Chapter

Development of Ellipsoidal Analysis and Filtering Methods for Nonlinear Control Stochastic Systems

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Abstract

The methods of the control stochastic systems (CStS) research based on the parametrization of the distributions permit to design practically simple software tools. These methods give the rapid increase of the number of equations for the moments, the semiinvariants, coefficients of the truncated orthogonal expansions of the state vector Y, and the maximal order of the moments involved. For structural parametrization of the probability (normalized and nonnormalized) densities, we shall apply the ellipsoidal densities. A normal distribution has an ellipsoidal structure. The distinctive characteristics of such distributions consist in the fact that their densities are the functions of positively determined quadratic form of the centered state vector. Ellipsoidal approximation method (EAM) cardinally reduces the number of parameters. For ellipsoidal linearization method (ELM), the number of equations coincides with normal approximation method (NAM). The development of EAM (ELM) for CStS analysis and CStS filtering are considered. Based on nonnormalized densities, new types of filters are designed. The theory of ellipsoidal Pugachev conditionally optimal control is presented. Basic applications are considered.

Keywords: conditionally optimal filtering and control, control stochastic system, ellipsoidal approximation method (EAM), ellipsoidal linearization method (ELM)

1. Introduction

The methods for the control stochastic systems (CStS) research based on the parametrization of the distributions permit to design practically simple software tools [1–6]. These methods give the rapid increase of the number of equations for the moments, the semiinvariants, and coefficients of the truncated orthogonal expansions of the state vector Y for the maximal order of the moments involved. For structural parametrization of the probability (normalized and nonnormalized) densities, we shall apply the ellipsoidal densities. A normal distribution has an ellipsoidal structure. The distinctive characteristics of such distributions consist in the fact that their densities are the functions of positively determined quadratic form $u = u(y) = (y^T - m^T)C(y - m)$ where m is an expectation of Y, C is some positively determined matrix. Ellipsoidal approximation method (EAM) cardinally reduces the number of parameters till $Q^{EAM} = Q^{NAM} + n_m - 1$ and $Q^{NAM} = r(r + 3)/2$ where $2n_m$ being the number of probabilistic moments. For ellipsoidal linearization method (ELM), we get $Q^{ELM} = Q^{NAM}$.

The theory of conditionally optimal filters (COF) is described in [7, 8] on the basis of methods of normal approximation (NAM), methods of statistical linearization (SLM), and methods of orthogonal expansions (OEM) for the differential stochastic systems on smooth manifolds with Wiener noise in the equations of observation and Wiener and Poisson noises in the state equations. The COF theory relies on the exact nonlinear equations for the normalized one-dimensional a posteriori distribution. The paper [9] considers extension of [7, 8] to the case where the a posteriori one-dimensional distribution of the filtration error admits the ellipsoidal approximation [4]. The exact filtration equations are obtained, as well as the OEM-based equation of accuracy and sensitivity, the elements of ellipsoidal analysis of distributions are given, and the equations of ellipsoidal COF (ECOF) using EAM and ELM are derived. The theory of analytical design of the modified ellipsoidal suboptimal filters was developed in [10, 11] on the basis of the approximate solution by EAM (ELM) of the filtration equation for the nonnormalized a posteriori characteristic function. The modified ellipsoidal conditionally optimal filters (MECOF) were constructed in [12] on the basis of the equations for nonnormalized distributions. It is assumed that there exist the Wiener and Poisson noises in the state equations and only Wiener noise being in the observation equations. At that, the observation noise can be non-Gaussian.

Special attention is paid to the conditional generalization of Pugachev optimal control [13] based on EAM (ELM).

Let us consider the development of EAM (ELM) for solving problems of ellipsoidal analysis and optimal, suboptimal, and conditionally optimal filtering and control in continuous CStS with non-Gaussian noises and stochastic factors.

2. Ellipsoidal approximation method

This method was worked out in [1–4] for analytical modeling of stochastic process (StP) in multidimensional nonlinear continuous, discrete and continuous-discrete (CStS). Let us consider elements of EAM.

Following [1–4] let us find ellipsoidal approximation (EA) for the density of r-dimensional random vector by means of the truncated expansion based on biorthogonal polynomials $\left\{p_{r,\nu}(u(y)), q_{r,\nu}(u(y))\right\}$, depending only on the quadratic form $u = u(y)$ $u = u(y)$ for which some probability density of the ellipsoidal structure $w(u(y))$ serves as the weight:

$$\int_{-\infty}^{\infty} w(u(y))p_{r,\nu}(u(y))q_{r,\mu}(u(y))dy = \delta_{\nu\mu}. \quad (1)$$

The indexes ν and μ at the polynomials mean their degrees relative to the variable u. The concrete form and the properties of the polynomials are determined further. But without the loss of generality, we may assume that $q_{r,0}(u) = p_{r,0}(u) = 1$. Then the probability density of the vector Y may be approximately presented by the expression of the form:

$$f(y) \approx f^*(u) = w(u)\left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u)\right]. \qquad (2)$$

Here the coefficients $c_{r,\nu}$ are determined by the formula:

$$c_{r,\nu} = \int_{-\infty}^{\infty} f(y) q_{r,\nu}(u) dy = E q_{r,\nu}(U), \quad (\nu = 1, \dots, N). \qquad (3)$$

As $p_{r,0}(u)$ and $q_{r,0}(u)$ are reciprocal constants (the polynomials of zero degree), then always $c_{r,0} p_{r,0} = 1$ and we come to the following results.

Statement 1. Formulae (2) and (3) express the essence of the EA of the probability density of the random vector$Y$.

For the control problems, the case when the normal distribution is chosen as the distribution $w(u)$ is of great importance

$$w(u) = w\left(x^T C x\right) = \frac{1}{\sqrt{(2\pi)^r |K|}} \exp\left(-x^T K^{-1} x/2\right); \qquad (4)$$

accounting that $C = K^{-1}$, we reduce the condition of the biorthonormality (1) to the form

$$\frac{1}{2^{r/2}\Gamma(r/2)} \int_0^{\infty} p_{r,\nu}(u) q_{r,\mu}(u) u^{r/2-1} e^{-u/2} du = \delta_{\nu\mu}, \qquad (5)$$

where $\Gamma(\cdot)$ is gamma function [5].

Statement 2. The problem of the choosing of the polynomial system $\left\{p_{r,\nu}(u) q_{r,\mu}(u)\right\}$ which is used at the EA of the densities (4) and (5) is reduced to finding a biorthonormal system of the polynomials for which the χ^2-distribution with r degrees of the freedom serves as the weigh.

A system of the polynomials which are relatively orthogonal to χ^2-distribution with r degrees of the freedom is described by series:

$$S_{r,\nu}(u) = \sum_{\mu=0}^{\nu} (-1)^{\nu+\mu} C_\nu^\mu \frac{(r+2\nu-2)!!}{(r+2\mu-2)!!} u^\mu. \qquad (6)$$

The main properties of polynomials $S_{r,\nu}$ are given in [2–4]. Between the polynomials $S_{r,\nu}(u)$ and the system of the polynomials $\left\{p_{r,\nu}(u),\ q_{r,\mu}(u)\right\}$, the following relations exist:

$$p_{r,\nu}(u) = S_{r,\nu}(u), q_{r,\nu}(u) = \frac{(r-2)!!}{(r+2\nu-2)!!(2\nu)!!} S_{r,\nu}(u), \quad r \geq 2. \qquad (7)$$

Example 1. Formulae for polynomials $p_{r,\nu}(u)$ and $q_{r,\nu}(u)$ and its derivatives for some r and ν are as follows [4]:

- At $r = 2$, $\nu \geq 2$,

$$p_{2,\nu}(u) = u^\nu, \quad q_{2,\nu}(u) \equiv 0, \quad q'_{2,\nu}(u) \equiv 0, \quad q''_{2,\nu}(u) \equiv 0;$$

- At $r \geq 2$, $\nu = 2$

$$p_{r,2}(u) = u^2, \quad q_{r,2}(u) = \frac{1}{8}u^2, \quad q'_{r,2}(u) = \frac{1}{4}u, \quad q''_{r,2}(u) = \frac{1}{4}.$$

For $r = 2$ at $\nu = 3$ we have

$$p_{2,3}(u) = u^3, \quad q_{2,3}(u) \equiv 0, \quad q'_{2,3}(u) \equiv 0, \quad q''_{2,3}(u) \equiv 0;$$

at $r = 3$

$$\left.\begin{array}{l} p_{3,3}(u) = S_{3,3}(u), \quad q_{3,3}(u) = \dfrac{1}{5040}S_{3,3}(u), \\ q'_{3,3}(u) = \dfrac{1}{5040}S'_{3,3}(u), \quad q''_{3,3}(u) = \dfrac{1}{5040}S''_{3,3}(u), \\ S_{3,3}(u) = -105 + 105u - 21u^2 + u^3, \\ S'_{3,3}(u) = 105 - 42u + 3u^2, \quad S''_{3,3}(u) = -42 + 6u; \end{array}\right\}$$

at $r = 4$:

$$\left.\begin{array}{l} p_{4,3}(u) = S_{4,3}(u), \quad q_{4,3}(u) = \dfrac{1}{9216}S_{4,3}(u), \\ q'_{4,3}(u) = \dfrac{1}{9216}S'_{4,3}(u), \quad q''_{4,3}(u) = \dfrac{1}{9216}S'_{4,3}(u), \\ S_{4,3}(u) = -197 + 144u - 24u^2 + u^3, \\ S'_{4,3}(u) = 144 - 48u + 3u^2, \quad S''_{4,3}(u) = -48 + 6u. \end{array}\right\}$$

Following [5] we consider the H-space $L_2(R^r)$ and the orthogonal system of the functions in them where the polynomials $S_{r,\nu}(u)$ are given by Formula (6), and $w(u)$ is a normal distribution of the r-dimensional random vector (4). This system is not complete in $L_2(R^r)$. But the expansion of the probability density $f(u) = f\left(\left(y^T - m^T\right)C(y - m)\right)$ of the random vector Y which has an ellipsoidal structure over the polynomials $p_{r,\nu}(u) = S_{r,\nu}(u)$, m.s. converges to the function $f(u)$ itself. The coefficients of the expansion in this case are determined by relation:

$$c_{r,\nu} = \int_{-\infty}^{\infty} f(u)p_{r,\nu}(u)dy / \frac{(2\nu)!!(r + 2\nu - 2)!!}{(r-2)!!}. \tag{8}$$

Statement 3. The system of the functions $\left\{\sqrt{w(u)}S_{r,\nu}(u)\right\}$ forms the basis in the subspace of the space $L_2(R^r)$ generated by the functions $f(u)$ of the quadratic form $u = (y - m)^T C(y - m)$.

At the probability density expansion over the polynomial $S_{r,\nu}(u)$, the probability densities of the random vector Y and all its possible projections are consistent. In other words, at integrating the expansions over the polynomials $S_{h+l,\nu}(u)$ and $h + l = r$, of the probability densities of the r-dimensional vector Y,

$$\begin{array}{l} f(y) = \dfrac{1}{\sqrt{(2\pi)^{h+l}|K|}}e^{-u/2}\left[1 + \displaystyle\sum_{\nu=2}^{N} c_{h+l,\nu}S_{h+l,\nu}(u)\right], \quad u = (y - m)^T K^{-1}(y - m), \\ \quad y = \left[y'^T y''^T\right]^T, \end{array} \tag{9}$$

on all the components of the l-dimensional vector y'', we obtain the expansion over the polynomials $S_{h,\nu}(u_1)$ of the probability density of the h-dimensional vector Y' with the same coefficients

$$f(y') = \frac{1}{\sqrt{(2\pi)^h |K_{11}|}} e^{-u_1/2} \left[1 + \sum_{\nu=2}^{N} c_{h,\nu} S_{h,\nu}(u_1)\right], \quad u_1 = (y' - m')^T K_{11}^{-1}(y' - m'),$$
$$c_{h,\nu} = c_{h+l,\nu}, \tag{10}$$

where K_{11} is a covariance matrix of the vector Y'.

But in approximation (10) the probability density of h-dimensional random vector Y' obtained by the integration of expansion (9) the density of $(h+l)$-dimensional vector is not optimal EA of the density.

For the random r-dimensional vector with an arbitrary distribution, the EA (2) of its distribution determines exactly the moments till the N^{th} order inclusively of the quadratic form $U = (Y-m)^T K^{-1}(Y-m)$, i.e.,

$$EU^\mu = E^{EA} U^\mu, \quad \mu \le N. \tag{11}$$

(E^{EA} stands for expectation relative to EA distribution).

In this case the initial moments of the order s and $s = s_1 + \dots + s_r$ of the random vector Y at the approximation (4) are determined by the formula:

$$\alpha_{s_1,\dots,s_r} = \alpha_s = EY_1^{s_1} \dots Y_r^{s_r} \approx \int_{-\infty}^{\infty} y_1^{s_1} \dots y_r^{s_r} w(u) dy + \sum_{\nu=2}^{N} c_{r,\nu} \int_{-\infty}^{\infty} y_1^{s_1} \dots y_r^{s_r} p_{r,\nu}(u) w(u) dy \tag{12}$$

Statement 4. At the EA of the distribution of the random vector, its moments are combined as the sums of the correspondent moments of the normal distribution and the expectations of the products of the polynomials $p_{r,\nu}(u)$ by the degrees of the components of the vector Y at the normal density $w(u)$.

3. EAM accuracy

For control problems the weak convergence of the probability measures generated by the segments of the density expansion to the probability measure generated by the density itself is more important than m.s. convergence of the segments of the density expansion over the polynomials $S_{r,\nu}(u)$ to the density, namely,

$$\int_A w(u) \left[1 + \sum_{\nu=2}^{N} c_{r,\nu} p_{r,\nu}(u)\right] \to \int_A f(u) dy$$

uniformly relative to A at $N \to \infty$ on the σ-algebra of Borel sets of the space R^r. Thus the partial sums of series (2) give the approximation of the distribution, i.e., the probability of any event A determined by the density $f(u)$ with any degree of the accuracy. The finite segment of this expansion may be practically used for an approximate presentation of $f(u)$ with any degree of the accuracy even in those cases when $f(u)/\sqrt{w(u)}$ does not belong to $L_2(R^r)$. In this case it is sufficient to substitute $f(u)$ by the truncated density. Expansion (2) is valid only for the densities which have the ellipsoidal structure. It is impossible in principal to approximate with any degree of the accuracy by means of the EA (2) the densities which arbitrarily depend on the vector y.

One is the way of the estimate of the accuracy of the distribution approximation in the comparison of the probability characteristics calculated by means of the known density and its approximate expression. The most complete estimate of the accuracy of the approximation may be obtained by the comparison of the probability occurrence on the sets of some given class. Besides that taking into consideration that the probability density is usually approximated by a finite segment of its orthogonal expansion for instance, over Hermite polynomials or by a finite segment of the Edgeworth series [1–5] which contain the moments till the fourth order, the accuracy may be characterized by the accuracy of the definition of the moments of the random vector or its separate components, in particular, of the fourth order moments.

Corresponding estimates for these two ways of approximation are given in [2, 3].

4. Ellipsoidal linearization method

Now we consider ellipsoidal linearization of nonlinear transforms of random vectors Y using mean square error (m.s.e.) criterion optimal m.s.e. regression of vector $Z = \varphi(Y)$ on vector Y is determined by the formula [4, 6]:

$$m_z(Y) = h_2 Y, \quad h_2 = \Gamma_{zy}\Gamma_y^{-1} \tag{13}$$

or

$$m_z(Y) = h_1 Y + a, \quad h_1 = K_{zy}K_y^{-1}, \quad a = m_z - h_1 m_y. \tag{14}$$

where h_1 and h_2 are equivalent linearization matrices and m_y and K_y are mathematical expectation and covariance matrix $\left(\det|\ K_y| \neq 0\right)$. In case (14) coefficient h_1 is equal to

$$h_1 = K_{zy}K_y^{-1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(z - m_z)\left(y - m_y\right)^T K_y^{-1} f(z, y)dzdy$$
$$= \int_{-\infty}^{\infty}[m_z(y) - m_z]\left(y - m_y\right)^T K_y^{-1} f_1(y)dy \tag{15}$$

where $f_1(y)$ is the density of random vector Y.
For ellipsoidal density $f_1(y)$ in (15) is defined by

$$f_1(y) = f_1^{\mathrm{EL}}(y) = \tilde{f}_1^{\mathrm{EL}}\left(u(y), m_y, K_y, c\right). \tag{16}$$

In case (14) we get Statement 5 for ELM:

$$m_z(Y) \approx m_{1z}^{\mathrm{EL}} + h_1^{\mathrm{EL}}\left(m_y, K_y, c\right)Y^0, \tag{17}$$

where

$$h_1^{\mathrm{EL}} = h_1^{\mathrm{EL}}(m_y, K_y, c) = \int_{-\infty}^{\infty} [m_z(y) - m_z](y - m_y)^T K_y^{-1} f_1^{\mathrm{EL}}(y) dy$$
$$= \int_{-\infty}^{\infty} [m_z(y) - m_z](y - m_y)^T K_y^{-1} \tilde{f}_1^{\mathrm{EL}}(u(y), m_y, K_y, c) dy. \tag{18}$$

In case (13) we have Statement 6 for ELM:

$$m_z(Y) \approx h_2^{\mathrm{EL}}(\Gamma_y, c) Y, \tag{19}$$

$$h_2^{\mathrm{EL}}(\Gamma_y, c) = \int_{-\infty}^{\infty} m_z(y) y^T \Gamma_y^{-1} f_1^{\mathrm{EL}}(y) dy = \int_{-\infty}^{\infty} m_z(y) y^T \Gamma_y^{-1} \tilde{f}_1(u(y), m_y, \Gamma_y, c) dy. \tag{20}$$

For control problems the following ELM new generalizations are useful:

1. Let us consider for fixed dimension $p = \dim y$ and N in (2) with normal $w(u)$ distinguish modifications of various orders ELM $_w^{p,2}$, ELM $_w^{p,3}$, …, ELM $_w^{p,N}$. In this case $c = \{c_{p,v}\}$ characterizes partial deviations from normal distributions of various orders v jointly for all p components of vector Y (be part of quadratic form $U(Y)$.

2. At decomposition of vector Y on $l_1, l_2, \dots, l_r$ random subvectors, $Y = \left[Y_{l_1}^T Y_{l_2}^T \dots Y_{l_r}^T\right]^T$, we distinguish $\mathrm{ELM}_w^{l_1,\dots,l_r,N}$. Coefficients $c_{l_1,v}, \dots, c_{l_2,v}$ characterize partial deviations of subvectors from normal distribution.

3. For matrix transforms $Z = \varphi(Y) = \left[\varphi_1(Y) \dots \varphi_q(Y)\right]^T$, $\varphi_i(Y) = \left[\varphi_{i1}(Y) \dots \varphi_{ip}(Y)\right]^T$ $\left(i = \overline{1, q}\right)$, $\dim\varphi = p \times q$, we have the following formulae for ELM:

$$m_z(Y) \approx m_{1z}^{\mathrm{EL}} + H_1^{\mathrm{EL}}(m_y, K_y, c)(Y - m); \tag{21}$$
$$m_z(Y) \approx H_2^{\mathrm{EL}} Y. \tag{22}$$

where

$$H_1^{\mathrm{EL}}(m_y, K_y, c) = \left[h_{11}^{\mathrm{EL}}(m_y, K_y, c) \dots h_{1q}^{\mathrm{EL}}(m_y, K_y, c)\right] \tag{23}$$
$$H_2^{\mathrm{EL}}(\Gamma_y, c) = \left[h_{21}^{\mathrm{EL}}(\Gamma_y, c) \dots h_{2q}^{\mathrm{EL}}(\Gamma_y, c)\right], \tag{24}$$

(h_{1i}^{EL} and h_{2i}^{EL} $\left(i = \overline{1, q}\right)$ are determined by formulae (18) and (19)).

4. For transforms depending on time process t, it is useful to work with overage ELM coefficients $\langle m_{iz} \rangle$ and $\langle h_i^{\mathrm{EL}} \rangle$ for time intervals.

5. EAM and ELM for nonlinear CStS analysis

Let us consider nonlinear CStS defined by the following Ito vector stochastic differential equation:

$$dY_t = a(Y_t,t)dt + b(Y_t,t)dW_0 + \int_{R_0^q} c(Y_t,t,v)P^0(dt,dv), \quad Y(t_0) = Y_0. \tag{25}$$

Here $Y_t \in \Delta^y$ is (Δ^y is a smooth state manifold) $W_0 = W_0(t)$ is an r – dimensional Wiener StP of intensity $v_0 = v_0(t)$, $P(\Delta, \mathscr{A})$ is simple Poisson StP for any set $\mathscr{A}$, $\Delta = (t_1, t_2]$, $P^0(\Delta, \mathscr{A}) = P^0(\Delta, \mathscr{A}) - \mu_P(\Delta, \mathscr{A})$, $\mu_P(\Delta, \mathscr{A}) = \mathrm{E}P^0(\Delta, \mathscr{A}) = \int_\Delta v_P(\tau, \mathscr{A})d\tau$. Integration by v extends to the entire space R^q with deleted origin, a and b are certain functions mapping $R^p \times R$, respectively, into R^p, R^{pr}, and c is for $R^p \times R^q$ into R^p.

Following [4] we use for finding the one-dimensional probability density $f_1(y;t)$ of the r-dimensional $Y(t)$ which is determined by Eq. (25). Suppose that we know a distribution of the initial value $Y_0 = Y(t_0)$ of the StP $Y(t)$. Following the idea of EAM, we present the one-dimensional density in the form of a segment of the orthogonal expansion in terms of the polynomials dependent on the quadratic form $u = (y^T - m^T)C(y - m)$ where m and $K = C^{-1}$ are the expectation and the covariance matrix of the StP$Y(t)$:

$$f_1(y;t) \cong f_1^{EAM}(u) = w_1(u)\left[1 + \sum_{\nu=2}^{N} c_{p,v}p_{p,\nu}(u)\right]. \tag{26}$$

Here $w_1(u)$ is the normal density of the p-dimensional random vector which is chosen in correspondence with the requirement $c_{p,1} = 0$. The optimal coefficients of the expansion $c_{p,v}$ are determined by the relation

$$c_{p,v} = \int_{-\infty}^{\infty} f_1(y;t)q_{p,v}(u)dy = Eq_{p,v}(U), \qquad (\nu = 1, \dots, N). \tag{27}$$

The set of the polynomials $\left\{p_{p,v}(u), q_{p,v}(u)\right\}$ is constructed on the base of the orthogonal set of the polynomials $\{S_{p,v}(u)\}$ according to the following rule which provides the biorthonormality of system at $p \geq 2$ given by (5). Thus the solution of the problem of finding the one-dimensional probability density by EAM is reduced to finding the expectation m, the covariance matrix K of the state vector, and the coefficients of the correspondent expansion $c_{p,v}$ also.

So we get the equations

$$\dot{m} = \varphi_{10}(m,K,t) + \sum_{\nu=2}^{N} c_{p,v}\varphi_{1\nu}(m,K,t), \tag{28}$$

$$\dot{K} = \varphi_{20}(m,K,t) + \sum_{\nu=2}^{N} c_{p,v}\varphi_{2\nu}(m,K,t), \tag{29}$$

$$\dot{c}_{p,\kappa} = -\left(\frac{c_{p,\kappa-1}}{2p} - \frac{\kappa c_{p,\kappa}}{p}\right) \times \mathrm{tr}\left\{K^{-1}\varphi_{20}(m,K,t) + K^{-1}\sum_{\nu=2}^{N} c_{p,v}\varphi_{2\nu}(m,K,t)\right\}$$
$$+\psi_{\kappa 0}(m,K,t) + \sum_{\nu=2}^{N} c_{p,v}\psi_{\kappa\nu}(m,K,t), \qquad \kappa = 2, \dots, N, \tag{30}$$

where the following indications are introduced:

$$\varphi_{10}(m,K,t) = \int_{-\infty}^{\infty} a(y,t)w_1(u)dy, \varphi_{1\nu}(m,K,t) = \int_{-\infty}^{\infty} a(y,t)p_{p,v}(u)w_1(u)dy,$$

$$\varphi_{20}(m,K,t) = \int_{-\infty}^{\infty} \left[a(y,t)\left(y^T - m^T\right) + (y-m)a(y,t)^T + \overline{\sigma}(y,t)\right]w_1(u)dy,$$

$$\varphi_{2v}(m,K,t) = \int_{-\infty}^{\infty} \left[a(y,t)\left(y^T - m^T\right) + (y-m)a(y,t)^T + \overline{\sigma}(y,t)\right]p_{p,v}(u)w_1(u)dy,$$

$$\overline{\sigma}(y,t) = \overline{\sigma}(y,t) + \int_{R_0^q} c(y,t,v)c(y,t,v)^T v_P(t,dv), \quad \sigma(y,t) = b(y,t)\nu_0(t)b(y,t)^T, \tag{31}$$

$$\begin{aligned}\psi_{\kappa 0}(m,K,t) = &\int_{-\infty}^{\infty} \left[q'_{p,\kappa}(u)\left(2(y-m)^T K^{-1}a(y,t) + \text{tr } K^{-1}\sigma(y,t)\right)\right.\\ &\left.+ 2q''_{\kappa}(u)(y-m)^T K^{-1}\sigma(y,t)(y-m)\right]w_1(u)dy\\ \psi_{\kappa v}(m,K,t) = &\int_{-\infty}^{\infty} q'_{p,\kappa}(u)\left[2(y-m)^T K^{-1}a(y,t) + \text{tr } K^{-1}\sigma(y,t)\right]\\ &+ 2q''_{\kappa}(u)(y-m)^T K^{-1}\sigma(y,t)(y-m)\}p_{p,v}(u)w_1(u)dy.\end{aligned} \tag{32}$$

Eqs. (28)–(30) at the initial conditions

$$m(t_0) = m_0, \quad K(t_0) = K_0, \quad c_{p,\kappa}(t_0) = c^0_{p,\kappa} \quad (\kappa = 2, \dots, N) \tag{33}$$

determine $m, K, c_{p,2}, \dots, c_{p,N}$ as time functions. For finding the variables $c^0_{p,\kappa}$, the density of the initial value Y_0 of the state vector should be approximated by Formula (26).

So we get the following result.

Statement 7. At sufficient conditions of existence and uniqueness of StP in Eq. (25), Eqs. (28)–(33) define EAM.

For stationary CStS we get the corresponding EAM equations putting in Eqs. (28)–(30) right-hand equal to zero.

Example 2. Following [4, 14, 15] in case of vibroprotection Duffing StS:

$$\ddot{X} + \delta\dot{X} + \omega^2 X + \mu X^3 = U + V, \quad X(t_0) = X_0, \quad \dot{X}(t_0) = \dot{X}_0,$$

(δ, ω^2, μ, U are constants, V is the white noise with intensity v) with accuracy till 4th probabilistic moments, ellipsoidal approximation of one-dimensional density is described by the set of parameters:

$$m_1 = EX, \quad m_2 = \dot{X}, \quad K_{11} = EX^{02}, \quad K_{12} = EX^0\dot{X}^0, \quad K_{22} = E\dot{X}^{02} \quad \text{and} \quad c_{2,2}.$$

These parameters satisfy the following ordinary differential equations:

$$\dot{m}_1 = m_2, \quad m_1(t_0) = m_1^0, \quad \dot{m}_2 = U - \omega^2 m_1 + \mu\left(m_1^3 + 3m_1K_{11}\right) - \delta m_2,$$
$$m_2(t_0) = m_2^0;$$

$$
\begin{aligned}
&\dot{K}_{11} = 2K_{12},\\
&\dot{K}_{12} = K_{22} - \omega^2 K_{11} + 3\mu K_{11}\left(K_{11} + m_1^2\right) - \delta K_{12} + 24\mu c_{2,2} K_{11}^2,\\
&\dot{K}_{22} = \nu - 2\left(\omega^2 K_{12} - 3\mu K_{12}\left(K_{11} + m_1^2\right) + \delta K_{22}\right) + 48\mu c_{2,2} K_{11} K_{12},\\
&K_{11}(t_0) = K_{11}^0, \quad K_{12}(t_0) = K_{12}^0, \quad K_{22}(t_0) = K_{22}^0;\\
&\dot{c}_{2,2} = c_{2,2}\left(\frac{K_{11}\nu}{|K|} - 5\delta\right) \quad \left(|K| = K_{11}K_{22} - K_{12}^2\right), \quad c_{2,2}(t_0) = c_{2,2}^0.
\end{aligned}
$$

At $U = 0$ stationary values are as follows:

$$
\overline{m}_1 = 0, \quad \overline{m}_2 = 0, \quad \overline{K}_{11} = \frac{\omega^2\sqrt{\omega^4 - 6\mu\nu/\delta}}{6\mu}, \quad \overline{K}_{12} = 0, \quad \overline{K}_{22} = \frac{\nu}{2\delta}, \quad \overline{c}_{2,2} = 0.
$$

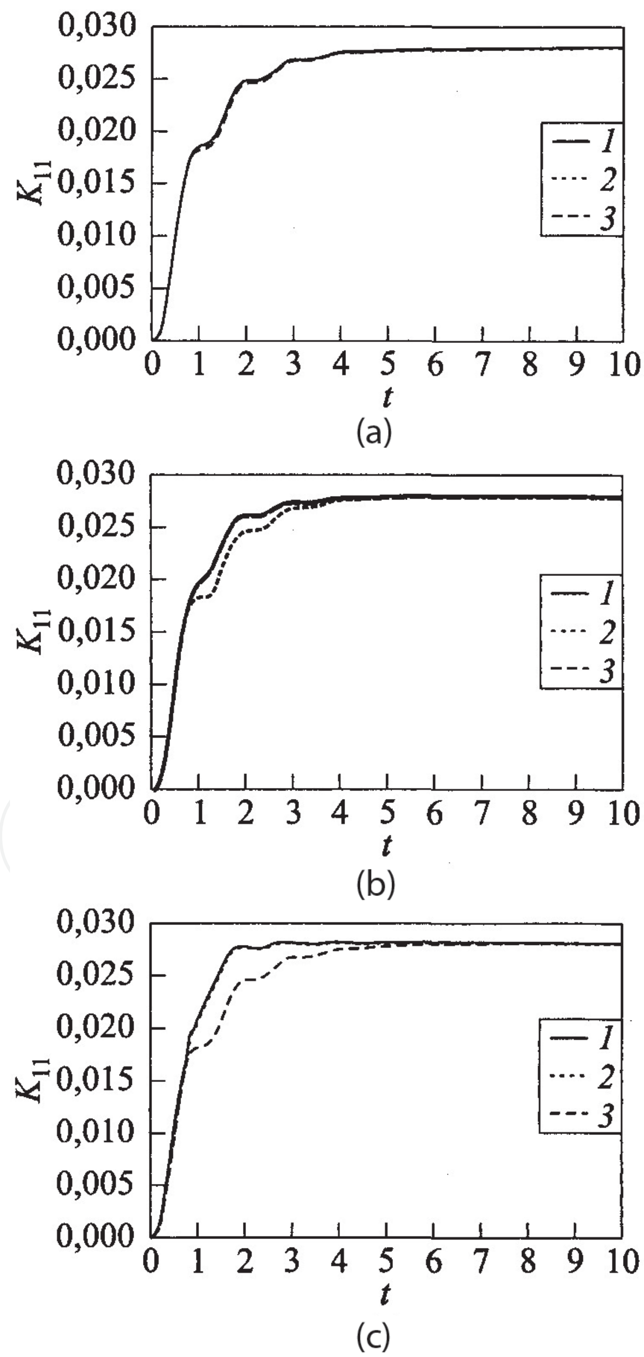

Figure 1.
K_{11} graphs for at 0,1 (a); 0,5 (b); 1,0 (c).

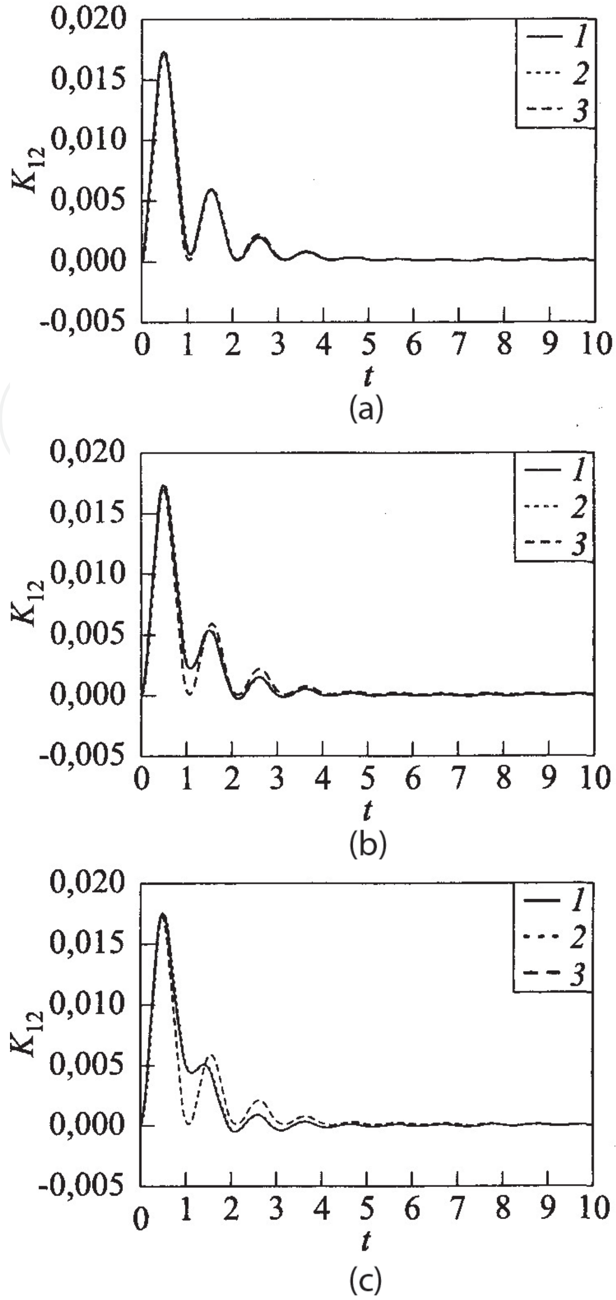

Figure 2.
K_{12} graphs for at 0,1 (a); 0,5 (b); 1,0 (c).

At conditions

$$1.\ U = 0; \quad \mu = 0.1; \quad \omega = 3; \quad \delta = 1; \quad \nu = 0.5;$$

$$2.\ U = 0; \quad \mu = 0.5; \quad \omega = 3; \quad \delta = 1; \quad \nu = 0.5;$$

$$3.\ U = 0; \quad \mu = 1; \quad \omega = 3; \quad \delta = 1; \quad \nu = 0.5$$

And at zero initial conditions, the results of analytical modeling for K_{11}, K_{12}, K_{22} are given in **Figures 1–3**. Mathematical expectations m_1 and m_n are equal to zero.

Graphs (1) are the results of integration of NAM equations at initial stage. Then for nongenerated covariance matrix K integration of EAM equations (2). Graphs are the results of EAM equation integration at the whole stage.

The results of investigations for $c_{2,2}$ are given in **Figure 4** for the following sets of conditions:

1. $U = 0$; $\mu = 1$; $\omega = 3$; $\delta = 0,5$; $\nu = 0,5$; $T = [0, 20]$ zero initial conditions;
2. $U = 0$; $\mu = 1$; $\omega = 3$; $\delta = 0,5$; $\nu = 1$; $T = [0, 20]$ zero initial conditions;
3. $U = 0$; $\mu = 1$; $\omega = 3$; $\delta = 0,5$; $\nu = 1$; $T = [0, 20]$ zero initial conditions except $m_1(0) = 0,2$;
4. $U = 0$; $\mu = 1$; $\omega = 3$; $\delta = 1$; $\nu = 1$; $T = [0, 20]$ zero initial conditions.

For the stationary CStS regimes, EAM gives the same results as NAM (MSL). EAM describes non-Gaussian transient vibro StP at initial stage.

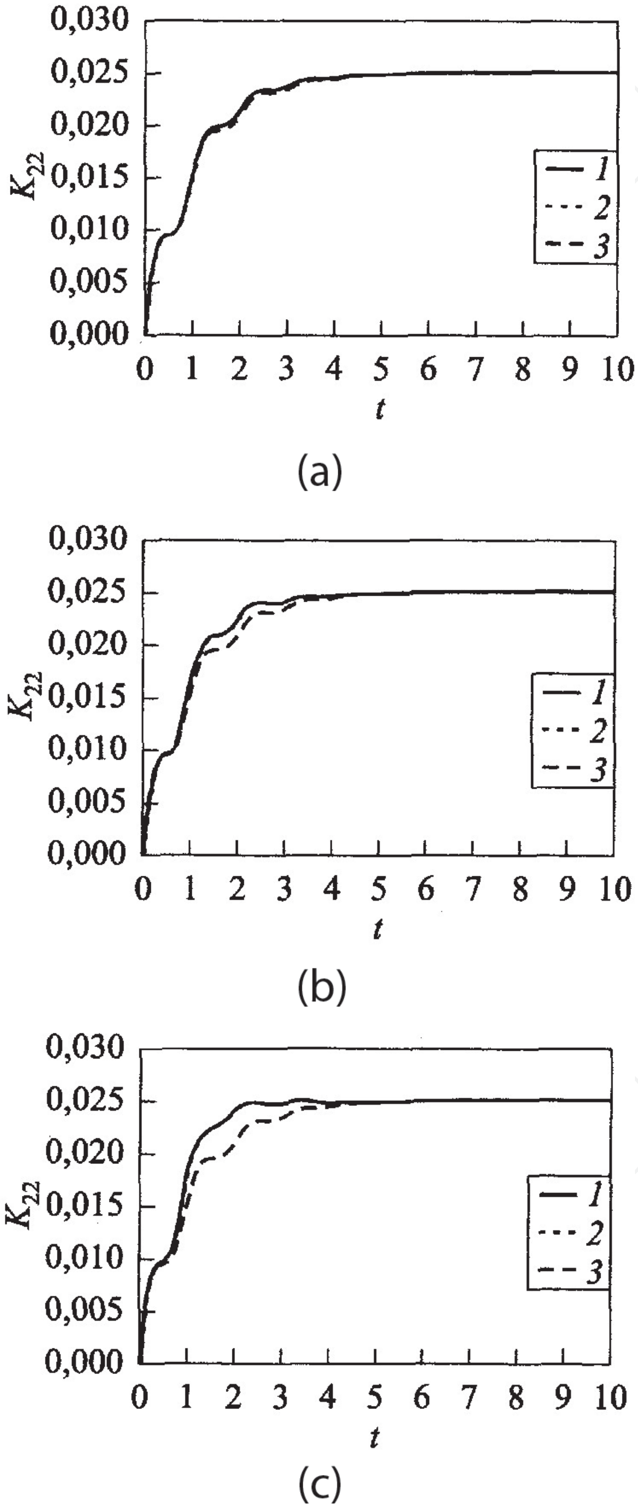

Figure 3.
K_{22} graphs for at 0,1 (a); 0,5 (b); 1,0 (c).

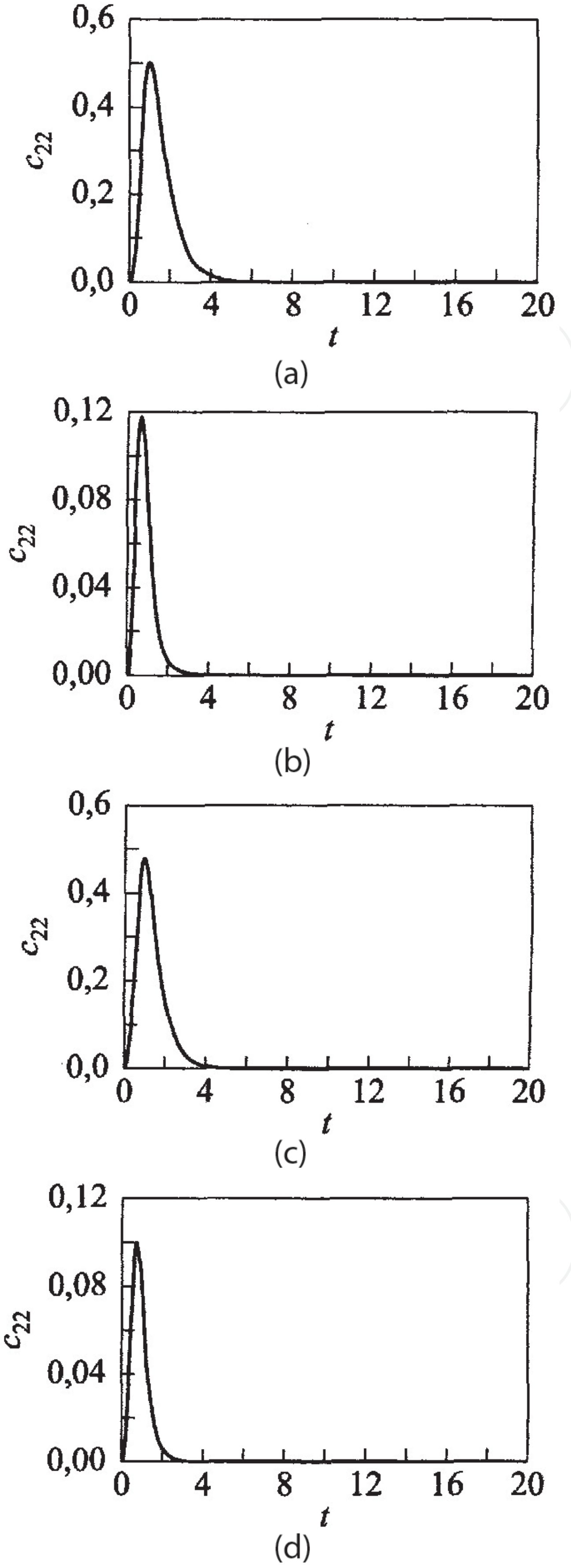

Figure 4.
C_{22} graphs for at sets № 1 (a); 2 (b); 3 (c); 4 (d).

Methodological and software support for analysis and filtering problem CStS for shock and vibroprotection is given in [4, 14].

6. Exact filtering equations for continuous a posteriori distribution

Following [7–9, 15], let the vector StP $\left[X_t^{\mathrm{T}} Y_t^{\mathrm{T}}\right]^{\mathrm{T}}$ be defined by a system on vector stochastic differential Ito equations:

$$\begin{aligned} dX_t &= \varphi(X_t, Y_t, \Theta, t)dt + \psi'(X_t, Y_t, \Theta, t)dW_0 \\ &\quad + \int_{R_0^q} \psi''(X_t, Y_t, \Theta, t, v)P^0(dt, dv), \quad X(t_0) = X_0, \end{aligned} \tag{34}$$

$$\begin{aligned} dY_t &= \varphi_1(X_t, Y_t, \Theta, t)dt + \psi_1'(X_t, Y_t, \Theta, t)dW_0 \\ &\quad + \int_{R_0^q} \psi_1''(X_t, Y_t, \Theta, t, v)P^0(dt, dv), \quad Y(t_0) = Y_0. \end{aligned} \tag{35}$$

where $Y_t = Y(t)$ is an n_y-dimensional observed StP $Y_t \in \Delta^y$ (Δ^y is a smooth manifold of observations); $X_t \in \Delta^x$ (Δ^x is a smooth state manifold), $W_0 = W_0(t)$ is an n_w-dimensional Wiener StP $\left(n_w \geq n_y\right)$ of intensity $\nu_0 = \nu_0(\Theta, t)$; $P^0(\Delta, A) = P(\Delta, A) - \mu_P(\Delta, A)$, $P(\Delta, A)$ for any set A represents a simple Poisson StP, and $\mu_P(\Delta, A)$ is its expectation,

$$\mu_P(\Delta, A) = \mathrm{E}P(\Delta, A) = \int_{\Delta} \nu_P(\tau, A)d\tau;$$

$v_P(\Delta, A)$ is the intensity of the corresponding Poisson flow of events, $\Delta = (t_1, t_2]$; integration by v extends to the entire space R^q with deleted origin; Θ is the vector of random parameters of size n_Θ; $\varphi = \varphi(X_t, Y_t, \Theta, t)$, $\varphi_1 = \varphi_1(X_t, Y_t, \Theta, t)$, $\psi' = \psi'(X_t, Y_t, \Theta, t)$, and $\psi_1' = \psi_1'(X_t, Y_t, \Theta, t)$ are certain functions mapping $R^{n_x} \times R^{n_y} \times R$, respectively, into $R^{n_x}, R^{n_y}, R^{n_x n_w}$, $R^{n_y n_w}$; $\psi'' = \psi''(X_t, Y_t, \Theta, t, v)$, and $\psi_1'' = \psi_1''(X_t, Y_t, \Theta, t, v)$ are certain functions mapping $R^{n_x} \times R^{n_y} \times R^q$ into R^{n_x}, R^{n_y}. Determine the estimate $\hat{X}_t$ StP X_t at each time instant t from the results of observation of StP $Y(\tau)$ until the instant t, $Y_{t_0}^t = \{Y(\tau) : t_0 \leq \tau < t\}$.

Let us assume that the state equation has the form (34); the observation Eq. (35), first, contains no Poisson noise ($\psi''_1 \equiv 0$); and, second, the coefficient at the Wiener noise ψ_1' in the observation equations is independent of the state $\left(\psi_1'(X_t, Y_t, \Theta, t) = \psi_1'(Y_t, \Theta, t)\right)$, and then the equations of the problem of nonlinear filtration are given by

$$\begin{aligned} dX_t &= \varphi(X_t, Y_t, \Theta, t)dt + \psi'(X_t, Y_t, \Theta, t)dW_0 \\ &\quad + \int_{R_0^q} \psi''(X_t, Y_t, \Theta, t, v)P^0(dt, dv), \quad X(t_0) = X_0, \end{aligned} \tag{36}$$

$$dY_t = \varphi_1(X_t, Y_t, \Theta, t)dt + \psi_1(Y_t, \Theta, t)dW_0, \quad Y(t_0) = Y_0. \tag{37}$$

The known sufficient conditions for the existence and uniqueness of StP defined by (36) and (37) under the corresponding initial conditions [1, 3, 16] are satisfied.

The optimal estimate $\hat{X}_t$ minimizing the mean square of the error at each time instant t is known [10–14] to represent for any StP X_t and Y_t.

An a posteriori expectation StP X_t: $\hat{X}_t = \mathrm{E}\left[X_t | Y_{t_0}^t\right]$. To determine this conditional expectation, one needs to know $p_t = p_t\,(x)$ and $g_t = g_t(\lambda)$, the a posteriori one-dimensional density, and the characteristic function of the distribution StP X_t.

Introduce the nonnormalized one-dimensional a posteriori density $\tilde{p}_t(x, \Theta)$ and a characteristic function $\tilde{g}_t(\lambda, \Theta)$ according to

$$\tilde{p}_t(x, \Theta) = \mu_t p_t(x, \Theta),\ \ \tilde{g}_t(\lambda, \Theta) = \mathrm{E}_{\Delta^x}^{p_t}\left[e^{i\lambda^{\mathrm{T}} X_t}\mu_t\right] = \mu_t g_t(\lambda, \Theta), \tag{38}$$

where μ_t is a normalizing function and $\mathrm{E}_{\Delta^x}^{p_t}$ is the symbol of expectation on the manifold Δ^x on the basis of density $p_t(x)$. Then, by generalizing [11] to the case of Eqs. (36) and (37), we get the following exact equation of the rms optimal nonlinear filtration:

$$\begin{aligned} d\tilde{g}_t(\lambda, \Theta) = \mathrm{E}_{\Delta^x}^{\tilde{p}_t}\Bigg\{&\left[i\lambda^{\mathrm{T}}\varphi(X, Y_t, \Theta, t) - \frac{1}{2}\left(\psi'\nu_0\psi'^{\mathrm{T}}\right)(X, Y_t, \Theta, t)\right. \\ &+ \left.\int_{R_0^q}\left[e^{i\lambda^{\mathrm{T}}\psi''(X, Y_t, \Theta, t, v)} - 1 - i\lambda^{\mathrm{T}}\psi''(X, Y_t, \Theta, t, v)\right]\nu_P(\Theta, t, dv)\right]e^{i\lambda^{\mathrm{T}} X}\Bigg\}dt \\ &+ \mathrm{E}_{\Delta^x}^{\tilde{p}_t}\left\{\left[\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}} + i\lambda^{\mathrm{T}}\left(\psi'\nu_0\psi'^{\mathrm{T}}\right)(X, Y_t, \Theta, t)\right]e^{i\lambda^{\mathrm{T}} X}\right\}\left(\psi'\nu_0\psi'^{\mathrm{T}}\right)^{-1}(Y_t, \Theta, t)dY_t. \end{aligned} \tag{39}$$

If by following [15, 17] the function ψ'' in (36) admits the representation

$$\psi'' = \psi'\omega(\Theta, v), \tag{40}$$

where $P^0(\Delta, A) = P^0((0, t], dv)$, then Eqs. (36) and (37) take the form

$$\dot{X}_t = \varphi(X_t, Y_t, \Theta, t) + \psi'(X_t, Y_t, \Theta, t)V(\Theta, t), \quad X(t_0) = X_0, \tag{41}$$

$$\dot{Y}_t = \varphi(X_t, Y_t, \Theta, t) + \psi_1(Y_t, \Theta, t)V_0(\Theta, t), \quad Y(t_0) = Y_0. \tag{42}$$

with $V_0(\Theta, t) = \dot{W}_0(\Theta, t); \quad V(\Theta, t) = \dot{\overline{W}}(\Theta, t)$,

$$\overline{W}(\Theta, t) = W_0(\Theta, t) + \int\limits_{R_0^q}\omega(\Theta, v)P^0((0, t], dv), \tag{43}$$

where $\nu_P(\Theta, t, v)dv = [\partial\mu(\Theta, t, v)/\partial t]dv$ is the intensity of the Poisson flow of discontinuities equal to $\omega(\Theta, t)$; the logarithmic derivatives of the one-dimensional characteristic functions obey certain formulas

$$\chi^{\overline{W}}(\rho; \Theta, t) = \chi^{W_0}(\rho; \Theta, t) + \int\limits_{R_0^q}\left[e^{i\rho^{\mathrm{T}}\omega(\Theta, v)} - 1 - i\rho^{\mathrm{T}}\omega(\Theta, v)\right]\nu_P(\Theta, t, v)dv, \tag{44}$$

where

$$\chi^{W_0}(\rho; \Theta, t) = -\frac{1}{2}\rho^{\mathrm{T}}\nu_0(\Theta, t)\rho.$$

In this case, the integral term in (39) admits the following notation:

$$\gamma = \int_{R_0^q} \left[e^{i\lambda^{\mathrm{T}}\psi''(X_t,Y_t,\Theta,t)\omega(\Theta,v)} - 1 - i\lambda^{\mathrm{T}}\psi''(X_t,Y_t,\Theta,t)\omega(\Theta,v) \right] \nu_P(\Theta,t,v)dv. \tag{45}$$

For the Gaussian CStS, the condition $\gamma \equiv 0$ is, obviously, true, and we come to the well-known statements [11, 15, 17].

Statement 8. Let the conditions for existence and uniqueness be satisfied for the non-Gaussian CStS (36) and (37). Then, the equation with a continuous rms of the optimal nonlinear filtration for the nonnormalized characteristic function (38) is given by (39).

Statement 9. Let the non-Gaussian CStS (41) and (42) the conditions for existence and uniqueness be satisfied. Then, the equation with continuous rms of optimal nonlinear filtration for the nonnormalized characteristic function is given by (39) provided that (45).

7. EAM (ELM) for nonlinear CStS filtering

EAM (ELM) for approximate conditionally optimal and suboptimal filtering (COF and SOF) in continuous CStS for normalized one-dimensional density is given in [11]. Let us consider the case of nonnormalized densities:

$$\tilde{p}_t(x,\Theta) \approx p_t^*(u,\Theta) = w(u,\Theta)\left[\mu_t + \sum_{\nu=1}^{N} c_\nu p_\nu(u)\right]. \tag{46}$$

Here, $w = w(u,\Theta)$ is the reference density and $\{p_\nu(u), q_\nu(u)\}$ is the biorthonormal system of polynomials, $C_t = K_t^{-1}$; K_t is the covariance matrix and c_ν is the coefficient of ellipsoidal expansion

$$c_\nu = \mu_t \mathrm{E}^{EA}\left[q_\nu(U_t)\right] = \left[q_\nu(U_\lambda)\tilde{g}_t^{EA}(\lambda,\Theta)\right]_{\lambda=0}, \tag{47}$$

with the notation

$$\begin{aligned} u &= \left(x^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\left(x - \hat{X}_t\right); \quad U_t = \left(X_t^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\left(X_t - \hat{X}_t\right); \\ U_\lambda &= \left(\partial^{\mathrm{T}}/i\partial\lambda - \hat{X}_t\right)C_t\left(\partial/i\partial\lambda - \hat{X}_t\right); \end{aligned} \tag{48}$$

E^{EA} is the expectation for the ellipsoidal distribution (46).

According to [11], in order to compile the stochastic differential equations for the coefficients c_ν, one has to find the stochastic Ito differential of the product $q_\chi(u)\tilde{g}_t(\lambda)$ bearing in mind that u depends on the estimate $\hat{X}_t = m_t/\mu_t$ and the expectation m_t and the normalizing function μ_t obey the stochastic differential equations. Therefore, one has to replace the variables x and u and the operators $\partial/i\partial\lambda$ and U_λ, carry out differentiation, and then assume that $\lambda = 0$.

So by repeating [11], we get that the equations for m_t and μ_t with the function $\hat{\varphi}_1$ obey the formula

$$\hat{\varphi}_1 = \mathrm{E}_{\Delta^x}^{p_t}[\varphi_1], \tag{49}$$

with regard to the notation

$$\sigma_0 = \psi\nu_0\psi^{\mathrm{T}}, \quad \sigma_1 = \psi\nu_0\psi_1^{\mathrm{T}}, \quad \sigma_2 = \psi_1\nu_0\psi_1^{\mathrm{T}} \tag{50}$$

and the equation for $\tilde{g}_t(\lambda, \Theta)$ is representable as

$$dm_t = fdt + hdY_t, \quad d\mu_t = bdY_t, \tag{51}$$

$$d\tilde{g}_t = Adt + BdY_t. \tag{52}$$

It is denoted here that

$$f = \mu_t f_0 + \sum_{\nu=1}^{N} c_\nu f_\nu, \quad h = \mu_t h_0 + \sum_{\nu=1}^{N} c_\nu h_\nu, \quad b = \mu_t b_0 + \sum_{\nu=1}^{N} c_\nu b_\nu,$$

$$f_0 = f_0(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^x}^{w}[\varphi], \ f_\nu = f_\nu(Y_t, \hat{X}_t,, \Theta, t) = \mathrm{E}_{\Delta^x}^{wp_\nu}[\varphi],$$

$$h_0 = h_0(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^U}^{w}\left[\sigma_1(Y_t, \Theta, t) + X\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}}\right]\sigma_2(Y_t, \Theta, t)^{-1},$$

$$h_\nu = h_\nu(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^U}^{wp_\nu}\left[\sigma_1(X, Y_t, \Theta, t) + X\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}}\right]\sigma_2(Y_t, \Theta, t)^{-1},$$

$$b_0 = b_0(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^U}^{w}\left[\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}}\right]\sigma_2(Y_t, \Theta, t)^{-1}, \quad b_\nu = b_\nu(Y_t, \hat{X}_t, \Theta, t)$$
$$= \mathrm{E}_{\Delta^U}^{wp_\nu}[\varphi_1(X, Y_t, \Theta, t)]\sigma_2(Y_t, \Theta, t)^{-1},$$

$$A = \mathrm{E}_{\Delta^x}^{\tilde{p}_t}\left\{i\lambda^{\mathrm{T}}\varphi(X, Y_t, \Theta, t) - \frac{1}{2}\lambda^{\mathrm{T}}\left(\psi'\nu_0\psi'^{\mathrm{T}}\right)(X, Y_t, \Theta, t)\lambda\right.$$
$$\left.\int_{R_0^q}\left[e^{i\lambda^{\mathrm{T}}\psi''(X, Y_t, \Theta, t, v)} - 1 - i\lambda^{\mathrm{T}}\psi''(X, Y_t, \Theta, t, v)\right]\nu_P(t, dv)e^{i\lambda^{\mathrm{T}}X}\right\},$$

$$B = \mathrm{E}_{\Delta^x}^{\tilde{p}_t}\left[\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}} + i\lambda^{\mathrm{T}}\left(\psi'\nu_0\psi_1'^{\mathrm{T}}\right)(X, Y_t, \Theta, t)\right]e^{i\lambda^{\mathrm{T}}X}\left(\psi_{1'}\nu_0\psi_1'^{\mathrm{T}}\right)^{-1}(X, Y_t, \Theta, t). \tag{53}$$

The equations for coefficient of MOE in (46) and (47) in virtue of [11] have the form

$$dc_\chi = \mathrm{E}_{\Delta^x}^{p^*}\{q_{\chi'}(u)\left(2\varphi^{\mathrm{T}}C_t(X - \hat{X}_t) + \mathrm{tr}[C_t\sigma_0]\right) + 2q_{\chi''}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\sigma_0C_t(X - \hat{X}_t)$$
$$+\int_{R_0^q}\left[q_\chi(\overline{u}) - q_\chi(u) - 2q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\psi''\right]\nu_P(t, dv) - q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t(h + \hat{X}_t b)\varphi_1/\mu_t$$
$$+q_{\chi'}(u)\mathrm{tr}\left[(h + \hat{X}_t b)\sigma_1^{\mathrm{T}}C_t\right]/\mu_t + 2q_{\chi''}(u)\mathrm{tr}\left[(h + \hat{X}_t b)\sigma_1^{\mathrm{T}}C_t(X - \hat{X}_t)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\right]/\mu_t\}dt$$
$$+\left\{\frac{1}{2n}(c_{\chi-1} + 2\chi c_\chi)\mathrm{tr}\left[\dot{C}_tK_t\right] + \frac{c_{\chi-1}}{2n}\left(\mathrm{tr}\left[C_th\sigma_2h^{\mathrm{T}}\right] - 2\hat{X}_t^{\mathrm{T}}C_th\sigma_2b^{\mathrm{T}} + \hat{X}_t^{\mathrm{T}}C_t\hat{X}_tb\sigma_2b^{\mathrm{T}}\right)/\mu_t^2\right\}dt$$
$$+\mathrm{E}_{\Delta^x}^{p^*}\left\{\left[q_\chi(u)\varphi_1^{\mathrm{T}} + q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)C_t\sigma_1\right]\sigma_2^{-1}\right\}dY_t. \tag{54}$$

In addition to the notation (54), we assume that

$$\gamma_{\chi 0} = \gamma_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) = E_{\Delta^x}^{w}\{q_{\chi'}(u)\left(2\varphi(X, Y_t, \Theta, t)^{\mathrm{T}} C_t (X - \hat{X}_t) + \mathrm{tr}[C_t \sigma_0(X, Y_t, \Theta, t)]\right)$$
$$+2q_{\chi''}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \sigma_0(X, Y_t, \Theta, t) C_t (X - \hat{X}_t) + \int_{R_0^q} \left[q_{\chi}(\overline{u}) - q_{\chi}(u) - 2q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \psi''(X, Y_t, \Theta, t, v)\right] \nu_P(t, dv)\},$$
$$\gamma_{\chi\nu} = \gamma_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t) = E_{\Delta^x}^{wp_\nu}\{q_{\chi'}(u)\left(2\varphi(X, Y_t, \Theta, t)^{\mathrm{T}} C_t (X - \hat{X}_t) + \mathrm{tr}[C_t \sigma_0(X, Y_t, \Theta, t)]\right)$$
$$+2q_{\chi''}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \sigma_0(X, Y_t, \Theta, t) C_t (X - \hat{X}_t)$$
$$+ \int_{R_0^q} \left[q_{\chi}(\overline{u}) - q_{\chi}(u) - 2q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \psi''(X, Y_t, \Theta, t, v)\right] \nu_P(t, dv)\},$$
$$\varepsilon_{\chi 0} = \varepsilon_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^x}^{w}\{q_{\chi'}(u)\left[\sigma_1(X, Y_t, \Theta, t)^{\mathrm{T}} - \varphi_1(X, Y_t, \Theta, t)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)\right]$$
$$+2q_{\chi''}(u)\sigma_1(X, Y_t, \Theta, t)^{\mathrm{T}} C_t (X - \hat{X}_t)\left(X^{T} - \hat{X}_t^{\mathrm{T}}\right)\},$$
$$\varepsilon_{\chi\nu} = \varepsilon_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t) = \mathrm{E}_{\Delta^x}^{wp_\nu}\{q_{\chi'}(u)\left[\sigma_1(X, Y_t, \Theta, t)^{\mathrm{T}} - \varphi_1(X, Y_t, \Theta, t)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)\right]$$
$$+2q_{\chi''}(u)\sigma_1(X, Y_t, \Theta, t)^{\mathrm{T}} C_t (X - \hat{X}_t)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right)\},$$
$$\eta_{\chi 0} = \eta_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) = E_{\Delta^x}^{w}\left\{q_{\chi}(u)\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}} + q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \sigma_1(X, Y_t, \Theta, t)\right\}\sigma_2(Y_t, \Theta, t)^{-1},$$
$$\eta_{\chi\nu} = \eta_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t) = E_{\Delta^x}^{wp_\nu}\left\{q_{\chi}(u)\varphi_1(X, Y_t, \Theta, t)^{\mathrm{T}} + q_{\chi'}(u)\left(X^{\mathrm{T}} - \hat{X}_t^{\mathrm{T}}\right) C_t \sigma_1(X, Y_t, \Theta, t)\right\}\sigma_2(Y_t, \Theta, t)^{-1}. \tag{55}$$

and then we can rearrange Eq. (54) in

$$dc_{\chi} = \Bigg\{\mu_t \gamma_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu \gamma_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t) + \mathrm{tr}\Bigg[\{\mu_t(h_0(Y_t, \hat{X}_t, \Theta, t) + \hat{X}_t b_0(Y_t, \hat{X}_t, \Theta, t)\}$$
$$+ \sum_{\nu=1}^{N} c_\nu\left(h_\nu(Y_t, \hat{X}_t, \Theta, t) + \hat{X}_t b_\nu(Y_t, \hat{X}_t, \Theta, t)\right)\Bigg\{\varepsilon_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu \varepsilon_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t)/\mu_t\Bigg\} C_t\Bigg]$$
$$+ \frac{1}{2n}(c_{\chi-1} + 2\chi c_\chi)\mathrm{tr}\left[\dot{C}_t K_t\right] + \frac{c_{\chi-1}}{2n}\mathrm{tr}\Bigg[C_t\Bigg(h_0(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu h_\nu(Y_t, \hat{X}_t, t)/\mu_t\Bigg)\sigma_2(Y_t, \Theta, t)$$
$$\times\Bigg(h_0(Y_t, \hat{X}_t, \Theta, t)^{\mathrm{T}} + \sum_{\nu=1}^{N} c_\nu h_\nu(Y_t, \hat{X}_t, \Theta, t)^{\mathrm{T}}/\mu_t\Bigg)\Bigg] - 2\hat{X}_t^{\mathrm{T}} C_t\Bigg(h_0(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu h_\nu(Y_t, \hat{X}_t, \Theta, t)/\mu_t\Bigg)$$
$$\times\sigma_2(Y_t, \Theta, t)\Bigg(b_0(Y_t, \hat{X}_t, \Theta, t)^{\mathrm{T}} + \sum_{\nu=1}^{N} c_\nu b_\nu(Y_t, \hat{X}_t, \Theta, t)^{\mathrm{T}}/\mu_t\Bigg)$$
$$+ X_t\hat{\mathrm{T}} C_t \hat{X}_t\Bigg(b_0(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu b_\nu(Y_t, \hat{X}_t, \Theta, t)/\mu_t\Bigg)\sigma_2(Y_t, \Theta, t)$$
$$\times\Bigg(b_0(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu b_\nu(Y_t, \hat{X}_t, \Theta, t)^{\mathrm{T}}/\mu_t\Bigg)\Bigg\}dt$$
$$+ \Bigg\{\mu_t \eta_{\chi 0}(Y_t, \hat{X}_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu \eta_{\chi\nu}(Y_t, \hat{X}_t, \Theta, t)\Bigg\}dY_t \quad (\chi = 1, \ldots, N). \tag{56}$$

The modified ellipsoidal suboptimal filter (MESOF) is defined by Eqs. (51), (52), and (56) and the relation $\hat{X}_t = m_t/\mu_t$ under the initial conditions

$$m(t_0) = \mathrm{E}[X_0|Y_0], \quad \mu(t_0) = 1, \quad c_\chi(t_0) = c_{\chi 0} \quad (\chi = 1, \ldots, N), \tag{57}$$

($c_{\chi 0} \quad (\chi = 1, \ldots, N)$ are the coefficients of the expansion (46) of the probability density $\tilde{p}_{t_0}(x) = p_0(x|Y_0)$ of the vector X_0 relative to Y_0).

Upon solution of Eqs. (51), (52), (56), and (57), the rms optimal estimate of the state vector and the covariance matrix of filtration error in MESOF obey the following approximate formulae:

$$\hat{X}_t = m_t/\mu_t;$$
$$R_t = \mathrm{E}^{w}_{\Delta^x}\left[\left(X - \frac{m_t}{\mu_t}\right)\left(X^{\mathrm{T}} - \frac{m_t^{\mathrm{T}}}{\mu_t}\right)\right] + \sum_{\nu=1}^{N}\frac{c_\nu}{\mu_t}\mathrm{E}^{wp_\nu}_{\Delta^x}\left[\left(X - \frac{m_t}{\mu_t}\right)\left(X^{\mathrm{T}} - \frac{m_t^{\mathrm{T}}}{\mu_t}\right)\right]. \tag{58}$$

Note that the order of the obtained MESOF, especially under high dimension n of the system state vector, is much lower than the order of other conditionally optimal filters. It is the case at allowing for the moments of up to the 10th order. Then, already for $n>3$ and $N = 5$, we have $n + N + 1 \leq n(n + 3)/2$. We conclude that for $n>3$ and $N = 5$, MECOF has a lower order than the filters of the method of normal approximation, generalized second-order Kalman-Bucy filters, and Gaussian filter. Thus, the following theorems underlie the algorithm of modified ellipsoidal conditionally optimal filtration.

Statement 10. Under the conditions of Statement 8, if there is MECOF, then it is defined by Eqs. (51), (52), and (56) under the conditions (57) and (58).

Statement 11. Under the conditions of Statement 9, if there is MESOF, then it is defined by the equations of Statement 10 under the conditions (45).

The aforementioned methods of MESOF construction offer a basic possibility of getting a filter close to the optimal-in-estimate one with any degree of accuracy. The higher the EA coefficient, the maximal order of the allowed for moments, the higher accuracy of approximation of the optimal estimate. However, the number of equations defining the parameters of the a posteriori one-dimensional ellipsoidal distribution grows rapidly with the number of allowed for parameters. At that, the information about the analytical nature of the problem becomes pivotal.

For approximate analysis of the filtration equations by following [11] and allowing for random nature of the parameters Θ, we come to the following equations for the first-order sensitivity functions [11]:

$$\begin{aligned} d\nabla^{\Theta}\hat{X}_s &= \nabla^{\Theta}A^{\hat{X}_s}dt + \nabla^{\Theta}B^{\hat{X}_s}dY_t, \quad \nabla^{\Theta}B^{\hat{X}_s}(t_0) = 0, \\ d\nabla^{\Theta}R_{sq} &= \nabla^{\Theta}A^{R_{sq}}dt + \nabla^{\Theta}B^{R_{sq}}dY_t, \quad \nabla^{\Theta}R_{sq}(t_0) = 0, \\ d\nabla^{\Theta}c_\kappa &= \nabla^{\Theta}A^{c_\kappa}dt + \nabla^{\Theta}B^{c_\kappa}dY_t, \quad \nabla^{\Theta}c_\kappa(t_0) = 0. \end{aligned} \tag{59}$$

Here the procedure of taking the derivatives is carried out over all input variables, and the coefficients of sensitivity are calculated for $\Theta = m^{\Theta}$. It is assumed at that the variance is small as compared with their expectations. Obviously, at differentiation with respect to Θ $\left(\nabla^{\Theta} = \partial/\partial\Theta\right)$, the order of the equations grows in proportion to the number of derivatives. The equations for the elements of the matrices of the second sensitivity functions are made up in a similar manner.

To estimate the MESOF (MECOF) performance, we follow [5, 8] and introduce for the Gaussian Θ with the expectation m^{Θ} and covariance matrix K^{Θ} the conditional loss function admitting quadratic approximation, the factor $\varepsilon = \varepsilon_2^{1/4}$, as well as

$$\rho^{\hat{X}_s} = \rho^{\hat{X}_s}(\Theta) = \rho\left(m^{\Theta}\right) + \sum_{ii=1}^{n^{\Theta}}\rho_i'\left(m^{\Theta}\right)\Theta_s^0 + \sum_{i,j=1}\sum\rho_{ij}''\left(m^{\Theta}\right)\Theta_i^0\Theta_j^0. \tag{60}$$

It is denoted here

$$
\begin{aligned}
&\varepsilon_2 = \mathrm{E}^{EA}\left[\rho(\Theta)^2\right] - \rho\left(m^\Theta\right)^2, \\
&\mathrm{E}^{EA}\left[\rho(\Theta)^2\right] = \rho\left(m^\Theta\right)^2 + \rho'\left(m^\Theta\right)^{\mathrm{T}} K^\Theta \rho'\left(m^\Theta\right) + 2\rho\left(m^\Theta\right)\mathrm{tr}\left[\rho''\left(m^\Theta\right)K^\Theta\right] \\
&+\left\{\mathrm{tr}\left[\rho''\left(m^\Theta\right)K^\Theta\right]\right\}^2 + 2\mathrm{tr}\left[\rho''\left(m^\Theta\right)K^\Theta\right]^2.
\end{aligned}
\tag{61}
$$

At that, in (61) the functions ρ' and ρ'' are determined through certain formulas on the basis of the first and second sensitivity functions. Therefore, we come to the following result.

Statement 12. Estimation of MESOF (MECOF) performance under the conditions of Statements 10 and 11 relies on Eqs. (59)–(61) under the corresponding derivatives in the right sides of Eq. (59).

8. New types of continuous MECOF

Based on Statements 10 and 11 in [18], continuous MECOF were described. We consider the problem of continuous conditionally optimal filtration for the general case of Eqs. (34) and (35) where it is desired to determine the optimal estimate $\hat{X}_t$ of process X_t at the instant $t>t_0$ from the results of observation of this process until the instant t, that is, over the interval $[t_0, t)$, in the class of permissible $\hat{X}_t = AZ_t$ estimates and with a stochastic differential equation given by

$$dZ_t = [\alpha_t \xi(Y_t, Z_t, \Theta, t) + \gamma_t]dt + \beta_t \eta(Y_t, Z_t, \Theta, t)dY_t \tag{62}$$

under the given vector and matrix structural functions ξ and η and every possible time functions $\alpha_t, \beta_t, \gamma_t$ (α_t and β_t are matrices and γ_t is a vector). The criterion for minimal rms error of the estimate Z_t is used as the optimality criterion. It is common knowledge that selection of the class of permissible filters defined by the structural functions ξ and η in Eq. (62) is the greatest challenge in practice of using the COF theory [1, 3, 11]. In principle they can be defined arbitrarily. One can select ξ and η at will so that the class of permissible filters contained an arbitrarily defined COF. In this case, COF is in practice more precise than the given COF. At the same time, by selecting a finite segment of some basis in the corresponding Hilbertian space L_2 as components of the vector function ξ and elements of the matrix function η, one can obtain an approximation with any degree of precision to the unknown optimal functions ξ and η. This technique of selecting the functions ξ and η on the basis of the equations of the theory of suboptimal filtration seems to be the most rational one. At that, the COF equations obtained from the equation for the nonnormalized a posteriori characteristic function open up new possibilities.

To use the equations obtained from nonnormalized equations for the a posteriori distribution, one needs to change the formulation of the COF problems [3, 11] so as to use the equation for the factor μ_t. For that, we take advantage of the following equations to determine the class of permissible continuous MECOF (62):

$$d\mu_t = \rho_t \chi(Y_t, Z_t, \Theta, t)dY_t, \tag{63}$$

$$\hat{X}_t = AZ_t/\mu_t, \tag{64}$$

where $\chi(Y_t, Z_t, \Theta, t)$ is a certain given structural matrix function and ρ_t is the row matrix of coefficients depending on t and subject to rms optimization along with the coefficients α_t, β_t, and γ_t in the filter Eq. (62).

Relying on the results of the last section and generalizing [7], one can specify the following types of the permissible MECOF:

1. This type of permissible MECOF can be obtained by assuming $Z_t = m_t, A = I_n$ and determining the functions ξ, η, χ in Eqs. (62) and (63) and obeying Eq. (51) which gives rise to the following expressions for the structural functions:

$$\xi = \xi(Y_t, Z_t, \Theta, t) = \left[\mu_t f_0(Y_t, Z_t/\mu_t, \Theta, t)^{\mathrm{T}} f_1\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}} \ldots f_N\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}}\right]^{\mathrm{T}}; \tag{65}$$

$$\begin{aligned} \eta &= \eta(Y_t, Z_t, \Theta, t) \\ &= \left[\mu_t h_0(Y_t, Z_t/\mu_t, \Theta, t)^{\mathrm{T}} h_1\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}} \ldots h_N\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}}\right]^{\mathrm{T}}; \end{aligned} \tag{66}$$

$$\begin{aligned} \chi &= \chi(Y_t, Z_t, \Theta, t) \\ &= \left[\mu_t b_0(Y_t, Z_t/\mu_t, \Theta, t)^{\mathrm{T}} b_1\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}} \ldots b_N\left(Y_t, Z_t/\mu_t, \Theta, t\right)^{\mathrm{T}}\right]^{\mathrm{T}}. \end{aligned} \tag{67}$$

At that, the order of MECOF defined by Eqs. (62) and (63) is equal to $n+1$. This type of MECOF may be designed for Z_t being constant Z_0 and $A = I_\ell(\ell < n)$.

2. To obtain a wider class of permissible MECOF, rearrange Eq. (56) in

$$\begin{aligned} dc_\chi = &\left\{F_{\chi 0}(Y_t, Z_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu F_{\chi\nu}(Y_t, Z_t, \Theta, t) + \sum_{\lambda,\nu=1}^{N} c_\lambda c_\nu F_{\chi\lambda\nu}(Y_t, Z_t, \Theta, t)\right. \\ &\left. + c_{\chi-1} \sum_{\lambda,\nu=1}^{N} c_\lambda c_\nu F_{\chi\lambda\nu'}(Y_t, Z_t, \Theta, t)\right\} dt \\ &+ \left\{\mu_t \eta_{\chi 0}(Y_t, Z_t, \Theta, t) + \sum_{\nu=1}^{N} c_\nu \eta_{\chi\nu}(Y_t, Z_t, \Theta, t)\right\} dY_t \end{aligned} \tag{68}$$

with the following notations:

$$F_{\chi 0}(Y_t, Z_t, \Theta, t) = \mu_t \gamma_{\chi 0}(Y_t, Z_t, \Theta, t) + \mu_t \mathrm{tr}\left[(h_0(Y_t, Z_t, \Theta, t) Z_t b_0(Y_t, Z_t, \Theta, t))\varepsilon_{\chi 0}(Y_t, Z_t, \Theta, t)\right];$$

$$\begin{aligned} F_{\chi\nu}(Y_t, Z_t, \Theta, t) &= \gamma_{\chi 0}(Y_t, Z_t, \Theta, t) + \mathrm{tr}[(h_\nu(Y_t, Z_t, \Theta, t) + Z_t b_\nu(Y_t, Z_t, \Theta, t))\varepsilon_{\chi 0}(Y_t, Z_t, \Theta, t) + (h_0(Y_t, Z_t, \Theta, t) \\ &+ Z_t b_0(Y_t, Z_t, \Theta, t))\varepsilon_{\chi\nu}(Y_t, Z_t, \Theta, t)] + \frac{1}{2n}\delta_{\chi-1,\nu}\{\mathrm{tr}\left[b_\nu K_t + C_t h_0(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) h_0(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\right] \\ &- 2Z_t^{\mathrm{T}} C_t h_0(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) b_0(Y_t, Z_t, \Theta, t)^{\mathrm{T}} + Z_t^{\mathrm{T}} C_t Z_t b_0(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) b_0(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\} + \frac{1}{n}\chi\delta_{\chi\nu}\mathrm{tr}\left[\dot{C}_t K_t\right]; \end{aligned}$$

$$\begin{aligned} F_{\chi\lambda\nu} &= \mathrm{tr}\left[C_t(h_\lambda(Y_t, Z_t, \Theta, t) + Z_t b_\lambda(Y_t, Z_t, \Theta, t))\varepsilon_{\chi\nu}(Y_t, Z_t, \Theta, t)\right]/\mu_t \\ &+ \frac{1}{n}\delta_{\chi-1,\lambda}\{\mathrm{tr}\left[C_t h_0(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) h_\nu(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\right] - Z_t^{\mathrm{T}} C_t (h_0(Y_t, Z_t, t)\sigma_2(Y_t, t) b_\nu(Y_t, Z_t\Theta, , t)^{\mathrm{T}} \\ &+ h_\nu(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) b_0(Y_t, Z_t, \Theta, t)^{\mathrm{T}} + Z_t^{\mathrm{T}} C_t Z_t b_0(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) b_\nu(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\}/\mu_t; \end{aligned}$$

$$\begin{aligned} F_{\chi\lambda\nu'} = \frac{1}{2n}&\left\{\mathrm{tr}\left[C_t(h_\lambda(Y_t, Z_t, \Theta, t) + \sigma_2(Y_t, \Theta, t) h_\nu(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\right]\right. \\ &- 2Z_t^{\mathrm{T}} C_t h_\lambda(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, \Theta, t) b_\nu(Y_t, Z_t, \Theta, t)^{\mathrm{T}} \\ &\left. + Z_t^{\mathrm{T}} C_t Z_t b_\lambda(Y_t, Z_t, \Theta, t)\sigma_2(Y_t, t) b_\nu(Y_t, Z_t, \Theta, t)^{\mathrm{T}}\right\}/\mu_t^2. \end{aligned} \tag{69}$$

By taking as the basis for the type of permissible MECOF Eqs. (51), (52), and (68), one has to regard all components of the vector Z_t as all components of the vector m_t and coefficients $c_1, \ldots, c_N$ so that $Z_t = \left[m_t^{\mathrm{T}} c_1 \ldots c_N\right]^{\mathrm{T}}$. At that, the order of all permissible filters is equal to n + N + 1. Putting $Z_t = Z_0$ and $A = I_l \quad (l < n)$, one gets the corresponding MECOF.

3. The widest class of permissible filters providing MECOF of the maximal reachable accuracy can be obtained if one takes the function ξ in (62) as the vector with all components of the vector functions $\mu f_{\chi 0}, c_\iota f_{\chi\nu} (\chi, \nu = 1, \ldots, N)$ in Eqs. (65) and (66) all addends involved in the scalar functions $F_{\chi 0}, c_\nu F_{\chi\nu}, c_\lambda c_\nu F_{\chi\lambda\nu}$ $(\chi, \lambda, \nu = 1, \ldots, N)$ $c_{\chi-1} c_\lambda c_\nu F_{\chi\lambda\nu'}$, $(\chi - 1, \lambda, \nu = 1, \ldots, N)$ in (68) and as the function η in (62), the matrix whose rows are the row matrices $\mu_t h_{\chi 0}, c_\nu h_{\chi\nu} (\chi, \nu = 1, \ldots, N)$ in (68) and all row matrices $\mu\eta_{\chi 0}, c_\nu \eta_{\chi\nu} (\chi, \nu = 1, \ldots, N)$ in (69). As for the function χ in Eq. (63), it is determined through (67) as in the case of the simplest types of permissible filters. The so-determined class of permissible filters has ECOF defined by Eqs. (51), (52), and (69), at that ECOF is more precise than ESOF. We notice that this class of permissible filters can give rise to an overcomplicated ECOF because of high dimension of the structural vector function ξ. So we distinguish the following new type of permissible filters.

4. Components of the vector function ξ are all components of the vector functions $\mu_t f_{\chi 0}, c_\iota f_{\chi\nu} (\chi, \nu = 1, \ldots, N)$ and all scalar functions $F_{\chi 0}, c_\nu F_{\chi\nu}, c_\lambda c_\nu F_{\chi\lambda\nu}$ $(\chi, \lambda, \nu = 1, \ldots, N)$, $c_{\chi-1} c_\lambda c_\nu F_{\chi\lambda\nu'}$ $(\chi - 1, \lambda, \nu = 1, \ldots, N)$ without decomposing them into individual addends. This class of permissible filters also includes ECOF (51), (52), and (69).

To determine the coefficients α_t, β_t, and γ_t of the equation MECOF (62), one needs to know the joint one-dimensional distribution of the random processes X_t and $\hat{X}_t$. It is determined by solving the problem of analysis of the system obeying the stochastic differential Eqs. (62) and (63). As always in the theory of conditionally optimal filtration, all complex calculations required to determine the optimal coefficients of the MECOF Eq. (62) or (63) are based only on the a priori data and therefore can be carried out in advance at designing MECOF. At that, the accuracy of filtration can be established for each permissible MECOF. The process of filtration itself comes to solving the differential equation, which enables one to carry out real-time filtration.

Consequently, we arrive to the following results.

Statement 13. Under the conditions of Statement 8, the MECOF equations like (62) and (63) coincide with the equations of continuous MECOF where the structural functions belong to the four aforementioned types.

Statement 14. The rms MECOF of the order $n_x + 1$ coinciding with MECOF is defined for CStS (34) and (35), Eqs. (62)–(64), and the structural functions of the first class.

Statement 15. The rms of MECOF of the order $n_x + N + 1$ coinciding with MECOF obeys for the CStS (34) and (35), Eqs. (62)–(64), and the structural functions of Statement 14.

Statement 16. If accuracy of MECOF determined according to Statement 14 is insufficient, then the functions of the Statement 15 can be used as structural ones.

Statement 17. The relations of Statement 12 underlie the estimate of quality of MECOF under the conditions of Statements 13–15, provided that there are corresponding derivatives in the right sides of the equations.

Example 3. The presented MECOF for linear CStS coincide with Kalman-Bucy filter [2–4, 11].

Example 4. MECOF for linear CStS with parametric noises coincide with linear Pugachev conditionally optimal filter.

Finally let us consider quasilinear CStS (36) and (37), reducible to the following differential one:

$$\dot{X}_t = \varphi(X_t, \Theta, t) + \psi(t, \Theta)V_1^{EL}, \tag{70}$$

$$\dot{Y}_t = \varphi_1(X_t, \Theta, t) + V_2^{EL} \tag{71}$$

where V_1 and V_2 are non-Gaussian white noises. In this case using ELM and Kalman-Bucy filters with parameters depending on m_t^x, K_t^x and c_{1t}^x, we get the following interconnected set of equations:

$$\dot{m}_1^x = \varphi_{00} \quad m_{t_0}^x = m_0^x, \quad \dot{m}_1^y = \varphi_{10} \quad m_{t_0}^y = m_0^y, \tag{72}$$

$$\dot{X}_t^0 = \varphi_{01}X_t^0 + \psi(t, \Theta)V_1, \quad \dot{Y}_t^0 = \varphi_{11}X_t^0 + V_2^{EL}, \quad Y_{t_0}^0 = Y_0^0, \tag{73}$$

$$\dot{K}_t^x = \varphi_{11}K_t^x + K_t^x\varphi_{11}^T + \psi(t, \Theta)G_1^{EL}(t, \Theta)\psi(t, \Theta)^T, \quad K_{t_0}^x = K_0^x, \tag{74}$$

$$\begin{aligned}&\dot{\hat{X}}_t = \varphi_{00} - \varphi_{01}m_t^x + \varphi_{01}\hat{X}_t + R_tG_2^{EL}(t, \Theta)^{-1}\left[\dot{Y}_t - \varphi_{11}\hat{X}_t - \varphi_{10} + \varphi_{11}m_t^x\right),\\ &\hat{X}_0 = M^{EL}\hat{X}_{t_0},\end{aligned} \tag{75}$$

$$\begin{aligned}&\dot{R}_t = \varphi_{01}R_t - R_t\varphi_{01} - R_t\varphi_{11}G_2^{EL}(t, \Theta)^{-1}\varphi_{11}R_t + \psi(t)G_1^{EL}(t, \Theta)\psi(t)^T,\\ &R_0 = M^{EL}\left[\left(X_0 - \hat{X}_0\right)\left(X_0 - \hat{X}_0\right)^T\right].\end{aligned} \tag{76}$$

Here the following notations are used:

$$\begin{aligned}&m_t^x = M^{EL}X_t, \quad m_t^y = M^{EL}Y_t \quad \text{and} \quad K_t^x = M^{EL}\left[X_t^0\, X_t^{0T}\right],\\ &R_t = M^{EL}\left[\left(X_t - \hat{X}_t\right)\left(X_t - \hat{X}_t\right)^T\right]\end{aligned} \tag{77}$$

being the mathematical expectations, state, and error covariance matrices

$$\begin{aligned}&\varphi_{00} = \varphi_{00}\left(m_t^x, K_t^x, c_{1t}^x, t, \Theta\right), \quad \varphi_{10} = \varphi_{10}\left(m_t^x, K_t^x, c_{1t}^x, t, \Theta\right),\\ &\varphi_{01} = \varphi_{01}\left(m_t^x, K_t^x, c_{1t}^x, t, \Theta\right) = \frac{\partial\varphi_{01}}{\partial m_t^x}, \quad \varphi_{11} = \varphi_{11}\left(m_t^x, K_t^x, c_{1t}^x, t, \Theta\right) = \frac{\partial\varphi_{10}}{\partial m_t^x}\end{aligned} \tag{78}$$

being ELM ecoefficiencies, G_i^{EL} $(i = 1, 2)$ are intensities of normal EL equivalent white noises. So Eqs. (72)–(78) define the corresponding Statement 18.

9. Ellipsoidal Pugachev conditionally optimal continuous control

The idea of conditionally optimal control (COC) was suggested by Pugachev (IFAC Workshop on Differential Games, Russia, Sochi, 1980) and developed [13]. The COC essence is in the search of optimal control among all permissible controls (as in classical control theory) but in the restricted class of permissible controls. These controls are computed in online regime. At practice the permissible continuous class of controls may be defined by the set of ordinary differential equations of the given structure.

So let us consider the following Ito equations:

$$dX = \varphi(X, Y, U, t)dt + \psi_1(X, Y, U, t)dW_0 + \int_{R_0^q} \psi_2(X, Y, U, t, v)P^0(dt, dv), \quad (79)$$

$$dY = \varphi'(X, Y, U, t)dt + \psi_1'(X, Y, U, t)dW_0 + \int_{R_0^q} \psi_2'(X, Y, U, t, v)P^0(dt, dv). \quad (80)$$

Here X is the nonobservable state vector; Y is the observable vector; $U \in \mathcal{D}$ is the control vector; W_0 being the Wiener StP, $P^0(A, B)$ being the independent of W_0 centered Poisson measure; φ, ψ_1, ψ_2 and $\varphi', \psi_1', \psi_2'$ being the known functions. Integration is realized in R^q space with the deleted origin. Initial conditions X_0 and Y_0 do not depend on X and Y. Functions φ, ψ_1, ψ_2 in (79) as a rule do not depend on Y, but depend on U components that are governed by Eq. (79). Functions $\varphi', \psi_1', \psi_2'$ in Eq. (80) depend on U components that govern observation.

The class of the admissible controls is defined by the equations

$$dU = [\alpha\xi(Y, U, t) + \gamma\,]dt + \beta\eta(Y, U, t)dY \quad (81)$$

without restrictions and with restrictions

$$\begin{aligned} dU = {} & [\alpha\xi(Y, U, t) + \gamma]dt + \beta\eta(Y, U, t)dY \\ & - \max\left\{0, n(U)^T[\alpha\xi(Y, U, t) + \gamma]dt + n(U)^T\beta\eta(Y, U, t)dY\right\}n(U)\mathbf{1}_{\partial\mathcal{D}}(U). \end{aligned} \quad (82)$$

Here $n(U)$ is the unit vector of external normal for boundary $\partial\mathcal{D}$ in point U; $1_{\partial\mathcal{D}}(U)$ is the set indicator.

Conditionally optimal criteria is taken in the form of mathematical expectation of some functional depending on $X_{t_0}^t = \{X(\tau) : \tau \in [t_0, t]\}$ and $U_{t_0}^t = \{U(\tau) : \tau \in [t_0, t]\}$:

$$\rho = \mathrm{E}\ell\left(X_{t_0}^t, U_{t_0}^t, t\right), \quad (83)$$

where E is the mathematical expectation and ℓ is the loss function at the given realizations $x_{t_0}^t, u_{t_0}^t$ of $X_{t_0}^t, U_{t_0}^t$.

So according to Pugachev we define COC as the control realized by minimization (83) by choosing α, β, γ and by satisfying (82) at every time moment and at a given α, β, γ for all preceding time moments. For the loss function (83) depending on X and U at the same time, moment t is necessary to compute ellipsoidal one-dimensional distribution of X and Y in Eqs. (79), (80), and (82) using EAM (ELM). This problem is analogous to COF and MCOF design (Section 8).

For high accuracy and high availability CStS especially functioning in real-time regime, software tools "StS-Analysis," "StS-Filtering," and "StS-Control" based on NAM, EAM, and ELM were developed for scientists, engineers, and students of Russian Technical Universities.

These tools were implemented for solving safety problems for system engineering [19].

In [18, 20] theoretical propositions of new probabilistic methodology of analysis, modeling, estimation, and control in stochastic organizational-technical-economical systems (OTES) based on stochastic CALS informational technologies (IT) are considered. Stochastic integrated logistic support (ILS) of OTES modeling life cycle (LC), stochastic optimal of current state estimation in stochastic media defined by internal and external noises including specially organized OTES-NS (noise support), and stochastic OTES optimal control according to social-technical--economical-support criteria in real time by informational-analytical tools (IAT) of global type are presented. Possibilities spectrum may be broaden by solving problems of OTES-CALS integration into existing markets of finances, goods, and services. Analytical modeling, parametric optimization and optimal stochastic processes regulation illustrate some technologies and IAT given plans. Methodological support based on EAM gives the opportunity to study infrequent probabilistic events necessary for deep CStS safety analysis.

10. Conclusion

Modern continuous high accuracy and availability control stochastic systems (CStS) are described by multidimensional differential linear, linear with parametric noises, and nonlinear stochastic equations. Right-hand parts of these equations also depend on stochastic factors being random variables defining the dispersion in engineering systems parameters. Analysis and synthesis CStS needs computation of non-Gaussian probability distributions of multidimensional stochastic processes. The known analytical parametrization modeling methods demand the automatic composing and the integration of big amount interconnected equations.

Two methods of analysis and analytical modeling of multidimensional non-Gaussian CStS were worked out: ellipsoidal approximation method (EAM) and ellipsoidal linearization method (ELM). In this case one achieves cardinal reduction the amount of distribution parameters.

Necessary information about ellipsoidal approximation methods is given and illustrated. Some important remarks for engineers concerning EAM accuracy are given. It is important to note that all complex calculations are performed on design stage. Algorithms for composition of EAM (ELM) equation are presented. Application to problems of shock and vibroprotection are considered.

For statistical CStS offline and online analysis approximate methods based on EAM (ELM) for a posteriori distributions are developed. In this case one has twice reduction of equation amount. Special bank of approximate suboptimal and Pugachev conditionally optimal filters for typical identification and calibration problems based on the normalized and nonnormalized was designed and implemented.

In theoretical propositions of new probabilistic methodology of analysis, modeling, estimation, and control in stochastic OTES based on stochastic CALS information technologies (IT) are considered. Stochastic integrated logistic support (ILS) of OTES modeling life cycle, stochastic optimal of current state estimation in stochastic media defined by internal and external noises including specially organized OTES-NS (noise support), and stochastic OTES optimal control according to social-technical-economical-support criteria in real time by informational-analytical tools (IAT) of global type are presented. Possibility spectrum may be broaden by solving problems of OTES-CALS integration into existing markets of finances, goods, and services. Methodological support based on EAM (ELM) gives the opportunity to study infrequent probabilistic events necessary for deep CStS safety analysis.

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