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Moments of the Discounted Aggregate Claims with Delay Inter-Occurrence Distribution and Dependence Introduced by a FGM Copula

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Abstract

In this chapter, with renewal argument, we derive higher simple moments of the Discounted Compound Delay Renewal Risk Process (DCDRRP) when introducing dependence between the inter-occurrence time and the subsequent claim size. To illustrate our results, we assume that the inter-occurrence time is following a delay-Poisson process and the claim amounts is following a mixture of Exponential distribution, we then provide numerical results for the first two moments. The dependence structure between the inter-occurrence time and the subsequent claim size is defined by a Farlie-Gumbel-Morgenstern copula. Assuming that the claim distribution has finite moments, we obtain a general formula for all the moments of the DCDRRP process.

Keywords: compound delay-Poisson process, discounted aggregate claims, moments, FGM copula, constant interest rate

1. Introduction

The classical Poisson model is attractive in the sense that the memoryless property of the exponential distribution makes calculations easy. Then the research was extended to ordinary Sparre-Andersen renewal risk models where the inter-claim times have other distributions than the exponential distribution. Dickson and Hipp [1, 2] considered the Erlang-2 distribution, Li and Garrido [3] the Erlang-n distribution, Gerber and Shiu [4] the generalized Erlang-n distribution (a sum of n independent exponential distributions with different scale parameters) and Li and Garrido [5] looked into the Coxian class distributions. One difficulty with these models is that we have to assume that a claim occurs at time 0, which is not the case in usual setting.

Albrecher and Teugels [6] considered modeling dependence with the use of an arbitrary copula. In a similar dependence model to Albrecher and Teugels as well, Asimit and Badescu [7] considered a constant force of interest and heavy tailed claim amounts.

Barges et al. [8] followed the idea of Albrecher and Teugels [6] and supposed that the dependence is introduced by a copula, the Farlie-Gumbel-Morgenstern (GGM) copula, between a claim inter-arrival time and its subsequent claim amount.

Adékambi and Dziwa [9] and Adékambi [10] provide a direct point of extension but assuming that the claim counting process to follow an unknown general distribution in a framework of dependence with random force of interest to calculate the first two moments of the present value of aggregate random cash flows or random dividends.

The discounted aggregate sum has also been applied in many other fields. For example, it can be used in health cost modeling, see Govorun and Latouche [11], Adékambi [12], or in reliability, in civil engineering, see Van Noortwijk and Frangopol [13].

The delayed or modified renewal risk model solves this problem by assuming that the time until the first claim has a different distribution than the rest of the inter-claim times. Not much research has been done for this model at this stage. Among the first works was Willmot [14] where a mixture of a "generalized equilibrium" distribution and an exponential distribution is considered for the distribution of the time until the first claim. Special cases of the model include the stationary renewal risk model and the delayed renewal risk model with the time until the first claim exponentially distributed. Our focus is to extend the work of Bargès et al. [8], Adékambi and Dziwa [9] and Adékambi [10] by allowing the counting process to follow a delay renewal risk process and thus derive a recursive formula of the moments of this subsequent Discounted Compound Delay Poisson Risk Value (DCDPRV).

For example, young performer companies typically have a high growth rate at the beginning, but as they mature their growth rate may decrease with the increasing scarcity of investment opportunities. That makes dividends dependent on the economic climate at the dividend occurrence time. Obviously the distribution of inter-dividends time in times of economic expansion and in times of economic crisis cannot be identically distributed. So it would be appropriate to use a delayed renewal model to model the distribution of the inter-dividend time. A delayed renewal process is just like an ordinary renewal process, except that the first arrival time is allowed to have a different distribution than the other inter-dividends times.

The chapter is organized as follows: In the second section, we present the model of the continuous time discounted compound delay-Poisson risk process that we use and give some notation. In Section 3, we present a general formula for all the moments of the DCDPRV process. A numerical example of the first two moments will then follow in Section 4.

2. The model

We use the same model as the one in Bargès et al. [8], where the instantaneous interest rate δ is constant.

Define our risk model as follows:

i. The number of claims $\{N(t), t \geq 0\}$ and $\{N_d(t), t \geq 0\}$ form, respectively, an ordinary and a delayed renewal process and, for $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$:

- the positive claim occurrence times are given by T_k,

- the positive claim inter-arrival times are given by $\tau_k = T_k - T_{k-1}, k \in \mathbb{N}$, and $T_0 = 0$.
- $(\tau_k)_{k \geq 2} \sim \tau_2$ are independent and identically distributed (i.i.d),

ii. The kth random claim is given by X_k, and

- $\{X_k, k \in \mathbb{N}\}$ are independent and identically distributed (i.i.d),

- $\{X_k, \tau_k, k \in \mathbb{N}\}$ are mutually independent; and the higher moments, $\mu_k = E[X_1^k]$ of X_1 exist.

iii. The discounted aggregate value at time $t = 0$ of the claims recorded over the period $[0, t]$ yields, respectively, for the ordinary and the delayed renewal case:

$$Z_0(t) = \sum_{k=1}^{N_o(t)} e^{-\delta T_k} X_k, \, Z_d(t) = \sum_{k=1}^{N_d(t)} e^{-\delta T_k} X_k, \tag{1}$$

where $Z_0(t) = Z_d(t) = 0$ if $N_0(t) = N_d(t) = 0$.

2.1 The dependence

We introduce a specific structure of dependence based on the Farlie-Gumbel-Morgenstern (FGM) copula. The advantage of using the FGM copula and its generalizations lies in its mathematical manageability. The joint cumulative distribution function (c.d.f.) of (X_i, τ_i), the ith claim and its occurrence time is

$$\begin{aligned} F_{X_i, \tau_i}(x, v) &= C(F_{X_i}(x), F_{\tau_i}(v)) \\ &= F_{X_i}(x)F_{\tau_i}(v) + \theta F_{X_i}(x)F_{\tau_i}(v)\,(1\text{-}F_{X_i}(x))\,(1\text{-}F_{\tau_i}(v)), \end{aligned} \tag{2}$$

for $(x, v) \in \mathbb{R}^+ * \mathbb{R}^+$ and where $F_{X_i}(x)$ and $F_{\tau_i}(v)$ are the marginals of X_i and τ_i respectively. Recall that the density of the FGM copula is

$$c_\theta^{FGM}(u, v) = 1 + \theta(1\text{-}2u)\,(1\text{-}2v), \tag{3}$$

for $(u, v) \in [0, 1] * [0, 1]$ so that the joint probability density function (p.d.f.) of (X_i, τ_i) is

$$\begin{aligned} f_{X_i, T_i}(x, v) &= c_\theta^{FGM}(F_{X_i}(x), F_{T_i}(v))f_{X_i}(x)f_{T_i}(v) \\ &= f_{X_i}(x)f_{T_i}(v) + \theta f_{X_i}(x)f_{T_i}(v)(1\text{-}2F_{X_i}(x))\,(1\text{-}2F_{T_i}(v)) \end{aligned}, \tag{4}$$

where f_{X_i} and f_{τ_i} are the p.d.f.'s of X_i and τ_i respectively.

With these hypotheses, we present in Section 3 recursive formula of the higher moments of this present value risk process, for a constant instantaneous interest rate.

3. Recursive expression for higher moments

It is often easier to calculate the moments of the random variable $\{Z_d(t), t \geq 0\}$ than finding its distribution. If the probability generation function of $\{Z_d(t), t \geq 0\}$ or its moment generating function (mgf) exists, it is possible to obtain the

corresponding distribution by inversion of its mgf. Since, there is relatively little research devoted to the study of the distribution of the discounted compound renewal sums. We could then think about another technique other than the one proposed by the above authors by studying the moments of $\{Z_d(t), t \geq 0\}$.

3.1 Delay renewal case

The mathematical expectation of total claims plays an important role in the determination of the pure premium, in addition to giving a measure of the central tendency of its distribution. The moments centered at the average of order 2, 3 and 4 are the other moments usually considered because they usually give a good indication of the pace of distribution, and these give us respectively a measure of the dispersion of the distribution around its mean, a measure of the asymmetry and flattening of the distribution considered.

Moments, whether simple, joined or conditional, may eventually be used to construct approximations of the distribution of the DCDPRV.

Theorem 3.1

The Laplace transform of the mth moment of $\{Z_d(t), t \geq 0\}$ is given by:

$$
\tilde{\pi}_{Z_d}^m(r) = \left(1 + \frac{\lambda_2}{r + m\delta} + \frac{\lambda_1 - \lambda_2}{r + m\delta + \lambda_1}\right) \tilde{u}_m(r)
$$

$$
= \lambda_1 \left(1 + \frac{\lambda_2}{r + m\delta} + \frac{\lambda_1 - \lambda_2}{r + m\delta + \lambda_1}\right)
$$

$$
\times \sum_{j=0}^{m-1} \binom{m}{j} \left\{ \frac{\left(\mu_{m-j} - \theta\left(\mu'_{m-j} - \mu_{m-j}\right)\right)}{\lambda_1 + m\delta + r} + \frac{2\theta\left(\mu'_{m-j} - \mu_{m-j}\right)}{2\lambda_1 + m\delta + r} \right\} \tilde{\pi}_{Z_o}^j(r)
$$

$$(5)$$

where

$$
\tilde{\pi}_{Z_d}^m(r) = \tilde{u}_m(r) + \frac{\lambda_2}{m\delta} \tilde{u}_m(r) \times L_{\tau_1}(m, \delta, r) + \frac{\lambda_1 - \lambda_2}{m\delta + \lambda_1} \tilde{u}_m \times L_{\tau_1}(m\delta + \lambda_1, r). \quad (6)
$$

Proof

Conditioning on the arrival of the first claim leads to

$$
\pi_{Z_d}^m(t) = E[Z^m(t)]
$$

$$
= E\left[E\left[\left(e^{-\delta s}X_1 + e^{-\delta s}Z_o(t-s)\right)^m | \tau_1 = s\right]\right]
$$

$$
= \sum_{j=0}^{m-1} \binom{m}{j} \int_0^t f_{\tau_1}(s) e^{-m\delta s} E\left[X^{m-j} | \tau_1 = s\right] \pi_{Z_o}^j(t-s) ds \quad (7)
$$

$$
+ \int_0^t f_{\tau_1}(s) e^{-m\delta s} \pi_{Z_o}^m(t-s) ds.
$$

We have

$$E[X^{m-j}|\tau_1 = s] = \int_0^\infty x^{m-j} f_{X|\tau_1=s}(x)dx$$

$$= \int_0^\infty x^{m-j}\{1 + \theta(1 - 2F_X(x))(1 - 2F_{\tau_1}(s))\}f_X(x)dx$$

$$= E[X^{m-j}] + \theta\int_0^\infty x^{m-j}(1 - 2F_X(x))(1 - 2F_{\tau_1}(s))f_X(x)dx$$

$$= E[X^{m-j}] + \theta\int_0^\infty x^{m-j}(2 - 2F_X(x))(1 - 2F_{\tau_1}(s))f_X(x)dx \qquad (8)$$

$$\qquad - \theta\int_0^\infty x^{m-j}(1 - 2F_{\tau_1}(s))f_X(x)dx$$

$$= E[X^{m-j}](1 - \theta(1 - 2F_{\tau_1}(s)))$$

$$\qquad + \theta(1 - 2F_{\tau_1}(s))\int_0^\infty (m-j)x^{m-j}(1 - F_{\tau_1}(s))dx.$$

We let,

$$\mu'_{m-j} = E[X'^{m-j}] = \int_0^\infty (m-j)x^{m-j-1}(1 - F_X(x))^2 dx \qquad (9)$$

$$< \int_0^\infty (m-j)x^{m-j-1}(1 - F_X(x))dx = E[X^{m-j}] < \infty$$

such that the above equation becomes

$$E[X^{m-j}|\tau_1 = s] = \mu_{m-j} + \theta(1 - 2F_{\tau_1}(s))\left(\mu'_{m-j} - \mu_{m-j}\right). \qquad (10)$$

$$\pi^m_{Z_d}(t) = E[Z^m(t)]$$

$$= E\left[E\left[\left(e^{-\delta s}X_1 + e^{-\delta s}Z_o(t-s)\right)^m|\tau_1 = s\right]\right]$$

$$= \sum_{j=0}^{m-1}\binom{m}{j}\int_0^t f_{\tau_1}(s)e^{-m\delta s}\left\{\mu_{m-j} + \theta(1 - 2F_{\tau_1}(s))\left(\mu'_{m-j} - \mu_{m-j}\right)\right\}\pi^j_{Z_o}(t-s)ds$$

$$\qquad + \int_0^t f_{\tau_1}(s)e^{-m\delta s}\pi^m_{Z_o}(t-s)ds.$$

Let us $\int_0^t f_{\tau_1}(s)e^{-m\delta s}ds = H_\delta(t)$, $\int_0^t f_{\tau_2}(s)e^{-m\delta s}ds = I_\delta(t)$ then

$$\pi_{Z_d}^m(t) = \sum_{j=0}^{m-1}\binom{m}{j}\int_0^t f_{\tau_1}(s)e^{-m\delta s}\left\{\mu_{m-j} + \theta(1 - 2F_{\tau_1}(s))\left(\mu'_{m-j} - \mu_{m-j}\right)\right\}\pi_{Z_o}^j(t-s)ds$$

$$+ H_{m\delta} * \pi_{Z_o(.)}^m$$

$$= u_m + H_{m\delta} * \left\{u_m + I_{m\delta} * \pi_{Z_o(.)}^m\right\}$$

$$= u_m + H_{m\delta} * u_m + H_{m\delta} * I_{m\delta} * \pi_{Z_o(.)}^m$$

$$= u_m + H_{m\delta} * u_m + H_{m\delta} * I_{m\delta} * \left\{u_m + I_{m\delta} * \pi_{Z_o(.)}^m\right\}$$

$$= u_m + H_{m\delta} * u_m + u_m * \sum_{k=1}^{\infty} H_{m\delta} * I_m^{\delta*(k)}(t) = u_m + u_m * \sum_{k=0}^{\infty} H_{m\delta} * I_m^{\delta*(k)}(t)$$

$$= u_m + \int_0^t u_m(t-s)e^{-m\delta s}dm_d(s),$$

(11)

where $u_m(t) = \sum_{j=0}^{m-1}\binom{m}{j}\int_0^t f_{\tau_1}(s)e^{-m\delta s}\left\{\mu_j + \theta(1 - 2F_{\tau_1}(s))\left(\mu'_j - \mu_j\right)\right\}\pi_{Z_o}^{m-j}(t-s)ds$.

We consider the case where the canonical random variable τ_2 has an Exponential distribution with parameter $\lambda_2 > 0$ and τ_1 has an Exponential distribution with parameter $\lambda_1 > 0$.

That is, we have:

$$f_{\tau_1}(t) = \lambda_1 e^{-\lambda_1 t}, f_{\tau_2}(t) = \lambda_2 e^{-\lambda_2 t}, L_{\tau_1}(\lambda_1, s) = \int_0^\infty e^{-sv}f_{\tau_1}(v)dv = \left(\frac{\lambda_1}{\lambda_1 + s}\right), L_{\tau_2}(\lambda_2, s) = \left(\frac{\lambda_2}{\lambda_2 + s}\right).$$

$$m_d(t) = \lambda_2 t + \frac{\lambda_1 - \lambda_2}{\lambda_1}\left(1 - e^{\lambda_1 t}\right)$$

(12)

The mth moment of $Z_d(t)$ is then given by,

$$\pi_{Z_d}^m(t) = u_m + \int_0^t u_m(t-s)e^{-m\delta s}dm_d(s)$$

$$= u_m + \lambda_2\int_0^t u_m(t-s)e^{-m\delta s}d(s) + (\lambda_1 - \lambda_2)\int_0^t u_m(t-s)e^{-(m\delta+\lambda_1)s}d(s)$$

$$= u_m + \frac{\lambda_2}{m\delta}\int_0^t u_m(t-s)m\delta e^{-m\delta s}d(s) + \frac{\lambda_1 - \lambda_2}{m\delta + \lambda_1}\int_0^t u_m(t-s)(m\delta + \lambda_1)e^{-(m\delta+\lambda_1)s}d(s)$$

(13)

Taking the Laplace transform of the above equation, we get:

$$\tilde{\pi}_{Z_d}^m(r) = \tilde{u}_m(r) + \frac{\lambda_2}{m\delta}\tilde{u}_m(r) \times L_{\tau_1}(m\delta, r) + \frac{\lambda_1 - \lambda_2}{m\delta + \lambda_1}\tilde{u}_m \times L_{\tau_1}(m\delta + \lambda_1, r)$$

(14)

But,

$$u_m(t) = \sum_{j=0}^{m-1} \binom{m}{j} \int_0^t f_{\tau_1}(s) e^{-m\delta s} \left\{ \mu_j + \theta(1 - 2F_{\tau_1}(s))\left(\mu_j' - \mu_j\right) \right\} \pi_{Z_o}^{m-j}(t-s) ds$$

$$= \sum_{j=0}^{m-1} \binom{m}{j} \int_0^t \lambda_1 e^{-\lambda_1 s} e^{-m\delta s} \left\{ \mu_j + \theta\left(2e^{-\lambda_1 s} - 1\right)\left(\mu_j' - \mu_j\right) \right\} \pi_{Z_o}^{m-j}(t-s) ds$$

$$= \frac{\lambda_1\left(\mu_j - \theta\left(\mu_j' - \mu_j\right)\right)}{\lambda_1 + m\delta} \sum_{j=0}^{m-1} \binom{m}{j} \int_0^t (\lambda_1 + m\delta) e^{-(\lambda_1 + m\delta)s} \pi_{Z_o}^{m-j}(t-s) ds$$

$$+ 2\theta \frac{\lambda_1\left(\mu_j' - \mu_j\right)}{2\lambda_1 + m\delta} \sum_{j=0}^{m-1} \binom{m}{j} \int_0^t (2\lambda_1 + m\delta) e^{-(2\lambda_1 + m\delta)s} \pi_{Z_o}^{m-j}(t-s) ds$$

(15)

Then the Laplace transform of $u_m(t)$, at r, will give:

$$\tilde{u}_m(r) = \lambda_1 \sum_{j=0}^{m-1} \binom{m}{j} \left\{ \frac{\left(\mu_j - \theta\left(\mu_j' - \mu_j\right)\right)}{\lambda_1 + m\delta + r} + \frac{2\theta\left(\mu_j' - \mu_j\right)}{2\lambda_1 + m\delta + r} \right\} \tilde{\pi}_{Z_o}^{m-j}(r) \qquad (16)$$

Substituting Eq. (14) into Eq. (13), we have:

$$\tilde{\pi}_{Z_d}^m(r) = \left(1 + \frac{\lambda_2}{r + m\delta} + \frac{\lambda_1 - \lambda_2}{r + m\delta + \lambda_1} \right) \tilde{u}_m(r)$$

$$= \lambda_1 \left(1 + \frac{\lambda_2}{r + m\delta} + \frac{\lambda_1 - \lambda_2}{r + m\delta + \lambda_1} \right) \sum_{j=0}^{m-1} \binom{m}{j} \left\{ \frac{\left(\mu_j - \theta\left(\mu_j' - \mu_j\right)\right)}{\lambda_1 + m\delta + r} + \frac{2\theta\left(\mu_j' - \mu_j\right)}{2\lambda_1 + m\delta + r} \right\} \tilde{\pi}_{Z_o}^{m-j}(r)$$

(17)

Solving the above equation for the ordinary case, where $(\tau_2)_{k \geq 2} \sim \tau_2$, we have:

$$\tilde{\pi}_{Z_o}^m(r) = \frac{\lambda_2 \mu_m}{r(r + \delta m + \lambda_2)} + \frac{\lambda_2}{(r + \delta m + \lambda_2)} \sum_{k=1}^{m-1} C_m^k \mu_k \tilde{\pi}_{Z_o}^{(m-k)}(r)$$

$$+ \theta\left(\mu_m' - \mu_m\right) \frac{\lambda_2(r + \delta m)}{r(r + \delta m + \lambda_2)(r + \delta m + 2\lambda_2)}$$

(18)

$$+ \theta \frac{\lambda_2(r + \delta m)}{(r + \delta m + \lambda_2)(r + \delta m + 2\lambda_2)} \sum_{k=1}^{m-1} C_m^k \left(\mu_k' - \mu_k\right) \tilde{\pi}_{Z_o}^{(m-k)}(r)$$

$$+ \frac{\lambda_2}{(r + \delta m + \lambda_2)} \tilde{\pi}_{Z_0}^m(r)$$

Rearranging the above equation, we will get

$$
\tilde{\pi}_{Z_o}^m(r) = \frac{\lambda_2 \mu_m}{r(r+\delta m)} + \frac{\lambda_2}{(r+\delta m)}\sum_{k=1}^{m-1} C_m^k \mu_k \tilde{\pi}_{Z_o}^{(m-k)}(r)
$$

$$
+ \theta \frac{\lambda_2\left(\mu_m' - \mu_m\right)}{r(r+\delta m + 2\lambda_2)} + \theta \frac{\lambda_2}{(r+\delta m + 2\lambda_2)}\sum_{k=1}^{m-1} C_m^k \left(\mu_k' - \mu_k\right)\tilde{\pi}_{Z_o}^{(m-k)}(r)
$$

(19)

Corollary 3.1

The first moment of $\{Z_d(t), t \geq 0\}$ is given by:

$$
\pi_{Z_d}(t) = \left(\theta\lambda_1\left(\mu_1' - \mu_1\right)\frac{\lambda_2+\delta}{(\delta+\lambda_1)(\delta+2\lambda_2)} + \frac{\lambda_1(\lambda_2+\delta)}{\delta(\delta+\lambda_1)}\mu_1 \right)
$$

$$
+ \left(\theta\lambda_1\left(\mu_1' - \mu_1\right)\left(\frac{\lambda_1-\lambda_2}{\lambda_1-2\lambda_2}\right) - (\lambda_1-\lambda_2)\mu_1 \right)\frac{1}{\delta+\lambda_1}e^{-(\delta+\lambda_1)t}
$$

(20)

$$
- \theta\lambda_1\frac{1}{\delta+2\lambda_2}\frac{\lambda_2}{\lambda_1-2\lambda_2}\left(\mu_1' - \mu_1\right)e^{-(\delta+2\lambda_2)t}
$$

$$
- 2\theta\lambda_1\frac{1}{\delta+2\lambda_1}\left(\mu_1' - \mu_1\right)e^{-(\delta+2\lambda_1)t} - \frac{\lambda_2}{\delta}\mu_1 e^{-\delta t}
$$

Proof:

From Theorem 3.1, we have:

$$
\tilde{\pi}_{Z_d}(r) = \frac{\lambda_1\mu_1}{r(r+\delta+\lambda_1)} + \frac{\lambda_1}{(r+\delta+\lambda_1)}\tilde{\pi}_{Z_o}(r)
$$

$$
+ \theta\left(\mu_1' - \mu_1\right)\frac{\lambda_1(r+\delta)}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_1)}
$$

(21)

From Bargès et al. [8], we have

$$
\tilde{\pi}_{Z_o}(r) = \frac{\lambda_2\mu_1}{r(r+\delta)} + \theta\frac{\lambda_2\left(\mu_1' - \mu_1\right)}{r(r+\delta+2\lambda_2)}
$$

(22)

Substituting Eq. (22) into Eq. (21), yields

$$
\tilde{\pi}_{Z_d}(r) = \frac{\lambda_1\mu_1}{r(r+\delta+\lambda_1)} + \frac{\lambda_1}{(r+\delta+\lambda_1)}\left\{ \frac{\lambda_2\mu_1}{r(r+\delta)} + \theta\frac{\lambda_2\left(\mu_1' - \mu_1\right)}{r(r+\delta+2\lambda_2)} \right\}
$$

$$
+ \theta\lambda_1\left(\mu_1' - \mu_1\right)\frac{(r+\delta)}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_1)}
$$

$$
= \left\{ \frac{\lambda_1\lambda_2}{r(r+\delta)(r+\delta+\lambda_1)} + \frac{\lambda_1}{r(r+\delta+\lambda_1)} \right\}\mu_1
$$

$$
+ \theta\lambda_1\left(\mu_1' - \mu_1\right)\left\{ \frac{\lambda_2}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_2)} + \frac{r+\delta}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_1)} \right\}
$$

(23)

with

$$
\frac{\lambda_1}{r(r+\delta+\lambda_1)} = \frac{\lambda_1}{(\delta+\lambda_1)}\cdot\frac{1}{r} - \frac{\lambda_1}{(\delta+\lambda_1)}\cdot\frac{1}{(r+\delta+\lambda_1)}
$$

(24)

$$\frac{\lambda_1\lambda_2}{r(r+\delta)(r+\delta+\lambda_1)} = \frac{\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\cdot\frac{1}{r} + \frac{\lambda_2}{(\delta+\lambda_1)}\cdot\frac{1}{(r+\delta+\lambda_1)} - \frac{\lambda_2}{\delta}\cdot\frac{1}{(r+\delta)} \qquad (25)$$

$$\frac{\lambda_2}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_2)} = \frac{\lambda_2}{(\delta+\lambda_1)(\delta+2\lambda_2)}\frac{1}{r}$$
$$+ \frac{\lambda_2}{\lambda_1-2\lambda_2}\left[\frac{1}{\delta+\lambda_1}\frac{1}{r+\delta+\lambda_1} - \frac{1}{\delta+2\lambda_2}\frac{1}{r+\delta+2\lambda_2}\right] \qquad (26)$$

$$\frac{r+\delta}{r(r+\delta+\lambda_1)(r+\delta+2\lambda_1)} = \frac{\delta}{(\delta+\lambda_1)(\delta+2\lambda_1)}\frac{1}{r}$$
$$+ \frac{1}{\delta+\lambda_1}\frac{1}{r+\delta+\lambda_1} - \frac{2}{\delta+2\lambda_1}\frac{1}{r+\delta+2\lambda_1} \qquad (27)$$

Substituting Eqs. (24), (25), (26) and (27) into Eq. (23), yields:

$$\tilde{\pi}_{Z_d}(r) = \mu_1\left\{\frac{\lambda_1}{(\delta+\lambda_1)}\cdot\frac{1}{r} - \frac{\lambda_1}{(\delta+\lambda_1)}\cdot\frac{1}{(r+\delta+\lambda_1)}\right\}$$

$$+ \theta(\mu_1'-\mu_1)\left\{\begin{array}{c}\dfrac{\lambda_2}{(\delta+\lambda_1)(\delta+2\lambda_2)}\dfrac{1}{r}\\[2mm] + \dfrac{\lambda_2}{\lambda_1-2\lambda_2}\left[\dfrac{1}{\delta+\lambda_1}\dfrac{1}{r+\delta+\lambda_1} - \dfrac{1}{\delta+2\lambda_2}\dfrac{1}{r+\delta+2\lambda_2}\right]\end{array}\right\}$$

$$+ \theta\lambda_1(\mu_1'-\mu_1)\left\{\begin{array}{c}\dfrac{\delta}{(\delta+\lambda_1)(\delta+2\lambda_1)}\dfrac{1}{r}\\[2mm] + \dfrac{1}{\delta+\lambda_1}\dfrac{1}{r+\delta+\lambda_1} - \dfrac{2}{\delta+2\lambda_1}\dfrac{1}{r+\delta+2\lambda_1}\end{array}\right\}$$

$$+ \mu_1\left\{\frac{\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\cdot\frac{1}{r} + \frac{\lambda_2}{(\delta+\lambda_1)}\cdot\frac{1}{(r+\delta+\lambda_1)} - \frac{\lambda_2}{\delta}\cdot\frac{1}{(r+\delta)}\right\} \qquad (28)$$

Rearranging the above equation, will give

$$\pi_{Z_d}(t) = \left(\theta\lambda_1(\mu_1'-\mu_1)\frac{\lambda_2+\delta}{(\delta+\lambda_1)(\delta+2\lambda_2)} + \frac{\lambda_1(\lambda_2+\delta)}{\delta(\delta+\lambda_1)}\mu_1\right)\frac{1}{r}$$

$$+ \left(\theta\lambda_1(\mu_1'-\mu_1)\left(\frac{\lambda_1-\lambda_2}{\lambda_1-2\lambda_2}\right) - (\lambda_1-\lambda_2)\mu_1\right)\frac{1}{\delta+\lambda_1}\frac{1}{r+\delta+\lambda_1}$$

$$- \theta\lambda_1\frac{1}{\delta+2\lambda_2}\frac{\lambda_2}{\lambda_1-2\lambda_2}(\mu_1'-\mu_1)\frac{1}{r+\delta+2\lambda_2} \qquad (29)$$

$$- 2\theta\lambda_1\frac{1}{\delta+2\lambda_1}(\mu_1'-\mu_1)\frac{1}{r+\delta+2\lambda_1} - \frac{\lambda_2}{\delta}\mu_1\frac{1}{r+\delta}$$

Remark 1

If $\lambda_1 = \lambda_2$ then Eq. (29) becomes

$$
\begin{aligned}
\tilde{\pi}_{Z_d}(r) &= \left(\theta\lambda(\mu_1' - \mu_1)\left(\frac{1}{\delta + 2\lambda}\right) + \frac{\lambda}{\delta}\mu_1\right)\frac{1}{r} \\
&\quad - \frac{1}{\delta + 2\lambda}\theta\lambda(\mu_1' - \mu_1)\frac{1}{r + \delta + 2\lambda} - \frac{\lambda}{\delta}\mu_1\frac{1}{(r + \delta)} \\
&= \frac{\lambda}{\delta}\mu_1\left[\frac{1}{r} - \frac{1}{r + \delta}\right] + \theta\lambda(\mu_1' - \mu_1)\left(\frac{1}{\delta + 2\lambda}\right)\left[\frac{1}{r} - \frac{1}{r + \delta + 2\lambda}\right] \\
&= \frac{\lambda\mu_1}{r(r + \delta)} + \frac{\lambda\theta(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda)},
\end{aligned}
\tag{30}
$$

which is exactly the result of Bargès et al. [8].

The inverse of the Laplace transform in Eq. (29) will give

$$
\begin{aligned}
\pi_{Z_d}(t) &= \left(\theta\lambda_1(\mu_1' - \mu_1)\frac{\lambda_2 + \delta}{(\delta + \lambda_1)(\delta + 2\lambda_2)} + \frac{\lambda_1\lambda_2 + \lambda_1\delta}{\delta(\delta + \lambda_1)}\mu_1\right) \\
&\quad + \left(\theta\lambda_1(\mu_1' - \mu_1)\left(\frac{\lambda_2}{\lambda_1 - 2\lambda_2} + 1\right) + (\lambda_2 - \lambda_1)\mu_1\right)\frac{1}{\delta + \lambda_1}e^{-(\delta + \lambda_1)t} \\
&\quad - \theta\lambda_1\frac{1}{\delta + 2\lambda_2}\frac{\lambda_2}{\lambda_1 - 2\lambda_2}(\mu_1' - \mu_1)e^{-(\delta + 2\lambda_2)t} \\
&\quad - 2\theta\lambda_1\frac{1}{\delta + 2\lambda_1}(\mu_1' - \mu_1)e^{-(\delta + 2\lambda_1)t} - \frac{\lambda_2}{\delta}\mu_1 e^{-\delta t}
\end{aligned}
\tag{31}
$$

Remarks 2

If $\theta = 0$ and $\lambda_1 \neq \lambda_2$ then

$$
\begin{aligned}
\pi_{Z_d}(t) &= \left[\frac{\lambda_1}{\delta}\left(\frac{\lambda_2 + \delta}{\delta + \lambda_1}\right) - \frac{\lambda_2}{\delta}e^{-\delta t}\right]\mu_1 + \left(\frac{\lambda_2 - \lambda_1}{\delta + \lambda_1}\right)\mu_1 e^{-(\delta + \lambda_1)t} \\
&= \lambda_2\left(\frac{1 - e^{-\delta t}}{\delta}\right)\mu_1 + (\lambda_1 - \lambda_2)\left(\frac{1 - e^{-(\delta + \lambda_1)t}}{\delta + \lambda_1}\right)\mu_1 \\
&= \left\{\lambda_2\bar{a}_{t|\delta} + (\lambda_1 - \lambda_2)\bar{a}_{t|\delta + \lambda_1}\right\}\mu_1
\end{aligned}
\tag{32}
$$

which is exactly the result of Léveillé et al. [15].

If $\lambda_1 = \lambda_2$ and $\theta \neq 0$ then

$$
\begin{aligned}
\pi_{Z_o}(t) &= \left(\theta\lambda(\mu_1' - \mu_1)\frac{1}{(\delta + 2\lambda_2)} + \frac{\lambda}{\delta}\mu_1\right) \\
&\quad - \theta\lambda\frac{1}{\delta + 2\lambda}(\mu_1' - \mu_1)e^{-(\delta + 2\lambda)t} - \frac{\lambda}{\delta}\mu_1 e^{-\delta t} \\
&= \frac{\lambda}{\delta}(1 - e^{-\delta t})\mu_1 + \theta\lambda\left(\frac{1 - e^{-(\delta + 2\lambda)t}}{\delta + 2\lambda}\right)(\mu_1' - \mu_1)
\end{aligned}
\tag{33}
$$

which is exactly the result of Bargès et al. [8].

If $\lambda_1 = \lambda_2$ and $\theta = 0$ then

$$
\pi_{Z_o}(t) = \frac{\lambda}{\delta}(1 - e^{-\delta t})\mu_1 = \lambda\bar{a}_{t|\delta}\mu_1,
\tag{34}
$$

which is exactly the result of Léveillé et al. [15].

Corollary 3.2

The second moment of $\{Z_d(t), t \geq 0\}$ is given by the following development:
The result in Theorem 3.1 when $n = 2$ gives:

$$\tilde{\pi}_{Z_d}^2(r) = \frac{2\lambda_1(\lambda_1 - \lambda_2)}{r(r + 2\delta + \lambda_1)(r + 2\delta + 2\lambda_1)} \left(\frac{1}{r} \mu_2 + 2\mu_1 \tilde{\pi}_{Z_o}(r) \right)$$

$$+ \frac{\lambda_1}{(r + 2\delta + \lambda_1)} \left(\frac{(r + 2\delta)(r + 2\delta + 2\lambda_2)}{\lambda_2(r + 2\delta + 2\lambda_1)} + 1 \right) \tilde{\pi}_{Z_0}^2(r), \tag{35}$$

From Bargès et al. [8], we have.

$$\tilde{\pi}_{Z_o}(r) = \frac{\lambda_2 \mu_1}{r(r + \delta)} + \theta \frac{\lambda_2(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda_2)} \tag{36}$$

and

$$\tilde{\pi}_{Z_o}^2(r) = \frac{\lambda_2 \mu_2}{r(r + 2\delta)} + \theta \frac{\lambda_2(\mu_2' - \mu_2)}{r(r + 2\delta + 2\lambda_2)}$$

$$+ \frac{2\lambda_2^2 \mu_1^2}{r(r + \delta)(r + 2\delta)} + \frac{2\theta\lambda_2^2 \mu_1(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda_2)(r + 2\delta)} + \frac{2\theta\lambda_2^2 \mu_1(\mu_1' - \mu_1)}{r(r + 2\delta + 2\lambda_2)(r + \delta)}$$

$$+ \frac{2\theta^2\lambda_2^2(\mu_1' - \mu_1)^2}{r(r + \delta + 2\lambda_2)(r + 2\delta + 2\lambda_2)} \tag{37}$$

Substituting Eqs. (39) and (40) into Eq. (38), yields:

$$\tilde{\pi}_{Z_d}^2(r) = \frac{2\lambda_1(\lambda_1 - \lambda_2)}{r(r + 2\delta + \lambda_1)(r + 2\delta + 2\lambda_1)} \left(\frac{1}{r} \mu_2 + 2\mu_1 \left(\frac{\lambda_2 \mu_1}{r(r + \delta)} + \theta \frac{\lambda_2(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda_2)} \right) \right)$$

$$+ \frac{\lambda_1}{(r + 2\delta + \lambda_1)} \left(\frac{(r + 2\delta)(r + 2\delta + 2\lambda_2)}{\lambda_2(r + 2\delta + 2\lambda_1)} + 1 \right)$$

$$\times \left\{ \begin{array}{l} \dfrac{\lambda_2 \mu_2}{r(r + 2\delta)} + \theta \dfrac{\lambda_2(\mu_2' - \mu_2)}{r(r + 2\delta + 2\lambda_2)} \\[2em] + \dfrac{2\lambda_2^2 \mu_1^2}{r(r + \delta)(r + 2\delta)} + \dfrac{2\theta\lambda_2^2 \mu_1(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda_2)(r + 2\delta)} + \dfrac{2\theta\lambda_2^2 \mu_1(\mu_1' - \mu_1)}{r(r + 2\delta + 2\lambda_2)(r + \delta)} \\[2em] + \dfrac{2\theta^2\lambda_2^2(\mu_1' - \mu_1)^2}{r(r + \delta + 2\lambda_2)(r + 2\delta + 2\lambda_2)} \end{array} \right\}, \tag{38}$$

and rearranging Eq. (38), will give:

$$\tilde{\pi}_{Z_d}^2(r) = \frac{\lambda_1\mu_2}{r(r+2\delta+\lambda_1)} + \frac{2\lambda_1\lambda_2\mu_1^2}{r(r+\delta)(r+2\delta+\lambda_1)} + \frac{2\theta\lambda_1\lambda_2\mu_1(\mu_1'-\mu_1)}{r(r+\delta+2\lambda_2)(r+2\delta+\lambda_1)}$$

$$+ \theta\lambda_1(\mu_2'-\mu_2)\frac{(r+2\delta)}{r(r+2\delta+\lambda_1)(r+2\delta+2\lambda_1)}$$

$$+ 2\theta\lambda_1\lambda_2\mu_1(\mu_1'-\mu_1)\frac{(r+2\delta)}{r(r+\delta)(r+2\delta+\lambda_1)(r+2\delta+2\lambda_1)}$$

$$+ 2\theta^2\lambda_1\lambda_2(\mu_1'-\mu_1)^2\frac{(r+2\delta)}{r(r+\delta+2\lambda_2)(r+2\delta+\lambda_1)(r+2\delta+2\lambda_1)} + \lambda_1\lambda_2\mu_2\frac{1}{r(r+2\delta)(r+2\delta+\lambda_1)}$$

$$+ \theta\lambda_1\lambda_2(\mu_2'-\mu_2)\frac{1}{r(r+2\delta+2\lambda_2)(r+2\delta+\lambda_1)} + 2\lambda_1\lambda_2^2\mu_1^2\frac{1}{r(r+\delta)(r+2\delta)(r+2\delta+\lambda_1)}$$

$$+ 2\theta\lambda_1\lambda_2^2\mu_1(\mu_1'-\mu_1)\frac{1}{r(r+\delta+2\lambda_2)(r+2\delta)(r+2\delta+\lambda_1)}$$

$$+ 2\theta\lambda_2^2\lambda_1\mu_1(\mu_1'-\mu_1)\frac{1}{r(r+2\delta+2\lambda_2)(r+\delta)(r+2\delta+\lambda_1)}$$

$$+ 2\theta^2\lambda_2^2\lambda_1(\mu_1'-\mu_1)^2\frac{1}{r(r+\delta+2\lambda_2)(r+2\delta+2\lambda_2)(r+2\delta+\lambda_1)},$$

$$(39)$$

which can be simplified to

$$\tilde{\pi}_{Z_d}^2(r) = \frac{\gamma_0}{r} + \frac{\gamma_1}{r+\delta} + \frac{\gamma_2}{r+2\delta} + \frac{\gamma_3}{r+2\delta+\lambda_1} + \frac{\gamma_4}{r+\delta+2\lambda_2} + \frac{\gamma_5}{r+2\delta+2\lambda_1} + \frac{\gamma_6}{r+2\delta+2\lambda_2},$$

$$(40)$$

with,

$$\gamma_0 = \left\{ \begin{array}{l} \dfrac{\lambda_1\mu_2}{(2\delta+\lambda_1)} + \dfrac{2\lambda_1\lambda_2\mu_1^2}{\delta(2\delta+\lambda_1)} + \dfrac{2\theta\lambda_1\lambda_2\mu_1(\mu_1'-\mu_1)}{(2\delta+\lambda_1)(\delta+2\lambda_2)} + \dfrac{\theta\delta\lambda_1(\mu_2'-\mu_2)}{(2\delta+\lambda_1)(\delta+\lambda_1)} \\[3mm] + \dfrac{2\theta\lambda_1\lambda_2\mu_1(\mu_1'-\mu_1)}{(2\delta+\lambda_1)(\delta+\lambda_1)} + \dfrac{2\delta\theta^2\lambda_1\lambda_2(\mu_1'-\mu_1)^2}{(2\delta+\lambda_1)(\delta+\lambda_1)(\delta+2\lambda_2)} + \dfrac{\lambda_1\lambda_2\mu_2}{2\delta(2\delta+\lambda_1)} \\[3mm] + \dfrac{\theta\lambda_1\lambda_2(\mu_2'-\mu_2)}{2(\delta+\lambda_2)(2\delta+\lambda_1)} + \dfrac{\lambda_1\lambda_2^2\mu_1^2}{\delta^2(2\delta+\lambda_1)} + \dfrac{\theta\lambda_1\lambda_2^2\mu_1(\mu_1'-\mu_1)}{\delta(\delta+2\lambda_2)(2\delta+\lambda_1)} \\[3mm] + \dfrac{\theta\lambda_1\lambda_2^2\mu_1(\mu_1'-\mu_1)}{\delta(\delta+\lambda_2)(2\delta+\lambda_1)} + \dfrac{\theta^2\lambda_1\lambda_2^2(\mu_1'-\mu_1)^2}{2(\delta+2\lambda_2)(\delta+\lambda_2)(2\delta+\lambda_1)} \end{array} \right\} \quad (41)$$

$$\gamma_1 = \left\{ -\frac{2\lambda_1\lambda_2\mu_1^2}{\delta(\delta+\lambda_1)} - \frac{2\theta\lambda_1\lambda_2\mu_1(\mu_1'-\mu_1)}{(\delta+\lambda_1)(\delta+2\lambda_1)} - \frac{2\lambda_1\lambda_2^2\mu_1^2}{\delta^2(\delta+\lambda_1)} - \frac{2\theta\lambda_1\lambda_2^2\mu_1(\mu_1'-\mu_1)}{\delta(\delta+2\lambda_2)(\delta+\lambda_1)} \right\} \quad (42)$$

$$\gamma_2 = \left\{ -\frac{\lambda_2\mu_2}{2\delta} + \frac{\lambda_2^2\mu_1^2}{\delta^2} + \frac{\theta\lambda_2^2\mu_1(\mu_1' - \mu_1)}{\delta(\delta - 2\lambda_2)} \right\} \tag{43}$$

$$\gamma_3 = \left\{ \begin{array}{l} -\dfrac{\lambda_1\mu_2}{2\delta + \lambda_1} + \dfrac{2\lambda_1\lambda_2\mu_1^2}{(2\delta + \lambda_1)(\delta + \lambda_1)} + \dfrac{2\theta\lambda_1\lambda_2\mu_1(\mu_1' - \mu_1)}{(2\delta + \lambda_1)(\lambda_1 + \delta - 2\lambda_2)} + \dfrac{\theta\lambda_1(\mu_2' - \mu_2)}{2\delta + \lambda_1} \\[3mm] -\dfrac{2\theta\lambda_1\lambda_2\mu_1(\mu_1' - \mu_1)}{(2\delta + \lambda_1)(\delta + \lambda_1)} - \dfrac{2\theta^2\lambda_1\lambda_2(\mu_1' - \mu_1)^2}{(2\delta + \lambda_1)(\lambda_1 + \delta - 2\lambda_2)} + \dfrac{\lambda_2\mu_2}{2\delta + \lambda_1} \\[3mm] +\dfrac{\theta\lambda_1\lambda_2(\mu_2' - \mu_2)}{(2\delta + \lambda_1)(\lambda_1 - 2\lambda_2)} - \dfrac{2\lambda_2^2\mu_1^2}{(2\delta + \lambda_1)(\delta + \lambda_1)} - \dfrac{2\theta\lambda_2^2\mu_1(\mu_1' - \mu_1)}{(2\delta + \lambda_1)(\lambda_1 + \delta - 2\lambda_2)} \\[3mm] +\dfrac{2\theta\lambda_1\lambda_2^2\mu_1(\mu_1' - \mu_1)}{(2\lambda_2 - \lambda_1)(\delta + \lambda_1)(2\delta + \lambda_1)} + \dfrac{2\theta^2\lambda_1\lambda_2^2(\mu_1' - \mu_1)^2}{(2\delta + \lambda_1)(\lambda_1 + \delta - 2\lambda_2)(2\lambda_2 - \lambda_1)} \end{array} \right\}$$

$$\tag{44}$$

$$\gamma_4 = \left\{ \begin{array}{l} \dfrac{-2\theta\lambda_1\lambda_2\mu_1(\mu_1' - \mu_1)}{(\delta + 2\lambda_2)(\lambda_1 + \delta - 2\lambda_2)} + \dfrac{2\theta^2\lambda_1\lambda_2(\mu_1' - \mu_1)^2(2\lambda_2 - \delta)}{(\delta + 2\lambda_2)(\lambda_1 + \delta - 2\lambda_2)(2\lambda_1 - 2\lambda_2 + \delta)} \\[3mm] -\dfrac{2\theta\lambda_1\lambda_2^2\mu_1(\mu_1' - \mu_1)}{(\delta + 2\lambda_2)(\delta - 2\lambda_2)(\lambda_1 + \delta - 2\lambda_2)} - \dfrac{2\theta^2\lambda_1\lambda_2^2(\mu_1' - \mu_1)^2}{\delta(\delta + 2\lambda_2)(\lambda_1 + \delta - 2\lambda_2)} \end{array} \right\} \tag{45}$$

$$\gamma_5 = \left\{ -\frac{\theta\lambda_1(\mu_2' - \mu_2)}{\delta + \lambda_1} + \frac{2\theta\lambda_1\lambda_2\mu_1(\mu_1' - \mu_1)}{(\delta + \lambda_1)(\delta + 2\lambda_1)} - \frac{2\theta^2\lambda_1\lambda_2(\mu_1' - \mu_1)^2}{(\delta + \lambda_1)(2\lambda_1 - \delta - 2\lambda_2)} \right\} \tag{46}$$

$$\gamma_6 = -\frac{\theta\lambda_1\lambda_2(\mu_2' - \mu_2)}{2(\delta + \lambda_2)(\lambda_1 - 2\lambda_2)} + \frac{\theta\lambda_1\lambda_2^2\mu_1(\mu_1' - \mu_1)}{(\delta + \lambda_2)(\delta + 2\lambda_2)(\lambda_1 - 2\lambda_2)} + \frac{\theta^2\lambda_1\lambda_2^2(\mu_1' - \mu_1)^2}{\delta(\delta + \lambda_2)(\lambda_1 - 2\lambda_2)}$$

$$\tag{47}$$

Remark 2
When

$$\lambda_1 = \lambda_2$$

$$\tilde{\pi}_{Z_d}^2(r) = \frac{2\lambda^2\mu_2}{r(r + \delta)(r + 2\delta)} + \frac{2\theta\lambda^2\mu_1(\mu_1' - \mu_1)}{r(r + \delta + 2\lambda)(r + 2\delta)}$$

$$+ \frac{\lambda\mu_2}{r(r + 2\delta)} + \frac{\theta\lambda(\mu_2' - \mu_2)}{r(r + 2\delta + 2\lambda)} \tag{48}$$

$$\frac{2\theta\lambda^2\mu_1(\mu_1' - \mu_1)}{r(r + \delta)(r + 2\delta + 2\lambda)} + \frac{2\theta^2\lambda^2(\mu_1' - \mu_1)^2}{r(r + \delta + 2\lambda)(r + 2\delta + 2\lambda)},$$

which is exactly the result of Bargès et al. [8].

The Laplace transform in Eq. (49), is a combination of terms of the form:

$$\tilde{g}(r) = \frac{1}{r(\alpha_1 + r)(\alpha_2 + r)...(\alpha_n + r)}, \tag{49}$$

with g a function defined for all non-negative real numbers. As described in the proof of Theorem 1.1 in Baeumer [16], each of these terms can be expressed as a combinations of partial fraction such as $\tilde{g}(r) = \gamma_0 \frac{1}{r} + \gamma_1 \frac{1}{\alpha_1 + r} + ... + \gamma_n \frac{1}{\alpha_n + r}$ where.

$\gamma_0 = \frac{1}{\alpha_1...\alpha_n}$, for $i = 1, ..., n$, $\gamma_i = -\frac{1}{\alpha_i} \prod_{j=1, j \neq i} \frac{1}{\alpha_j - \alpha_i}$.

Since the inverse Laplace transform of $\frac{1}{\alpha_i + r}$ is $e^{-\alpha_i t}$, it is easy to invert $\tilde{g}$ and obtain

$$g(t) = \gamma_0 + \gamma_1 e^{-\alpha_1 t} + \gamma_2 e^{-\alpha_2 t} + ... + \gamma_n e^{-\alpha_n t}. \tag{50}$$

Using Eq. (49) in Eq. (53), it results that

$$\pi^2_{Z_d}(t) = \gamma_0 + \gamma_1 e^{-\delta t} + \gamma_2 e^{-2\delta t} + \gamma_3 e^{-(2\delta + \lambda_1)t} + \gamma_4 e^{-(\delta + 2\lambda_2)t} + \gamma_5 e^{-(2\delta + 2\lambda_1)t} + \gamma_6 e^{-(2\delta + 2\lambda_2)t}, t \geq 0, \tag{51}$$

where $(\gamma_i)_{i \in \{0,1,2,...,6\}}$ are given by equation Eq. (50).

Remarks

If $\theta = 0$ then

$$\gamma_0^{\theta=0} = \frac{\lambda_1 \mu_2}{2\delta + \lambda_1} + \frac{2\lambda_1 \lambda_2 \mu_1^2}{\delta(2\delta + \lambda_1)} + \frac{\lambda_1 \lambda_2 \mu_2}{2\delta(2\delta + \lambda_1)} + \frac{\lambda_1 \lambda_2^2 \mu_1^2}{\delta^2(2\delta + \lambda_1)}$$

$$= \frac{\lambda_1}{2\delta + \lambda_1}\left(\frac{2\delta + \lambda_2}{2\delta}\right)\mu_2 + \frac{\lambda_1 \lambda_2}{\delta(2\delta + \lambda_1)}\left(\frac{2\delta + \lambda_2}{\delta}\right)\mu_1^2 \tag{52}$$

$$= \frac{\lambda_1(2\delta + \lambda_2)}{2\delta(2\delta + \lambda_1)}\left\{\mu_2 + \frac{2\lambda_2 \mu_1^2}{\delta}\right\},$$

$$\gamma_1^{\theta=0} = -\frac{2\lambda_1 \lambda_2 \mu_1^2}{\delta(\delta + \lambda_1)} - \frac{2\lambda_1 \lambda_2^2 \mu_1^2}{\delta^2(\delta + \lambda_1)} = -\frac{2\lambda_1 \lambda_2 \mu_1^2}{\delta(\delta + \lambda_1)}\left(1 + \frac{\lambda_2}{\delta}\right), \gamma_2^{\theta=0} = \frac{\lambda_2^2 \mu_1^2}{\delta^2} - \frac{\lambda_2 \mu_2}{2\delta} \tag{53}$$

$$\gamma_3^{\theta=0} = -\frac{\lambda_1 \mu_2}{2\delta + \lambda_1} + \frac{2\lambda_1 \lambda_2 \mu_1^2}{(2\delta + \lambda_1)(\delta + \lambda_1)} + \frac{\lambda_2 \mu_2}{2\delta + \lambda_1} - \frac{2\lambda_2^2 \mu_1^2}{(2\delta + \lambda_1)(\delta + \lambda_1)}$$

$$\tag{54}$$

$$= \frac{(\lambda_2 - \lambda_1)\mu_2}{2\delta + \lambda_1} + \frac{2\lambda_2(\lambda_1 - \lambda_2)\mu_1^2}{(2\delta + \lambda_1)(\delta + \lambda_1)},$$

$$\gamma_4^{\theta=0} = \gamma_5^{\theta=0} = \gamma_6^{\theta=0}. \tag{55}$$

Then,

$$
\begin{aligned}
\pi^2_{Z_d}(t) &= \frac{\lambda_1(2\delta+\lambda_2)}{2\delta(2\delta+\lambda_1)}\left\{\mu_2+\frac{2\lambda_2\mu_1^2}{\delta}\right\}-\frac{2\lambda_1\lambda_2\mu_1^2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)e^{-\delta t}\\
&\quad+\left(\frac{\lambda_2^2\mu_1^2}{\delta^2}-\frac{\lambda_2\mu_2}{2\delta}\right)e^{-2\delta t}+\left(\frac{1}{2\delta+\lambda_1}(\lambda_2-\lambda_1)\mu_2+\frac{2\lambda_2(\lambda_1-\lambda_2)\mu_1^2}{(2\delta+\lambda_1)(\delta+\lambda_1)}\right)e^{-(2\delta+\lambda_1)t}\\
&=\left(\frac{\lambda_1(2\delta+\lambda_2)}{2\delta(2\delta+\lambda_1)}-\frac{\lambda_2}{2\delta}e^{-2\delta t}+\frac{(\lambda_2-\lambda_1)}{2\delta+\lambda_1}e^{-(2\delta+\lambda_1)t}\right)\mu_2\\
&\quad+\left(\frac{\lambda_2\lambda_1(2\delta+\lambda_2)}{\delta^2(2\delta+\lambda_1)}-\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)e^{-\delta t}+\frac{\lambda_2^2}{\delta^2}e^{-2\delta t}+\frac{2\lambda_2(\lambda_1-\lambda_2)}{(2\delta+\lambda_1)(\delta+\lambda_1)}e^{-(2\delta+\lambda_1)t}\right)\mu_1^2\\
&=\left(\frac{\lambda_1(2\delta+\lambda_2)}{2\delta(2\delta+\lambda_1)}-\frac{\lambda_2}{2\delta}\left(1-2\delta\overline{a}_{t|2\delta}\right)+\frac{(\lambda_2-\lambda_1)}{2\delta+\lambda_1}\left(1-(2\delta+\lambda_1)\overline{a}_{t|2\delta+\lambda_1}\right)\right)\mu_2\\
&\quad+\left(\frac{\lambda_2\lambda_1(2\delta+\lambda_2)}{\delta^2(2\delta+\lambda_1)}-\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)e^{-\delta t}+2\frac{\lambda_2^2}{\delta^2}e^{-2\delta t}-\frac{\lambda_2^2}{\delta^2}e^{-2\delta t}\right)\mu_1^2\\
&\quad+\left\{\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(\delta+\lambda_1)}-\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(2\delta+\lambda_1)}\right\}\mu_1^2 e^{-(2\delta+\lambda_1)t}\\
&=\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2\\
&\quad+\left(\frac{\lambda_2\lambda_1(2\delta+\lambda_2)}{\delta^2(2\delta+\lambda_1)}-\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)e^{-\delta t}+2\frac{\lambda_2^2}{\delta^2}e^{-2\delta t}-\frac{\lambda_2^2}{\delta^2}\left(1-2\delta\overline{a}_{t|2\delta}\right)\right)\mu_1^2\\
&\quad+\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(\delta+\lambda_1)}e^{-(2\delta+\lambda_1)t}\mu_1^2-\frac{2\lambda_2(\lambda_1-\lambda_2)\left(1-(2\delta+\lambda_1)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_1^2}{\delta(2\delta+\lambda_1)}\\
&=\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_1^2\\
&\quad-e^{\delta t}\left\{\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)-2\frac{\lambda_2^2}{\delta^2}e^{-\delta t}+\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(\delta+\lambda_1)}e^{-(\delta+\lambda_1)t}\right\}
\end{aligned}
$$

(56)

$$
\begin{aligned}
\pi^2_{Z_d}(t) &= \left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_1^2\\
&\quad-e^{\delta t}\left\{\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)-2\frac{\lambda_2^2}{\delta^2}e^{-\delta t}+\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(\delta+\lambda_1)}e^{-(\delta+\lambda_1)t}\right\}\\
&=\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_1^2\\
&\quad-e^{\delta t}\left\{\frac{2\lambda_1\lambda_2}{\delta(\delta+\lambda_1)}\left(1+\frac{\lambda_2}{\delta}\right)-2\frac{\lambda_2^2}{\delta^2}\left(1-\delta\overline{a}_{t|2\delta}\right)+\frac{2\lambda_2(\lambda_1-\lambda_2)}{\delta(\delta+\lambda_1)}\left(1-(\delta+\lambda_1)\overline{a}_{t|\delta+\lambda_1}\right)\right\}\\
&=\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_1^2\\
&\quad-\frac{2\lambda_2}{\delta}e^{\delta t}\left\{\left(\lambda_2\overline{a}_{t|\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|\delta+\lambda_1}\right)\right\}\\
&=\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2\\
&\quad+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}-e^{\delta t}\left\{\left(\lambda_2\overline{a}_{t|\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|\delta+\lambda_1}\right)\right\}\right)\mu_1^2.
\end{aligned}
$$

(57)

To finally have:

$$
\begin{aligned}
\pi^2_{Z_d}(t) &= \left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}\right)\mu_2\\
&\quad+\frac{2\lambda_2}{\delta}\left(\lambda_2\overline{a}_{t|2\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|2\delta+\lambda_1}-e^{\delta t}\left\{\left(\lambda_2\overline{a}_{t|\delta}+(\lambda_1-\lambda_2)\overline{a}_{t|\delta+\lambda_1}\right)\right\}\right)\mu_1^2,
\end{aligned}
$$

(58)

which is exactly the result of Léveillé et al.[15].
If $\lambda_1 = \lambda_2$ then

$$\gamma_0^{\lambda_1=\lambda_2} = \frac{\lambda\mu_2}{2\delta} + \frac{\lambda^2\mu_1^2}{\delta^2} + \left(\frac{\theta\lambda^2}{\delta(\delta+\lambda)} + \frac{\theta\lambda^2}{\delta(\delta+2\lambda)}\right)\mu_1(\mu_1'-\mu_1) + \frac{\theta\lambda(\mu_2'-\mu_2)}{2(\delta+\lambda)}$$
$$+ \frac{\theta^2\lambda^2(\mu_1'-\mu_1)^2}{(\delta+\lambda)(\delta+2\lambda)}, \tag{59}$$

$$\gamma_1^{\lambda_1=\lambda_2} = -\frac{2\lambda^2\mu_1^2}{\delta^2} - \frac{2\theta\lambda^2\mu_1(\mu_1'-\mu_1)}{\delta(\delta+2\lambda)}, \gamma_2^{\lambda_1=\lambda_2} = -\frac{\lambda\mu_2}{2\delta} + \frac{\lambda^2\mu_1^2}{\delta^2} + \frac{\theta\lambda^2\mu_1(\mu_1'-\mu_1)}{\delta(\delta-2\lambda)}, \gamma_3^{\lambda_1=\lambda_2} = 0 \tag{60}$$

$$\gamma_4^{\lambda_1=\lambda_2} = -\frac{2\theta\lambda^2\mu_1(\mu_1'-\mu_1)}{(\delta+2\lambda)(\delta-2\lambda)} - \frac{2\theta^2\lambda^2(\mu_1'-\mu_1)^2}{\delta(\delta+2\lambda)}, \tag{61}$$

$$\gamma_5^{\lambda_1=\lambda_2} = -\frac{\theta\lambda(\mu_2'-\mu_2)}{\delta+\lambda} + \frac{2\theta\lambda^2\mu_1(\mu_1'-\mu_1)}{(\delta+\lambda)(\delta+2\lambda)} + \frac{2\theta^2\lambda^2(\mu_1'-\mu_1)^2}{\delta(\delta+\lambda)}, \tag{62}$$

$$\gamma_6^{\lambda_1=\lambda_2} = \frac{\theta\lambda(\mu_2'-\mu_2)}{2(\delta+\lambda)} - \frac{\theta\lambda^2\mu_1(\mu_1'-\mu_1)}{(\delta+\lambda)(\delta+2\lambda)} - \frac{\theta^2\lambda^2(\mu_1'-\mu_1)^2}{\delta(\delta+\lambda)}. \tag{63}$$

Then,

$$\pi_{Z_d}^2(t) = \lambda\mu_2\left(\frac{1}{\delta} - \frac{e^{-\delta t}}{2\delta}\right) + \theta\lambda(\mu_2'-\mu_2)\left(\frac{1}{2\lambda+2\delta} - \frac{e^{-(2\lambda+2\delta)t}}{2\lambda+2\delta}\right)$$
$$+ 2\lambda^2\mu_1^2\left(\frac{1}{2\delta^2} - \frac{e^{-\delta t}}{\delta^2} + \frac{e^{-2\delta t}}{2\delta^2}\right)$$
$$+ 2\theta\lambda^2\mu_1(\mu_1'-\mu_1)\left(\frac{1}{2\delta(2\lambda+\delta)} - \frac{e^{-(2\lambda+\delta)t}}{(2\lambda+\delta)(-2\lambda+\delta)} + \frac{e^{-2\delta t}}{2\delta(-2\lambda+\delta)}\right)$$
$$+ 2\theta\lambda^2\mu_1(\mu_1'-\mu_1)\left(\frac{1}{\delta(2\lambda+2\delta)} - \frac{e^{-\delta t}}{\delta(2\lambda+\delta)} + \frac{e^{-(2\lambda+2\delta)t}}{(2\lambda+2\delta)(2\lambda+\delta)}\right)$$
$$+ 2\theta^2\lambda^2(\mu_1'-\mu_1)^2\left(\frac{1}{(2\lambda+\delta)(2\lambda+2\delta)} - \frac{e^{-(2\lambda+\delta)t}}{\delta(2\lambda+\delta)} + \frac{e^{-(2\lambda+2\delta)t}}{\delta(2\lambda+2\delta)}\right) \tag{64}$$

which is exactly the result of Bargès et al. [8].
If $\lambda_1 = \lambda_2$ and $\theta = 0$ then

$$\pi_{Z_o}^2(t) = \lambda\bar{a}_{t|2\delta}\mu_2 + (\lambda\bar{a}_{t|2\delta}\mu_1)^2, \tag{65}$$

which is exactly the result of Léveillé et al. [15].
Remark 3.1
Noting for $i = 1, 2, ..., m, j = 1, 2, ..., m, p = 0, 1$ and $k \in \mathbb{N} - \{0\}$

$$\eta_k(i,j,p) = \frac{\binom{i}{j}\theta^p\lambda^k(E[X^j])^{1-p}(E[X^{ij}] - E[X^j])^p}{(r + p \times 2\lambda + i\delta)^k}. \tag{66}$$

We can rewrite $\tilde{\pi}_{Z_o}(r)$ and $\tilde{\pi}^2_{Z_o}(r)$ as

$$\tilde{\pi}_{Z_o}(r) = \frac{1}{r}[\eta_k(1,1,0) + \eta_k(1,1,1)], \tag{67}$$

$$\tilde{\pi}^2_{Z_o}(r) = \frac{1}{r}[\eta_k(2,2,0) + \eta_k(2,2,1) + (\eta_k(2,1,0) + \eta_k(2,1,1))(\eta_k(1,1,0) + \eta_k(1,1,1))]$$

$$= \frac{1}{r}\left[\begin{array}{c} \eta_k(2,2,0) + \eta_k(2,2,1) + \eta_k(2,1,0)\eta_k(1,1,0) + \eta_k(2,1,0)\eta_k(1,1,1) \\ + \eta_k(2,1,1)\eta_k(1,1,0) + \eta_k(2,1,1)\eta_k(1,1,1) \end{array} \right] \tag{68}$$

The term $\tilde{\pi}^m_{Z_o}(r)$ can also be expressed using

$$\tilde{\pi}^m_{Z_o}(r) = \frac{1}{r}\sum_{i=1}^{m} (i_1,j_1,p_1)...(i_n,j_n,p_n) \in \wp_{m,n}\eta_k(i_n,j_n,p_n) \times ... \times \eta_k(i_1,j_1,p_1), \tag{69}$$

where

$$\wp_{m,n} = \left\{ \begin{array}{c} (i_1,j_1,p_1), ..., (i_n,j_n,p_n) : i_1 = m, i_1 + ... + i_n = m - 1 + n, i_1 > ... > i_n, \\ j_1 = m - 1 + n, j_1 + ... + j_n = m, j_1 > ... > j_n, p \in \{0,1\} \end{array} \right\}, \tag{70}$$

and

$$\tilde{\pi}^m_{Z_d}(r) = \frac{2\lambda_1(\lambda_1 - \lambda_2)}{r(r + m\delta + \lambda_1)(r + m\delta + 2\lambda_1)}\left(\frac{1}{r}\mu_m + \sum_{k=1}^{m-1}C^k_m E[X^k_1]\tilde{\pi}^{(m-k)}_{Z_o}(r)\right)$$

$$+ \frac{\lambda_1}{(r + m\delta + \lambda_1)}\left(\frac{(r + m\delta)(r + m\delta + 2\lambda_2)}{\lambda_2(r + m\delta + 2\lambda_1)} + 1\right)\tilde{\pi}^m_{Z_0}(r), \tag{71}$$

4. Application

4.1 First two moments

For the numerical illustration, suppose that $X \sim pExp\left(\beta_1 = \frac{1}{80}\right) +$ $(1 - p) Exp\left(\beta_2 = \frac{1}{200}\right)$, the inter-claim time distribution parameters $\lambda_1 = 2; 4$ and $\lambda_2 = 1$, the interest rate $\delta = 3\%$ (**Tables 1–4**). We use three different values for the copula parameter $\theta = -1; 0; 1$, $p = 1/3$ and fix the time $t = 1; 10; 100$. The mth moment of X is

$$\mu_m = p\frac{m!}{\beta^m_1} + (1 - p)\frac{m!}{\beta^m_2} \text{ and } \mu'_m = \int_0^\infty mx^{m-1}(1 - F_X(x))^2 dx = \mu_m = p\frac{m!}{(2\beta_1)^m} +$$

$$(1 - p)\frac{m!}{(2\beta_2)^m}. \tag{72}$$

θ	$t = 1$	$t = 10$	$t = 100$
−1	482.3375	4450	16,428
0	438.1057	4407.1	16,385
1	393.874	4364.2	16,342

Table 1.
$E[Z_d(t)]$ for $\lambda_1 = 1, \lambda_2 = 10, \delta = 3\%$.

θ	$t = 1$	$t = 10$	$t = 100$
−1	597.1633	4578.7	16,557
0	554.5237	4535.5	16,513
1	511.8841	4492.3	16,470

Table 2.
$E[Z_d(t)]$ for $\lambda_1 = 5, \lambda_2 = 10, \delta = 3\%$.

θ	$t = 1$	$t = 10$	$t = 100$
−1	33900.85	29646.71	7928.481
0	23972.65	20976.12	5711.548
1	359.8795	1036.223	1542.412

Table 3.
$Std[Z_d(t)]$ for $\lambda_1 = 1, \lambda_2 = 10, \delta = 3\%$.

θ	$t = 1$	$t = 10$	$t = 100$
−1	34027.1	29756.44	7954.606
0	24061.85	21053.72	5731.564
1	366.1484	1035.491	1545.671

Table 4.
$Std[Z_d(t)]$ for $\lambda_1 = 5, \lambda_2 = 10, \delta = 3\%$.

4.2 Premium calculation

From the results in Section 4.1, we can compute the premium related to the risk of an insurance portfolio represented by $G(t)$, depending on the premium calculation principles adopted by the insurance company. The loaded premium $Z_d(t)$ consists in the sum of the pure premium $E[Z_d(t)]$, the expected value of the costs related to the portfolio, and a loading for the risk $M(t)$ as

$$G(t) = E[Z_d(t)] + M(t) \tag{73}$$

The loading for the risk differs according to the premium calculation principles.

4.2.1 The expected value principle

Denote by $\theta > 0$ the safety loading. The expected value principle defines the loaded premium as:

$$G(t) = E[Z_d(t)] + \theta E[Z_d(t)], \tag{74}$$

where $M(t) = \theta E[Z_d(t)]$.

4.2.2 The variance principle

Denote by $\theta > 0$ the safety loading. The variance principle defines the loaded premium as:

$$G(t) = E[Z_d(t)] + \theta Var[Z_d(t)], \tag{75}$$

where $M(t) = \theta Var[Z_d(t)]$.

4.2.3 The standard deviation principle

Denote by $\theta > 0$ the safety loading. The standard deviation principle defines the loaded premium as:

$$G(t) = E[Z_d(t)] + \theta\sqrt{Var[Z_d(t)]}, \tag{76}$$

where $M(t) = \theta\sqrt{Var[Z_d(t)]}$.

4.2.4 The quantile principle

The standard deviation principle defines the loaded premium as:

$$G(t) = F^{-1}_{Z_d(t)}(1 - \varepsilon), \tag{77}$$

where ε is smallest (for example: $\varepsilon = 0.5\%, 1\%, 2.5\%, 5\%$).
In this case, the safety loading $M(t)$ is given by

$$M(t) = F^{-1}_{Z_d(t)}(1 - \varepsilon) - E[Z_d(t)] \tag{78}$$

The principles of standard deviation and variance only require partial information on the distribution of the random variable, $Z_d(t)$, i.e., its expectation and its variance.

Often, the actuary only has this information for different reasons (time constraints, information ...).

If the actuary has more information about the random variable, $Z_d(t)$ i.e., he knows the form of $F_{Z_d(t)}$, then he can apply the quantile principle.

But he does not know much about $F_{Z_d(t)}$, then he can approximate the distribution of $Z_d(t)$ using the matching moments technique.

5. Conclusion

We have derived exact expressions for all the moments of the DCDPRV process using renewal arguments, again disproving the popular belief that renewal techniques cannot be applied in the presence of economic factors. Our results, for the DCDPRV process, are consistent: (i) with the results of Léveillé et al. [15] for $\theta = 0, \lambda_1 \neq \lambda_2$ and for $\theta = 0, \lambda_1 = \lambda_2$, (ii) with the results of Bargès et al. [8] for $\theta \neq 0, \lambda_1 = \lambda_2$.

Within this framework, further research is needed to get exact expressions (or approximations) of certain functional of the $\{Z_d(t), t \geq 0\}$ process, as stop-loss premiums and ruin probabilities.

Our models have applications in reinsurance, house insurance and car insurance. They can also be used in evaluation of health programs, finance, and other areas.

For example, consider the case of a male currently aged 25 who is starting a defined contribution (DC) pension plan and is planning to retire in, say, 40 years at the age of 65. He anticipates that when he reaches that age he will convert his accumulated pension fund into a life annuity in order to hedge his own longevity

risk and avoid outliving his own financial resources. The value of his retirement income will depend not only on the value of his pension fund, but also on the price of annuities at the time. Other things being equal, this means that his retirement income prospects will be affected by the distribution on future annuity value: the greater the dispersion of that distribution, the riskier his retirement income will be. For the assessment of the accumulated pension fund and its variability our models can be used. We can suppose that this man makes a deposit to a bank account, and that the time between successive deposits follows a renewal process and the force of interest is stochastic. Our model allows us to calculate the accumulated pension fund and its variability at the age of 65.

Another possible application is in reliability, to model the net present value of aggregate equipment failures costs until its total breakdown. A piece of equipment is deemed to be beyond repair when the repair time exceeds a predetermined gap. Of course, another possible definition of total breakdown is when the cost of repair exceeds a predetermined gap. But, since the cost of repair is defined per unit time, the two definitions are somewhat equivalent.

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