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Global Indeterminacy and Invariant Manifolds Near Homoclinic Orbit to a Real Saddle

Beatrice Venturi

Abstract

In this paper we investigate the dynamic properties of the Romer model. We determine the whole set of conditions which lead to global indeterminacy and the existence of a homoclinic orbit that converges in both forward and backward time to a real saddle equilibrium point. The dynamics near this homoclinic orbit have been investigated. The economic implications are discussed in the conclusions.

Keywords: externality, endogenous growth model, homoclinic orbits to real saddle, stable cycle, global indeterminacy

1. Introduction

In this note we prove the existence of a homoclinic orbit in an extension (see [1]) of well-known endogenous growth two-sector technological change Romer model [2], introduced in [3]. It represents the first attempt to make formal a model of endogenous growth, through research and development (R&D) activities.

In this model, the knowledge is composed of two components, human capital, which defines the specific knowledge of every person, and the so-called technology, that, in general, is available for everybody.

The first component ascribes the rivalry feature to its employment because it is incorporated in the physical person. Indeed, a human resource used by a firm cannot be used by another firm. The second component of knowledge ascribes the feature of non-rivalry good, because it can be used by different firms in the same time. The consequence of the human capital rivalry, who invests in human capital accumulation, receives the profit of this accumulation too, while the non-rivalry feature implies the spillover effects diffusion, that is, the inventor of a new technology will not be the only beneficiary of the positive effects related to this discovery. It is impossible to him to take total possession of his fruits. This fact implies the development of an externality that, in turn, reduces the single person efforts to improve the productive technology under the level that should be socially advisable.

An externality is an economic action effect that involves another subject, not directly implicated in this action (change, production, or consumption action).

The market equilibrium in the presence of externalities is not optimal, because expenses and private utilities do not coincide to expenses and social utilities (e.g., pollution). Therefore, the external effects, positive and negative, are not,

respectively, remunerated or compensated. Both the human capital and the technology are fruits of conscious human choices.

Moreover the research activity is intensive in the use of human capital and technology too, without physical capital K and skilled labor is employed in research.

In several papers the question of the uniqueness of the equilibrium trajectory has been studied for this model. Benhabib, Perli, and Xie (BPX), in [3], used numerically analysis for proving the existence of stable periodic solutions in a generalized version of the Romer model in which they consider the complementarity between different intermediary capital goods.

Finally, [2] extends BPX model by tightening the parameter restrictions necessary to obtain an interior steady state and studies the stability of the steady state in BPX model without unskilled labor.

Following [4], we consider the three-dimensional reduced version of the model, obtained by a standard change of variables, related with the growth rate of the fundamental variables.

By using the method of undetermined coefficients (see [5]), we are able to prove the existence of a homoclinic orbit that converges both forward and backward to the unique equilibrium point whose linearization matrix admits two positive and one negative real eigenvalues. The stable and unstable manifolds are locally governed by real eigenvalues. The concept of homoclinic bifurcations is very important from a dynamic point of view. Such phenomena causes global rearrangements in phase space, including changes to basins of attractions and generation of chaotic dynamics [6].

We show that such a homoclinic orbit gives rise to global indeterminacy in a parameter set commonly investigated by means of the instruments of the local analysis.

The paper develops as follows. In the second section, we analyze the optimal control model, and we introduce the equivalent three-dimensional continuous time abstract stationary system. The third section is devoted to the steady-state analysis of the model in reduced form. In the fourth section, we apply the procedure developed by [5], and we show that a homoclinic loop emerges as a solution trajectory. In the last section, we consider a homoclinic bifurcation of dimension one. The economic implications are discussed in the conclusions.

2. The model

We consider now the three-dimensional reduced version of the Romer model. In the original optimal control model, the state variables are k , the physical capital; A , the level of knowledge currently available [1, 2]; and

$$\dot{k} = Y - C = \eta^\gamma k^\gamma A^{\xi-\gamma} h^\alpha L^\beta \quad (1)$$

$\xi \geq 1$ is the degree of complementarity, γ is a positive externality parameter in the production of physical capital, β is the share of capital, and α is defined by the following relationship $\alpha = 1 - \gamma - \beta$. The control variables are h the human capital, the skilled labor employed in the final sector Y , C is the consumption, ρ is a positive discount factor, and σ is the inverse of the intertemporal elasticity of substitution. The only consumption good C is measured in units of the final output Y . The final output is produced with two capital goods, the physical capital k and human capital h .

We consider the following substitution:

$$r = kA^{\frac{\alpha+\beta+\gamma}{\alpha+\beta}}; h = h; q = \frac{c}{k} \quad (2)$$

where r is the interest rate. We set $\Lambda = \alpha\xi/(\gamma - \xi)$

The deterministic reduced form of this model is given by

$$\begin{cases} (r/r) = (1/(1-\alpha))((\xi-1+\beta)\delta(1-h) - \beta((\xi/\gamma^2)r - q) - \alpha(r - (\delta/\Lambda)h)) \\ (h/h) = (1/(1-\alpha))((\xi-1-\gamma)\delta(1-h) - \gamma((\xi/\gamma^2)r - q) - \alpha(r - (\delta/\Lambda)h)) \\ (q/q) = ((r-\rho)/\sigma) - (\xi/\gamma^2)r + q \end{cases} \quad (3)$$

The set of parameters $\omega \equiv (\beta, \xi, \alpha, \delta, \gamma, \rho, \sigma)$ lives inside a significant economic parameter set $\Omega \equiv \{(0, 1) \times \mathbb{R}_{++} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_{++} - \{1\}\}$.

3. Steady-states analysis

A stationary (equilibrium) point $P^* = P(r^*, h^*, q^*)$ of the system (S) is any solution of the following system:

$$\begin{cases} 1/(1-\alpha)(\xi-1+\beta)\delta(1-h) - \beta((\xi/\gamma^2)r - q) - \alpha(r - (\delta/\Lambda)h) = 0 \\ 1/(1-\alpha)(\xi-1-\gamma)\delta(1-h) - \gamma((\xi/\gamma^2)r - q) - \alpha(r - (\delta/\Lambda)h) = 0 \\ (r-\rho)/\sigma - (\xi/\gamma^2)r + q = 0 \end{cases} \quad (4)$$

We get only one admissible steady state:

$$\begin{cases} h^* = \left(\frac{\Lambda}{\delta}\right) \left(\frac{((\sigma(\xi-\gamma) - (\xi-1)) - \rho(1-\gamma))}{(\Lambda(\sigma(\xi-\gamma) - (\xi-1)) - (1-\gamma))}\right) \\ r^* = \left(\frac{1}{1 - \left(\frac{1}{\sigma}\right)}\right) \left(\left(\frac{\delta}{\Lambda}\right)h^* - \delta(1-h^*) - \left(\frac{\rho}{\sigma}\right)\right) \sigma \neq 1 \\ r^* = \left(\frac{1}{(1-\gamma)}\right) \left((\xi-1-\gamma) * \delta(1-h^*) + \left(\frac{\delta}{\Lambda}\right)h^* - \left(\frac{\rho}{\sigma}\right)\right) \sigma = 1 \\ q^* = ((\xi/\gamma^2)) - (1/\sigma)r^* - (\rho/\sigma) \end{cases} \quad (5)$$

We denote $J = J(P^*)$ the Jacobian matrix J of the system (S) evaluated in the equilibrium point P^* , given by

$$J(P^*) = \begin{pmatrix} \left(\frac{r^*}{(1-\alpha)}\right)\alpha + \beta\left(\frac{\xi}{\gamma^2}\right) & \left(\frac{\delta}{(1-\alpha)}\right)r^*((\alpha/\Lambda) - (\xi-1+\beta)) & \frac{\beta}{(1-\alpha)}r^* \\ \left(\frac{h^*}{(1-\alpha)}\right)\left(\frac{\xi}{\gamma} - 1\right) & \left(\frac{\delta}{(1-\alpha)}\right)h^*((1/\Lambda) - (\xi-1-\gamma)) & \left(\frac{\gamma}{(1-\alpha)}\right)h^* \\ q^*(1/\sigma) - \frac{\xi}{\gamma^2} & 0 & q^* \end{pmatrix} \quad (6)$$

Lemma: We consider the following two subsets of the parameters space Ω :

$$\Omega_1 = \left\{ \begin{array}{l} \omega \in \Omega : \delta H(\sigma(\xi - \gamma) - (\xi - 1)) - \rho(1 - \gamma) < 0 \ \rho > (\delta/\Lambda)H \\ ((1 - \sigma)(\xi - \gamma)\delta H - (1 + \Lambda)) - \rho(1 - \gamma) > 0 \end{array} \right. \quad (7)$$

$$\Omega_2 = \left\{ \begin{array}{l} \omega \in \Omega : \delta H(\sigma(\xi - \gamma) - (\xi - 1)) - \rho(1 - \gamma) < 0 \ \rho < (\delta/\Lambda)H \\ ((1 - \sigma)(\xi - \gamma)\delta H - (1 + \Lambda)) - \rho(1 - \gamma) < 0 \end{array} \right. \quad (8)$$

(a) If the model parameters belong to set Ω_2 , then as shown in [1], the unique interior steady state is determinate.

(b) If the model parameters belong to set Ω_1 , then as shown in [1], the unique interior steady state is indeterminate or unstable.

4. The existence of a saddle with three purely real eigenvalue state analyses

We are interested in the special case in which $J = J(P^*)$ has three real eigenvalues. To this end, we analyze the dynamics of the model around the equilibrium point: P^* in Ω^1 .

Lemma: We consider the following subsets of the parameters space Ω_1 :

$$\Omega_{IR} = \{\omega \in \Omega_1: J(P^*) \text{ possesses real eigenvalues}\}. \quad (9)$$

Let $\omega \in \Omega_{IR}$ be. Then $J(P^*)$ has one positive and two negative purely real eigenvalues.

Proof: We apply the Cardano's formula to (S) and we get the result.

Example: Set $(\beta, \xi, \alpha, \delta, \gamma, \rho, \sigma) \equiv (0.6, 2.7, 0.3, 0.02, 0.1, 0.03, 0.01)$.

This economy has.

$P^* = P(r^*, h^*, q^*) \simeq (0.03, 0, 0.9981, 0.2250000000)$.

The computation of the eigenvalues of $J(P^*)$ leads to $\lambda_1 \simeq 0.1970612707$, $\lambda_2 \simeq -0.2753036128$, $\lambda_3 \simeq 0.3412714849$ with $|\lambda_1| > \lambda_3$, and $|\lambda_2| > \lambda_3$.

5. The existence of a homoclinic orbit

The second step of our calculations is the explicit calculus of the homoclinic orbit in $J(P^*)$.

Theorem: Existence of homoclinic orbits to the real saddle in P^* . Let $\omega \in \Omega_{IR}$. Then

$$\Omega_{1H} = \{\omega \in \Omega_{IR} : (S) \text{ possesses a homoclinic orbit } \Gamma(P^*)\} \neq \emptyset. \quad (10)$$

In order to construct the homoclinic orbit analytically, we apply the procedure developed by [5]. We compute the stable and unstable manifolds, of the saddle equilibrium point $J(P^*)$, respectively, W^s and W^u , with the undetermined coefficients method. We show that a homoclinic loop emerges as a solution trajectory of system (S) for parameter values belonging to the set $\Omega_{1H} \subset \Omega_{IR}$. The application of the method leads to the following relationship:

$$\phi(\xi) = \psi^2 \left(\frac{F_{3d}}{\lambda_3 + 2\lambda_1} \right) + \left(\frac{1}{\lambda_3} \right) \psi^2 \left(\frac{F_{2d}(\lambda^2 + 2\lambda^3)}{(F_{2f}(\lambda^2 - 2\lambda^1))} \right) F_{2f} = 0 \quad (11)$$

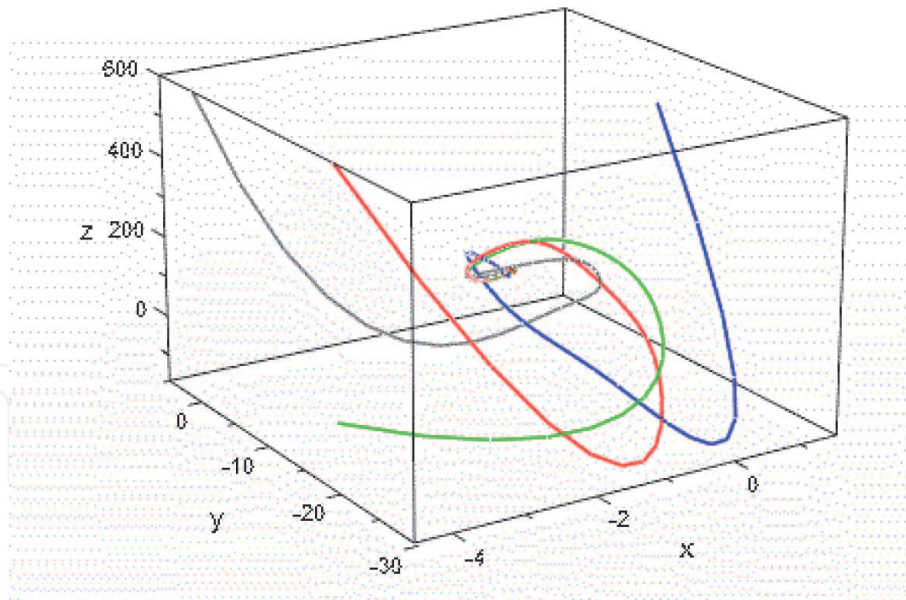


Figure 1.
 A neighborhood of the homoclinic orbit.

where ζ and ψ are arbitrary constants with $(\xi, \psi) \in (0,1)^2$, the $F_{i,j}$ coefficients, $i = 1, 2, 3$ and $j = e, d, f$, and $F_{i,j}$ are intricate combinations of the original parameters of the model and of three scaling factors (C_1, C_2, C_3) associated with the choice of the eigenvectors. We now introduce a normal topological form for homoclinic bifurcation (see [7]). We consider a two-dimensional cross-section Σ with coordinates (ζ, ψ) of W^u , the unidimensional unstable manifold. Suppose that $\zeta = 0$ corresponds to the intersection of Σ with the stable manifold W^s of P^* . Let conversely the point with coordinates (ζ^u, ψ^u) correspond to the intersection of W^u , with Σ . Then, the following occurs.

Definition: A split function can be defined as $\phi = \zeta^u$. Its zero $\phi = 0$ gives a condition for the homoclinic bifurcation in \mathbb{R}^3 .

It might be impossible to characterize the system for a full set of parameter spaces and the boundary of the homoclinic orbit region. Using σ as a bifurcation parameter, a homoclinic orbit can emerge as solution trajectories of the system (S). We observe a parameter set that remains inside Ω_{1H} .

We found this result after many maple simulations (see **Figure 1**).

6. Conclusions

In this paper from an economic point of view, we show that a low inverse intertemporal elasticity of substitution plays a crucial role in determining global indeterminacy. We have focused on the parameter regions around a saddle equilibrium point with purely real eigenvalues.

We have applied the procedure developed by [5], and we have shown that a homoclinic loop emerges as a solution trajectory of the reduced system for an economic set of parameter values.

In order to get a homoclinic bifurcation, we have introduced a normal topological form. By varying the exponent of the inverse of elasticity of substitution, we consider a homoclinic bifurcation of dimension one.

As clearly pointed out in the literature [8], the homoclinic orbit connecting the unique steady state to itself implies the existence of a tubular neighborhood of the original homoclinic orbit. Any initial condition starting inside this tubular

neighborhood gives rise to perfect-foresight equilibrium. Finally, with similar arguments introduced in [9], we are able to show global indeterminacy of the equilibrium for the model, since the result is valid beyond the small neighborhood relevant for the local analysis.

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