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Quasinormal Modes of Dirac Field in Generalized Nariai Spacetimes

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Abstract

The exact electrically charged solutions to the Dirac equation in higher-dimensional generalized Nariai spacetimes are obtained. Using these solutions, the boundary conditions leading to quasinormal modes of the Dirac field are analyzed, and their correspondent quasinormal frequencies are analytically calculated.

Keywords: quasinormal modes, generalized Nariai spacetimes, Dirac field, boundary conditions

1. Introduction

Quasinormal modes (QNMs) are eigenmodes of dissipative systems. For instance, if a spacetime with an event or cosmological horizon is perturbed from its equilibrium state, QNMs arise as damped oscillations with a spectrum of complex frequencies that do not depend on the details of the excitation. In fact, these frequencies depend just on the charges of the black hole, such as the mass, electric charge, and angular momentum [1, 2]. QNMs have been studied for a long time, and its interest has been renewed by the recent detection of gravitational waves, inasmuch as these are the modes that survive for a longer time when a background is perturbed and, therefore, these are the configurations that are generally measured by experiments [3–26, 42–44]. Mathematically, this discrete spectrum of QNMs stems from the fact that certain boundary conditions must be imposed to the physical fields propagating in such background [27]. In this chapter, we consider a higher-dimensional generalization of the charged Nariai spacetime [28], namely, $dS_2 \times S^2 \times \dots \times S^2$, and investigate the dynamics of perturbations of the electrically charged Dirac field (spin 1/2). In such a geometry, the spinorial formalism [29–31] is used to show that the Dirac equation is separable [32] and can be reduced to a Schrödinger-like equation [33] whose potential is contained in the Rosen-Morse class of integrable potentials, which has the so-called Pöschl-Teller potential as a particular case [34, 35]. Finally, the boundary conditions leading to QNMs are analyzed, and the quasinormal frequencies (QNFs) are analytically obtained [5, 36].

2. Presenting the problem

In D dimensions, the dynamics of general relativity in spacetimes with a cosmological constant Λ is described by the Einstein-Hilbert action¹

$$S = \frac{1}{16\pi} \int d^D x \sqrt{|g|} [\mathcal{R} - (D-2)\Lambda] + S_m, \quad (1)$$

where \mathcal{R} is the Ricci scalar and S_m stands for the action of all matter fields $\{\Phi_i\}$ coupled to gravity appearing in the theory, which can be scalar, spinorial, vectorial, and so on. The least action principle allows to find the equations of motion for the fields $g_{\mu\nu}$ and Φ_i which are given, respectively, by

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \frac{(D-2)}{2}\Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \frac{\delta S_m}{\delta \Phi_i} = 0, \quad (2)$$

where $T_{\mu\nu}$ is the symmetric stress-energy tensor associated to Φ_i defined by the equation

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \quad (3)$$

Since any symmetry has been imposed, the general solution of the system of Eq. (2) is some metric and fields in the background this metric

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad \Phi_i = \Phi_i(x). \quad (4)$$

Now, let the pair $g_{\mu\nu}^{(0)}$ and $\Phi_i^{(0)}$ be a solution for the equations of motion Eq. (2). Then, in order to study the perturbations around this particular solution, we write our fields as a sum of the unperturbed fields $g_{\mu\nu}^{(0)}$ and $\Phi_i^{(0)}$ and the small perturbations $h_{\mu\nu}$ and Ψ_i

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad \Phi_i = \Phi_i^{(0)} + \Psi_i, \quad (5)$$

where by “small” we mean that we neglect the quadratic and higher-order powers of the perturbation fields. Inserting the above equation into Eq. (2), we are left with a set of linear equations satisfied by the perturbed fields $h_{\mu\nu}$ and Ψ_i . In general, these equations are coupled, namely, Ψ_i is a source for $h_{\mu\nu}$ and vice versa. However, in the special case in which $\Phi_i^{(0)} = 0$, the equations governing the perturbed fields Ψ_i can be decoupled from the metric perturbation $h_{\mu\nu}$ and vice versa. The reason why this happens is that when $\Phi_i^{(0)} = 0$, the stress-energy tensor $T_{\mu\nu}$ can be set to zero at first order in the perturbation, since $T_{\mu\nu}$ is typically quadratic or of higher order in the matter fields and, therefore, can be neglected. Therefore, investigating the linear dynamics of generic small perturbations of the matter fields with $T_{\mu\nu} = 0$ is equivalent to studying the test fields Ψ_i in the background $g_{\mu\nu}^{(0)}$.

In what follows, let us consider a specific matter field Ψ propagating in a generalized version of the Nariai spacetime described in Ref. [28]. Here, Ψ is an

¹ The coefficient of Λ in S can be chosen of several manners. In particular, for any dimension D , in order to insure that the pure dS or pure AdS spacetimes are described by $g_{tt} = 1 - (\Lambda/3)r^2$, as occurs in the case $D = 4$, this coefficient should be $(D-1)(D-2)$.

electrically charged spinorial field of mass m that obeys the Dirac equation minimally coupled to an electromagnetic field in such spacetime. In $D = 2d$, this spacetime is formed from the direct product of the de Sitter space dS_2 with $(d - 1)$ copies of the unit spheres S^2 possessing different radii R_j . Thus, the natural line element of the higher-dimensional version of the Nariai spacetime is given by

$$ds^2 = g_{\mu\nu}^0 dx^\mu dx^\nu = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \sum_{j=2}^d R_j^2 d\Omega_j^2, \quad (6)$$

where $f(r)$ is a function of the coordinate r and $d\Omega_j^2$ is the line element of the j th unit sphere S^2 as follows

$$f(r) = 1 - \frac{r^2}{R_1^2}, \quad d\Omega_j^2 = d\theta_j^2 + \sin^2\theta_j d\phi_j^2. \quad (7)$$

The radii R_1 and R_j are given by

$$R_1 = \left[\Lambda - \frac{1}{2}Q_1^2 + \frac{Q}{2(D-2)} \right]^{-1/2}, \quad R_j = \left[\Lambda + \frac{1}{2}Q_j^2 + \frac{Q}{2(D-2)} \right]^{-1/2}, \quad (8)$$

where Q_1 is an electric charge and Q_j are magnetic charges, while Q is defined by

$$Q = Q_1^2 - \sum_{j=2}^d Q_j^2. \quad (9)$$

This spacetime is a locally static solution of Einstein's equation with a cosmological constant Λ and electromagnetic field $\mathcal{F} = d\mathcal{A}$ whose gauge field \mathcal{A} is given by

$$\mathcal{A} = Q_1 r dt + \sum_{j=2}^d Q_j R_j^2 \cos\theta_j d\phi_j. \quad (10)$$

The coordinates in the metric are also called static, because they do not depend explicitly on the time coordinate t . One may notice that, in this coordinate system, this background has a local Killing vector ∂_t whose norm vanishes at $r = \pm R_1$. Indeed, $r = \pm R_1$ define closed null surfaces that surround the observer at all times, known as event horizons. The boundary conditions defining QNMs in our spacetime will be posed at these surfaces, as discussed in [36]. For this reason, the dependence of all the components of the field Ψ on the coordinates along the Killing vector ∂_t is assumed to be of the form $e^{-i\omega t}$. Usually, the articles consider that the coordinate r in de Sitter space assume values in the interval $r \in (0, R_1)$ [37–39]. However, this is just justified for de Sitter with $D > 2$, but not for $D = 2$; see [36] for more details. By this reason, our domain of interest will be $r \in (-R_1, R_1)$. In such domain, it is useful to introduce the tortoise coordinate x defined by the equation

$$dx = \frac{1}{f(r)} dr \Rightarrow x = R_1 \operatorname{arctanh}\left(\frac{r}{R_1}\right), \quad (11)$$

in terms of which the line element Eq. (6) becomes

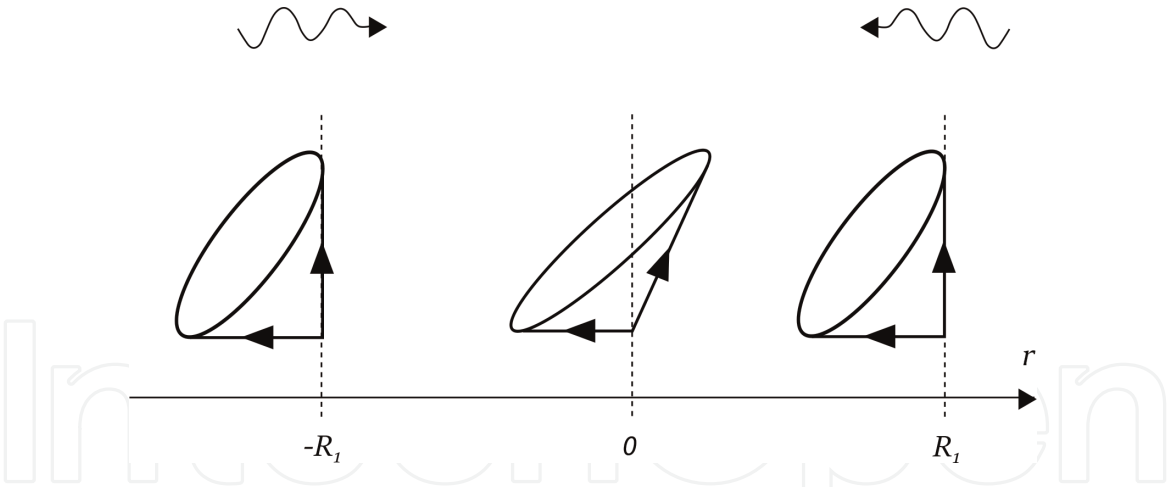

Figure 1.

Illustration of the boundary condition associated to QNMs in our spacetime. The wavy arrows represent the direction of the perturbation field at the boundaries $r = \pm R_1$, while the cones are the local light cones. Mathematically, the wavy arrow pointing to the right represents $e^{-i\omega(t-x)}$, while the wavy arrow pointing to the left represents $e^{-i\omega(t+x)}$. For more details, see Ref. [36].

$$ds^2 = \frac{1}{\cosh^2(x/R_1)} (-dt^2 + dx^2) + \sum_{j=2}^d R_j^2 d\Omega_j^2, \quad (12)$$

and the gauge field can be rewritten as

$$\mathcal{A} = Q_1 R_1 \tanh(x/R_1) dt + \sum_{j=2}^d Q_j R_j^2 \cos \theta_j d\phi_j. \quad (13)$$

In particular, note that the tortoise coordinate maps the domain between two horizons, $r \in (-R_1, R_1)$, into the interval $x \in (-\infty, \infty)$.

The QNMs accounting for an important class of fields are associated to Ψ which are solutions to the equations of motion that satisfy specific boundary conditions imposed at the horizons of the spacetime in which the field is propagating; see [5, 6, 40, 41] for more details. In this chapter, we will use the boundary conditions as illustrated in **Figure 1**.

From the mathematical of view, since we are assuming that the time dependence of Ψ is $e^{-i\omega t}$, this boundary condition means that near the horizons $r = \pm R_1$, that is, as $x \rightarrow \pm\infty$, the radial component of the field Ψ should behave as $e^{-i\omega(t+x)}$ at $x \rightarrow \infty$, while it should go as $e^{-i\omega(t-x)}$ at $x \rightarrow -\infty$. The eigenfrequencies of this problem are complex, the reason why they are called QNFs. The real part of the QNFs is associated with the oscillation frequencies of the signal, while the imaginary part is related to its decay in time. This decay in time is closely related to the fact that the event horizon has a dissipative nature.

One interesting feature of this spacetime is that we can compute exactly the QNMs. The exactly solvable systems are usually limits of more realistic systems and allow us to study in detail some properties of a physical process and test some methods which can be used to analyze more complicated systems. Thus they are powerful tools in many research lines. Therefore we expect that the exactly computed QNFs for D -dimensional generalized Nariai spacetime may play an important role in future research [42].

3. Dirac equation in D -dimensional generalized Nariai spacetime

Let us present the construction of a solution to the Dirac equation minimally coupled to the electromagnetic field of D -dimensional generalized Nariai spacetime.

A field of spin 1/2 with electric charge q and mass m propagating in such spacetime is a spinorial field obeying the following version of the Dirac equation:

$$\Gamma^\alpha (\nabla_\alpha - iq\mathcal{A}_\alpha)\Psi = m\Psi, \quad (14)$$

where \mathcal{A}_α stands for the components of the background gauge field. In $D = 2d$ dimensions, the Dirac matrices Γ^α represent faithfully the Clifford algebra by $2^d \times 2^d$ matrices obeying the relation

$$\Gamma_\alpha \Gamma_\beta + \Gamma_\beta \Gamma_\alpha = g(\mathbf{e}_\alpha, \mathbf{e}_\beta) \mathbb{I}_d, \quad (15)$$

with \mathbb{I}_d standing for the $2^d \times 2^d$ identity matrix. The index α, β, γ run from 1 to $2d$ and label the vector fields of an orthonormal frame $\{\mathbf{e}_\alpha\}$. In order to solve the Dirac equation, we must introduce a suitable orthonormal frame of vector fields, which in the case of our background is given by

$$\begin{aligned} \mathbf{e}_1 &= -i \cosh(x/R_1) \partial_t, & \mathbf{e}_j &= \frac{1}{R_j \sin \theta_j} \partial_{\phi_j}, \\ \mathbf{e}_{\bar{1}} &= \cosh(x/R_1) \partial_x, & \mathbf{e}_{\bar{j}} &= \frac{1}{R_j} \partial_{\theta_j}, \end{aligned} \quad (16)$$

where the index j ranges from 2 to d . In particular, note that

$$g(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \delta_{\alpha\beta} \leftrightarrow \begin{cases} g(\mathbf{e}_a, \mathbf{e}_b) = \delta_{ab}, \\ g(\mathbf{e}_a, \mathbf{e}_{\bar{b}}) = 0, \\ g(\mathbf{e}_{\bar{a}}, \mathbf{e}_{\bar{b}}) = \delta_{\bar{a}\bar{b}}, \end{cases} \quad (17)$$

where a and \bar{a} are indices that range from 1 to d . The index a labels the first d vector fields of the orthonormal frame $\{\mathbf{e}_a\}$, while the index \bar{a} labels the remaining d vectors of the frame $\{\mathbf{e}_a\}$. The derivatives of the frame vector fields determine the spin connection according to the following relation:

$$\nabla_\alpha \mathbf{e}_\beta = \omega_{\alpha\beta}{}^\gamma \mathbf{e}_\gamma. \quad (18)$$

Since the metric g is a covariantly constant tensor, it follows that the coefficients of the spin connection with all low indices $\omega_{\alpha\beta\gamma} = \omega_{\alpha\beta}{}^\epsilon \delta_{\epsilon\gamma}$ are antisymmetric in their two last indices, $\omega_{\alpha\beta\gamma} = -\omega_{\alpha\gamma\beta}$. Note that the indices of the spin connection are raised and lowered with $\delta_{\alpha\beta}$ and $\delta^{\alpha\beta}$, respectively, so that frame indices can be raised and lowered unpunished. In particular, $\omega_\alpha{}^{\beta\gamma} = \omega_\alpha{}^{[\beta\gamma]}$, where indices inside the square brackets are antisymmetrized. The covariant derivative of a spinorial field Ψ is, then, given by

$$\nabla_\alpha \Psi = \partial_\alpha \Psi - \frac{1}{4} \omega_\alpha{}^{\beta\gamma} \Gamma_\beta \Gamma_\gamma \Psi, \quad (19)$$

with ∂_α denoting the partial derivative along the vector field \mathbf{e}_α .

Our aim is to separate the Dirac Eq. (14). In order to accomplish this, it is necessary to use a suitable representation for the Dirac matrices. We recall that

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (20)$$

are the Hermitian Pauli matrices and \mathbb{I} denote the 2×2 identity matrix. Using this notation, a convenient representation of the Dirac matrices is the following:

$$\begin{aligned}\Gamma_a &= \underbrace{\sigma_3 \otimes \dots \otimes \sigma_1}_{(a-1) \text{ times}} \otimes \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{(d-a) \text{ times}}, \\ \Gamma_{\tilde{a}} &= \underbrace{\sigma_3 \otimes \dots \otimes \sigma_2}_{(a-1) \text{ times}} \otimes \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{(d-a) \text{ times}},\end{aligned}\tag{21}$$

where \mathbb{I} stands for the 2×2 identity matrix. Indeed, we can easily check that the Clifford algebra given in Eq. (15) is properly satisfied by the above matrices.² In this case, spinorial fields are represented by the column vectors on which these matrices act. We can introduce a basis of this representation by the direct products of spinors ξ^s given by

$$\xi^+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi^- = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\tag{22}$$

which, under the action of the Pauli matrices, satisfy concisely the relations

$$\sigma_1 \xi^s = \xi^{-s}, \quad \sigma_2 \xi^s = i s \xi^{-s}, \quad \sigma_3 \xi^s = s \xi^s.\tag{23}$$

Indeed, in $D = 2d$ dimensions, a general spinor field has 2^d degrees of freedom and can be written as

$$\Psi = \sum_{\{s\}} \Psi^{s_1 s_2 \dots s_d} \xi^{s_1} \otimes \xi^{s_2} \otimes \dots \otimes \xi^{s_d},\tag{24}$$

where each of the indices s_a can take the values “+1” and “−1.” Since every s_a can take just two values, it follows that the sum over $\{s\} \equiv \{s_1, s_2, \dots, s_d\}$ comprises 2^d terms, which is exactly the number of components of a spinorial field in $D = 2d$ dimensions.

In the representation (Eq. (21)), the operator $\Gamma^\alpha \nabla_\alpha$, called Dirac operator, is then represented by

$$\Gamma^\alpha \nabla_\alpha = \sum_{a=1}^d (\Gamma_a \nabla_a + \Gamma_{\tilde{a}} \nabla_{\tilde{a}}) = \sum_{a=1}^d \underbrace{\sigma_3 \otimes \dots \otimes D_a}_{(a-1) \text{ times}} \otimes \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{(d-a) \text{ times}},\tag{25}$$

where

$$D_a = \sigma_1 \nabla_a + \sigma_2 \nabla_{\tilde{a}},\tag{26}$$

is the Dirac operator on \mathbb{R}^2 with coordinates $\{x^a, y^a\}$. The spinorial basis introduced previously is very convenient, since the action of the Dirac matrices on the spinor fields can be easily computed. Indeed, using Eqs. (21), (23), and (24), we eventually arrive at the following equation

² In $D = 2d + 1$, besides the $2d$ Dirac matrices Γ_a and $\Gamma_{\tilde{a}}$, we need to add one further matrix, which will be denoted by Γ_{d+1} given by $\Gamma_{d+1} = \underbrace{\sigma_3 \otimes \sigma_3 \dots \otimes \sigma_3}_{d \text{ times}}$.

$$\begin{aligned} \Gamma_a \Psi &= \sum_{\{s\}} (s_1 s_2 \dots s_{a-1}) \Psi^{s_1 s_2 \dots s_d} \xi^{s_1} \otimes \xi^{s_2} \otimes \dots \otimes \xi^{s_{a-1}} \otimes \xi^{-s_a} \otimes \xi^{s_{a+1}} \otimes \\ &\dots \otimes \xi^{s_d} = \sum_{\{s\}} (s_1 s_2 \dots s_a) \Psi^{s_1 s_2 \dots s_{a-1} (-s_a) s_{a+1} \dots s_d} \xi^{s_1} \otimes \xi^{s_2} \otimes \\ &\dots \otimes \xi^{s_{a-1}} \otimes \xi^{s_a} \otimes \xi^{s_{a+1}} \otimes \dots \otimes \xi^{s_d}, \end{aligned} \quad (27)$$

where from the first to the second line we have changed the index s_a to $-s_a$, which does not change the final result, since we are summing over all values of s_a , which comprise the same list of the values of $-s_a$. Moreover, we have used $(s_a)^2 = 1$. Analogously, we have:

$$\begin{aligned} \Gamma_{\hat{a}} \Psi &= \sum_{\{s\}} (s_1 s_2 \dots s_{a-1}) (i s_a) \Psi^{s_1 s_2 \dots s_d} \xi^{s_1} \otimes \xi^{s_2} \otimes \dots \otimes \xi^{s_{a-1}} \otimes \xi^{-s_a} \otimes \xi^{s_{a+1}} \otimes \\ &\dots \otimes \xi^{s_d} = -i \sum_{\{s\}} (s_1 s_2 \dots s_a) s_a \Psi^{s_1 s_2 \dots s_{a-1} (-s_a) s_{a+1} \dots s_d} \xi^{s_1} \otimes \xi^{s_2} \otimes \\ &\dots \otimes \xi^{s_{a-1}} \otimes \xi^{s_a} \otimes \xi^{s_{a+1}} \otimes \dots \otimes \xi^{s_d}. \end{aligned} \quad (28)$$

All that was seen above are necessary tools to attack our initial problem of separating the general Eq. (14). In order to solve such an equation, we need to separate the degrees of freedom of the field, which can be quite challenging in general. Fortunately, the spacetime considered here is the direct product of two-dimensional spaces of constant curvature, which is exactly the class of spaces studied in Ref. [36]. Indeed, in this latter paper, it is shown that the Dirac equation minimally coupled to an electromagnetic field is separable in such backgrounds. In particular, assuming that the components of the spinor field Eq. (24) can be decomposed in the form

$$\Psi^{s_1 s_2 \dots s_d} = \Psi_1^{s_1}(t, x) \Psi_2^{s_2}(\Phi_2, \theta_2) \dots \Psi_d^{s_d}(\Phi_d, \theta_d), \quad (29)$$

where each index s_a can take the values $s_a = \pm 1$, the fields $\Psi_1^{s_1}(t, x)$ satisfy the following differential equation (the reader is invited to demonstrate the equation below or consult more details in [36]):

$$\left[\partial_{\hat{1}} + \frac{\omega_{1\hat{1}\hat{1}}}{2} - iq \mathcal{A}_{\hat{1}} - is_1 \left(\partial_1 + \frac{\omega_{1\hat{1}\hat{1}}}{2} - iq \mathcal{A}_1 \right) \right] \Psi_1^{s_1} = (L - is_1 m) \Psi_1^{-s_1}. \quad (30)$$

The separation constant L in the above equation depends on the angular modes. In particular, in the special case of vanishing magnetic charges Q_j , it is determined by the eigenvalues λ_j of the Dirac operator on unit sphere S^2 according to the following relation

$$L = \sqrt{\lambda_2^2 + \lambda_3^2 + \dots + \lambda_d^2}, \quad \lambda_j = \pm 1, \pm 2, \pm 3, \dots, \quad (31)$$

as demonstrated in Appendix A of Ref. [36]. In our frame of vectors, the only components of the spin connection that are potentially nonvanishing are

$$\begin{cases} \omega_{1\hat{1}\hat{1}} = -\omega_{\hat{1}\hat{1}\hat{1}} = -\frac{1}{R_1} \sinh(x/R_1), \\ \omega_{\hat{j}\hat{j}\hat{j}} = -\omega_{\hat{j}\hat{j}\hat{j}} = \frac{1}{R_j} \cot \theta_j, \end{cases} \quad (32)$$

and the nonzero components of the gauge field can be written as

$$\mathcal{A}_1 = -iQ_1R_1 \sinh(x/R_1), \quad \mathcal{A}_j = Q_jR_j \cot \theta_j. \quad (33)$$

Now, since the components of the metric are independent of the coordinate t , the vector ∂_t is a Killing vector for this metric. So, it is useful to assume the following time dependence for the field $\Psi_1^{s_1}(t, x)$

$$\Psi_1^{s_1}(t, x) = e^{-i\omega t} \psi^{s_1}(x). \quad (34)$$

Inserting this field along with the gauge field Eq. (33), and taking into account the first relation of the Eq. (32) into the Eq. (30), we end up with the following coupled system of differential equations:

$$\left[\frac{d}{dx} + is_1\omega + \left(isqQ_1R_1 - \frac{1}{2R_1} \right) \tanh(x/R_1) \right] \psi^{s_1} = \frac{(L - is_1m)}{\cosh(x/R_1)} \psi^{-s_1}. \quad (35)$$

In order to solve these equations, we should first decouple the fields ψ^{s_1} and ψ^{-s_1} . Eliminating ψ^{-s_1} we obtain a second-order equation for ψ^{s_1} . Indeed, we can prove that the fields ψ^{s_1} satisfy the following second-order ordinary differential equation

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x) \right] \psi^{s_1} = 0, \quad (36)$$

which is a Schrödinger-like equation with V being a potential of the form

$$V(x) = A + B \tanh(x/R_1) + \frac{C}{\cosh^2(x/R_1)}, \quad (37)$$

where the parameters A , B , and C are given by

$$\begin{cases} A = \frac{1}{4R_1^2} - qQ_1(is_1 + qQ_1R_1^2), \\ B = -\frac{\omega}{R_1}(is_1 + 2qQ_1R_1^2), \\ C = m^2 + L^2 + \frac{1}{4R_1^2} + q^2Q_1^2R_1^2. \end{cases} \quad (38)$$

These are known as potentials of Rosen-Morse type, which are generalizations of the Pöschl-Teller potential [34, 35]. It is straightforward to see that this potential satisfies the following properties:

$$V \rightarrow \begin{cases} A + B & \text{at } x \rightarrow +\infty, \\ A - B & \text{at } x \rightarrow -\infty. \end{cases} \quad (39)$$

In many cases, the potential function V is regular at $r = 0$ ($x = 0$), in particular V can be equal to a constant different from zero. In fact, in our case, we find that

$$V \rightarrow A + C \quad \text{at } x \rightarrow 0, \quad (40)$$

which clearly is regular. So, we point out that for this potential both limits (Eqs. (39) and (40)) are finite, and thus there is no reason to demand for a regular solution in this point.

Thus, the problem of finding the QNMs is reduced to the searching of the corresponding spectrum of QNFs ω of Eq. (36). Most of the problems concerning the QNMs fall into Schrödinger-like equation with real potentials which vanish at both horizons [5], highlighting the fact that the solutions can be taken to be plane waves. However, clearly this is not the case. Although it is possible to make field redefinitions in order to make the potential real, we shall not do this here. For such procedure we refer the reader to [33]. Once an analytical form for the QNFs of Rosen-Morse type potential is not known, we must find an analytical exact solution of Eq. (36) and impose physically appropriate boundary conditions at the horizons, $x \rightarrow \pm\infty$, which define the QNFs in a unique way.

In order to solve Eq. (36), let us make the following change of variable

$$y = \frac{1}{2} + \frac{1}{2} \tanh(x/R_1). \quad (41)$$

In particular, notice that y is defined on the domain $y \in (0, 1)$ with the boundaries $x \rightarrow \pm\infty$ being given by $y = 0$ and $y = 1$. In addition to this change of independent variable, if we now set the Ansatz

$$\psi^{s_1}(x) = y^\alpha(1-y)^\beta H^{s_1}(y), \quad (42)$$

with the parameters α and β being constants conveniently chosen as

$$\alpha = \frac{R_1}{2} \sqrt{A - B - \omega^2} \quad , \quad \beta = -\frac{R_1}{2} \sqrt{A + B - \omega^2}, \quad (43)$$

the functions H^{s_1} must be solutions of the following differential equation

$$y(1-y) \frac{d^2 H^{s_1}}{dy^2} + [2\alpha + 1 - (2 + 2\alpha + 2\beta)y] \frac{dH^{s_1}}{dy} - [CR_1^2 + (\alpha + \beta)(1 + \alpha + \beta)] H^{s_1}. \quad (44)$$

This new variable as well as the Ansatz that we have been using are really interesting because in terms of these, it is immediate to see that the functions H^{s_1} satisfy a hypergeometric equation. Indeed, comparing with the standard hypergeometric differential equation

$$y(1-y) \frac{d^2 H^{s_1}}{dy^2} + [c - (1 + a + b)y] \frac{dH^{s_1}}{dy} - abH^{s_1} = 0, \quad (45)$$

we find that the constants a , b , and c are given by

$$\begin{cases} a = \frac{1}{2} + \alpha + \beta + \sqrt{\frac{1}{4} - CR_1^2}, \\ b = \frac{1}{2} + \alpha + \beta - \sqrt{\frac{1}{4} - CR_1^2}, \\ c = 2\alpha + 1. \end{cases} \quad (46)$$

Such an equation admits two linearly independent solutions whose linear combination furnishes the following general solution:

$$H^{s_1}(y) = D {}_2F_1(a, b, c; y) + Ey^{(1-c)} {}_2F_1(1+a+c, 1+b+c, 2-c; y), \quad (47)$$

where ${}_2F_1$ is the hypergeometric function and D and E are arbitrary integration constants. Given the hypergeometric solution for H^{s_1} is known, one can immediately find the general solution for ψ^{s_1} . Indeed, from Eqs. (42), (46), and (47), we conclude that the solution of Eq. (36), which is regular at the origin, can be written as

$$\begin{aligned} \psi^{s_1} = & (1-y)^{\frac{1}{2}(a+b-c)} [Dy^{\frac{1}{2}(c-1)} {}_2F_1(a, b, c; y) \\ & + Ey^{-\frac{1}{2}(c-1)} {}_2F_1(1+a-c, 1+b-c, 2-c; y)]. \end{aligned} \quad (48)$$

In order to fix the integration constants D and E , we need to apply the appropriate boundary conditions. Inverting the Eq. (41) we find that, near the boundaries $x \rightarrow \pm\infty$, the relation between the coordinates x and y assumes the simpler form

$$\begin{aligned} y & \simeq e^{+2x/R_1} \quad \text{at } x \rightarrow -\infty, \\ 1-y & \simeq e^{-2x/R_1} \quad \text{at } x \rightarrow +\infty. \end{aligned} \quad (49)$$

Thus, taking into account the latter relation and using the fact that at $y = 0$ ($x \rightarrow -\infty$) the hypergeometric function ${}_2F_1(a, b, c; 0) = 1$, one eventually obtains that near the boundary $x \rightarrow -\infty$ the field ψ^{s_1} behaves as

$$\psi^{s_1}|_{x \rightarrow -\infty} \simeq D e^{(c-1)x/R_1} + E e^{-(c-1)x/R_1}. \quad (50)$$

On the other hand, in order to apply the boundary conditions at $y = 1$ ($x \rightarrow \infty$), it is useful to write the hypergeometric functions as functions of $(1-y)$, so that they become united at the boundary. This can be done by rewriting the hypergeometric functions appearing in Eq. (48) by means of the following identity [45]:

$$\begin{aligned} {}_2F_1(a, b, c; y) = & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-y) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-y)^{(c-a-b)} {}_2F_1(c-a, c-b, c-a-b+1; 1-y), \end{aligned} \quad (51)$$

where Γ stands for the gamma function. Doing so, and using Eq. (49), we eventually arrive at the following behavior of the solution at $x \rightarrow +\infty$:

$$\begin{aligned} \psi^{s_1}|_{x \rightarrow +\infty} \simeq & \left[D \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} + E \frac{\Gamma(c-a-b)\Gamma(2-c)}{\Gamma(1-a)\Gamma(1-b)} \right] e^{-(a+b-c)x/R_1} \\ & + \left[D \frac{\Gamma(a+b-c)\Gamma(c)}{\Gamma(a)\Gamma(b)} + E \frac{\Gamma(a+b-c)\Gamma(2-c)}{\Gamma(a-c+1)\Gamma(b-c+1)} \right] e^{(a+b-c)x/R_1}. \end{aligned} \quad (52)$$

Now, from parameters Eqs. (38) and (43), we find that the constants appearing in the hypergeometric equation can be written as

$$\begin{aligned}
 a &= iR_1\sqrt{\mu^2 + q^2Q_1^2R_1^2 + L^2} + (1 + s_1)\left(\frac{1}{4} - i\omega\frac{R_1}{2}\right) - i(1 - s_1)\frac{qQ_1R_1^2}{2}, \\
 b &= -iR_1\sqrt{\mu^2 + q^2Q_1^2R_1^2 + L^2} + (1 + s_1)\left(\frac{1}{4} - i\omega\frac{R_1}{2}\right) - i(1 - s_1)\frac{qQ_1R_1^2}{2}, \\
 c &= \frac{1}{2} + is_1(qQ_1R_1^2 - \omega R_1).
 \end{aligned} \tag{53}$$

In particular, the following relations hold

$$(c - 1)/R_1 = -is_1\omega + is_1qQ_1R_1 - \frac{1}{2R_1}, \tag{54}$$

$$(a + b - c)/R_1 = -i\omega - iqQ_1R_1 + s_1\frac{1}{2R_1}. \tag{55}$$

Now we are ready to impose the boundary conditions. Obviously, without loss of generality, we can consider that the spin s_1 is already chosen and fixed at $s_1 = +$ or $s_1 = -$ since the QNFs should not depend on the choice of $s_1 = \pm$. Let us impose, for instance, the boundary conditions for the component $s_1 = +$ of the spinorial field. In this case, using the identity Eq. (54) along with the Eq. (34), we eventually arrive at the following behavior of the solution at $x \rightarrow \infty$:

$$\Psi_1^+(t, x)|_{x \rightarrow -\infty} = D e^{-i\omega(t+x)} e^{\left(iqQ_1R_1 - \frac{1}{2R_1}\right)x} + E e^{-i\omega(t-x)} e^{-\left(iqQ_1R_1 - \frac{1}{2R_1}\right)x}. \tag{56}$$

Now, **Figure 1** tells us that the field is assumed to move toward higher values of x at the boundary $x \rightarrow -\infty$, while at the boundary $x \rightarrow +\infty$ it should move toward lower values of x . Then, since the time dependence of the field Ψ_1^+ is of the type $e^{-i\omega t}$, this means that Ψ_1^+ should behave as $e^{-i\omega(t-x)}$ at $x \rightarrow -\infty$, while it should go as $e^{-i\omega(t+x)}$ at $x \rightarrow +\infty$. Thus, from Eq. (55), we conclude that we must set $D = 0$. In such a case, from Eq. (52), the field Ψ_1^+ becomes

$$\begin{aligned}
 \Psi_1^+|_{x \rightarrow +\infty} &\simeq E \left[\frac{\Gamma(c - a - b)\Gamma(2 - c)}{\Gamma(1 - a)\Gamma(1 - b)} \right] e^{-i\omega(t-x)} e^{\left(iqQ_1R_1 - \frac{1}{2R_1}\right)x} \\
 &+ E \left[\frac{\Gamma(a + b - c)\Gamma(2 - c)}{\Gamma(a - c + 1)\Gamma(b - c + 1)} \right] e^{-i\omega(t+x)} e^{\left(-iqQ_1R_1 + \frac{1}{2R_1}\right)x}.
 \end{aligned} \tag{57}$$

Finally, to satisfy the QNM boundary condition near the boundary at $x \rightarrow \infty$, we must eliminate the term $e^{-i\omega(t-x)}$ of the above equation. Since E cannot be zero (as otherwise the field would vanish identically), we need the combination of the gamma functions to be zero. Now, once the gamma function has no zeros, the way to achieve this is to let the gamma functions in the denominator diverge, $\Gamma(1 - a) = \infty$ or $\Gamma(1 - b) = \infty$. Since the gamma functions diverge only at nonpositive integers, we are led to the following constraint:

$$1 - a = -n \quad \text{or} \quad 1 - b = -n, \quad \text{where} \quad n \in \{0, 1, 2, \dots\}. \tag{58}$$

Using the Eq. (53), we find that these constraints translate to

$$\omega = \pm \sqrt{m^2 + q^2Q_1^2R_1^2 + L^2} + \frac{i}{R_1} \left(n + \frac{1}{2} \right), \tag{59}$$

which are the QNFs of the Dirac field propagating in D -dimensional generalized Nariai spacetimes. The real part of a QNF is associated with the oscillation frequency, while the imaginary part is related to its decay rate. At this point, it is worth recalling that L is a separation constant of the Dirac equation that is related to the angular mode of the field.

Likewise, imposing the boundary condition to the component $s_1 = -$ of the spinorial field, we find that we must set $E = 0$ at Eq. (50) and then $c - a = -n$ or $c - b = -n$, with n being a nonnegative integer. This, in its turn, leads to the same spectrum obtained for the component $s_1 = +$ as expected, namely, Eq. (59).

4. Conclusions

In this chapter we have investigated the perturbations on a spinorial field propagating in a generalized version of the charged Nariai spacetime. Besides the separability of the degrees of freedom of these perturbations, one interesting feature of this background is that the perturbations can be analytically integrated. They all obey a Schrödinger-like equation with an integrable potential that is contained in the Rosen-Morse class of integrable potentials. Such an equation admits two linearly independent solutions given in terms of standard hypergeometric functions. This is a valuable property, since even the perturbation potential associated to the humble Schwarzschild background is nonintegrable, despite the fact that it is separable. We have also investigated the QNMs associated to this spinorial field. Analyzing the Eq. (59), namely,

$$\omega_D = \pm \sqrt{m^2 + q^2 Q_1^2 R_1^2 + L^2} + \frac{i}{R_1} \left(n + \frac{1}{2} \right), \quad (60)$$

it is interesting to note that the imaginary parts of the QNFs, which represent the decay rates, do not depend on any details of the perturbation; rather, they only depend on the charges of the gravitational background through the dependence on R_1 . On the other hand, the real parts of the QNFs depend on the mass of the field and on the angular mode of the perturbations. Another fact worth pointing out is that the fermionic field always has a real part in its QNFs spectrum, meaning that it always oscillates. This is not reasonable. Indeed, for Klein-Gordon and Maxwell perturbations in the D -dimensional Nariai spacetime, their QNFs are equal to [36].

$$\begin{aligned} \omega_{KG} &= \pm \sqrt{m^2 + \sum_{j=2}^d \frac{\ell_j(\ell_j + 1)}{R_j^2} - \frac{1}{4R_1^2} - \frac{i}{R_1} \left(n + \frac{1}{2} \right)}, \\ \omega_M &= \pm \sqrt{\sum_{j=2}^d \frac{\ell_j(\ell_j + 1)}{R_j^2} - \frac{1}{4R_1^2} - \frac{i}{R_1} \left(n + \frac{1}{2} \right)}, \end{aligned} \quad (61)$$

where ℓ_j and m_j are integers, $|m_j| \leq \ell_j$, and $\ell \geq 0$. Due to the negative factor $-1/(4R_1^2)$ inside the square root appearing in the bosonic spectrum, it follows that for small enough R_1 , along with small enough mass and angular momentum, the argument of the square root can be negative, so that this term becomes imaginary.

To finish, we believe that a good exercise is to calculate the QNFs of the gravitational field in D -dimensional generalized charged Nariai spacetime. Research on the latter problem is still ongoing and, due to the great number of degrees of freedom in the gravitational field, shall be considered in a future work. The next interesting step is the investigation of superradiance phenomena for the spin $1/2$

field. Although bosonic fields like scalar, electromagnetic, and gravitational fields can exhibit superradiant behavior in four-dimensional Kerr spacetime [46], curiously, this is not the case for the Dirac field [33]. Thus, it would be interesting to investigate whether an analogous thing happens in the background considered here.

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