{(2)} G^{\nu_2}_{M\times M} + C_2 F_M^{(1)} + D_2 F_M^{(2)} - F_M^{(4)},
\end{aligned}
$$

the above system (46) becomes

$$
\begin{aligned}
&K_M B_M^T(t) + K_M \hat{Q}^{\alpha}_{M\times M} B_M^T(t) + L_M \hat{Q}^{\beta}_{M\times M} B_M^T(t) + F_M B_M^T(t) = 0 \\
&L_M B_M^T(t) + K_M \hat{R}^{\alpha}_{M\times M} B_M^T(t) + L_M \hat{R}^{\beta}_{M\times M} B_M^T(t) + G_M B_M^T(t) = 0 \\
&[K_M \;\; L_M] \begin{bmatrix} B_M^T(t) & 0 \\ 0 & B_M^T(t) \end{bmatrix} + [K_M \quad L_M] \begin{bmatrix} \hat{Q}^{\alpha}_{M\times M} & 0 \\ 0 & \hat{R}^{\beta}_{M\times M} \end{bmatrix} \begin{bmatrix} B_M^T(t) & 0 \\ 0 & B_M^T(t) \end{bmatrix} \\
&+[K_M \quad L_M] \begin{bmatrix} 0 & \hat{R}^{\alpha}_{M\times M} \\ \hat{Q}^{\beta}_{M\times M} & 0 \end{bmatrix} \begin{bmatrix} B_M^T(t) & 0 \\ 0 & B_M^T(t) \end{bmatrix} + [F_M \quad G_M] \begin{bmatrix} B_M^T(t) & 0 \\ 0 & B_M^T(t) \end{bmatrix} = 0 \\
&[K_M \quad L_M] + [K_M \quad L_M] \begin{bmatrix} \hat{Q}^{\alpha}_{M\times M} & \hat{R}^{\alpha}_{M\times M} \\ \hat{Q}^{\beta}_{M\times M} & \hat{R}^{\beta}_{M\times M} \end{bmatrix} + [F_M \quad G_M] = 0,
\end{aligned}
\tag{47}
$$

which is an algebraic equation that can be easily solved by using Matlab functional solver or Mathematica for unknown matrix $[K_M \quad L_M]$. Calculating the coefficient matrix K_M, L_M and putting it in equations

$$x(t) = K_M P^{\alpha}_{M\times M} B^T_M(t) + F^{(1)}_M B^T_M(t) - K_M Q^{\alpha,\phi}_{M\times M} B^T_M(t)$$
$$y(t) = L_M P^{\beta}_{M\times M} B^T_M(t) + F^{(2)}_M B^T_M(t) - L_M Q^{\beta,\phi}_{M\times M} B^T_M(t),$$

we get the required approximate solution.

4. Applications of our method to some examples

Example 4.1. *Consider*

$$D^{\alpha} y(t) + c_1 D^{\nu} y(t) + c_2 y(t) = f(t), \quad 1 < \alpha < 2 \tag{48}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 0.$$

Solution: *We solve this problem under the following parameters sets defined as* $S_1 = \{\alpha = 2, \nu = 1, c_1 = 1, c_2 = 1\}$, $S_2 = \{\alpha = 1.8, \nu = 0.8, c_1 = 10, c_2 = 100\}$, $S_3 = \{\alpha = 1.5, \nu = 0.5, c_1 = 1/10, c_2 = 1/100\}$, *and select source term for* S_1 *as*

$$f_1(t) = t^6 (t-1)^3 + t^6((72t - 168) + 126) - 30t^4 + 3t^5 (3t-2)(t-1)^2 \tag{49}$$

$$\begin{aligned} f_2(t) = &\frac{111476825837237031 25 t^{\frac{21}{5}} (1750t^3 - 4200t^2 + 3255t - 806)}{406548945561989414912} \\ &+ \frac{2786920645930925781 25 t^{\frac{26}{5}} (5250t^3 - 14350t^2 + 12915t - 3813)}{25002760152062349017088} \\ &+ 100t^6 (t-1)^3, \end{aligned} \tag{50}$$

$$\begin{aligned} f_3(t) = &\frac{5081767996463981 t^{\frac{9}{2}} (1344t^3 - 3360t^2 + 2730t - 715)}{264146673456906240} \\ &+ \frac{5081767996463981 t^{\frac{11}{2}} (1344t^3 - 3808t^2 + 3570t - 1105)}{22452467243837030400} + \frac{t^6 (t-1)^3}{100}. \end{aligned} \tag{51}$$

The exact solution of the above problem is

$$y(t) = t^6 (t-1)^3.$$

We solve this problem with the proposed method under different sets of parameters as defined in S_1, S_2, S_3. *The observation and simulation demonstrate that the solution obtained with the proposed method is highly accurate. The comparison of exact solution with approximate solution obtained using the parameters set* S_1 *is displayed in **Figure 1** subplot (a), while in **Figure 1** subplot (b) we plot the absolute difference between the exact and approximate solutions using different scale levels. One can easily observe that the absolute error is much less than* 10^{-12}. *The order of derivatives in this set is an integer.*

By solving the problem under parameters set S_2 *and* S_3, *we observe the same phenomena. The approximate solution matches very well with the exact solution. See **Figures 2** and **3** respectively.*

Example 4.2. *Consider*

$$D^{\alpha} y(t) - 2D^{0.9} y(t) - 3y(t) = -4\cos(2t) - 7\sin(2t) \tag{52}$$

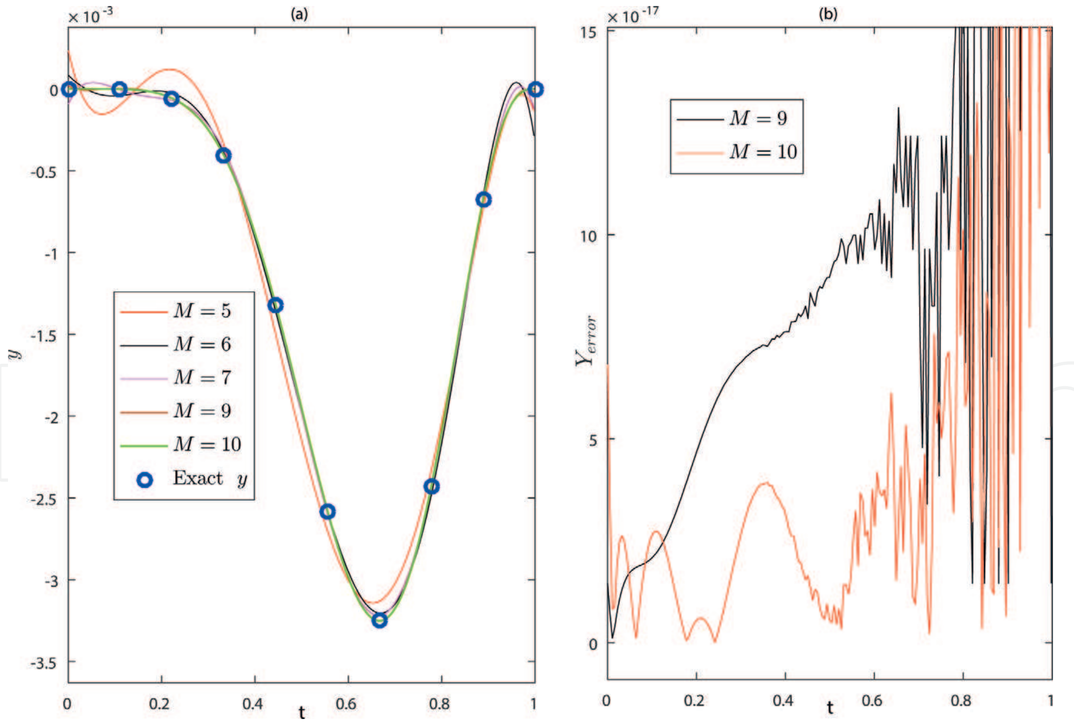

Figure 1.
(a) Comparison of exact and approximate solution of Example 4.1, under parameters set S_1. *(b) Absolute error in the approximate solution of Example 4.1, under parameters set* S_1.

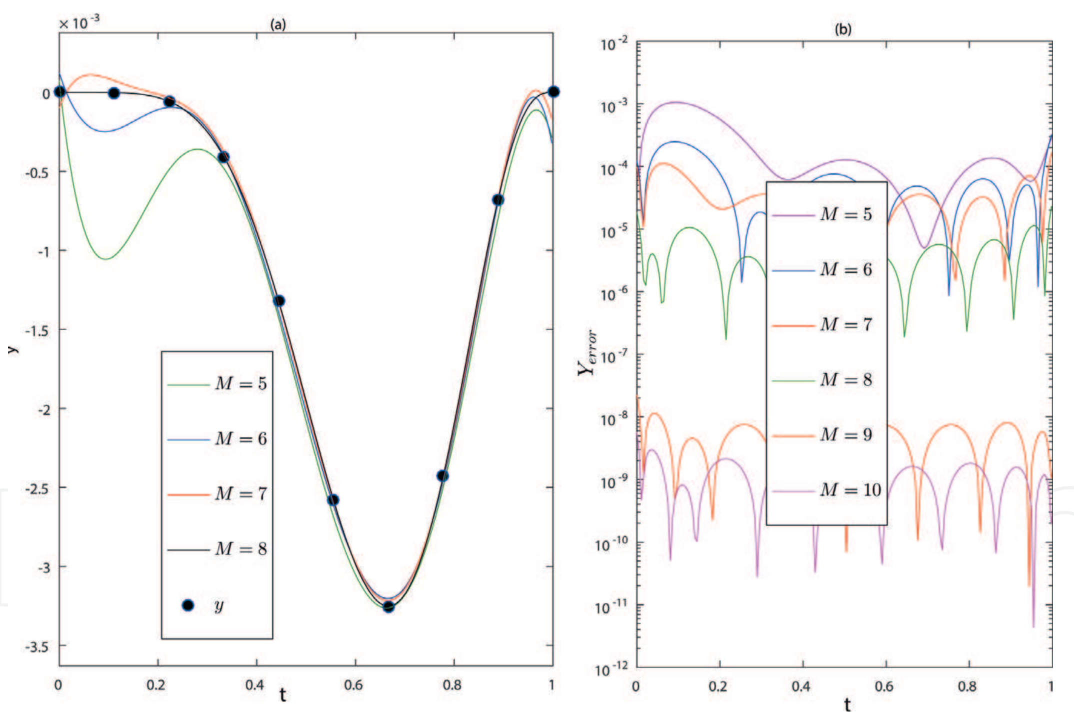

Figure 2.
(a) Comparison of exact and approximate solution of Example 4.1, under parameters set S_2. *(b) Absolute error in the approximate solution of Example 4.1, under parameters set* S_2.

subject to the boundary conditions

$$y(0) = 0, \qquad y(1) = \sin(2).$$

Solution: *The exact solution of the above problem is* $y(t) = \sin(2t)$, *when* $\alpha = 2$. *However the exact solution at fractional order is not known. We use the well-known property of FDEs that when* $\alpha \to 2$, *the approximate solution approaches the exact solution for the evaluations of approximate solutions and check the accuracy by using different scale levels. By increasing the scale level M, the accuracy is also increased. By the*

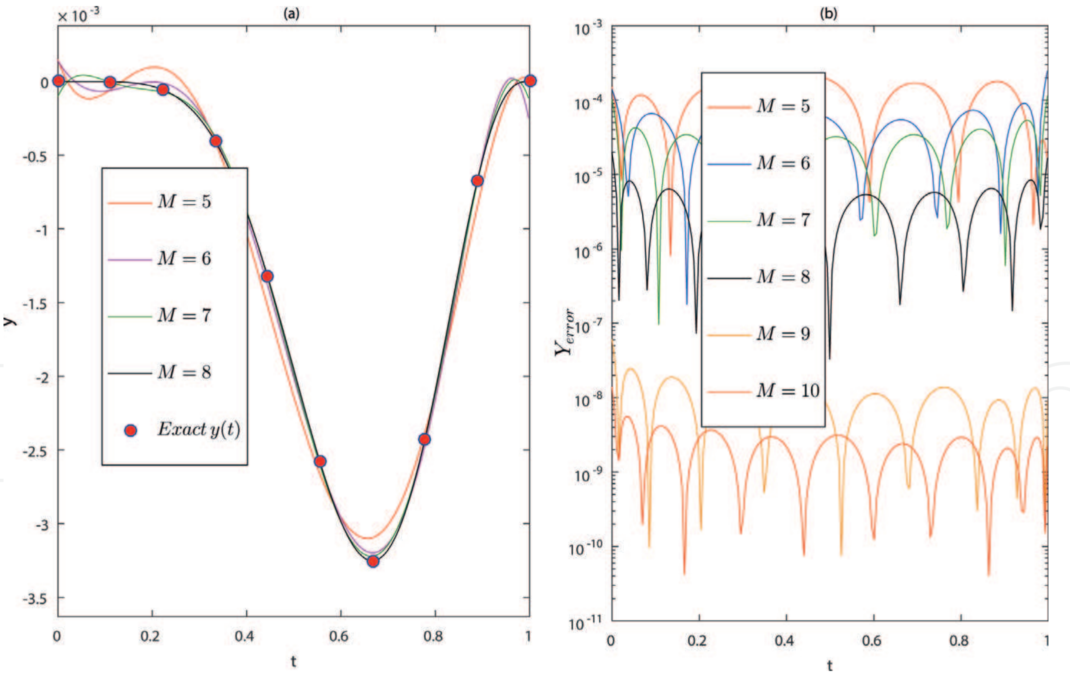

Figure 3.
(a) Comparison of exact and approximate solution of Example 4.1, under parameters set S_3. (b) Absolute error in the approximate solution of Example 4.1, under parameters set S_3.

proposed method, the graph of exact and approximate solutions for different values of M and at $\alpha = 1.7$ is shown in **Figure 4***. From the plot, we observe that the approximate solution becomes equal to the exact solution at $\alpha = 2$. We approximate the error of the method at different scale levels and record that when scale level increases the absolute error decreases as shown in* **Figure 4** *subplot (b) and accuracy approaches 10^{-9}, which is a highly acceptable figure. For convergence of our proposed method, we examined the quantity $\int_0^1 |y_{exact} - y_{approx}| dt$ for different values of M and observed that the norm of error decreases with a high speed with the increase of scale level M as shown in* **Figure 4b**.

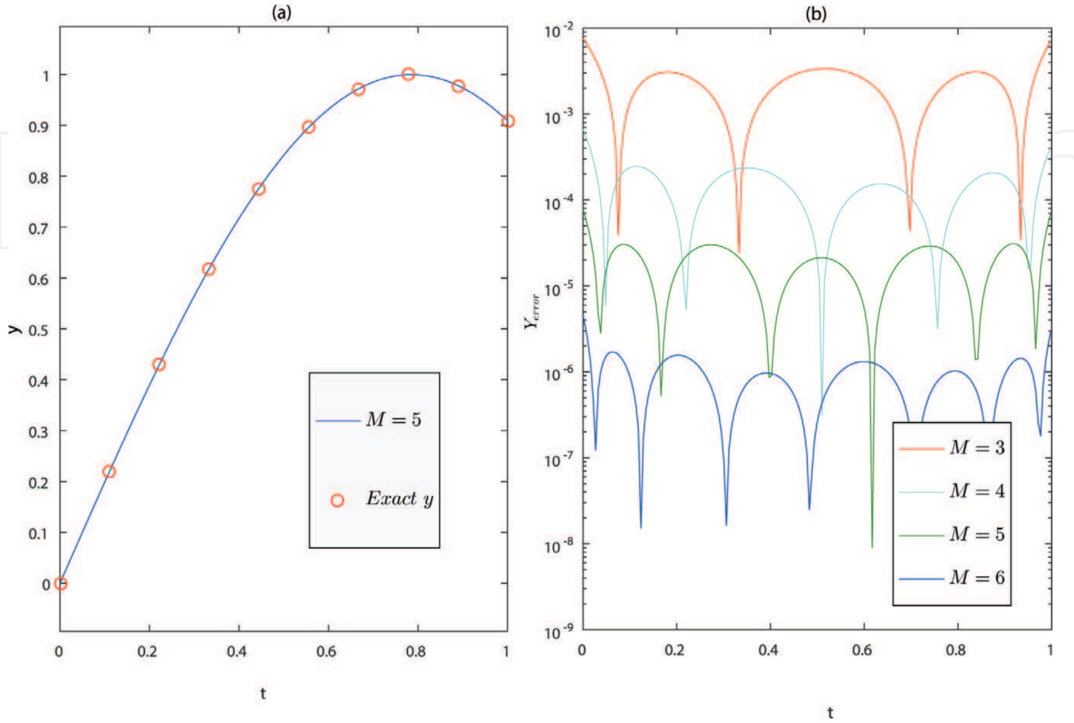

Figure 4.
(a) Comparison of exact and approximate solution of Example 4.2. (b) Absolute error for different scale level M of Example 4.2.

Example 4.3. *Consider the following coupled system of fractional differential equations*

$$\begin{aligned} D^{1.8}x(t) + Dx(t) + 9D^{0.8}y(t) + 2x(t) - y(t) = f(t) \\ D^{1.8}y(t) - 6D^{0.8}x(t) + Dy(t) - x(t) = g(t) \end{aligned} \tag{53}$$

subject to the boundary conditions

$$x(0) = 1, \quad x(1) = 2 \quad and \quad y(0) = 2, \quad y(1) = 2.$$

Solution: *The exact solution is*

$$x(t) = t^5(1-t), \qquad y(t) = t^4(1-t).$$

We approximate the solution of this problem with this new method. The source terms are given by

$$f(t) = 2t^5(t-1) - t^4(t-1) + t^4(6t-5) - \frac{2229536516744740625t^{\frac{16}{5}}(10t-7)}{1008806316530991104} + \frac{1337721910046844375t^{\frac{16}{5}}(25t-21)}{1008806316530991104} \tag{54}$$

$$g(t) = t^3(5t-4) - t^5(t-1) - \frac{1114768258372370312 5t^{\frac{21}{5}}(15t-13)}{6557241057451442176} - \frac{891814606697896 25t^{\frac{11}{5}}(25t-16)}{144115188075855872}. \tag{55}$$

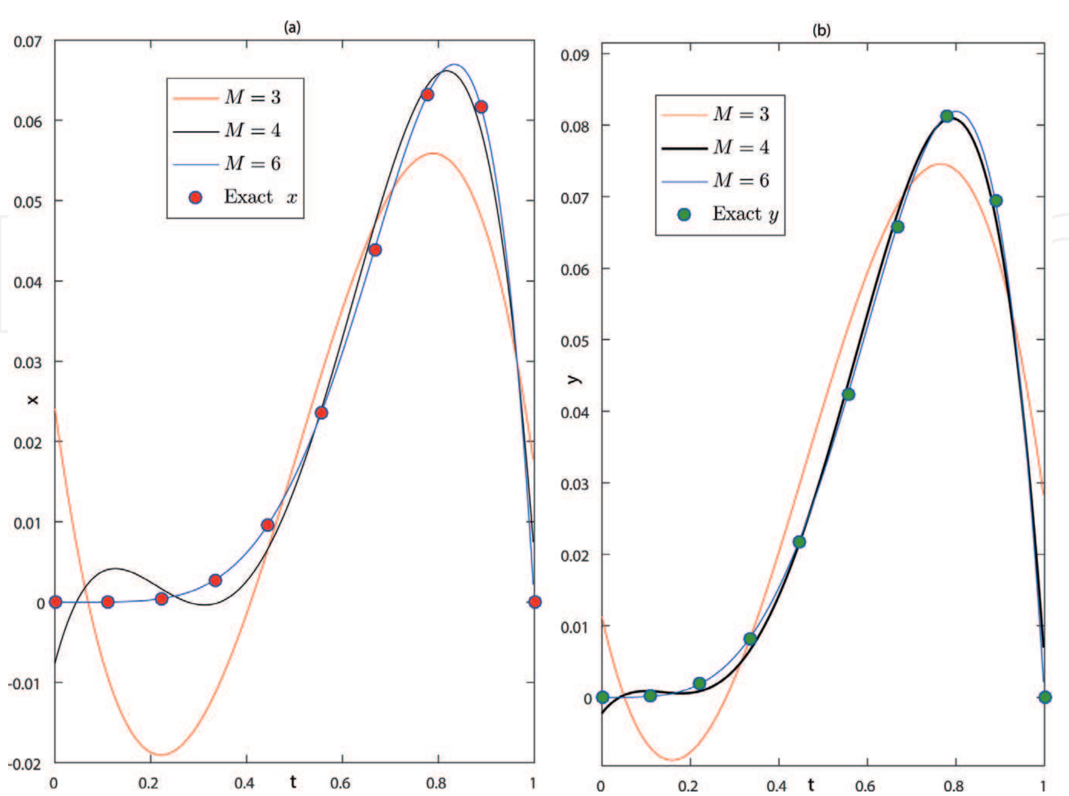

Figure 5.
Comparison of exact and approximate solution of Example 4.3 for different scale level M.

In the given **Figure 5**, *we have shown the comparison of exact* $x(t), y(t)$ *and approximate* $x(t), y(t)$ *in subplot (a) and (b) respectively.*

As expected, the method provides a very good approximation to the solution of the problem. At first, we approximate the solutions of the problem at $\alpha = 2$ *because the exact solution at* $\alpha = 2$ *is known. We observe that at very small scale levels, the method provides a very good approximation to the solution. We approximate the absolute error by the formula*

$$X_{error} = |x_{exact} - x_{approx}|.$$

and

$$Y_{error} = |y_{exact} - y_{approx}|.$$

We approximate the absolute error at different scale level of M, and observe that the absolute error is much less than 10^{-10} *at scale level* $M = 7$, *see* **Figure 6**. *We also approximate the solution at some fractional value of* α *and observe that as* $\alpha \to 2$ *the approximate solution approaches the exact solution, which guarantees the accuracy of the solution at fractional value of* α. **Figure 6** *shows this phenomenon. In* **Figure 6**, *the subplot (a) represents the absolute error of* $x(t)$ *and subplot (b) represents the absolute error of* $y(t)$.

Example 4.4. *Consider the following coupled system*

$$\begin{aligned} D^{1.8}x(t) - x(t) + 3y(t) &= f(t) \\ D^{1.8}y(t) + 4x(t) - 2y(t) &= g(t), \end{aligned} \tag{56}$$

subject to the boundary conditions

$$x(0) = -1, \quad x(1) = -1 \quad and \quad y(0) = -1, \quad y(1) = -1.$$

Solution: *The exact solution for* $\alpha = \beta = 2$ *is*

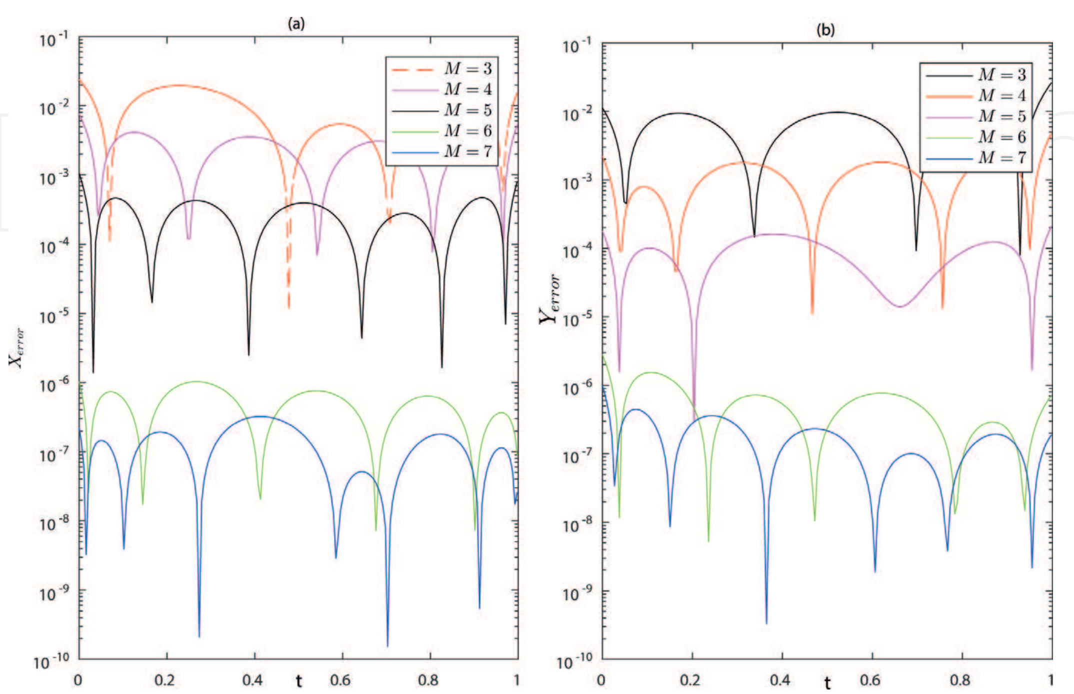

Figure 6.
Absolute error in approximate solutions at different scale level M = *3:7 for Example 4.3.*

$$x(t) = t^5 - t^4 - 1, \quad and \quad y(t) = t^4 - t^3 - 2. \tag{57}$$

The source terms are given by

$$f(t) = \frac{445907303348948125t^{3.2}(25t-21)}{302641894959297312} + t^3 - 4t^4 + 3t^5 - 2,$$
$$g(t) = \frac{89181460669789625t^{2.5}(5t-4)}{14411518807585872} - 4t^3 + 6t^4 - 2t^5 - 2.$$

Approximating the solution with the proposed method, we observe that our scheme gives high accuracy of approximate solution. In **Figure 7**, *we plot the exact solutions together with the approximate solutions in* **Figure 7(a)** and **(b)** *for $x(t)$ and $y(t)$, respectively. We see from the subplots (a) and (b) that our approximations have close agreement to that of exact solutions. This accuracy may be made better by increasing scale level. Further, one can observe that absolute error is below 10^{-10} in* **Figure 8**, *which indicates better accuracy of our proposed method for such types of practical problems of applied sciences.*

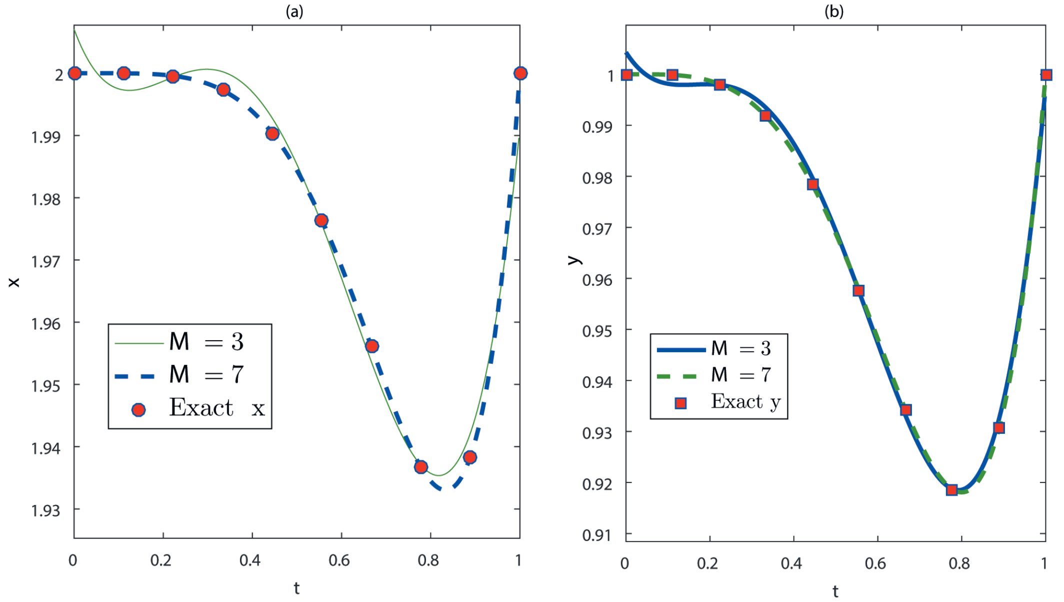

Figure 7.
Comparison of exact and approximate solution at scale level M = *3, 7 for Example 4.4.*

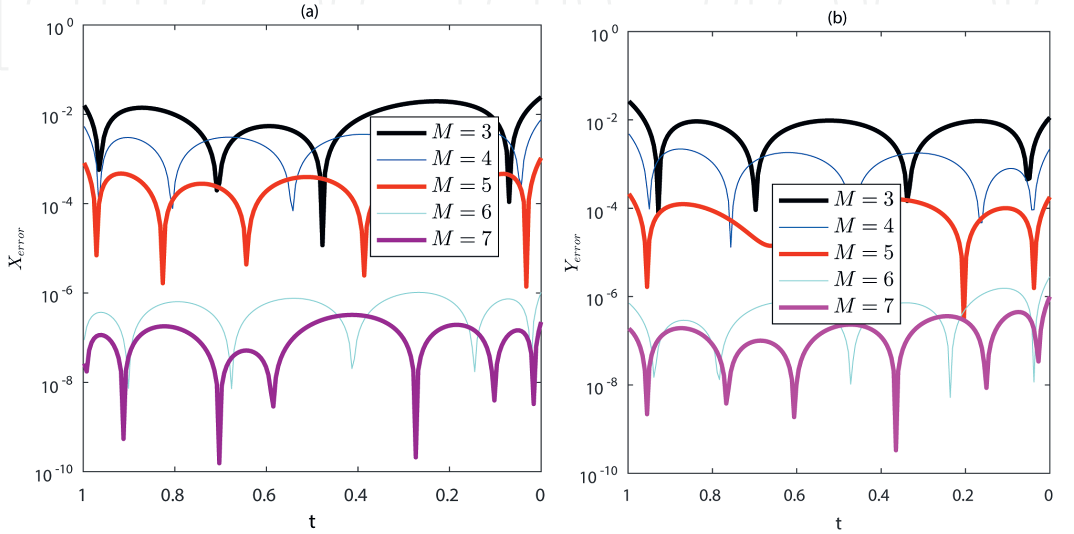

Figure 8.
Absolute error for different scale level M = *3:7 for Example 4.4.*

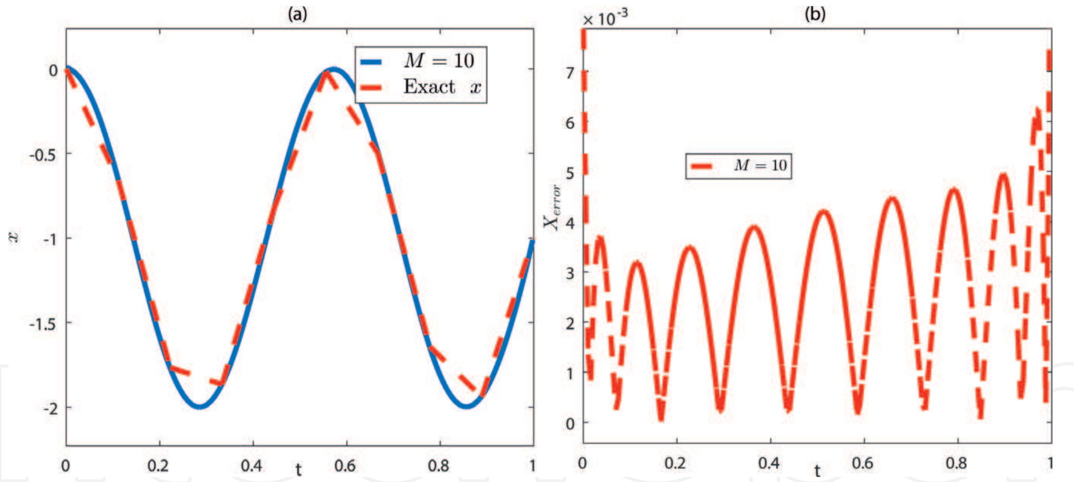

Figure 9.
(a) Comparison of exact and approximate solution at scale level $M = 10, \omega = 3.5, \alpha = 2, \nu = 1$ *for Example 4.5. (b) Absolute error at* $M = 10$.

ω	M	α	ν	$\|x_{app} - x_{ex}\|$ at BM	$\|x_{app} - x_{ex}\|$ at WM	$\|x_{app} - x_{ex}\|$ at JM
0.5	10	2	1	7.000(−3)	2.966(−1)	1.500(−2)
1.5	15	1.6	0.9	6.091(−3)	4.918(−2)	1.623(−1)
2.0	20	1.8	0.8	1.237(−3)	2.108(−2)	2.723(−2)
3.5	25	1.9	0.7	1.008(−3)	5.795(−2)	1.813(−3)

Table 1.
Comparison of solution between Legendre wavelet method (LWM) [48], Jacobi polynomial method (JM) and Bernstein polynomials method (BM) for Example 4.5.

In **Figure 8**, *the subplot (a) represents absolute error for* $x(t)$ *while subplot (b) represents the same quantity for* $y(t)$. *From the subplots, we see that maximum absolute error for our proposed method for the given problem (4.4) is below* 10^{-10}. *This is very small and justifies the efficiency of our constructed method.*

Example 4.5. *Consider the boundary value problem*

$$\begin{aligned} &D^{\alpha}x(t) + (\omega\pi)^2 D^{\nu}x(t) + x(t) = -\omega\pi(\sin(\omega\pi t) + \omega\pi) \\ &x(0) = 0, \quad x(1) = -2. \end{aligned} \tag{58}$$

Taking $\alpha = 2, \nu = 1$ *and* $\omega = 1, 3, 5, \ldots,$ *the exact solution is given by*

$$x(t) = \cos(\omega\pi t) - 1.$$

We plot the comparison between exact and approximate solutions to the given example at $M = 10$ *and corresponding to* $\omega = 3.5, \alpha = 2, \beta = 1$. *Further, we approximate the solution through Legendre wavelet method (LWM)* [48], *Jacobi polynomial method (JM) and Bernstein polynomials method (BM), as shown in* **Figure 9**.

From **Table 1**, we see that Bernstein polynomials also provide excellent solutions to fractional differential equations.

5. Conclusion and future work

The above analysis and discussion take us to the conclusion that the new method is very efficient for the solution of boundary value problems as well as initial value problems including coupled systems of fractional differential equations. One can easily extend the method for obtaining the solution of such types of problems with other kinds of boundary and initial conditions. Bernstein polynomials also give best approximate solutions to fractional order differential equations like Legendre wavelet method (LWM), approximation by Jacobi polynomial method (JPM), etc. The new operational matrices obtained in this method can easily be extended to two-dimensional and higher dimensional cases, which will help in the solution of fractional order partial differential equations. Also, we compare our result to that of approximate methods for different scale levels. We observed that the proposed method is also an accurate technique to handle numerical solutions.

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