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Chapter

Applications of the Abelian Vortex Model to Cosmic Strings and the Universe Evolution

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Abstract

Due to the wide range of applications and effects of the Abelian vortex model of Nielsen and Olesen in the many areas of physics, ranging from condensed matter to astrophysical effects, some work in the literature is necessary to approach this topic in a succinct form that the undergraduate student in both physics and related areas has the possibility to know and understand. The mechanisms associated with this vortex model indicate him as a strong candidate for the source for the topological defects proposed by Vilenkin.

Keywords: cosmic string, curved space-time, relativity, field theory

1. Introduction

According to the Big Bang theory, the universe is expanding and cooling. During its expansion, the spontaneous breaks of fundamental symmetries led the universe to undergo a series of phase transitions. In high-energy physics models, the formation of topological defects, caused by transitions, such as domain walls, monopoles, and cosmic strings, among others, is predicted to occur according to the reference [1, 2].

The cosmic string is among the most studied types of topological defects, although recent observations of cosmic background radiation have discarded it as the primary source for primordial density perturbations. Such a defect still serves as one of the contributions of this disturbance. This type of defect also serves as a possible source for explaining a considerable number of astrophysical effects, such as: bursts of gamma rays, where the energy scale of the string in which the symmetry is broken, on an energy scale of the order of 10^{14} GeV, explains the rate, duration, and fluency of gamma ray bursts [3]; high-frequency gravitational wave emissions, which have as a consequence of these emissions the stochastic set of gravitational waves generated by a cosmological network of non-Gaussian loops [4]; and the generation of high-energy cosmic rays [5]. The cosmic rays of high-energy particles may have originated during the process of collapse and/or annihilation of topological defects associated with the great unification theories.

In condensed matter physics, it is well known that superconductors almost completely exclude any external magnetic field if it is less than a critical value (Meissner effect) [6]. However, for type 2 superconducting, which are formed by materials in which the transition to the superconducting state is gradual, in the

presence of an intermediate state, if the external field is increased to a certain value greater than the critical value, such field. This superconductor passes through a magnetic flux tube form. These phenomena are called magnetic flux vortices which, in turn, are quantized.

The possibility of the theoretical existence of such vortices was first demonstrated by Abrikosov [7]. He showed that these naturally occur as solutions to the Ginsburg-Landau theory of superconductivity in the presence of an external magnetic field. Following this theory, the existence of such objects was verified experimentally, and many of their properties were rigorously investigated in [6]. Some years later, Nielsen and Olesen [8] showed, starting from the relativistic field theory model with spontaneous break of symmetry, more specifically of the Abelian Higgs model interacting with a field of gauge, that this system presents solutions with cylindrical symmetry carrying a magnetic flux. These configurations correspond to vortex solutions.

The analysis of the influence of this system on space-time geometry was performed by Garfinkle [9] and Laguna [10]. In their works, the authors coupled the energy-momentum tensor, associated to the Nielsen-Olesen model, with the Einstein field equations. In this sense, they have shown that the vortex has an internal structure characterized by the nonzero magnetic flux that runs along it, the extent of which is determined by the energy scale at which the symmetry is broken. Two scale lengths appear naturally, one related to the extent of the magnetic flux which, in turn, is proportional to the inverse of the vector field mass, m_v, east field, which acquires mass due to the Higgs mechanism; and the other associated with the inverse of the scalar field mass, m_s, the latter, as a measure of the point where the scalar field decreases to its vacuum value. Moreover, the authors also analyzed the geometry of space-time and verified that asymptotically the surface perpendicular to the vortex corresponds to Minkowski's space-time minus a slice, resulting in a space with an angular deficit.

A special vortex solution satisfying the Bogomolny-Prasad-Sommerfield (BPS) boundary [11, 12] shows the masses of the scalar field and of the same caliber field, that is, $m_s = m_v$. For this case, Linet [13] was able to find an exact solution for the metric tensor, which is determined in terms of the energy density of the cosmic string. In this limit, the surface perpendicular to the line of the solution of vortex has a conical structure and, the space-time surrounding, corresponds to the space-time of an idealized cosmic string.

At great distances, the space-time generated by a cosmic string has, in its origin and in the orthogonal plane to the disposition of this object, a conical topology with a planar angle deficit proportional to the linear density of mass of this cosmic string. In quantum field theory, the nontrivial topology of this object induces non-vanish vacuum expected values for physical observables. These vacuum polarization effects can be interpreted as a modification in the quantum levels of the lower energy state of a theory. In quantum field theory, induced by a conic structure, they were the targets of many works published. For example, we can observe several published works, taking into account the case for scalar fields [14–19] and fermionic fields [20–22] interacting with vector fields. Another induced physical observable, due to the presence of this defect, is the current and charge density, which will serve as the source for Maxwell's equations. Such an object considering fermionic fields is seen in [23–26].

2. The general relativity and the space-time

The general relativity theory is a geometric theory of gravitation published by Albert Einstein in 1915 and the current description of gravitation in modern physics.

It is a set of hypotheses that generalizes Newton's special relativity and the universal gravitation law providing a unified description of gravity as a geometric property of the space-time. In particular, the "curvature of space-time" is directly related to the energy and moment of any matter and radiation present. The relation is specified by Einstein's field equations, a system of partial differential equations.

All geometric information about the space-time would be contained in this mathematical object called, formally, metric tensor, $g_{\mu\nu}$. In other words, the distribution of matter and energy tells how the geometry of space-time [27] must be. The equation proposed by Einstein for the theory of General Relativity is given by the expression below

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{1}$$

Here, $R_{\mu\nu}$ is the Ricci tensor that is obtained from the Riemann tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar of curvature, and $T_{\mu\nu}$ is the energy-momentum tensor. In order to introduce the idea of the metric structure of the space-time, we will briefly review the necessary basic concepts, such as inertial frame and interval of events [27].

Let us suppose that an inertial frame S is described in Cartesian coordinates (t, x, y, z). In this frame, we have the line element ds being infinitesimal and having its own time interval (event) given by

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{2}$$

But if we consider a non-inertial reference system, S', for example, the line element will not be given, in general, by the sum of the squares of the coordinate differentials. In this case, for a better understanding, let us consider an event in a rotating frame, around the z axis, whose angular frequency of rotation is ω. Let (t', x', y', z') be the coordinates of this new S' referential. The relation between both reference frames may be illustrated by **Figure 1**.

The general coordinate transformations between the both reference frames S and S' are given as follow,

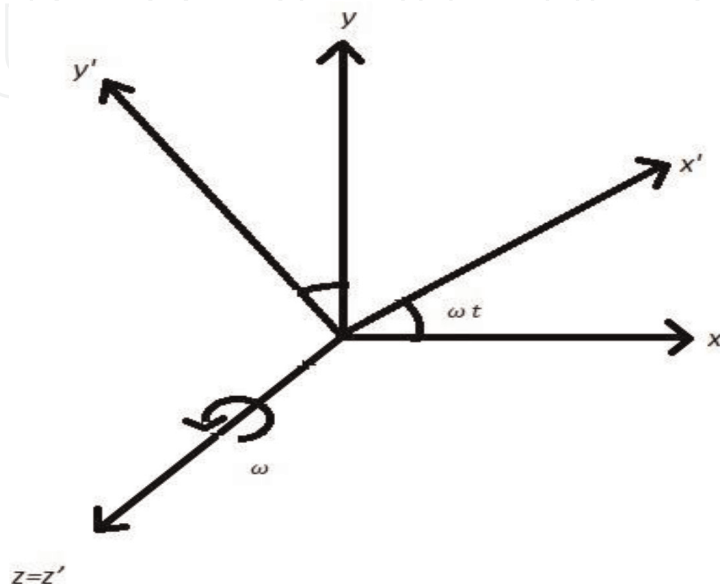$$x = x' \cos(\omega t) - y' sen(\omega t) \tag{3}$$

Figure 1.
The relation between S and S' reference frame with angular velocity ω around the z = z' axis.

$$y = x' sen(\omega t) + y' \cos(\omega t) \tag{4}$$

$$z = z'. \tag{5}$$

In this way, taking into account the derivative of the Eqs. (3)–(5) and putting them in the Eq. (2), the line element will take the form expressed by

$$ds^2 = \left[1 - \omega^2 \left(x'^2 + y'^2\right)\right] dt^2 + 2\omega dt(y'dx' - x'dy') - d\vec{r}\,'2. \tag{6}$$

We see, therefore, that the line element is not only the sum or difference of the squares of the differential coordinates.

Looking to the Eq. (2), we identify that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$, where we have the metric signature given by $\eta_{\mu\nu} = (1, -1, -1, -1)$ being the four-vector position $x^\mu = \left(t, -\vec{r}\right)$. On the other hand, looking into Eq. (6), when non-inertial coordinate systems are used, the line element will include terms that are products of the different coordinate differentials. So, we can write the line element as follows

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{7}$$

Now, $g_{\mu\nu}(x)$ represents a set of ten functions of the space and time coordinates and it is symmetric, i.e., $g_{\mu\nu}(x) = g_{\nu\mu}(x)$. The system described by Eq. (7) is called "curved system" and corresponds to an accelerated reference system. The functions $g_{\mu\nu}(x)$ contain all the geometric properties of the space-time. For the case where we deal with inertial frames, we just have $g_{\mu\nu}(x) = \eta_{\mu\nu}$.

Einstein showed that accelerated referential are equivalent to gravitational fields so that gravitational effects will be described by the metric tensor, $g_{\mu\nu}(x)$. In this case, the gravitation can be understood as a deviation in the metric of the space-time plane. Moreover, this metric is not fixed arbitrarily but will depend on the local distribution of matter.

In fact, this equivalence is verified only locally. In a non-inertial system, given a metric $g_{\mu\nu}(x)$, we can always reduce it globally to the Galileo form, Eq. (2), by means of a suitable coordinate transformation. On the other hand, a gravitational field cannot be eliminated globally by a coordinate transformation, and the metric can only be reduced to the flat form (Minkowski) only in a very small finite region of the space, i.e., locally. When such a situation occurs, the space-time is called pseudo-Riemannian space-time.

3. Cosmic strings

It is believed that fluctuations that gave rise to the large-scale structures of the Universe must have a primordial origin, that is, they are associated with the first moments after the Big Bang. The existing theories for structure formation in the Universe fall into two categories.

One of them based on amplification of quantum fluctuations in a scalar field during inflation. The other one based on a phase transition with symmetry breaking in the primordial universe that gives rise to the formation of topological defects.

Seen from the moment of creation, the Universe goes through phase successions. The transitions between the first of these phases occur when the Universe is dominated by a quantum gravitation whose exact contours are unknown but during

which the interactions are thought to be unified and characterized by a high degree of symmetry. These transitions imply symmetry breaks and can have important implications including the formation of topological defects such as the formation of cosmic strings or initiation of a period of exponential inflation.

A cosmic string is an object that can be obtained from an infinitely concentrated distribution of matter, with linear density of mass μ [2]. In the case of a certain distribution being located on the z-axis, the energy-momentum tensor, in cylindrical coordinates, is given by

$$T_\nu^\beta = \mu \, diag(1, 0, 0, 1)\delta^{(2)}\left(\vec{r}\right). \tag{8}$$

Here, $\delta^{(2)}\left(\vec{r}\right)$ is a two-dimensional Dirac delta function. Geometrically, a topological defect can be characterized by a space-time whose metric associated with this defect has the corresponding Riemann-Christoffel tensor null at all points, except for the defect, i.e., the space-time has conical singularity. In other words, it may be characterized by a bending tensor, which is proportional to a delta function supported on the defect.

We want that the Eq. (8) generates a geometry with cylindrical symmetry. For that, our goal is to find a solution to Einstein's equations describing the gravitational field of an ideal cosmic string with linear mass density μ along the z-axis. In this sense, the string will have no dependence over time, so it is a temporal invariant. We will also admit a symmetry of the string in relation to the azimuth angle, and finally that it remains invariant by boosts. Thus, the most general line element, in cylindrical coordinates, which exhibits such symmetry and maintains invariance by boosts transformations along the z-axis, is given by

$$ds^2 = A^2(r)dt^2 - dr^2 - B^2(r)d\phi^2 - A^2(r)dz^2. \tag{9}$$

Using Eq. (1), taking into account the metric tensor given in Eq. (9), we can calculate the Christoffel symbols and obtain a set of non-linear differential equations given by

$$R_t^t = R_z^z = \frac{A''(r)}{A(r)} + \frac{A'(r)B'(r)}{A(r)B(r)} - \left(\frac{A'(r)}{A(r)}\right)^2, \tag{10}$$

$$R_r^r = 2\frac{A''(r)}{A(r)} + \frac{B'(r)}{B(r)}, \tag{11}$$

$$R_r^r = \frac{B''(r)}{B(r)} + 2\frac{A'(r)B'(r)}{A(r)B(r)}. \tag{12}$$

Solving these equations, we get the following solutions

$$A'(r) = \frac{d}{dr}A(r) = 0; \quad \frac{B\prime(r)}{B(r)} = \frac{1}{B(r)}\frac{d^2}{dr^2}B(r) = -8\pi\mu. \tag{13}$$

The above solution provides the following line element [2, 14]

$$ds^2 = dt^2 - dr^2 - (1 - 4\mu)d\phi^2 - dz^2. \tag{14}$$

Redefining the angular coordinate in Eq. (14), where we use the substitution $\phi' = \phi/q$ with $q^{-1} = (1 - 4\mu)$, we have

$$ds^2 = dt^2 - dr^2 - d\phi'^2 - dz^2, \qquad (15)$$

where the angular coordinate varies in the range $\left(0, \frac{2\pi}{q}\right)$, so that space-time is now locally flat except for $r = 0$, which means except under the defect. This line element, from a global point of view, corresponds to Minkowski's space-time minus one piece subtended by the angle $8\pi\mu$. The quantity μ has great importance in string theory since it characterizes the intensity of the gravitational interaction and its value obtained from the Great Unification Theories is comprised in the order of 10^{-6} [28, 29]. Then, space-time generated by a cosmic string has the shape of a cone in the perpendicular plane to the string. Being flat itself, it satisfies Einstein's equations in every region where $T_\nu^\beta = 0$.

The effect of the string is therefore to introduce a deficit in the azimuthal angle given by $\Delta\phi = 8\pi\mu$, generating in the surface (t, z) = constants, a conical geometry instead of a flat geometry, which will be pointed in the limit of the string internal structure going to zero. In this case, the corresponding space-time is conic and best described in cylindrical coordinates due to the symmetry of the problem. The geometry described above has many interesting features, such as:

- Absence of Newtonian gravitational potential although this does not imply the absence of gravitational effects, that is, a particle placed in the presence of a cosmic string will not be attracted to it, whatever the order of magnitude of the mass density of the string, which is quite different from that predicted by Newton's gravitational string of matter; in other words, the cosmic strings have zero gravitational potential [30].

- It can act as a gravitational lens as shown in **Figure 2**, that is, due to the conic nature of space-time around the cosmic string, double images of objects located behind the string can be formed in relation to an observer [2].

- Gravitational analog of the Aharonov-Bohm effect, due to the movement of test particles in space-time of cosmic strings through the study of geodesics [31].

- Electrostatic self-interaction [13] that arises due to the gravitational field inducing a curvature in space-time, and this curvature causes distortions in the field lines of the electrostatic potential generated by a charged particle, causing this particle to undergo a finite force upon itself.
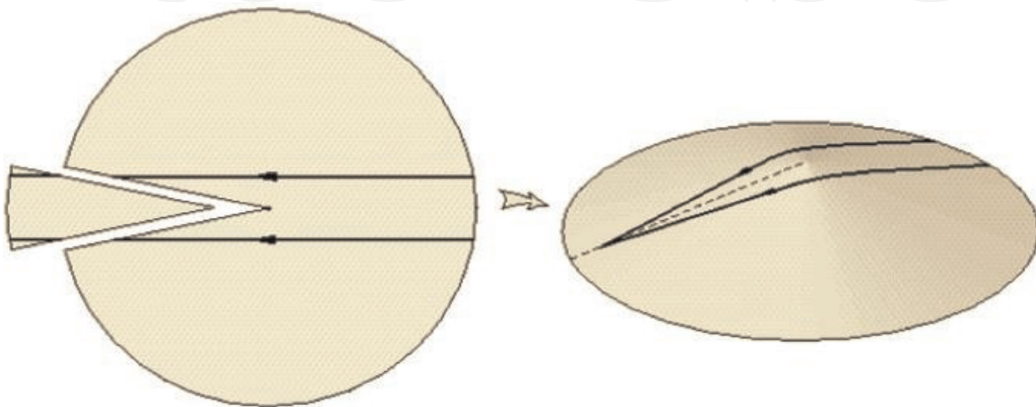

Figure 2.
Representation of the light way coming from the infinity and "curving" due to the presence of a cosmic string.

4. The Higgs mechanism

Most of the symmetries observed in nature are not exact. For example, Isospin is not an exact symmetry of nature, because the proton and the neutron do not have the same mass. One way to study symmetry breaks in field theory with symmetry breaking is to introduce the Lagrangian terms with small coefficients that explicitly perform the break. In this section, we will be interested in a symmetry breaking which the Lagrangian is symmetric under the action of a group of transformations but the state of less energy is not.

To understand how spontaneous symmetry breaking appears in many Abelian field theories, we will start considering the simple case, that is, the Lagrangian for a complex scalar field given by

$$L = \partial_\nu \varphi \partial^\nu \varphi^* - V(|\varphi|), \tag{16}$$

where $V(|\varphi|) = \mu^2 \varphi \varphi^* + \lambda(\varphi^* \varphi)^2$, being λ the self-coupling constant. To this theory, making the transformation over the scalar field is as follows

$$\varphi \to \varphi' = \varphi \, e^{iq\alpha} \text{ and } \varphi^* \to \varphi'^* = \varphi^* e^{-iq\alpha}. \tag{17}$$

Here, the parameter α does not depend on the point, and we can see that the derivative in Eq. (16) goes to

$$\partial_\nu \varphi \to \partial_\nu \varphi' = (\partial_\nu \varphi) \, e^{iq\alpha} \text{ and } \partial_\nu \varphi^* \to \partial_\nu \varphi'^* = (\partial_\nu \varphi^*) \, e^{-iq\alpha\wedge} = U(\partial_\nu \varphi). \tag{18}$$

Putting Eqs. (17) and (18) into Eq. (14), we see that

$$L \to L' = L. \tag{19}$$

As we may see, these transformations under the fields keep the Lagrangian unchanged. The transformations over the fields and their derivatives that do not depend on the point are named global gauge transformation.

On the other hand, let us consider that the parameter α now depends on the point, it means, $\alpha \equiv \alpha(x) \to U(x) = e^{iq\alpha(x)}$. These kinds of transformation are called local gauge transformation.

This way the transformation over the derivatives, Eq. (18), becomes

$$\partial_\nu \varphi \to \partial_\nu \varphi' = \partial_\nu \left(\varphi \, e^{iq\alpha(x)} \right) = (\partial_\nu \varphi) e^{iq\alpha(x)} + iq(\partial_\nu \alpha(x)) \varphi. \tag{20}$$

As we can see, the field derivative does not transform as the field itself. The second term that appears in Eq. (20) turns the Lagrangian as not invariant by these transformations over the fields. This way, to turn this theory unchanged by transformations where the parameter now depends on the point, we have to add new fields called "compensating fields," $A_\nu(x)$. Doing this we also have to redefine the derivative concept, and this way, we have

$$\partial_\nu \to D_\nu = \partial_\nu + iq A_\nu(x). \tag{21}$$

In Eq. (21), we have the covariant derivative. Now, under transformations over the fields, the fields derivative will transform itself like the own fields, which means

$$D_\nu \varphi \to (D_\nu \varphi)' = U(x) D_\nu \varphi. \tag{22}$$

Hence, the total Lagrangian will change by the addiction of the dynamic of these "compensating fields" and its dynamic is given by the term $L(A_\nu)$ where we have only A_ν interacting among itself, this way we get

$$L = L(\varphi, \varphi^*, D_\nu\varphi, (D_\nu\varphi)^*) + L(A_\nu), \tag{23}$$

where

$$L(\varphi, \varphi^*, D_\nu\varphi, (D_\nu\varphi)^*) = D_\nu\varphi(D^\nu\varphi)^* - V(|\varphi|) \tag{24}$$

$$L(A_\nu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{25}$$

Note that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell electromagnetic tensor, and the "compensating field" is the four-vector potential of the electromagnetism, and this way, the parameter q is the electron charge. In Eq. (23), we have a U(1) invariant theory that couples photons with the charged matter. This theory is the known quantum electrodynamics theory.

In general, the Higgs-Kibble mechanism is a process that generates mass for the gauge fields in this theory. Taking into account Eq. (23) with the parameters $\lambda > 0$ and $\mu^2 < 0$, this theory presents the spontaneous symmetry breaking. In this case, there exist a "ring" of degenerated vacuum states given by the minimal potential. This "ring" of degenerated vacuum values is parameterized as

$$\varphi_0 = \sqrt{\frac{|\mu^2|}{2\lambda}}e^{i\Lambda}. \tag{26}$$

The study around a vacuum value state can be done by taking the scalar field

$$\varphi = \frac{1}{\sqrt{2}}[v + \eta(x) + i\xi(x)]. \tag{27}$$

Being $v = \sqrt{|\mu^2|/\lambda}$, substituting Eq. (27) in Eq. (23), we have

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{q^2v^2}{2}A_\nu A^\nu + \frac{1}{2}(\partial_\nu\eta)^2 + \frac{1}{2}(\partial_\nu\xi)^2 - \lambda v^2\eta^2 - qvA_\nu\xi. \tag{28}$$

The term $A_\nu A^\nu$ that appears in Eq. (28) shows that the gauge field now acquires mass. Besides that we also can see in Eq. (28) that a massive scalar field, η, with mass $m_\eta^2 = 2\lambda v^2$ and a Goldstone scalar field appear. However, the Goldstone scalar field does not present physical relevance and may be reabsorbed through a gauge field redefinition. Taking the gauge field redefinition given by

$$B_\nu = A_\nu - q\partial_\nu\xi, \tag{29}$$

we may rewrite Eq. (28) as

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{q^2v^2}{2}B_\nu B^\nu + \frac{1}{2}(\partial_\nu\eta)^2 - \lambda v^2\eta^2, \tag{30}$$

where

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{31}$$

The B_ν field presents mass $m_B = qv$, non-vanishing.

4.1 Topological defects

Topological defects are stable configurations of matter formed during phase transitions in the primordial universe. As already mentioned, during the early phases of the Universe, the material components are in physical states characterized by high degrees of symmetry and it is thought that the interactions will be unified. The cooling of the Universe, due to expansion, promotes the conditions for some of these symmetries to break, it is said, spontaneously.

This happens in much the same way as a pencil which, standing vertically and only resting on its sharp beak, drops down on a flat, oriented surface in any direction. The symmetry of rotation that exists around the axis of the pencil vanishes and, furthermore, the point where the tip was supported separates all possible positions from the topped pencil and is said to be a topological defect. (A classic example of a break in symmetry is the ferromagnetic transitions in Landau theory.) According to the types of symmetries that are broken, various types of topological defects may form, including walls, cosmic strings, monopoles, and textures. The type of defect formed is determined by the symmetry properties of the material and the nature of the phase transition.

To describe the idealized cosmic strings, i.e., static cosmic strings with infinite matter distribution along the z-axis and whose internal structure may be negligible, we will use the Nielsen and Olesen model. In this sense, by coupling the energy-momentum tensor associated with this theory to the Einstein field equations of general relativity, we study the influence of this model on space-time geometry. In fact, Laguna [10] and Garfinkle [9] did this, and in their works, they had shown that the space-time generated by the Nielsen-Olesen model was equivalent to space-time generated by a cosmic string. Thus, for a better understanding of the nature of a cosmic string, it is necessary to understand a little about models in field theory with spontaneous break of symmetry, as with the model proposed by Nielsen and Olesen.

Domain walls are two-dimensional objects that form when a discrete symmetry is broken during a phase transition. A network of walls effectively divides the Universe into several "cells." This type of defect has some very peculiar properties, one being that the gravitational field of a wall is repulsive rather than attractive. These objects may be represented as follow in **Figure 3**.

Cosmic strings are one-dimensional objects that form when an axial or cylindrical symmetry is broken. They are very thin and can extend along the visible Universe. These objects may be represented as follow in **Figure 4**.
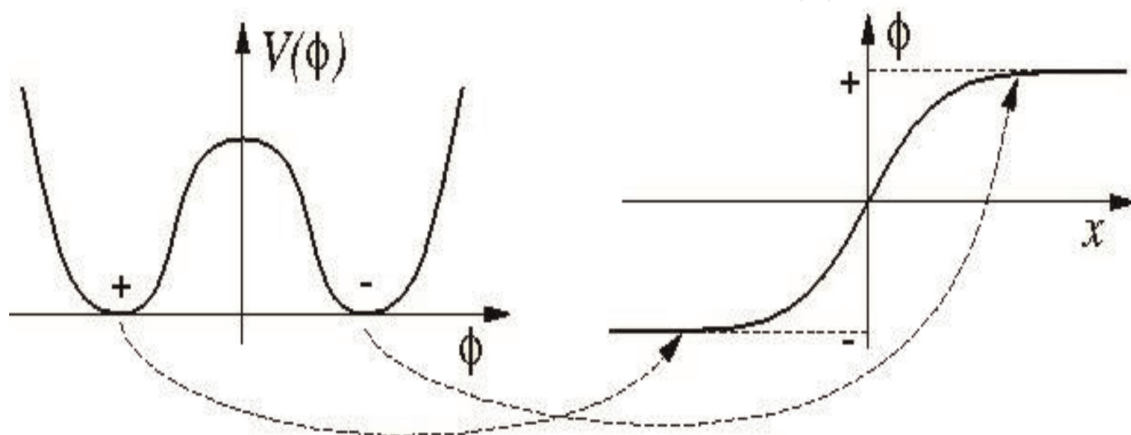

Figure 3.
Domain walls associated with models where there is more than a minimum.

Monopoles have dimension zero, that is, are punctual, and form when a spherical symmetry is broken. In field theory with non-abelian gauge symmetriy broken may appear defects like magnetic monopole. These objects may be represented as follow in **Figure 5**.

Whenever there is the possibility that cosmic strings or other topological defects form in a cosmological phase transition, they actually form. This circumstance had been first pointed out by Kibble, and therefore, in a cosmological context, the process of the formation of defects became to be known as the "Kibble mechanism" [1].

One fact regarding the universe inflation period is that the causal effects in the early universe can only propagate at the speed of light c. This means that in the instant t, regions of the Universe separated more than a distance $d = ct$ cannot know anything about each other. In a phase transition with symmetry breaking, different regions of the Universe will fall into different minimum potentials. This way, we
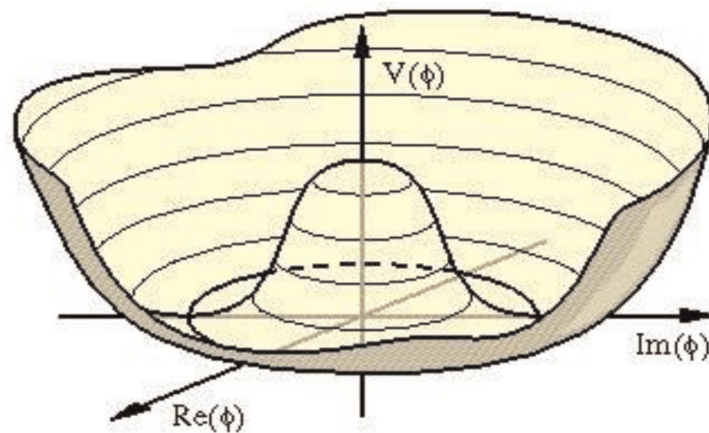

Figure 4.
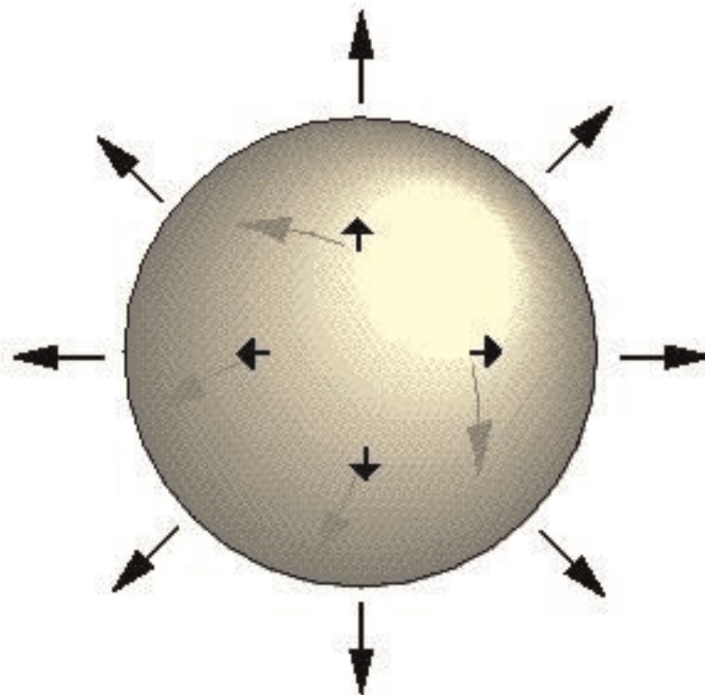*Cosmic strings associated with models in which a set of minimums is not connected.*

Figure 5.
Representation of a magnetic monopole defect. They are expected to be supermassive and have a magnetic charge.

actually think that topological defects are precisely the "boundaries" between these regions corresponding to different minimum potentials and their formation is thus an inevitable consequence of the phase transition.

4.2 Vortex model in field theory

The model proposed by Nielsen and Olesen for Abelian vortices, in the context of general relativity, generates a geometric structure similar to that of a cosmic string. In this sense, this object is a strong candidate to describe mathematically the cosmic strings; that is, they are strong candidates for the sources for this type of defect. However, Nielsen and Olesen, starting from a relativistic theory of fields, in 1973, have shown that it is possible to obtain solutions of vortices [8] starting from the Lagrangian density of the Abelian Higgs model, which is expressed by

$$L = -\frac{1}{4}F^{\beta\nu}F_{\beta\nu} + D_\nu\varphi(D^\nu\varphi)^* - \mu^2\varphi\varphi^* - \lambda(\varphi\varphi^*)^2. \tag{32}$$

Note that $D_\nu = \partial_\nu + ieA_\nu$ is the covariant derivative, $F_{\beta\nu} = \partial_\beta A_\nu - \partial_\nu A_\beta$ is the electromagnetism Maxwell's tensor, and $\lambda(\varphi\varphi^*)^2$ is the auto-interaction term; when this term is put, this theory starts to present a infinity degenerated vacuum, i.e., the theory has infinite states of lower energy, which satisfies the condition $|\varphi|^2 = m^2/(2\lambda)$. This way, for a particular choice vacuum configuration $\varphi = \sqrt{\frac{m^2}{2\lambda}}$, the local gauge symmetry is broken.

It is known that the action for this theory is written as

$$S = \int d^4x\, L\left(\varphi, \varphi^*, D_\nu\varphi, (D_\nu\varphi)^*, A_\mu, A_\mu^*\right). \tag{33}$$

In Eq. (17) using the Hamilton's principle, we get the following equations of motion. For $\varphi(x)$, we get

$$\frac{\partial L}{\partial\varphi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\varphi)}\right) = 0. \tag{34}$$

For A_μ, we have

$$\frac{\partial L}{\partial A^\mu} - \partial^\alpha\left(\frac{\partial L}{\partial(\partial^\alpha A^\mu)}\right) = 0. \tag{35}$$

Now using Eq. (16) into Eqs. (18) and (35), we have the following system of differential equations

$$\partial^\nu F_{\beta\nu} = j_\beta = -\frac{ie}{2}\left(\varphi^*\partial_\beta\varphi - \varphi\partial_\beta\varphi^*\right) - e^2A_\beta\varphi\varphi^* \tag{36}$$

$$D_\nu D^\nu\varphi = \lambda\varphi\left(\varphi\varphi^* - \frac{m^2}{\lambda}\right). \tag{37}$$

For a vortex in the z-direction, the components associated with the vector potential, in the Cartesian coordinate system, are $A^\mu = (0, A_x, A_y, 0)$. For this configuration, the component of the tensor $F_{\beta\nu}$ that interests us is F_{12}, because from it we can calculate the flux that passes through the plane (x, y). Parametrizing the

Higgs field by $\varphi = |\varphi| \exp(i\chi)$, the flux, Φ, passing through an area bounded by a closed curve C, is given by

$$\Phi = \int dxdy F_{12} = \oint_C dx^i A_i = -\frac{1}{q} \oint_C dx^i \partial_i \chi. \tag{38}$$

Here, we use, in Eq. (38), the fact that the line integral is carried out on the closed curve C, very far from the magnetic flux and that $j_\mu = 0$. The equations of motion presented in Eqs. (36) and (37) are coupled differential equations in first order that are hard to find solutions. However, the standard procedure to solve these equations, at least numerically, is to assume the following cylindrical *ansatz*, with symmetry along the z-axis for the fields [8]

$$A_\mu = \left(0, 0, A(r)\hat{\theta}, 0\right) \; and \; \varphi(r,\theta) = f(r)e^{in\theta}. \tag{39}$$

This procedure reduces Eqs. (36) and (37) to

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}f(r)\right) + \left[\left(\frac{n}{r} - qA(r)\right)^2 + \lambda\left(f^2(r) - \frac{m^2}{\lambda}\right)\right] = 0, \tag{40}$$

$$-\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}A(r)\right) + \left(q^2 A(r) - \frac{nq}{r}\right)f(r) = 0. \tag{41}$$

There exist no analytical solutions to these equations. On the other hand, we can find many vortex properties by general and numerical considerations under both equations. From the general point of view, it is possible to show that these equations present solutions as asymptotically well-defined. For points closer to its nucleus, we have $f(r) \approx A(r) \to 0$. For points pretty distant from the vortex nucleus, it is observed that the functions $f(r)$ and $A(r)$ may be approximated in first order to

$$f(r) \to \frac{m}{\sqrt{\lambda}} \; and \; A(r) \to \frac{n}{qr}. \tag{42}$$

By using computational methods we can solve numerically Eqs. (40) and (41), and in **Figure 6**, we can see their behavior.
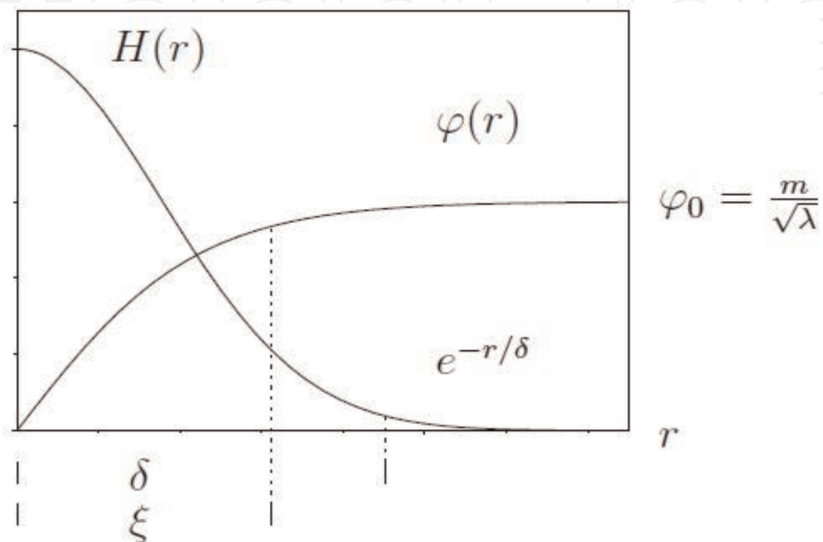

Figure 6.
H(r) and φ(r) represent, respectively, the behavior of the magnetic field and the scalar field.

From **Figure 6**, we can see that two mass scale come up, first of them is $m_s = \sqrt{2}m$ that is related with the mass of the scalar field dislocated; it means $\varphi' = \varphi - m/\sqrt{\lambda}$. The second one is related with the photon mass, $m_v = qm/\sqrt{\lambda}$, remember that the photon acquires mass because of the Higgs mechanism. Note that two length scales also appear in **Figure 6**. The first one $\delta = 1/m_v$ is related with the range of the electromagnetic field. The second latter, $\xi = 1/m_s$, is related with the space scale for the Higgs field arrive its own vacuum value.

In the literature Eqs. (23) and (24) form a system of coupled-equations and this system do not have exact solutions, but asymptotically we may solve these equations. The solutions that present finite linear density of energy, follow reference [32], are given by

$$f(r) \rightarrow \frac{m}{\sqrt{\lambda}}\left(1 - k\, e^{-\frac{r}{\xi}}\right) \; and \; A(r) \rightarrow \frac{n}{qr}\left(1 - k\, e^{-\frac{r}{\delta}}\right), \tag{43}$$

where k is a constant of proportionality.

On the other hand, Garfinkle [9], in 1985, studied the gravitational effects associated with the vortices of Nielsen and Olesen. For this purpose, he used the energy tensor, $T_{\beta\nu}$, obtained from the Lagrangian of the Abelian Higgs model, Eq. (16). In the context of the general relativity, he used this tensor as source of the Einstein equations. In this case, a static metric, with cylindrical symmetry, can be written as

$$ds^2 = e^a dt^2 - dr^2 - e^c d\phi^2 - e^b dz^2, \tag{44}$$

where a, b, and c are functions of the radio r satisfying the relations

$$a(0) = b(0) = 0 \; and \; \lim_{r \to 0} \frac{e^c}{r^2} = 1. \tag{45}$$

Given the metric, Eq. (44), solving the Einstein field equations for the energy-momentum tensor of Nielsen and Olesen, Garfinkle had found, as in flat space-time, symmetrically cylindrical static solutions which he represented as vortices. It also showed that, asymptotically, the space-time around a vortex become the Minkowski space-time minus a slice corresponding that one shown in **Figure 2**. This means that, asymptotically, the vortex can be seen as a cosmic string containing a magnetic field around it.

5. Conclusions

Throughout this work, we introduced some reasons why cosmic string-like topological defects have been studied in energy physics and condensed matter. In fact the quantum effects on the fields of matter are caused due to the non-trivial topology of these objects giving rise to polarization effects. By understanding the vacuum as a state of lower energy, the effects of vacuum polarization can be understood as changes in the scale of this lower energy. Such effects in quantum field theory are seen by calculating the vacuum expected values, VEV, of certain observables, such as the induced current density [23] and the energy-momentum tensor of the matter fields [19] induced. These observables serve as sources for the Maxwell equations in the case of induced current density and for the Einstein equations in the case of the energy-momentum tensor. In the latter case, the source of the Einstein equations no longer consists of the classical energy-momentum

tensor, $T_{\mu\nu}$, but rather the energy-momentum quantizer, $\langle T_{\mu\nu} \rangle$, which will result in certain fixes in the metric tensor [33].

We have also seen that in an inertial frame, the space-time is described by the Minkowski metric, Eq. (1), which consists of a singular and diagonal metric. However, when we move to accelerated frames, the metric becomes point dependent, consisting of a set of ten space-time coordinate functions, containing all information about the geometry of the range. In this way, we can see that accelerated frames are equivalent to gravitational fields, so that gravitational effects can be described by the metric tensor, $g_{\mu\nu}(x)$. Thus, the gravitation may be understood as a deviation in the metric of the flat space-time. Moreover, this metric is not fixed arbitrarily but will depend on the distribution of local matter.

Furthermore, the cosmic string is an object whose density of matter is infinitely concentrated in a line whose mass density is μ. With this object, which can be described by the energy-momentum tensor given in Eq. (7), the deformation caused in the space-time is conical and the metric described by this density of matter is given by Eq. (15), which consists of a Minkowskian metric with cylindrical symmetry, less than a slice equal to $8\pi\mu$, which corresponds to the planar angle deficit orthogonal to the axis of symmetry of the cosmic string.

Finally, we have seen that such idealized objects can be described through the Abelian vortices models proposed by Nielsen and Olesen. They showed that by the abelian Higgs model, Eq. (32), assuming a cylindrical *ansatz*, Eq. (39), It is possible the obtaining a set of two coupled second order differential equation, as it was showed in Eqs. (40) and (41), although they do not have a closed analytic form, but that may be obtained numerical and asymptotic solutions, Eq. (27). In this way, it is observed that two length scales appear naturally from this theory. One associated with the inverse of the mass of the scalar field, $\xi \equiv 1/m_s$, and the other one related to the inverse of the mass of the vector field, $\delta \equiv 1/m_v$, which acquires mass due to the mechanism of Higgs. Also in the scope of the Abelian vortices, Linet [13] and Garfinkle [9], starting from the energy-momentum tensor associated to the Nielsen and Olesen model as the source of the Einstein field equations, they obtained a metric associated to this model, and they found a metric described by a cosmic string. The internal structure of this object is delimited by the scale of energy in which the Higgs field decays to its vacuum value.

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Conflict of interest

The authors declare no conflict of interest.

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