# We are IntechOpen, the world's leading publisher of Open Access books
# Built by scientists, for scientists

**4,800**
Open access books available

**122,000**
International authors and editors

**135M**
Downloads

Our authors are among the

**154**
Countries delivered to

**TOP 1%**
most cited scientists

**12.2%**
Contributors from top 500 universities

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

## Interested in publishing with us?
## Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Chapter

# Quantized Field of Single Photons

*Constantin Meis*

## Abstract

We present theoretical developments expressing the physical characteristics of a single photon in conformity with the experimental evidence. The quantization of the electromagnetic field vector potential amplitude is enhanced to a free of cavity photon state. Coupling the Schrödinger equation with the relativistic massless particle Hamiltonian to a symmetrical vector potential relation, we establish the *vector potential - energy* equation for the photon expressing the simultaneous *wave-particle* nature of a single photon, an intrinsic physical property. It is shown that the vector potential can be naturally considered as a real wave function for the photon entailing a coherent localization probability. We deduce directly the electric and magnetic field amplitudes of the cavity-free single photon, which are revealed to be proportional to the square of the angular frequency. The zero-energy electromagnetic field ground state (EFGS), a quantum vacuum real component, issues naturally from Maxwell's equations and the vector potential quantization procedure. The relation of the quantized amplitude of the photon vector potential to the electron-positron charge results directly showing the physical relationship between photons and electrons-positrons that might be at the origin of their mutual transformations. It becomes obvious that photons, as well as electrons-positrons, are issued from the same quantum vacuum field.

**Keywords:** single photon, vector potential, photon wave-particle equation, photon wave function, photon electric field, electromagnetic field ground state, electron-positron charge

## 1. Introduction

During the last decades, an impressive technological development has been achieved permitting the manipulation of single photons with a high degree of statistical accuracy. However, despite the significant experimental advances, we still do not have a clear physical picture of a single photon state universally accepted by the scientific community, especially involved in quantum electrodynamics. In this chapter, based on the present state of knowledge, we make a synthesis of the physical characteristics of a single photon put in evidence by the experiments, and we advance theoretical developments for its representation. Accordingly, the concept of the wave-particle nature of a single photon becomes physically comprehensive and in agreement with the experimental evidence.

However, before advancing in the theoretical developments, we consider that it is important starting with a brief historical review on the efforts carried out previously for understanding the nature of light while simultaneously making a synthesis of the main experimental results which are of crucial importance for the comprehension of the birth of the photon concept.

The very first scientific publications on the nature of light are due to ancient Greeks who believed light is composed of corpuscles [1, 2]. Around 300 BC Euclid published the book *Optica* in which he developed the laws of reflection based on the rectilinear propagation of light. Two centuries later, Ptolemy of Alexandria published the book *Optics*, in which he included extensively all the previous knowledge on light. In this book, colours as well as refraction of the moonlight and sunlight by the earth's atmosphere were analysed. After Ptolemy of Alexandria, almost no progress has been reported until the seventeenth century.

In the year of 1670, Newton revived the ideas of ancient Greeks and advanced the theory following that light is composed of corpuscles that travel rectilinearly [3]. Ten years later, Huygens developed the principles of the wave theory of light [1, 4, 5]. Huygens' wave theory was a hard opponent to Newton's corpuscle concept. In the beginning of the nineteenth century, Young obtained experimentally interference patterns using different sources of light and explained some polarisation observations by assuming that light oscillations are perpendicular to the propagation axis [1, 6]. Euler and Fresnel explained the diffraction patterns observed experimentally by applying the wave theory [6]. In 1865, Maxwell published his theory on the electromagnetic waves establishing the relations between the electric and magnetic fields and showing that light is composed of electromagnetic waves [7]. A few years later, Hertz confirmed Maxwell's theory by discovering the long-wavelength electromagnetic radiation [1, 7]. Thus, at the end of the nineteenth century, the scientific community started to accept officially the wave nature of light replacing Newton's theory.

Nevertheless, new events supporting the particle nature of light occurred in the beginning of the twentieth century. Stefan and Wien discovered the direct relationship between the thermal radiation energy and the temperature of a black body [8, 9]. However, the emitted radiation energy density as a function of the temperature calculated by Rayleigh failed to describe the experimental results at short wavelengths. Scientists had given the name "UV catastrophe" to this problem revealing the necessity of a new theoretical approach. Planck managed to establish the correct energy density expression for the radiation emitted by a black body with respect to temperature, in excellent agreement with the experiment [8]. For that purpose, he assumed that the bodies are composed of "oscillators" which have the particularity of emitting the electromagnetic energy in "packets" of $h\nu$, where $\nu$ is the frequency and $h$ is a constant that was later called Planck's constant. During the same period, the experiments carried out by Michelson et al. [10] demonstrated that the speed of light in vacuum is a universal physical constant corresponding to the product of the frequency $\nu$ times the wavelength $\lambda$, that is, $c = \lambda \nu$.

In 1902, Lenard pointed out that the photoelectric effect, discovered by Hertz 15 years earlier [11], occurs beyond a threshold frequency of light and the kinetic energy of the emitted electrons does not depend on the incident light intensity. Based on Planck's works, Einstein proposed a simple interpretation of the photoelectric effect assuming that the electromagnetic radiation is composed of quanta with energy $h\nu$ [12]. He advanced that *the energy of a light ray when spreading from a point consists of a finite number of energy quanta localised in points in space, which move without dividing and are only absorbed and emitted as a whole.* Although that was a decisive step towards the particle theory of light, the concept of the light quanta was still not generally accepted, and Bohr, who was strongly opposed to the particle concept of light [13], announced in his Nobel lecture (1922) that *the light quantum hypothesis is not compatible with the interference phenomena and consequently it cannot throw light in the nature of radiation.* Bohr's statement was rather surprising because Taylor's experiments, consisting of repeating Young's double slit diffraction at extremely low light intensities, had already demonstrated since 1909 that light rays

are composed of discrete parts whose spots compose the diffraction patterns by gradual accumulation on the detection screen [14]. Compton published his studies on X-rays scattered by free electrons in 1923 advancing that the experimental results could only be interpreted based on the light quanta model [15].

Thus, the photoelectric effect and Compton scattering have been initially considered as the undoubtable demonstrations of the particle nature of light and historically were the strongest arguments in favour of the light quanta concept, which started to be universally accepted, and Lewis introduced the word "photon", from the Greek word *phos* (Φωs, which means light) [1, 4].

Therein, it is extremely important to mention that Wentzel in 1926 [16] and Beck in 1927 [17], as well as much later Lamb and Scully in the 1960s [18], demonstrated that the photoelectric effect can be interpreted remarkably well by only considering the wave nature of light, without referring to photons at all [19]. Furthermore, the Compton scattering has been fully interpreted by Klein and Nishina in 1929 [20] also by considering the electromagnetic wave nature of light without invoking the photon concept. On the other hand, Young's experiment, initially presented as the most convincing argument for the wave nature of light, was applied by Taylor at very low intensities to demonstrate the particle concept of light [14]. Indeed, much later Jin et al. [21] published an excellent theoretical interpretation of Young's diffraction experiments based only on the particle representation of light.

Thus, the picture on the nature of light in the 1930s was rather confusing since both opposite sides defending the wave or the particle nature advanced equally strong arguments. Hence, Bohr, inspired by de Broglie's thesis on the simultaneous wave character of particles, announced the *complementarity principle* according to which *light has both wave and particle natures appearing mutually exclusively in each specific experimental condition* [1, 2, 19].

The development of lasers [23] in the 1960s and the revolutionary parametric down-convertion techniques [24, 25] in the 1970s, have made it possible to realise conditions in which, with a convenient statistical confidence, only a single photon may be present in the experimental apparatus. In this way, the double-prism experiment [26] realised in the 1990s contradicted for the first time Bohr's *mutual exclusiveness* demonstrating that a *single photon exhibits both the wave and particle natures in the same experimental conditions*.

According to the experimental investigations, it has been always stated that a photon has circular, left or right, polarisation with spin $\pm h/2\pi$ and cannot be conceived along the propagation axis within a length shorter than its wavelength [27]. Indeed, since Mandel's experiments in the 1960s [28, 29], all the efforts to localise precisely a single photon remained fruitless yielding the conclusion is that a photon cannot be better localised than within a volume of the order of the cube of its wavelength [30, 31]. Furthermore, Grangier et al. demonstrated experimentally the indivisibility of photons [19, 32], while in recent years the entangled state experiments [30, 31] have shown that the photon should be locally an integral entity during the detection procedure but with a real non-local wave function.

The lateral expansion of a single photon, considered locally as an indivisible entity, was always an intriguing part of physics. With the purpose of studying the lateral expansion of the electromagnetic rays, Robinson in 1953 [33] and Hadlock in 1958 [34] carried out experiments using microwaves crossing small apertures and deduced that no energy is transmitted through apertures whose dimensions are smaller than roughly $\sim\lambda/4$. In 1986, for the same purpose, Hunter and Wadlinger [35, 36] used X-band microwaves with $\lambda$ = 28.5 mm and measured the transmitted power through rectangular or circular apertures of different dimensions. They concluded that no energy is transmitted when the apertures are

smaller than $\sim\lambda/\pi$ confirming that the lateral expansion of the photons is a fraction of the wavelength.

Thus, the experiments have shown that the single photon is not a point and cannot be localised at a coordinate, as stated by Einstein, while it can exhibit both the wave and particle natures in the same experimental conditions contradicting Bohr's mutual exclusiveness. However, quantum electrodynamics (QED) has been developed during the 1930s to 1960s based upon the point particle model for the photon [22, 37–39]. In fact, the point photon concept has permitted to establish an efficient mathematical approach for describing states before and after an interaction processes [19, 39–41], but it is naturally inappropriate for the description of the real nature of a single photon.

Finally, what we can essentially draw out by summing up the experimental evidence is that a single photon is a minimum, local, indivisible part of the electromagnetic field with precise energy $h\nu$ and momentum $h\nu/c$, having circular left or right polarisation with spin $\pm h/2\pi$, respectively. It is not a point particle since it expands over a wavelength $\lambda$ along the propagation axis and is detectable within a volume of the order of $\lambda^3$ entailing that its lateral expansion is a fraction of its wavelength. Hence, it appears to be a local "wave-corpuscle" guided by a non-local wave function, absorbed and emitted as a whole and capable of interacting with charges increasing or decreasing its frequency and consequently its energy.

In what follows, we present first the standard theoretical representation of the electromagnetic field quantization resulting in photons, and next we proceed to recent advances based on the vector potential quantization enhanced to a single photon state.

## 2. The electromagnetic field vector potential

### 2.1 Reality of the vector potential

Since the formulation of Maxwell's equations, the vector potential $\vec{A}\left(\vec{r},t\right)$ was considered as a pure mathematical artefact [4, 5, 7] used to calculate the electric field:

$$\vec{E}\left(\vec{r},t\right) = -\partial\vec{A}\left(\vec{r},t\right)/\partial t - \vec{\nabla}\Phi\left(\vec{r},t\right) \tag{1}$$

where $\Phi\left(\vec{r},t\right)$ is the scalar potential, as well as the magnetic field:

$$\vec{B}\left(\vec{r},t\right) = \vec{\nabla}\times\vec{A}\left(\vec{r},t\right) \tag{2}$$

In 1949, Ehrenberg and Siday were the first to put in evidence the influence of the vector potential on charged particles [42] deducing that it is a real physical field. Ten years later, Aharonov and Bohm re-infirmed the influence of the vector potential on electrons in complete absence of electric and magnetic fields [43]. That was confirmed experimentally by Chambers [44], Tonomura et al. [45], and Osakabe et al. [46] demonstrating without any doubt the reality of the vector potential field end its direct influence on charges.

From a theoretical point of view [43], the behaviour of a particle with charge $q$ and mass $m$ in the vicinity of a solenoid where the vector potential is present is described by the Hamiltonian:

$$H = \frac{1}{2m}\left(-i\hbar\,\vec{\nabla} - q\vec{A}\left(\vec{r},t\right)\right)^2 + q\Phi\left(\vec{r},t\right) \tag{3}$$

with $\hbar = h/2\pi$ as Planck's reduced constant.

If the solenoid is extremely long along the $z$ axis, then the magnetic field is uniform in the region inside and zero outside. The scalar potential $\Phi$ can be put to zero by assuming that the solenoid is not charged. In this case, in the outside region, the electric and magnetic fields are zero, but the vector potential is not zero and depends on the magnetic field flux in the solenoid:

$$\vec{A} = \frac{1}{2\pi r}\int_s \vec{B}\cdot d\vec{S}\,\hat{e}_\theta \tag{4}$$

where $r$ is the radial distance from the $z$ axis of the solenoid, $S$ is the surface of the circle with radius $r$ perpendicular to $z$, and $\hat{e}_\theta$ is the angular unit vector in cylindrical coordinates.

The Schrödinger equation for a charged particle outside the solenoid, where the vector potential is not zero, writes in complete absence of any other external potential:

$$i\hbar\frac{\partial}{\partial t}\Psi\left(\vec{r},t\right) = \frac{1}{2m}\left(i\hbar\,\vec{\nabla} + q\vec{A}\right)^2\Psi\left(\vec{r},t\right) \tag{5}$$

with $\vec{A}$ given by Eq. (4). The solutions of the last equation are the wave functions:

$$\Psi\left(\vec{r},t\right) = \Psi_p\left(\vec{r},t\right)e^{\frac{iq}{\hbar}\int_0^{\vec{r}}\vec{A}\left(\vec{r}'\right)\cdot d\vec{r}'} \tag{6}$$

where $\Psi_p\left(\vec{r},t\right)$ is the solution of Schrödinger's equation in absence of the vector potential:

$$i\hbar\frac{\partial}{\partial t}\Psi_p\left(\vec{r},t\right) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\Psi_p\left(\vec{r},t\right) \tag{7}$$

The exponential part of the wave function of Eq. (6) entails that two particles have equal charge and mass moving both outside the solenoid at the same distance from the axis, but the first in the same direction with the vector potential $\vec{A}$ and the second in the opposite direction will suffer a phase difference:

$$\delta\Theta = \frac{q}{\hbar}\int_s \vec{B}\cdot d\vec{S} \tag{8}$$

Interference patterns for electrons in analogue conditions have been observed experimentally [44–46] demonstrating that the vector potential is a real physical field and interacts directly with charged particles in complete absence of magnetic and electric fields and of any other potential.

### 2.2 The radiation vector potential: classical to quantum link

The vector potential, being a real field, is considered as the fundamental link between the electromagnetic wave theory issued from Maxwell's equations and the

particle concept in quantum electrodynamics (QED) [19, 22, 39]. We will show analytically how this link is established.

In the classical theory [5, 7], the energy density of a mode $k$ of the electromagnetic wave writes:

$$W_k\left(\vec{r},t\right) = \frac{1}{2}\left(\varepsilon_0\left|\vec{E}_k\left(\vec{r},t\right)\right|^2 + \frac{1}{\mu_0}\left|\vec{B}_k\left(\vec{r},t\right)\right|^2\right) \tag{9}$$

where $\varepsilon_0$ and $\mu_0$ are the electric permittivity and magnetic permeability of the vacuum, respectively, related to the speed of light in vacuum $c$ by $\varepsilon_0\mu_0 c^2 = 1$.

In the case of a monochromatic plane wave with angular frequency $\omega_k$, the electric $\vec{E}_k\left(\vec{r},t\right)$ and magnetic $\vec{B}_k\left(\vec{r},t\right)$ fields are proportional to the vector potential amplitude $A_{0k}(\omega_k)$:

$$\vec{E}_k\left(\vec{r},t\right) = -2\omega_k A_{0k}(\omega_k)\,\hat{\varepsilon}\,\sin\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{10}$$

$$\vec{B}_k\left(\vec{r},t\right) = -\frac{1}{c}2\omega_k A_{0k}(\omega_k)\left(\hat{k}\times\hat{\varepsilon}\right)\sin\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{11}$$

where $\hat{\varepsilon}$ is a unit vector perpendicular to the propagation axis, $\left|\vec{k}\right| = 2\pi/\lambda_k$ is the wave vector along the propagation axis, and $\lambda_k$ is the wavelength of the mode $k$.

Introducing Eqs. (10) and (11) in Eq. (9), the energy density now depends on the square of the vector potential amplitude:

$$W_k\left(\vec{r},t\right) = 4\varepsilon_0\omega_k^2|A_{0k}(\omega_k)|^2\sin^2\left(\vec{k}\cdot\vec{r}-\omega_k t\right) \tag{12}$$

The mean value over a period, thus over a wavelength, is time independent:

$$W_k = 2\varepsilon_0\omega_k^2|A_{0k}(\omega_k)|^2 \tag{13}$$

Note that the last equation expressing the mean energy density over a period of the mode $k$ of the electromagnetic wave is independent on any external volume yielding that in the classical description, a free of cavity electromagnetic radiation mode expands naturally within a minimum volume. In a given cavity, this volume corresponds roughly to that imposed by the boundary conditions and the cut-off wave vectors [4, 5, 7].

On the other hand, in the quantum description, the energy density for a number $N(\omega_k)$ of $k$-mode photons with angular frequency $\omega_k$ and energy $\hbar\omega_k$ in a volume $V$ writes simply:

$$W_k = \frac{N(\omega_k)\hbar\omega_k}{V} \tag{14}$$

In order to link the classical to the quantum description [4, 9, 19], the classical mean energy density over a period, expressed by Eq. (13), is imposed to be equivalent to the quantum mechanics expression of Eq. (14) for $N(\omega_k) = 1$ getting the vector potential amplitude for a single $k$-mode photon:

$$|A_{0k}(\omega_k)| = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} \tag{15}$$

The last relation is the fundamental link between the classical and quantum theory of light which is used to define in QED the vector potential amplitude operators for a single photon [19, 22, 27, 30, 37–41]:

$$\tilde{A}_{k\lambda} = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} a_{k\lambda} \qquad \tilde{A}^*_{k\lambda} = \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k V}} a^+_{k\lambda} \tag{16}$$

where $a_{k\lambda}$ and $a^+_{k\lambda}$ are, respectively, the annihilation and creation non-Hermitian operators for a $k$-mode and $\lambda$-polarisation photon.

Therein, it is worth noting that an external arbitrary volume parameter $V$ appears in the vector potential amplitude of the single photon, expressed by Eq. (15), which is supposed to be an intrinsic physical property. This could entail the unphysical interpretation that a single photon in an infinite cavity has zero vector potential, thus zero electric and magnetic fields and consequently zero energy. This ambiguity, which is scarcely quoted in the literature, is lifted by considering that, in the case of a single photon, the volume $V$ in Eq. (15) is equivalent to that defined by the boundary conditions in a cavity for the single radiation mode $k$.

## 3. Electromagnetic field quantization and the photon description

### 3.1 Harmonic oscillator representation of the electromagnetic field

The energy of the electromagnetic field in a volume $V$ considered as a superposition of different $k$-modes and $\lambda$-polarisations is obtained directly from Eq. (13):

$$E_{EM} = 2\varepsilon_0 V \sum_{k,\,\lambda} \omega_k^2 |A_{k\lambda}(\omega_k)|^2 \tag{17}$$

where the summation over the $\lambda$-polarisations takes only two values corresponding to circular left and right [19, 22, 37–41].

Replacing in Eq. (17) the vector potential amplitude and its conjugate by the relations of the vector potential amplitude operators defined in Eq. (16), we get the "normal ordering" radiation Hamiltonian corresponding to the order $a^+_{k\lambda} a_{k\lambda}$ of the creation and annihilation operators:

$$\tilde{H}_{NO} = \sum_{k,\,\lambda} a^+_{k\lambda} a_{k\lambda} \hbar\omega_k \tag{18}$$

and the "anti-normal ordering" Hamiltonian corresponding to the order $a_{k\lambda} a^+_{k\lambda}$

$$\tilde{H}_{ANO} = \sum_{k,\,\lambda} \left(a^+_{k\lambda} a_{k\lambda} + 1\right) \hbar\omega_k \tag{19}$$

where we have used the fundamental commutation relation in quantum electrodynamics:

$$\left[a_{k\lambda}, a^+_{k\lambda}\right] = 1 \tag{20}$$

In Dirac's representation the eigenfunctions take the simple expression $|n_{k\lambda}\rangle$, and the action of the creation and annihilation operators of a single $k$-mode and $\lambda$-polarisation photon writes:

$$a_{k\lambda}^{+}|n_{k\lambda}\rangle = \sqrt{n_{k\lambda}+1}\,|n_{k\lambda}+1\rangle; \quad a_{k\lambda}|n_{k\lambda}\rangle = \sqrt{n_{k\lambda}}\,|n_{k\lambda}-1\rangle \tag{21}$$

The successive action of both operators in the normal order corresponds to the photon number Hermitian operator $\tilde{n}_{k\lambda} = a_{k\lambda}^{+}a_{k\lambda}$ having the eigenvalue $n_{k\lambda}$ representing the number of $k$-mode and $\lambda$-polarisation photons:

$$a_{k\lambda}^{+}a_{k\lambda}|n_{k\lambda}\rangle = \tilde{n}_{k\lambda}|n_{k\lambda}\rangle = n_{k\lambda}|n_{k\lambda}\rangle \tag{22}$$

In this representation the normal and anti-normal ordering radiation Hamiltonians write, respectively:

$$\tilde{H}_{NO} = \sum_{k,\,\lambda}\tilde{n}_{k\lambda}\hbar\omega_k; \quad \tilde{H}_{ANO} = \sum_{k,\,\lambda}\left(\tilde{n}_{k\lambda}+1\right)\hbar\omega_k \tag{23}$$

We obtain a harmonic oscillator Hamiltonian for the electromagnetic field by considering the mean value of the normal ordering and anti-normal ordering Hamiltonians:

$$\tilde{H}_{EM} = \frac{1}{2}\left(\tilde{H}_{NO}+\tilde{H}_{ANO}\right) = \sum_{k,\,\lambda}\left(\tilde{n}_{k,\,\lambda}+\frac{1}{2}\right)\hbar\omega_k \tag{24}$$

Thus, in QED the electromagnetic field is considered to be an ensemble of harmonic oscillators each represented by a point particle, the photon, whose eigenfunction is denoted simply by $|1_{k\lambda}\rangle$ [19, 39, 41].

Although we have no experimental facts showing the harmonic oscillator nature of a single photon, this representation has been adopted since the 1930s [37].

In a different way, a harmonic oscillator representation for the electromagnetic field can be obtained by the intermediate of the canonical variables of position $Q_{k\lambda}$ and momentum $P_{k\lambda}$. For that purpose we introduce the definitions expressing the vector potential amplitude and its complex conjugate with respect to $Q_{k\lambda}$ and $P_{k\lambda}$ [19, 30, 41]:

$$A_{k\lambda} = \frac{(\omega_k Q_{k\lambda}+iP_{k\lambda})}{2\omega_k\sqrt{\varepsilon_0 V}}; \qquad A_{k\lambda}^{*} = \frac{(\omega_k Q_{k\lambda}-iP_{k\lambda})}{2\omega_k\sqrt{\varepsilon_0 V}} \tag{25}$$

Introducing the last expressions in Eq. (17), we get the electromagnetic field energy:

$$E_{EM} = \frac{1}{2}\sum_{k,\,\lambda}\left(P_{k\lambda}^{2}+\omega_k^{2}Q_{k\lambda}^{2}\right) \pm i\omega_k[Q_{k\lambda},P_{k\lambda}] \tag{26}$$

where the (+) sign is obtained when Eq. (17) is considered initially to be in the "normal order", $A_{k\lambda}^{*}A_{k\lambda}$, and the (−) one when in the "anti-normal order" $A_{k\lambda}A_{k\lambda}^{*}$.

With the purpose of establishing a harmonic oscillator representation for the electromagnetic field, it is generally considered that $[Q_{k\lambda},P_{k\lambda}] = 0$ in Eq. (26),

because $Q_{k\lambda}$ and $P_{k\lambda}$ are simply canonical variables, getting the energy of an ensemble of harmonic oscillators:

$$E_{EM} = \frac{1}{2}\sum_{k,\,\lambda}\left(P_{k\lambda}^2 + \omega_k^2 Q_{k\lambda}^2\right) \tag{27}$$

Replacing in the last equation the classical canonical variables of position and momentum with the corresponding Hermitian operators [19, 30, 41]:

$$\tilde{P}_{k\lambda} = i\sqrt{\frac{\hbar\omega_k}{2}}\left(a_{k\lambda}^{+} - a_{k\lambda}\right); \quad \tilde{Q}_{k\lambda} = \sqrt{\frac{\hbar}{2\omega_k}}\left(a_{k\lambda}^{+} + a_{k\lambda}\right) \tag{28}$$

and putting $a_{k\lambda}^{+}a_{k\lambda} = \tilde{n}_{k\lambda}$, one gets the harmonic oscillator Hamiltonian for the radiation field:

$$\tilde{H}_{EM} = \sum_{k,\,\lambda}\left(\tilde{n}_{k,\,\lambda} + \frac{1}{2}\right)\hbar\omega_k \tag{29}$$

At that level it is important to note that, for a harmonic oscillator of a particle with mass $m$ and momentum $\vec{p} = m\,d\vec{q}/dt$, with canonical variables of position $Q = \left|\vec{q}\right|\sqrt{m}$ and momentum $P = \left|\vec{p}\right|/\sqrt{m}$, the transition from the classical expression of energy:

$$E = \frac{1}{2}\left(P^2 + \omega^2 Q^2\right) \tag{30}$$

to the quantum mechanics Hamiltonian:

$$\tilde{H} = \frac{1}{2}\left(\tilde{P}^2 + \omega^2\tilde{Q}^2\right) = \left(a^{+}a + \frac{1}{2}\right)\hbar\omega = \left(\tilde{n} + \frac{1}{2}\right)\hbar\omega \tag{31}$$

where $\tilde{P} = i\sqrt{\frac{\hbar\omega}{2}}(a^{+} - a)$, $\tilde{Q} = \sqrt{\frac{\hbar}{2\omega}}(a^{+} + a)$, and $\tilde{n} = a^{+}a$ is direct and needs no commutation operations between the canonical variables $P$ and $Q$ [19, 39].

Consequently, the harmonic oscillator Hamiltonian for a particle of mass $m$ expressed by Eq. (31) is a quite physical result (e.g., phonons in solid-state physics) obtained with a perfect correspondence between the classical canonical variables of momentum and position $P$ and $Q$, respectively, and the corresponding Hermitian operators $\tilde{P}$ and $\tilde{Q}$.

Conversely, this is not the case for the electromagnetic field [19, 30, 39] because commutations between the canonical variables $Q_{k\lambda}$ and $P_{k\lambda}$ occur during the mathematical transition from Eq. (17) to Eq. (26). It is then considered that $[Q_{k\lambda}, P_{k\lambda}] = 0$ in order to obtain Eq. (27) just before replacing the canonical variables by the corresponding quantum mechanics operators. Therein, it is important to remark that Heisenberg's commutation relation $\left[\tilde{Q}_{k\lambda}, \tilde{P}_{k'\lambda'}\right] = i\hbar\delta_{kk'}\delta_{\lambda\lambda'}$ is a fundamental concept of quantum mechanics, which should not be ignored when replacing classical variables by the corresponding quantum mechanics operators [19]. In fact, without dropping $[Q_{k\lambda}, P_{k\lambda}]$ in Eq. (26) and replacing the canonical variables by the corresponding quantum operators of Eq. (28), we get naturally the same normal ordering and anti-normal ordering radiation Hamiltonians as in Eq. (23):

$$\begin{pmatrix} \tilde{H}_{EM}^{(+)} = \frac{1}{2}\sum_{k,\lambda}\left(\tilde{P}_{k\lambda}^2 + \omega_k^2\tilde{Q}_{k\lambda}^2\right) + i\omega_k\left[\tilde{Q}_{k\lambda}, \hat{P}_{k'\lambda'}\right] \\ \tilde{H}_{EM}^{(-)} = \frac{1}{2}\sum_{k,\lambda}\left(\tilde{P}_{k\lambda}^2 + \omega_k^2\tilde{Q}_{k\lambda}^2\right) - i\omega_k\left[\tilde{Q}_{k\lambda}, \tilde{P}_{k'\lambda'}\right] \end{pmatrix} = \begin{pmatrix} \tilde{H}_{NO} = \sum_{k,\lambda}\tilde{n}_{k\lambda}\hbar\omega_k \\ \tilde{H}_{ANO} = \sum_{k,\lambda}(\tilde{n}_{k\lambda}+1)\ \ \hbar\omega_k \end{pmatrix} \tag{32}$$

Obviously, as frequently quoted [2, 19, 39], the fundamental mathematical ambiguity consisting of cancelling the commuting classical variable term $[Q_{k\lambda}, P_{k\lambda}] = 0$ before the substitution by non-commuting quantum mechanics operators $\left[\tilde{Q}_{k\lambda}, \tilde{P}_{k'\lambda'}\right] = i\hbar\delta_{kk'}\delta_{\lambda\lambda'}$ leads to the harmonic oscillator Hamiltonian for the electromagnetic field.

In fact, since no experiment has yet demonstrated that a single photon is a harmonic oscillator, the main reason for considering the electromagnetic field as an ensemble of harmonic oscillators lies in the importance of the zero-point energy (ZPE) issued in absence of photons from the eigenvalue $n_{k\lambda} = 0$ of Eq. (29) corresponding to the vacuum energy:

$$E_{ZPE} = \sum_{k,\lambda}\frac{1}{2}\hbar\omega_k \tag{33}$$

The summation of the last expression over all modes and polarisations is infinite and represents the principal singularity in the QED formalism [19, 22, 27, 30, 39].

Nevertheless, the zero-point energy is very important because it is considered to be the basis for the explanation of the vacuum effects such as the spontaneous emission, the Lamb shift and the Casimir effect. However, as pointed out by many authors [19, 27, 39, 41], it is important to underline that the explanation of the spontaneous emission and the Lamb shift in QED is not due to Eq. (33) but precisely to the commutation properties of the photon creation and annihilation operators, $a_{k\lambda}^{+}$ and $a_{k\lambda}$, respectively. It has to be emphasized that in quantum mechanics theory Eq. (33), being a constant, commutes with all Hermitian operators corresponding to physical observables and consequently has absolutely no influence to any quantum process.

Conversely, the zero-point energy expressed by Eq. (33) is useful for the explanation of the spontaneous emission and the Lamb shift in the classical description of radiation [2, 39, 47].

Regarding the Casimir effect, it is often commented that caution has to be taken concerning the interpretation of its physical origin because it has been demonstrated by different methods [48–50] that it can be easily explained using classical electrodynamics without invoking at all the zero-point energy.

Hence, in view of the above, the normal ordering Hamiltonian is the one mainly used in QED, casting aside the vacuum singularity issued from the harmonic oscillator formalism, while the zero-point energy issued from the harmonic oscillator Hamiltonian is principally useful in the classical formalism for the interpretation of the vacuum effects [2, 19, 39, 47].

### 3.2 Electromagnetic field vector potential quantization in QED

We have analysed in Section 3.1 the electromagnetic field energy quantization according to the harmonic oscillator representation. Now, we will analyse the vector potential field quantization following the second quantisation process.

Considering the natural units ($\hbar = c = 1$), the radiation vector potential writes within the frame of the quantum field theory (QFT) [27, 38]:

$$\vec{A}\ (x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^{2} \hat{\varepsilon}_{k\lambda} \left[\alpha^{(\lambda)}(k) e^{-ikx} + \alpha^{(\lambda)+}(k) e^{ikx}\right] \tag{34}$$

with

$$\left[\alpha^{(\lambda)}(k), \alpha^{(\lambda')+}\left(k'\right)\right] = 2k_0 (2\pi)^3 \delta_{\lambda\lambda'} \delta^3\left(\vec{k} - \vec{k'}\right) \tag{35}$$

where $\delta_{\lambda\lambda'}$ is the Kronecker delta, $\delta^3\left(\vec{k} - \vec{k'}\right)$ is the Dirac delta function, and

$$\alpha^{(\lambda)}(k) = (2k_0)^{1/2} (2\pi)^{3/2} \alpha_{k\lambda}; \quad \alpha^{(\lambda)+}(k) = (2k_0)^{1/2} (2\pi)^{3/2} \alpha^{+}_{k\lambda} \tag{36}$$

Using Eq. (36) in Eq. (34), the vector potential becomes:

$$\vec{A}\ (x) = \int \frac{d^3k}{(2\pi)^{3/2} (2k_0)^{1/2}} \sum_{\lambda=1}^{2} \hat{\varepsilon}_{k\lambda} \left[\alpha_{k\lambda} e^{-ikx} + \alpha^{+}_{k\lambda} e^{ikx}\right] \tag{37}$$

with $\omega_k = \left(\vec{k}^2 + l^2\right)^{1/2}$ and $k^2 = k_0^2 - \left(\vec{k}\right)^2 = l^2$.

For $k = l = 0$ then $k_0 = \left|\vec{k}\right| = \omega_k$ and Eq. (37) writes:

$$\vec{A}\ (x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{1}{2\omega_k}} \sum_{\lambda=1}^{2} \hat{\varepsilon}_{k\lambda} \left[\alpha_{k\lambda} e^{-ikx} + \alpha^{+}_{k\lambda} e^{ikx}\right] \tag{38}$$

Suppressing the natural units (i.e., introducing $c$ and $\hbar$) and transforming the last equation in the SI system, which is generally used in QED, we get [38, 41, 51].

$$\vec{A}\ (x) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_k}} \sum_{\lambda=1}^{2} \hat{\varepsilon}_{k\lambda} \left[\alpha_{k\lambda} e^{-ikx} + \alpha^{+}_{k\lambda}(k) e^{ikx}\right] \tag{39}$$

On the basis of the density of state theory, the quantization of a field in a cavity of volume $V$ permits to transform the continuous summation over the modes to a discrete one [19, 51]:

$$\int \frac{d^3k}{(2\pi)^{3/2}} = \sqrt{\frac{1}{V}} \sum_{k} \tag{40}$$

The last transformation is only valid for an ensemble of modes $k$ whose wavelengths $\lambda_k$ are much shorter than the characteristic dimensions of the volume $V$ [19, 30, 39, 41].

Switching now to Heisenberg's representation:

$$\alpha_{k\lambda}(t) = \alpha_{k\lambda} e^{-i\omega_k t} \quad ; \quad \alpha^{+}_{k\lambda}(t) = \alpha^{+}_{k\lambda} e^{i\omega_k t} \tag{41}$$

Generalizing the coordinate system, adapting the phase and using Eq. (40), the vector potential of the electromagnetic field writes in QED [19, 30, 39, 41, 51]:

$$\vec{A}\left(\vec{r},t\right)=\sum_{k}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\sum_{\lambda=1}^{2}\hat{\varepsilon}_{k\lambda}\left[\alpha_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}+\alpha_{k\lambda}^{+}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right] \tag{42}$$

Considering the scalar potential to be constant, the electric field is:

$$\vec{E}\left(\vec{r},t\right)=i\sum_{k}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sum_{\lambda=1}^{2}\hat{\varepsilon}_{k\lambda}\left[\alpha_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}-\alpha_{k\lambda}^{+}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right] \tag{43}$$

The last expressions represent in a given volume $V$ the quantized vector potential and the electric field of the electromagnetic radiation composed of a large number of modes $k$ each with angular frequency $\omega_k$ and wavelength $\lambda_k=2\pi c/\omega_k$ much smaller than $V^{1/3}$:

$$\lambda_k \ll V^{1/3} \quad (\forall k) \tag{44}$$

The amplitudes in Eqs. (42) and (43) have been obtained using the density of state theory and are valid only on the condition of Eq. (44). Furthermore, the boundary conditions of the electromagnetic waves considered in cavities and waveguides impose the wave vectors $k$ of the modes to be higher than a characteristic cut-off value $k>k_{cut-off}$ $\left(\lambda_k<\lambda_{cut-off}\right)$ depending on the dimensions and the shape of the volume containing the radiation field [4–7]. Consequently, for a volume $V$ with finite dimensions, the summation in Eqs. (42) and (43) runs only over the modes $k$ with wave vectors higher than the minimum cut-off value $k_{cut-off}(V)$ imposed by the shape and dimensions of $V$ so that we can write more precisely:

$$\vec{A}\left(\vec{r},t\right)=\sum_{k>k_{cut-off}(V)}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}}\sum_{\lambda=1}^{2}\hat{\varepsilon}_{k\lambda}\left[\alpha_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}+\alpha_{k\lambda}^{+}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right] \tag{45}$$

$$\vec{E}\left(\vec{r},t\right)=i\sum_{k>k_{cut-off}(V)}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\sum_{\lambda=1}^{2}\hat{\varepsilon}_{k\lambda}\left[\alpha_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}-\alpha_{k\lambda}^{+}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right] \tag{46}$$

The last equations represent the vector potential and the electric field of a large number of modes $k$ of the quantized electromagnetic field in a finite volume $V$ with $\lambda_k \ll V^{1/3}$ $(\forall k)$.

## 4. Quantized vector potential of a single photon

We have seen in Section 3.1 that according to the energy quantization procedure, a $k$-mode and $\lambda$-polarisation photon is considered to be a point harmonic oscillator represented by the simplified eigenfunction $|1_{k\lambda}\rangle$. On the other hand, following the field quantization procedure in Section 3.2, it appears clearly that the established vector potential and electric field expressions in Eqs. (42) and (43),

under the condition of Eq. (44), concern a large number of modes in a considerably big volume compared to their wavelengths. Thus, with the aim of obtaining a clearer picture of the single photon, we will now complement the previous descriptions by enhancing the vector potential amplitude quantization to the photon level.

### 4.1 Photon vector potential amplitude and quantization volume

As mentioned in Section 2.2, the classical expression of the mean energy density over a period for a single electromagnetic mode $k$, represented by Eq. (13), can be considered equivalent to that for a single photon in the quantum representation, given by Eq. (14) for $N(\omega_k) = 1$, on the condition that the volume $V$ is not arbitrary but corresponds to that defined by the boundary conditions in a cavity for the considered electromagnetic mode. In fact, the physical properties of a free photon are independent on any surrounding volume unless the characteristic dimensions of the last one are of the order of the photon wavelength [52].We recall again that the experimental evidence has shown [19, 28, 30, 51] that a single photon with angular frequency $\omega_k$ and wavelength $\lambda_k = 2\pi c/\omega_k$ can only be localised within a volume $V_k$ whose dimensions are roughly the cube of its wavelength:

$$V_k \propto \lambda_k^3 \qquad \Rightarrow \qquad V_k \propto \omega_k^{-3} \tag{47}$$

From a theoretical point of view, this is also compatible with the density of state theory according to which the spatial volume corresponding to a single state of the quantized field is proportional to $\omega_k^{-3}$ [19, 30, 39, 41].

On the other hand, the dimension analysis of the vector potential issued from the general solution of Maxwell's equations yields that it is proportional to an angular frequency [5, 7, 9]:

$$\vec{A}\left(\vec{r},t\right) = \frac{\mu}{4\pi}\int \frac{\vec{J}\left(\vec{r}\,', t - \frac{|\vec{r}-\vec{r}'|}{c}\right)}{|\,\vec{r} - \vec{r}\,'|}\, d^3r' \quad \propto \quad \omega \tag{48}$$

where $J$ is the current density ($C\,m^{-2}\,s^{-1}$) and $\mu$ the magnetic permeability.

Indeed, it is well established experimentally that the energy density radiated by a dipole is proportional to $\omega^4$ entailing from Eq. (12) that the vector potential amplitude is normally proportional to $\omega$ [4, 5, 7].

This result is gauge independent since it concerns the natural units of the vector potential.

According to the previous considerations, for a free single $k$-mode photon with $\lambda$-polarisation (left or right circular), the vector potential can be written in quantum and classical formalism:

$$\begin{aligned}
\tilde{\alpha}_{k\lambda}\left(\vec{r},t\right) &= \alpha_{0k}(\omega_k)\left[\hat{\varepsilon}_{k\lambda}\alpha_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} + \hat{\varepsilon}^*_{k\lambda}\alpha^+_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right] \\
\vec{\alpha}_{k\lambda}\left(\vec{r},t\right) &= \alpha_{0k}(\omega_k)\left[\hat{\varepsilon}_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)} + \hat{\varepsilon}^*_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t\right)}\right]
\end{aligned} \tag{49}$$

where, following to the above analysis, the amplitude writes:

$$\alpha_{0k}(\omega_k) = |\xi|\omega_k \tag{50}$$

with $\xi$ being a constant [2, 53–55].

We can evaluate $\xi$ [2, 53] by using Eqs. (49) and (50) in Eq. (13) and normalising the energy to that of a single photon, $\hbar\omega_k$, by integrating over a wavelength along the propagation direction while taking into account the experimental results on the lateral expansion of the photon [33–36, 56]. We get:

$$|\xi| = \left|\frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\hbar}{8\alpha\varepsilon_0 c^3}}\right| = \left|\frac{\hbar}{4\pi e c}\right| = 1.747 \quad 10^{-25} \ \ Volt \ \ m^{-1}s^2 \tag{51}$$

where $\alpha = e^2/4\pi\varepsilon_0\hbar c \approx 1/137.036$ is the fine structure constant and $e$ is the electron charge. Obviously, when introducing Eq. (50) in Eq. (17) for a single $k$-mode photon, an appropriate volume $V_k$ has to be considered for the equation to hold:

$$E_k = \hbar\omega_k = 2\varepsilon_0 V_k \xi^2 \omega_k^4 \tag{52}$$

Thus, the characteristic volume of a free single photon writes in agreement with Eq. (47):

$$V_k = \left(\frac{\hbar}{2\varepsilon_0\xi^2}\right)\omega_k^{-3} \tag{53}$$

Replacing $\xi$ expressed by Eq. (51) in Eq. (53), we obtain the photon quantization volume:

$$V_k \ \approx \ 4\alpha\lambda_k^3 \tag{54}$$

Equations (50) and (53) express the quantized vector potential amplitude and the spatial extension of a single photon with the constant $\xi$ evaluated to be $|\xi| = \hbar/4\pi|e|c$.

### 4.2 Photon classical-quantum (wave-particle) physical properties

For a free $k$-mode photon, the volume $V_k$ corresponds to the space in which the quantized vector potential oscillates at the angular frequency $\omega_k$ over a period along the propagation axis generating orthogonal electric and magnetic fields whose amplitudes are, respectively:

$$\left|\vec{\varepsilon_k}\right| = \left|-\partial\vec{\alpha}_{k\lambda}\left(\vec{r},t\right)/\partial t\right| \propto |\xi|\omega_k^2; \qquad \left|\vec{\beta_k}\right| \propto \sqrt{\varepsilon_0\mu_0}|\xi|\omega_k^2 \tag{55}$$

which are independent on any external arbitrary volume parameter and are directly proportional to the square of the angular frequency [2, 54, 55].

We can now express the quantum properties of the photon, energy, momentum, and spin by integrating the classical electromagnetic expressions over the volume $V_k$ and by using the vector potential amplitude obtained in Eq. (50), linking in this way the classical (wave) to the quantum (particle) representations [2]. The energy writes:

$$E_k = \int_{V_k} 2\varepsilon_0\alpha_{0k}^2\omega_k^2 d^3r = \int_{V_k} 2\varepsilon_0\xi^2\omega_k^4 d^3r = 2\varepsilon_0\xi^2\omega_k^4 V_k = \hbar\omega_k \tag{56}$$

With the same token considering circular polarisation [4, 5, 7, 9] for the amplitudes of the electric and magnetic fields in Eq. (55), the momentum is:

$$\vec{p_k} = \int_{V_k} \varepsilon_0 \vec{\varepsilon_{k\lambda}} \times \vec{\beta_{k\lambda}} \, d^3r = \varepsilon_0 \left(\sqrt{2}\omega_k \alpha_{0k}\right)\left(\sqrt{2}\omega_k \alpha_{0k}/c\right) V_k \, \vec{k} \, / \left|\vec{k}\right| = \hbar \, \vec{k} \qquad (57)$$

According to the classical electromagnetic theory, the spin can be written through the electric and magnetic field components; hence, using again the circular polarisation, we get:

$$\left|\vec{S}\right| = \int_{V_k} \varepsilon_0 \left|\vec{r} \times \left(\vec{\varepsilon_{k\lambda}} \times \vec{\beta_{k\lambda}}\right)\right| d^3r = \varepsilon_0 (c/\omega_k)\left(\sqrt{2}\omega_k \alpha_{0k}\right)\left(\sqrt{2}\omega_k \alpha_{0k}/c\right) V_k = \hbar \qquad (58)$$

where we have taken the mean value $< \left|\vec{r}\right|>_k = c/\omega_k$ obtained for a single photon state [57].

The fact that the quantum properties, energy, momentum, and spin, of the photon can be expressed through the classical electromagnetic fields integrated over the volume $V_k$ signifies that the photon has naturally a spatial extension, and consequently when employing the term "wave-particle", one must have in mind that a single photon is a "three-dimensional particle".

We can now obtain Heisenberg's uncertainty relation for position and momentum using $V_k$. Indeed, replacing $V$ in Eq. (16) by $V_k$, we get the photon vector potential amplitude operators:

$$\tilde{\alpha}_{0k\lambda} = |\xi| \omega_k \alpha_{k\lambda} \, ; \qquad \tilde{\alpha}^*_{0k\lambda} = |\xi| \omega_k \alpha^+_{k\lambda} \qquad (59)$$

The corresponding position $\tilde{Q}_{k\lambda}$ and momentum $\tilde{P}_{k\lambda}$ Hermitian operators [19, 30, 51] write:

$$\tilde{Q}_{k\lambda} = \sqrt{\varepsilon_0 V_k}\left(\tilde{\alpha}_{0k\lambda} + \tilde{\alpha}^*_{0k\lambda}\right); \quad \tilde{P}_{k\lambda} = -i\omega_k \sqrt{\varepsilon_0 V_k}\left(\tilde{\alpha}_{0k\lambda} - \tilde{\alpha}^*_{0k\lambda}\right) \qquad (60)$$

Thus, introducing Eq. (59) in Eq. (60) and using Eq. (20) with Eq. (53), Heisenberg's commutation relation, a fundamental concept in quantum theory, results directly [2]:

$$\left[\tilde{Q}_{k\lambda}, \tilde{P}_{k'\lambda'}\right] = -i\varepsilon_0 \omega^2_{k'} \omega_k \sqrt{V_k V_{k'}}\left[\left(\xi a_{k\lambda} + \xi a^+_{k\lambda}\right), \left(\xi a_{k'\lambda'} - \xi a^+_{k'\lambda'}\right)\right] = i\hbar \delta_{kk'} \delta_{\lambda\lambda'} \qquad (61)$$

The fundamental properties of the photon, energy $E_k$, momentum $\vec{p_k}$, and wave vector $\vec{k}$, are complemented by the vector potential amplitude $\alpha_{0k}$ expressing its electromagnetic nature:

$$E_k/\hbar = \left|\vec{p_k}\right| c/\hbar = \left|\vec{k}\right| c = \alpha_{0k}/|\xi| = \omega_k \qquad (62)$$

Considering Heisenberg's *energy-time* uncertainty principle:

$$\delta E_k \, \delta t \geq \hbar \qquad (63)$$

we directly deduce from Eq. (62) the *vector potential-time* uncertainty:

$$\delta \alpha_{0k} \delta t \geq |\xi| \qquad (64)$$

The energy and vector potential uncertainties with respect to time are intrinsic physical properties of the wave-particle nature of the photon.

### 4.3 Photon wave-particle equation and wave function

Obviously, the photon vector potential function $\vec{\alpha}_{k\lambda}\left(\vec{r},t\right)$ expressed in Eq. (49) satisfies the wave propagation equation in vacuum issued from Maxwell's equations:

$$\vec{\nabla}^2\vec{\alpha}_{k\lambda}\left(\vec{r},t\right)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\alpha}_{k\lambda}\left(\vec{r},t\right)=0 \tag{65}$$

as well as the *vector potential energy* (*wave-particle*) equation for the photon [2, 54]:

$$i\begin{pmatrix}\xi\\ \hbar\end{pmatrix}\frac{\partial}{\partial t}\vec{\alpha}_{k\lambda}\left(\vec{r},t\right)=\begin{pmatrix}\tilde{\alpha}_{0k}\\ \tilde{H}\end{pmatrix}\vec{\alpha}_{k\lambda}\left(\vec{r},t\right) \tag{66}$$

where the vector potential operator $\tilde{\alpha}_{0k}=-i\xi c\,\vec{\nabla}$ and the relativistic Hamiltonian for a massless particle $\tilde{H}=-i\hbar c\,\vec{\nabla}$ have the eigenvalues $\xi\omega_k$ and $\hbar\omega_k$, respectively [2, 53].

It is worth remarking the symmetry between the pairs $\{E_k,\hbar\}$ and $\{\alpha_{0k},\xi\}$ for a single photon characterising, respectively, the particle (energy) and electromagnetic wave (vector potential) natures, having in mind that the energy corresponds to the integration of the single-mode electromagnetic field energy density over the volume $V_k$.

Now, when considering the propagation of a $k$-mode photon with wavelength $\lambda_k$, the difficulties for defining a position operator are widely commented in the literature [28, 30, 31, 39–41, 56]. At the same time, the efforts for defining a wave function for the photon based on the electric and magnetic fields were rather fruitless [58–62]. It has been emphasized several times [19, 27, 28, 31, 56] that a photon cannot be localised along the propagation axis in a length shorter than the wavelength $\lambda_k$ and within a volume smaller than roughly $\lambda_k^3$.

In fact, from a theoretical point of view, for a photon propagating in the $z$ direction, Heisenberg's uncertainty for the position $z$ and momentum $P_z=\hbar k_z=h/\lambda_k$ writes:

$$\delta z\;\;\delta P_z\geq h\quad\rightarrow\quad\delta z\;\;\delta(1/\lambda_k)\geq 1 \tag{67}$$

Notice that the momentum uncertainty along the propagation axis is expressed through the uncertainty over the inverse of the wavelength.

Considering now the vector potential function with the quantized amplitude $\alpha_{0k}=|\xi|\,\omega_k$ as a real wave function for the photon, then when a $k$-mode photon is emitted at a coordinate $z_e$ at an instant $t_e$ and propagates in vacuum along the $z$ axis, the probability $\Pi_k(z)$ to be localised at time $t$ and at the coordinate $z = z_e + c\,(t\text{-}t_e)$ corresponds to the square of the modulus of the vector potential and is consequently proportional to the square of the angular frequency [54, 55]:

$$\Pi_k(z)\propto\left|\vec{\alpha}_{k\lambda}\,(z,t)\right|^2=\xi^2\omega_k^2\propto\lambda_k^{-2} \tag{68}$$

Obviously, the shorter the wavelength of the photon, the higher the localization probability in agreement with Heisenberg's uncertainty and the experimental evidence.

### 4.4 Electromagnetic field ground state, photons, and electrons-positrons

The photon vector potential is composed of a fundamental function $\Xi_{k\lambda}$ times the angular frequency $\omega_k$ and writes in the classical (wave) and quantum (particle) formalisms:

$$\vec{\alpha}_{k\lambda} = |\xi|\,\omega_k\left[\hat{\varepsilon}_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + \hat{\varepsilon}^{*}_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] = \omega_k\vec{\Xi}_{k\lambda}\left(\omega_k,\vec{r},t\right) \tag{69}$$

$$\tilde{\alpha}_{k\lambda} = |\xi|\,\omega_k\left[a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + a^{+}_{k\lambda}\hat{\varepsilon}^{*}_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] = \omega_k\tilde{\Xi}_{k\lambda}\left(a_{k\lambda},a^{+}_{k\lambda}\right) \tag{70}$$

In this way, the general equation for the vector potential of the electromagnetic wave considered as a superposition of plane wave modes writes:

$$\vec{A}\left(\vec{r},t\right) = \sum_{k,\,\lambda}|\xi|\,\omega_k\left[\hat{\varepsilon}_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + \hat{\varepsilon}^{*}_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] = \sum_{k,\,\lambda}\omega_k\vec{\Xi}_{k\lambda}\left(\omega_k,\vec{r},t\right) \tag{71}$$

and that of a large number of cavity-free photons in quantum electrodynamics is:

$$\tilde{A} = \sum_{k,\,\lambda}|\xi|\,\omega_k\left[a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)} + a^{+}_{k\lambda}\hat{\varepsilon}^{*}_{k\lambda}e^{-i\left(\vec{k}\cdot\vec{r}-\omega_k t+\phi\right)}\right] = \sum_{k,\,\lambda}\omega_k\tilde{\Xi}_{k\lambda}\left(a_{k\lambda},a^{+}_{k\lambda}\right) \tag{72}$$

According to Eqs. (55) and (62), for $\omega_k \to 0$ all the physical properties of the photon vanish entailing that the photon exists only for a non-zero frequency of the vector potential oscillation. However, the zero-frequency level does not correspond to perfect inexistence because the fundamental field $\Xi_{k\lambda}$ does not vanish for $\omega_k = 0$ but reduces to $\Xi^{0}_{k\lambda}$ involving the amplitude $\xi$ and the general expression of the polarisation vectors $\hat{\varepsilon}_{k\lambda}$ [63, 64] and writes in the classical and quantum representations:

$$\vec{\Xi}^{0}_{k\lambda} = |\xi|\left[\hat{\varepsilon}_{k\lambda}e^{i\phi} + \hat{\varepsilon}^{*}_{k\lambda}e^{-i\phi}\right] \quad ; \quad \tilde{\Xi}^{0}_{k\lambda} = |\xi|\left[a_{k\lambda}\hat{\varepsilon}_{k\lambda}e^{i\phi} + a^{+}_{k\lambda}\hat{\varepsilon}^{*}_{k\lambda}e^{-i\phi}\right] \tag{73}$$

The field $\Xi^{0}_{k\lambda}$ is the *electromagnetic field ground state* (EFGS) permeating all the space ($\lambda_k \to \infty$) and having zero energy and zero vector potential as well as zero electric and magnetic fields. This physical state lies beyond the Bohm-Aharonov situation in which the energy and the electric and magnetic fields are zero but a vector potential is present in space [43]. Thus, in complete absence of energy and vector potential, the field $\Xi^{0}_{k\lambda}$ can be assimilated to a quantum vacuum component constituting the main "skeleton" of any photon which now clearly appears to be a vacuum oscillation [2, 63, 64].

Combination of the expression $|\xi| = \hbar/4\pi|e|c$ to the fine structure constant definition $\alpha = e^2/4\pi\varepsilon_0\hbar c$ permits to draw directly the electron-positron elementary charge $e = \pm 1.602 \quad 10^{-19} C$, a fundamental physical constant, which now is expressed exactly through the EFGS amplitude $\xi$ [64, 65]:

$$e = \pm(4\pi)^2 \alpha \frac{|\xi|}{\mu_0} \tag{74}$$

Using again Eq. (51) and recalling that the electron mass may be written as $m_e = e\hbar/2\mu_B$, where $\mu_B$ is the Bohr magneton, we deduce that the electron mass is also expressed as a function of the EFGS amplitude $\xi$ [64]:

$$m_e = 2\pi c\, e^2 \frac{|\xi|}{\mu_B} \tag{75}$$

entailing that the mass derives also from the EFGS and is proportional to the charge square.

Equations (50), (74), and (75) show the strong physical relationship between photons and electrons-positrons which are all related directly to the EFGS through the amplitude $\xi$. Obviously, photons and electrons-positrons, also probably leptons-antileptons, are issued from the same quantum vacuum field. This may be at the origin of the physical mechanism governing the photon generation during the electron-positron (and probably lepton-antilepton) annihilation and that of the electron-positron (lepton-antilepton) pair creation during the annihilation of high-energy gamma photons in the vicinity of very heavy nucleus.

## 5. Conclusion

In this chapter we have presented recent theoretical developments complementing the standard formalism with the purpose of describing a single photon state in conformity with the experiments. We resume below the principal features.

The quantization of the vector potential amplitude $\alpha_{0k}$, a real physical entity, for a single free of cavity $k$-mode photon with angular frequency $\omega_k$ is expressed by $\alpha_{0k} = |\xi|\,\omega_k$, where $|\xi| = \hbar/4\pi|e|c$, and leads to the establishment of a *vector potential - energy* (electromagnetic wave-particle) formalism (Eq. (66)) expressing the simultaneous wave-particle nature of the photon. A single photon state is a local indivisible entity of the electromagnetic field extending over a wavelength $\lambda_k$ and consisting of the quantized vector potential oscillating at the angular frequency $\omega_k$, with circular polarisation, giving birth to orthogonally oscillating electric and magnetic fields whose amplitudes are proportional to the square of the angular frequency $|\xi|\,\omega_k^2$ (Eq. (55)). Its lateral expansion, confirmed experimentally, yields a minimum photon volume $V_k$ which is proportional to $\lambda_k^3$. The quantum properties of the photon, energy, momentum, and spin are obtained directly from the classical electromagnetic expressions integrated over the volume $V_k$ (Eqs. (56)–(58)). It is also shown (Eq. (61)) that the Heisenberg uncertainty can be readily obtained through the use of the volume $V_k$.

A single photon, as a local three-dimensional entity of the electromagnetic field, is absorbed and emitted as a whole and propagates guided by the non-local vector potential function (Eq. (49)), which appears to be a natural wave function for the photon satisfying both the propagation equation (Eq. (65)) and the *vector potential - energy* equation (Eq. (66)). The probability for detecting a photon around

a given point on the propagation axis is obtained by the square modulus of the vector potential and is proportional to the square of the angular frequency $\xi^2\omega_k^2$ (Eq. (68)) which signifies that the higher the frequency, the better the localization, in agreement with the experiments.

Finally, the electromagnetic field ground state (EFGS) at zero frequency, a real quantum vacuum component, issues naturally from the vector potential wave function putting in evidence that photons are oscillations of the vacuum field. Furthermore, the electron-positron charge and mass are directly proportional to the vector potential amplitude quantization constant showing the strong physical relationship with the photons. Obviously, the origin of the mechanisms governing the transformations of photons to electrons-positrons and inversely lies in the nature of the electromagnetic field ground state.

## Author details

Constantin Meis[1,2]

1 National Institute for Nuclear Science and Technology, CEA - Saclay, France

2 Department of Physics, University of Versailles, UVSQ, France

*Address all correspondence to: constantin.meis@cea.fr

IntechOpen

© 2019 The Author(s). Licensee IntechOpen. This chapter is distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. CC BY

## References

[1] Ronchi V. The Nature of Light: An Historical Survey. Cambridge (MA) USA: Harvard University Press; 1970

[2] Meis C. Light and Vacuum. 2nd ed. Singapore: World Scientific; 2017

[3] Newton I. Opticks. New York: Dover; 1952

[4] Saleh BEA, Teich MC. Fundamentals of Photonics. New York: John Wiley & Sons; 2007

[5] Read FH. Electromagnetic Radiation. New York: John Wiley & Sons; 1980

[6] Born M, Wolf E. Principles of Optics. Cambridge (MA) USA: Harvard University Press; 1999

[7] Jackson JD. Classical Electrodynamics. New York: John Wiley & Sons; 1998

[8] Planck M. The Theory of Heat Radiation. New York: Dover; 1959

[9] Bransden BH, Joachain CJ. Physics of Atoms and Molecules. 2nd ed. London: Longman; 2003

[10] Mickelson AA et al. Conference on Michelson-Morley Experiment. Astrophysical Journal. 1928;**68**:341

[11] Hertz H. Ueber sehr schnelle electrische Schwingungen. Annalen der Physik. 1887;**33**:983

[12] Einstein A. In: Stachel J, Cassidy DC, Renn J, Schulmann R, editors. The collected papers of Albert Einstein. New Jersey: Princeton University Press; 1987

[13] Bohr N, Kramers HA, Slater JC. The quantum theory of radiation. Philosophical Magazine. 1924;**47**:785

[14] Taylor GI. Interference fringes with feeble light. Proceedings of the Cambridge Philosophical Society. 1909; **15**:114

[15] Compton AH. A Quantum Theory of the Scattering of X-rays by Light Elements. Physical Review. 1923;**21**:483

[16] Wentzel G. Zur Theorie des photoelektrischen Effekts. Zeitschrift für Physik. 1926;**40**:574

[17] Beck G. Zur Theorie des photoeffekts. Zeitschrift für Physik. 1927;**41**:443

[18] Lamb WE Jr, Scully MO. Polarization, Matière et Rayonnement, Volume Jubilaire en L'honneur d'Alfred Kastler. Paris: French Physical Society, Press Universitaires de France; 1969

[19] Garrison JC, Chiao RY. Quantum Optics. New York: Oxford University Press; 2008

[20] Klein O, Nishina Y. über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen. Zeitschrift für Physik. 1929;**52**:853

[21] Jin F, Yuan S, De Raedt H, Michielsen K, Mayiashita S. Corpuscular Model of Two-Beam Interference and Double-Slit Experiments with Single Photons. Journal of the Physical Society of Japan. 2010;**79**(7):074401

[22] Feynman R. The Strange Theory of Light and Matter. New Jersey: Princeton University Press; 1988

[23] Schawlow A, Townes C. Infrared and optical masers. Physical Review. 1958;**112**(6):1940-1949

[24] Klyshko DN, Penin AN, Polkovnikov BF. Parametric luminescence and light scattering by polaritons. JETP Letters. 1970;**11**:05

[25] Burnham DC, Weinberg DL. Observation of simultaneity in parametric production of optical photon pairs. Physical Review Letters. 1970; **25**(2):84

[26] Ghose P, Home D. The two-prism experiment and wave-particle duality of light. Foundations of Physics. 1996;**26** (7):943

[27] Akhiezer AI, Berestetskii BV. Quantum Electrodynamics. New York: Interscience; 1965

[28] Mandel L. Configuration-Space Photon Number Operators in Quantum Optics. Physical Review. 1966;**144**:1071

[29] Pfleegor RL, Mandel L. Interference of Independent Photon Beams. Physical Review. 1967;**159**(5):1084

[30] Grynberg G, Aspect A, Fabre C. Quantum Optics. New York: Cambridge University Press; 2010

[31] Auletta G. Foundations and Interpretation of Quantum Mechanics. Singapore: World Scientific; 2001

[32] Grangier P, Roger G, Aspect A. Experimental evidence for a photon anticorrelation effect on a beam splitter: A new light on single-photon interferences. Europhysics Letters. 1986;**1**(4):173

[33] Robinson HL. Diffraction patterns in circular apertures less than one wavelength. Journal of Applied Physics. 1953;**24**(1):35

[34] Hadlock RK. Diffraction Patterns at the Plane of a Slit in a Reflecting Screen. Journal of Applied Physics. 1958;**29**:918

[35] Hunter G, Wadlinger RLP. Photons and Neutrinos as Electromagnetic Solitons. Physics Essays. 1989;**2**:158

[36] Hunter G. Quantum uncertainties. NATO ASI Series B. In: Proceedings of NATO Advanced Research Workshop; June 23–27, 1986. p. 331

[37] Heitler W. The Quantum Theory of Radiation. Oxford: Clarendon Press; 1954

[38] Ryder LH. Quantum field theory. Cambridge: Cambridge University Press; 1987

[39] Milonni PW. The Quantum Vacuum. London: Academic Press Inc.; 1994

[40] Mittleman MH. Introduction to the Theory of Laser-Atom Interactions. New York: Plenum Press; 1982

[41] Weissbluth M. Photon-Atom Interactions. London: Academic Press Inc.; 1988

[42] Ehrenberg W, Siday RE. The refractive index in electron optics. Proceedings of the Physical Society. Section B. 1949;**62**:8-2

[43] Aharonov Y, Bohm D. Significance of electromagnetic potentials in the quantum theory. Physics Review. 1959; **115**:485-491

[44] Chambers RG. Shift of an electron interference pattern by enclosed magnetic flux. Physical Review Letters. 1960;**5**:3-5

[45] Tonomura A et al. Evidence for aharonov-bohm effect with magnetic field completely shielded from electron wave. Physical Review Letters. 1986;**56**: 792

[46] Osakabe N et al. Experimental confirmation of Aharonov-Bohm effect using a toroidal magnetic field confined by a superconductor. Physical Review A. 1986;**34**:815

[47] Welton TA. Physics Review. 1948; **74**:1157

[48] Milonni PW. Physical Review A. 1982;**25**:1315

[49] Schwinger J, DeRaad LL Jr, Milton KA. Annals of Physics. 1978;**115**:1

[50] Ezawa H, Nakamura K, Watanabe K. The Casimir force from Lorentz's. Frontiers in Quantum Physics. In: Lim SC, Abd-Shukor R, Kwek KH, editors. Proceedings of the 2nd International Conference; Kuala Lumpur, Malaysia; July 9–11, 1997. Springer; 1998. pp. 160-169

[51] Tannoudji CC, Diu B, Laloë F. Mécanique Quantique. Vol. III. Paris: EDP Sciences; 2017

[52] Walther H, Varcoe BTH, Englert B-G, Becker T. Reports on Progress in Physics. 2006;**69**(5):1325

[53] Meis C. Photon wave-particle duality and virtual electromagnetic waves. Foundations of Physics. 1997;**27**: 865

[54] Meis C, Dahoo PR. Vector potential quantization and the photon intrinsic electromagnetic properties: Towards nondestructive photon detection. International Journal of Quantum Information. 2017;**15**(8):1740003

[55] Meis C, Dahoo PR. Vector potential quantization and the photon wave function. Journal of Physics: Conference Series. 2017;**936**:012004

[56] Jeffers S, Roy S, Vigier JP, Hunter G. The Present Status of the Quantum Theory of Light. Boston: Kluwer Academic Publishers; 1997

[57] Rybakov YP, Saha B. Soliton model of atom. Foundations of Physics. 1995; **25**(12):1723-1731

[58] Bialynicki-Birula I. Photon wave function. In: Wolf E, editor. Progress in Optics XXXVI. Amsterdam: Elsevier; 1996

[59] Chandrasekar N. Quantum mechanics of photons. Advanced Studies in Theoretical Physics. 2012;**6** (8):391-397

[60] Sipe JE. Photon wave functions. Physical Review A. 1995;**52**:1875

[61] Smith BJ, Raymer MG. Photon wave functions, wave packet quantization of light and coherence theory. New Journal of Physics. 2007;**9**:414

[62] Khokhlov DL. Spatial and temporal wave functions of photons. Applied Physics Research. 2010;**2**(2):50-54

[63] Meis C. Vector potential quantization and the quantum vacuum. Physics Research International. 2014; **2014**:187432

[64] Meis C. The electromagnetic field ground state and the cosmological evolution. Journal of Physics: Conference Series. 2018;**1141**:012072

[65] Meis C, Dahoo PR. The single photon state, the quantum vacuum and the elementary electron-positron charge. American Institute of Physics: Conference Proceedings. 2018;**2040**: 020011