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# Contraction Mappings and Applications

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## Abstract

The aim of the chapter is to find the existence results for the solution of non-homogeneous Fredholm integral equations of the second kind and non-linear matrix equations by using the fixed point theorems. Here, we derive fixed point theorems for two different type of contractions. Firstly, we utilize the concept of manageable functions to define multivalued  $\alpha_*$  –  $\eta_*$  manageable contractions and prove fixed point theorems for such contractions. After that, we use these fixed point results to find the solution of non-homogeneous Fredholm integral equations of the second kind. Secondly, we introduce weak  $\mathcal{F}$  contractions named as  $\alpha$ - $\mathcal{F}$ -weak-contraction to prove fixed point results in the setting of metric spaces and by using these results we find the solution for non-linear matrix equations.

**Keywords:** contraction mapping, fixed point, integral equations, matrix equations, manageable function

## 1. Introduction

Let  $H(n)$  denote the set of all  $n \times n$  Hermitian matrices,  $P(n)$  the set of all  $n \times n$  Hermitian positive definite matrices,  $S(n)$  the set of all  $n \times n$  positive semidefinite matrices. Instead of  $X \in P(n)$  we will write  $X > 0$ . Furthermore,  $X \geq 0$  means  $X \in S(n)$ . Also we will use  $X \geq Y$  ( $X \leq Y$ ) instead of  $X - Y \geq 0$  ( $Y - X \geq 0$ ). The symbol  $\|\cdot\|$  denotes the spectral norm, that is,

$$\|A\| = \sqrt{\lambda^+(A^*A)},$$

where  $\lambda^+(A^*A)$  is the largest eigenvalue of  $A^*A$ . We denote by  $\|\cdot\|_1$  the Ky Fan norm defined by

$$\|A\|_1 = \sum_{i=1}^n s_i(A),$$

where  $s_i(A)$ ,  $i = 1, \dots, n$ , are the singular values of  $A$ . Also,

$$\|A\|_1 = \text{tr}\left((A^*A)^{1/2}\right),$$

which is  $\text{tr}(A)$  for (Hermitian) nonnegative matrices. Then the set  $H(n)$  endowed with this norm is a complete metric space. Moreover,  $H(n)$  is a partially

ordered set with partial order  $\leq$ , where  $X \leq Y \Leftrightarrow Y \leq X$ . In this section, denote  $d(X, Y) = \|Y - X\|_1 = \text{tr}(Y - X)$ . Now, consider the non-linear matrix equation

$$X = Q + \sum_{i=1}^m A_i^* \gamma(X) A_i, \quad (1)$$

where  $Q$  is a positive definite matrix,  $A_i, i = 1, \dots, m$ , are arbitrary  $n \times n$  matrices and  $\gamma$  is a mapping from  $H(n)$  to  $H(n)$  which maps  $P(n)$  into  $P(n)$ . Assume that  $\gamma$  is an order-preserving mapping ( $\gamma$  is order preserving if  $A, B \in H(n)$  with  $A \leq B$  implies that  $\gamma(A) \leq \gamma(B)$ ). There are various kinds of problems in control theory, dynamical programming, ladder networks, etc., where the matrix equations plays a crucial role. Matrix Eq. (1) have been studied by many authors see [1–3].

At the same time, integral equations have been developed to solve boundary value problems for both ordinary and partial differential equations and play a very important role in nonlinear analysis. Many problems of mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of mechanics of continuous media reduce to the Fredholm integral Eq. A rich literature on existence of solutions for nonlinear integral equations, which contain particular cases of important integral and functional equations can be found, for example, see [4–14]. An important technique to solve integral equations is to construct an iterative procedure to generate approximate solutions and find their limit, a host of attractive methods have been proposed for the approximate solutions of Fredholm integral equations of the second kind, see [15–19]. We consider a non-homogeneous Fredholm integral equation of second kind of the form

$$z(r) = \int_b^c \mathcal{B}(r, s, z(s)) ds + g(r), \quad (2)$$

where  $t \in [b, c]$ ,  $\mathcal{B} : [b, c] \times [b, c] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

An advancement in this direction is to find the solution of such mathematical models by using fixed point theorems. In this technique, we generate a sequence by iterative procedure for some self-map  $T$  and then look for a fixed point of  $T$ , that is actually the solution of given mathematical model. The simplest case is when  $T$  is a contraction mapping, that is a self-mapping satisfying

$$d(Tx, Ty) \leq kd(x, y),$$

where  $k \in [0, 1)$ . The contraction mapping principle [20] guarantees that a contraction mapping of a complete metric space to itself has a unique fixed point which may be obtained as the limit of an iteration scheme defined by repeated images under the mapping of an arbitrary starting point in the space. The multivalued version of contraction mapping principle can be found in [21]. In general, fixed point theorems allow us to obtain existence theorems concerning investigated functional-operator equations.

In this chapter, we prove the existence of solution for matrix Eq. (1) and integral Eq. (2) by using newly developed fixed point theorems.

## 2. Background material from fixed point theory

Let  $\mathcal{X}$  be a set of points, a distance function on  $\mathcal{X}$  is a map  $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  that is symmetric, and satisfies  $d(i, i) = 0$  for all  $i \in \mathcal{X}$ . The distance is said to be a metric if the triangle inequality holds, i.e.,

$$d(i, j) \leq d(i, k) + d(k, j),$$

for all  $i, j, k \in \mathcal{X}$  and  $(\mathcal{X}, d)$  is called metric space.

Denote by  $2^{\mathcal{X}}$ , the family of all nonempty subsets of  $\mathcal{X}$ ,  $CL(\mathcal{X})$ , the family of all nonempty and closed subsets of  $\mathcal{X}$ ,  $CB(\mathcal{X})$ , the family of all nonempty, closed, and bounded subsets of  $\mathcal{X}$  and  $K(\mathcal{X})$ , the family of all nonempty compact subsets of  $\mathcal{X}$ . It is clear that,  $K(\mathcal{X}) \subseteq CB(\mathcal{X}) \subseteq CL(\mathcal{X}) \subseteq 2^{\mathcal{X}}$ , let

$$H(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{x \in \mathcal{A}} D(x, \mathcal{B}), \sup_{y \in \mathcal{B}} D(y, \mathcal{A}) \right\},$$

where  $D(x, \mathcal{B}) = \inf \{d(x, y) : y \in \mathcal{B}\}$ . Then  $H$  is called generalized Pompeiu Hausdorff distance on  $CL(\mathcal{X})$ . It is well known that  $H$  is a metric on  $CB(\mathcal{X})$ , which is called Pompeiu Hausdorff metric induced by  $d$ .

If  $T : \mathcal{X} \rightarrow \mathcal{X}$  is a single valued self-mapping on  $\mathcal{X}$ , then  $T$  is said to have a fixed point  $x$  if  $Tx = x$  and if  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  is multivalued mapping, then  $T$  is said to have a fixed point  $x$  if  $x \in Tx$ . We denote by  $Fix\{T\}$ , the set of all fixed points of mapping  $T$ .

**Definition 2.1** [22] Let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a multivalued map on a metric space  $(\mathcal{X}, d)$ ,  $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$  be two functions where  $\eta$  is bounded, then  $T$  is an  $\alpha_*$ -admissible mapping with respect to  $\eta$ , if

$$\alpha(y, z) \geq \eta(y, z) \text{ implies that } \alpha_*(Ty, Tz) \geq \eta_*(Ty, Tz), \quad y, z \in \mathcal{X},$$

where

$$\alpha_*(\mathcal{A}, \mathcal{B}) = \inf_{y \in \mathcal{A}, z \in \mathcal{B}} \alpha(y, z), \quad \eta_*(\mathcal{A}, \mathcal{B}) = \sup_{y \in \mathcal{A}, z \in \mathcal{B}} \eta(y, z).$$

Further, Definition 2.1 is generalized in the following way.

**Definition 2.2** [23] Let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a multivalued map on a metric space  $(\mathcal{X}, d)$ ,  $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  be two functions. We say that  $T$  is generalized  $\alpha_*$ -admissible mapping with respect to  $\eta$ , if

$$\alpha(y, z) \geq \eta(y, z) \text{ implies that } \alpha(u, v) \geq \eta(u, v), \quad \text{for all } u \in Ty, v \in Tz.$$

If  $\eta(y, z) = 1$  for all  $y, z \in \mathcal{X}$ , then  $T$  is said to be generalized  $\alpha_*$ -admissible mapping.

### 3. Some fixed point results

Consistent with Du and Khojasteh [24], we denote by  $Man(\widehat{\mathbb{R}})$ , the set of all manageable functions  $\vartheta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  fulfilling the following conditions:

$$(\vartheta_1) \quad \vartheta(t, s) < s - t \text{ for all } s, t > 0;$$

( $\vartheta_2$ ) for any bounded sequence  $\{t_n\} \subset (0, +\infty)$  and any nondecreasing sequence  $\{s_n\} \subset (0, +\infty)$ , it holds that

$$\limsup_{n \rightarrow \infty} \frac{t_n + \vartheta(t_n, s_n)}{s_n} < 1. \quad (3)$$

**Example 3.1** [24] Let  $r \in [0, 1)$ . Then  $\vartheta_r : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\vartheta_r(t, s) = rs - t$  is a manageable function.

**Example 3.2** Let  $\vartheta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\vartheta(t, s) = \begin{cases} \psi(s) - t & \text{if } (t, s) \in [0, +\infty) \times [0, +\infty), \\ f(t, s) & \text{otherwise,} \end{cases}$$

where  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  satisfying  $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$  for all  $t > 0$  and  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is any function. Then  $\vartheta(t, s) \in \widehat{Man}(\mathbb{R})$ . Indeed, by using Lemma 1 of [25], we have for any  $s, t > 0$ ,  $\vartheta(t, s) = \psi(s) - t < s - t$ , so,  $(\vartheta_1)$  holds. Let  $\{t_n\} \subset (0, +\infty)$  be a bounded sequence and let  $\{s_n\} \subset (0, +\infty)$  be a nonincreasing sequence. Then  $\lim_{n \rightarrow \infty} s_n = \inf_{n \in \mathbb{N}} s_n = a$  for some  $a \in [0, +\infty)$ , we get

$$\limsup_{n \rightarrow \infty} \frac{t_n + \vartheta(t_n, s_n)}{s_n} = \limsup_{n \rightarrow \infty} \frac{\psi(s_n)}{(s_n)} < \lim_{n \rightarrow \infty} \frac{(s_n)}{(s_n)} = 1,$$

so,  $(\vartheta_2)$  is also satisfied.

**Definition 3.3** Let  $(\mathcal{X}, d)$  be a metric space and  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a closed valued mapping. Let  $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$  be two functions and  $\vartheta \in \widehat{Man}(\mathbb{R})$ . Then  $T$  is called a multivalued  $\alpha_*$ - $\eta_*$ -manageable contraction with respect to  $\vartheta$  if for all  $y, z \in \mathcal{X}$

$$\alpha_*(Ty, Tz) \geq \eta_*(Ty, Tz) \quad \text{implies} \quad \vartheta(H(Ty, Tz), d(y, z)) \geq 0. \quad (4)$$

Now we prove first result of this section.

**Theorem 3.4** Let  $(\mathcal{X}, d)$  be a complete metric space and let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a closed valued map satisfying following conditions:

1.  $T$  is  $\alpha_*$ -admissible map with respect to  $\eta$ ;
2.  $T$  is  $\alpha_* - \eta_*$  manageable contraction with respect to  $\vartheta$ ;
3. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in Tz_0$  such that  $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$ ;
4. for a sequence  $\{z_n\} \subset \mathcal{X}$ ,  $\lim_{n \rightarrow \infty} \{z_n\} = x$  and  $\alpha(z_n, z_{n+1}) \geq \eta(z_n, z_{n+1})$  for all  $n \in \mathbb{N}$ , implies  $\alpha(z_n, x) \geq \eta(z_n, x)$  for all  $n \in \mathbb{N}$ .

Then  $Fix\{T\} \neq \emptyset$ .

*Proof.* Let  $z_1 \in Tz_0$  be such that  $\alpha(z_0, z_1) \geq \eta(z_0, z_1)$ . Since  $T$  is  $\alpha_*$ -admissible map with respect to  $\eta$ , then  $\alpha_*(Tz_0, Tz_1) \geq \eta_*(Tz_0, Tz_1)$ . Therefore, from (4) we have

$$\vartheta(H(Tz_0, Tz_1), d(z_0, z_1)) \geq 0. \quad (5)$$

If  $z_1 = z_0$ , then  $z_0 \in Fix\{T\}$ , also if  $z_1 \in Tz_1$ , then  $z_1 \in Fix\{T\}$ . So, we adopt that  $z_0 \neq z_1$  and  $z_1 \notin Tz_1$ . Thus  $0 < d(z_1, Tz_1) \leq H(Tz_0, Tz_1)$ . Define  $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$\lambda(t, s) = \begin{cases} \frac{t + \vartheta(t, s)}{s} & \text{if } t, s > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

By  $(\vartheta_1)$ , we know that

$$0 < \lambda(t, s) < 1 \quad \text{for all } t, s > 0. \quad (7)$$

Also note that if  $\vartheta(t, s) \geq 0$ , then

$$0 < t \leq s\lambda(t, s). \quad (8)$$

So, from (5) and (7), we get

$$0 < \lambda(H(Tz_0, Tz_1), d(z_0, z_1)) < 1. \quad (9)$$

Let

$$\varepsilon_1 = \left( \frac{1}{\sqrt{\lambda(H(Tz_0, Tz_1), d(z_0, z_1))}} - 1 \right) d(z_1, Tz_1). \quad (10)$$

Since  $d(z_1, Tz_1) > 0$ . So, by using (9), we get  $\varepsilon_1 > 0$  and

$$\begin{aligned} d(z_1, Tz_1) &< d(z_1, Tz_1) + \varepsilon_1 \\ &= \left( \frac{1}{\sqrt{\lambda(H(Tz_0, Tz_1), d(z_0, z_1))}} \right) d(z_1, Tz_1). \end{aligned} \quad (11)$$

This implies that there exists  $z_2 \in Tz_1$  such that

$$d(z_1, z_2) < \left( \frac{1}{\sqrt{\lambda(H(Tz_0, Tz_1), d(z_0, z_1))}} \right) d(z_1, Tz_1). \quad (12)$$

By induction, we form a sequence  $\{z_n\}$  in  $\mathcal{X}$  satisfying for each  $n \in \mathbb{N}$ ,  $z_n \in Tz_{n-1}$ ,  $z_n \neq z_{n-1}$ ,  $z_n \notin Tz_n$ ,  $\alpha_*(z_{n-1}, z_n) \geq \eta_*(z_{n-1}, z_n)$ ,

$$0 < d(z_n, Tz_n) \leq H(Tz_{n-1}, Tz_n), \quad (13)$$

$$\vartheta(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n)) \geq 0, \quad (14)$$

and

$$d(z_n, z_{n+1}) < \left( \frac{1}{\sqrt{\lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n))}} \right) d(z_n, Tz_n), \quad (15)$$

by taking

$$\varepsilon_n = \left( \frac{1}{\sqrt{\lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n))}} - 1 \right) d(z_n, Tz_n). \quad (16)$$

By using (7), (8), (13), and (15), we get for each  $n \in \mathbb{N}$

$$d(z_n, Tz_n) \leq d(z_{n-1}, z_n) \lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n)) \leq d(z_{n-1}, z_n), \quad (17)$$

this implies that  $\{d(z_n, Tz_n)\}_{n \in \mathbb{N}}$  is a bounded sequence. By combining (15) and (17), for each  $n \in \mathbb{N}$ , we get

$$d(z_n, z_{n+1}) < \left( \sqrt{\lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n))} \right) d(z_{n-1}, z_n). \quad (18)$$

Which means that  $\{d(z_{n-1}, z_n)\}_{n \in \mathbb{N}}$  is a monotonically decreasing sequence of non-negative reals and so it must be convergent. So, let  $\lim_{n \rightarrow \infty} d(z_n, z_{n+1}) = c \geq 0$ . From  $(\vartheta_2)$ , we get



$$\limsup_{n \rightarrow \infty} \lambda(H(Tz_n, Tz_n), d(z_{n-1}, z_n)) < 1. \quad (19)$$

Now, if  $c > 0$ , then by taking the  $\lim_{n \rightarrow \infty} \sup$  in (18) and using (19), we have

$$c \leq \sqrt{\limsup_{n \rightarrow \infty} \lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n))} c < c. \quad (20)$$

This contradiction shows that  $c = 0$ . Hence,  $\lim_{n \rightarrow \infty} d(z_n, z_{n+1}) = 0$ . Next, we prove that  $\{z_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{X}$ . Let, for each  $n \in \mathbb{N}$ ,

$$\sigma_n = \sqrt{\lambda(H(Tz_{n-1}, Tz_n), d(z_{n-1}, z_n))}, \quad (21)$$

then from Eq. (9), we have  $\sigma_n \in (0, 1)$ . By (18), we obtain

$$d(z_n, z_{n+1}) < \sigma_n d(z_{n-1}, z_n). \quad (22)$$

(19) implies that  $\lim_{n \rightarrow \infty} \sigma_n < 1$ , so there exists  $\gamma \in [0, 1)$  and  $n_0 \in \mathbb{N}$ , such that

$$\sigma_n \leq \gamma \quad \text{for all } n \in \mathbb{N}, n \geq n_0. \quad (23)$$

For any  $n \geq n_0$ , since  $\sigma_n \in (0, 1)$  for all  $n \in \mathbb{N}$  and  $\gamma \in [0, 1)$ , (22, 23) implies that

$$d(z_n, z_{n+1}) < \sigma_n d(z_{n-1}, z_n) < \sigma_n \sigma_{n-1} d(z_{n-2}, z_{n-1}) \cdots \leq \gamma^{n-n_0+1} d(z_0, z_1). \quad (24)$$

Put  $\beta_n = \left(\frac{\gamma^{n-n_0+1}}{1-\gamma}\right) d(z_0, z_1)$ ,  $n \in \mathbb{N}$ . For  $m, n \in \mathbb{N}$  with  $m > n \geq n_0$ , we have from (24) that

$$d(z_n, z_m) \leq \sum_{j=n}^{m-1} d(z_j, z_{j+1}) < \beta_n. \quad (25)$$

Since  $\gamma \in [0, 1)$ ,  $\lim_{n \rightarrow \infty} \beta_n = 0$ . Hence  $\lim_{n \rightarrow \infty} \sup\{d(z_n, z_m) : m > n\} = 0$ . This shows that  $\{z_n\}$  is a Cauchy sequence in  $\mathcal{X}$ . Completeness of  $\mathcal{X}$  ensures the existence of  $z \in \mathcal{X}$  such that  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . Now, since  $\alpha(z_n, z) \geq \eta(z_n, z)$  for all  $n \in \mathbb{N}$ ,  $\alpha_*(Tz_n, Tz) \geq \eta_*(Tz_n, Tz)$ , and so from (4), we have  $\vartheta(H(Tz_n, Tz), d(z_n, z)) \geq 0$ . Then from (7, 8), we have

$$H(Tz_n, Tz) \leq \lambda(H(Tz_n, Tz), d(z_n, z)) d(z_n, z) < d(z_n, z). \quad (26)$$

Since  $0 < d(z, Tz) \leq H(Tz_n, Tz) + d(z_n, z)$ , so by using (26), we get

$$0 < d(z, Tz) < 2d(z_n, z). \quad (27)$$

Letting limit  $n \rightarrow \infty$  in above inequality, we get  $d(z, Tz) = 0$ . Hence  $z \in \text{Fix}\{T\}$ . □

Let  $\Delta(F)$  be the set of all functions  $\mathcal{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$  satisfying following conditions:

( $\mathcal{F}_1$ )  $\mathcal{F}$  is strictly increasing;

( $\mathcal{F}_2$ ) for all sequence  $\{\alpha_n\} \subseteq \mathbb{R}^+$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$  if and only if

$\lim_{n \rightarrow \infty} \mathcal{F}(\alpha_n) = -\infty$ ;

( $\mathcal{F}_3$ ) there exist  $0 < k < 1$  such that  $\lim_{n \rightarrow 0^+} \alpha^k \mathcal{F}(\alpha) = 0$ ,

$\Delta(F_*)$ , if  $\mathcal{F}$  also satisfies the following:

( $\mathcal{F}_4$ )  $\mathcal{F}(\inf A) = \inf \mathcal{F}(A)$  for all  $A \subset (0, \infty)$  with  $\inf A > 0$ ,

**Definition 3.5** [27] Let  $(\mathcal{X}, d)$  be a metric space. A mapping  $T : \mathcal{X} \rightarrow \mathcal{X}$  is said to be  $\mathcal{F}$ -contraction if there exists  $\tau > 0$  such that

$$d(Tx, Ty) > 0 \text{ implies } \tau + \mathcal{F}(d(Tx, Ty)) \leq \mathcal{F}(d(x, y)).$$

**Theorem 3.6** [26] Let  $(\mathcal{X}, d)$  be a complete metric space and let  $T : \mathcal{X} \rightarrow \mathcal{X}$  be an  $\mathcal{F}$ -contraction. Then  $T$  has a unique fixed point  $x^* \in \mathcal{X}$  and for every  $x_0 \in \mathcal{X}$  a sequence  $T^n x_0, n \in \mathbb{N}$  is convergent to  $x^*$ .

**Definition 3.7** ([27]). Let  $(\mathcal{X}, d)$  be a metric space and  $T : \mathcal{X} \rightarrow CB(\mathcal{X})$  be a mapping. Then  $T$  is a multivalued  $\mathcal{F}$ -contraction, if  $\mathcal{F} \in \Delta(F)$  and there exists  $\tau > 0$  such that for all  $x, y \in \mathcal{X}$ ,

$$H(Tx, Ty) > 0 \Rightarrow \tau + \mathcal{F}(H(Tx, Ty)) \leq \mathcal{F}(d(x, y)).$$

**Theorem 3.8** ([27]). Let  $(\mathcal{X}, d)$  be a complete metric space and  $T : \mathcal{X} \rightarrow K(\mathcal{X})$  be a multivalued  $\mathcal{F}$ -contraction, then  $T$  has a fixed point in  $\mathcal{X}$ .

**Theorem 3.9** ([27]). Let  $(\mathcal{X}, d)$  be a complete metric space and  $T : \mathcal{X} \rightarrow C(\mathcal{X})$  be a multivalued  $\mathcal{F}$ -contraction. Suppose  $\mathcal{F} \in \Delta(F_*)$ , then  $T$  has a fixed point in  $\mathcal{X}$ .

For more in this direction, see, [28–31]. Here, we give the concept of multivalued  $\alpha$ - $\mathcal{F}$ -weak-contractions and prove some fixed point results.

**Definition 3.10** Let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a multivalued mapping on a metric space  $(\mathcal{X}, d)$ , then  $T$  is said to be an multivalued  $\alpha$ - $\mathcal{F}$ -weak-contraction on  $\mathcal{X}$ , if there exists  $\sigma > 0$ ,  $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ ,  $\mathcal{F} \in \Delta(F)$  and  $\alpha : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$  such that for all  $z \in \mathcal{X}$ ,  $y \in F_\sigma^z$  with  $D(z, Tz) > 0$  satisfying

$$\tau(d(z, y)) + \mathcal{F}(\alpha(z, y)D(y, Ty)) \leq \mathcal{F}(M(z, y)), \quad (28)$$

where,

$$M(z, y) = \max \left\{ d(z, y), D(z, Tz), D(y, Ty), \frac{D(y, Tz) + D(z, Ty)}{2}, \frac{D(y, Ty)[1 + D(z, Tz)]}{1 + d(z, y)}, \frac{D(y, Tz)[1 + D(z, Ty)]}{1 + d(z, y)} \right\}. \quad (29)$$

and

$$F_\sigma^z = \{y \in Tz : \mathcal{F}(d(z, y)) \leq \mathcal{F}(D(z, Tz)) + \sigma\}.$$

Note that  $F_\sigma^z \neq \emptyset$  in both cases when  $\mathcal{F} \in \Delta(F)$  and  $\mathcal{F} \in \Delta(F_*)$  [32].

**Definition 3.11** Let  $T : \mathcal{X} \rightarrow P(\mathcal{X})$  be a multivalued mapping on a metric space  $(\mathcal{X}, d)$ , then  $T$  is said to be an multivalued  $\alpha$ - $\mathcal{F}$ -contraction on  $\mathcal{X}$ , if there exists  $\sigma > 0$ ,  $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ ,  $\mathcal{F} \in \Delta(F)$  and  $\alpha : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$  such that for all  $z \in \mathcal{X}$ ,  $y \in F_\sigma^z$  with  $D(z, Tz) > 0$  satisfying

$$\tau(d(z, y)) + \mathcal{F}(\alpha(z, y)D(y, Ty)) \leq \mathcal{F}(d(x, y)), \quad (30)$$

**Theorem 3.12** Let  $(\mathcal{X}, d)$  be a complete metric space and  $T : \mathcal{X} \rightarrow K(\mathcal{X})$  be an multivalued  $\alpha$ - $\mathcal{F}$ -weak-contraction satisfying the following assertions:

1.  $T$  is multivalued  $\alpha$ -orbital admissible mapping;
2. the map  $z \rightarrow D(z, Tz)$  is lower semi-continuous;
3. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in Tz_0$  such that  $\alpha(z_0, z_1) \geq 1$ ;
4.  $\tau$  satisfies  $\lim_{t \rightarrow s^+} \inf \tau(t) > \sigma$  for all  $s \geq 0$ .



Then  $T$  has a fixed point in  $\mathcal{X}$ .

*Proof.* Let  $z_0 \in \mathcal{X}$ , since  $Tz \in K(\mathcal{X})$  for every  $z \in \mathcal{X}$ , the set  $F_\sigma^z$  is non-empty for any  $\sigma > 0$ , then there exists  $z_1 \in F_\sigma^{z_0}$  and by hypothesis  $\alpha(z_0, z_1) \geq 1$ . Assume that  $z_1 \notin Tz_1$ , otherwise  $z_1$  is the fixed point of  $T$ . Then, since  $Tz_1$  is closed,  $D(z_1, Tz_1) > 0$ , so, from (28), we have

$$\tau(d(z_0, z_1)) + \mathcal{F}(\alpha(z_0, z_1)D(z_1, Tz_1)) \leq \mathcal{F}(M(z_0, z_1)), \quad (31)$$

where

$$M(z_0, z_1) = \max \left\{ d(z_0, z_1), D(z_0, Tz_0), D(z_1, Tz_1), \frac{D(z_1, Tz_0) + D(z_0, Tz_1)}{2}, \frac{D(z_1, Tz_1)[1 + D(z_0, Tz_0)]}{1 + d(z_0, z_1)}, \frac{D(z_1, Tz_0)[1 + D(z_0, Tz_1)]}{1 + d(z_0, z_1)} \right\}. \quad (32)$$

Since  $Tz_0$  and  $Tz_1$  are compact, so we have

$$\begin{aligned} M(z_0, z_1) &= \max \left\{ d(z_0, z_1), d(z_0, z_1), d(z_1, z_2), \frac{d(z_1, z_1) + d(z_0, z_2)}{2}, \frac{d(z_1, z_2)[1 + d(z_0, z_1)]}{1 + d(z_0, z_1)}, \frac{d(z_1, z_1)[1 + d(z_0, z_2)]}{1 + d(z_0, z_1)} \right\} \\ &= \max \left\{ d(z_0, z_1), d(z_1, z_2), \frac{d(z_0, z_2)}{2} \right\}. \end{aligned} \quad (33)$$

Since  $\frac{d(z_0, z_2)}{2} \leq \frac{d(z_0, z_1) + d(z_1, z_2)}{2} \leq \max\{d(z_0, z_1), d(z_1, z_2)\}$ , it follows that

$$M(z_0, z_1) \leq \max\{d(z_0, z_1), d(z_1, z_2)\}. \quad (34)$$

Suppose that  $d(z_0, z_1) < d(z_1, z_2)$ , then (31) implies that

$$\begin{aligned} \tau(d(z_0, z_1)) + \mathcal{F}(D(z_1, Tz_1)) &\leq \tau(d(z_0, z_1)) + \mathcal{F}(\alpha(z_0, z_1)D(z_1, Tz_1)) \\ &\leq \mathcal{F}(d(z_1, z_2)), \end{aligned} \quad (35)$$

consequently,

$$\tau(d(z_0, z_1)) + \mathcal{F}(d(z_1, z_2)) \leq \mathcal{F}(d(z_1, z_2)), \quad (36)$$

or,  $\mathcal{F}(d(z_1, z_2)) \leq \mathcal{F}(d(z_1, z_2)) - \tau(d(z_0, z_1))$ , which is a contradiction. Hence  $M(d(z_0, z_1)) \leq d(z_0, z_1)$ , therefore by using  $(F_1)$ , (31) implies that

$$\tau(d(z_0, z_1)) + \mathcal{F}(\alpha(z_0, z_1)d(z_1, z_2)) \leq \mathcal{F}(d(z_0, z_1)). \quad (37)$$

On continuing recursively, we get a sequence  $\{z_n\}_{n \in \mathbb{N}}$  in  $\mathcal{X}$ , where  $z_{n+1} \in F_\sigma^{z_n}$ ,  $z_{n+1} \notin Tz_{n+1}$ ,  $\alpha(z_n, z_{n+1}) \geq 1$ ,  $M(z_n, z_{n+1}) \leq d(z_n, z_{n+1})$  and

$$\tau(d(z_n, z_{n+1})) + \mathcal{F}(D(z_{n+1}, Tz_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})). \quad (38)$$

Since  $z_{n+1} \in F_\sigma^{z_n}$  and  $Tz_n$  and  $Tz_{n+1}$  are compact, we have

$$\tau(d(z_n, z_{n+1})) + \mathcal{F}(d(z_{n+1}, z_{n+2})) \leq \mathcal{F}(d(z_n, z_{n+1})) \quad (39)$$

and

$$\mathcal{F}(d(z_n, z_{n+1})) \leq \mathcal{F}(d(z_n, z_{n+1})) + \sigma. \quad (40)$$

Combining (39) and (40) gives

$$\mathcal{F}(d(z_{n+1}, z_{n+2})) \leq \mathcal{F}(d(z_n, z_{n+1})) + \sigma - \tau(d(z_n, z_{n+1})) \quad (41)$$

Let  $d_n = d(z_n, z_{n+1})$  for  $n \in \mathbb{N}$ , then  $d_n > 0$  and from (41)  $\{d_n\}$  is decreasing. Therefore, there exists  $\delta \geq 0$  such that  $\lim_{n \rightarrow \infty} d_n = \delta$ . Now let  $\delta > 0$ . From (41), we get

$$\begin{aligned} \mathcal{F}(d_{n+1}) &\leq \mathcal{F}(d_n) + \sigma - \tau(d_n) \leq \mathcal{F}(d_{n-1}) + 2\sigma - \tau(d_n) - \tau(d_{n-1}) \cdots \\ &\leq \mathcal{F}(d_0) + n\sigma - \tau(d_n) - \tau(d_{n-1}) - \cdots - \tau(d_0). \end{aligned} \quad (42)$$

Let  $\tau(d_{p_n}) = \min\{\tau(d_0), \tau(d_1), \dots, \tau(d_n)\}$  for all  $n \in \mathbb{N}$ . From (42), we get

$$\mathcal{F}(d_{n+1}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_n})). \quad (43)$$

From (38), we also get

$$\mathcal{F}(D(z_{n+1}, Tz_{n+1})) \leq \mathcal{F}(D(z_0, Tz_0)) + n(\sigma - \tau(d_{p_n})). \quad (44)$$

Now consider the sequence  $\{\tau(d_{p_n})\}$ . We distinguish two cases.

*Case 1.* For each  $n \in \mathbb{N}$ , there is  $m > n$  such that  $\tau(d_{p_n}) > \tau(d_{p_m})$ . Then we obtain a subsequence  $\{d_{p_{n_k}}\}$  of  $\{d_{p_n}\}$  with  $\tau(d_{p_{n_k}}) > \tau(d_{p_{n_{k+1}}})$  for all  $k$ . Since  $d_{p_{n_k}} \rightarrow \delta^+$ , we deduce that  $\lim_{k \rightarrow \infty} \inf \tau(d_{p_{n_k}}) > \sigma$ . Hence  $\mathcal{F}(d_{n_k}) \leq \mathcal{F}(d_0) + n(\sigma - \tau(d_{p_{n_k}}))$  for all  $k$ . Consequently,  $\lim_{k \rightarrow \infty} \mathcal{F}(d_{n_k}) = -\infty$  and by  $(\mathcal{F}_2)$ , we obtain  $\lim_{k \rightarrow \infty} d_{p_{n_k}} = 0$ , which contradicts that  $\lim_{n \rightarrow \infty} d_n > 0$ .

*Case 2.* There is  $n_0 \in \mathbb{N}$  such that  $\tau(d_{p_{n_0}}) > \tau(d_{p_m})$  for all  $m > n_0$ . Then  $\mathcal{F}(d_m) \leq \mathcal{F}(d_0) + m(\sigma - \tau(d_{p_{n_0}}))$  for all  $m > n_0$ . Hence  $\lim_{m \rightarrow \infty} \mathcal{F}(d_m) = -\infty$ , so  $\lim_{m \rightarrow \infty} d_m = 0$ , which contradicts that  $\lim_{m \rightarrow \infty} d_m > 0$ . Thus,

$$\lim_{n \rightarrow \infty} d_n = 0.$$

From  $(\mathcal{F}_3)$ , there exists  $0 < r < 1$  such that  $\lim_{n \rightarrow \infty} (d_n)^r \mathcal{F}(d_n) = 0$ . By (43), we get for all  $n \in \mathbb{N}$

$$(d_n)^r \mathcal{F}(d_n) - (d_n)^r \mathcal{F}(d_0) \leq (d_n)^r n(\sigma - \tau(d_{p_n})) \leq 0. \quad (45)$$

Letting  $n \rightarrow \infty$  in (45), we obtain  $\lim_{n \rightarrow \infty} n(d_n)^r = 0$ . This implies that there exists  $n_1 \in \mathbb{N}$  such that  $n(d_n)^r \leq 1$ , or,  $d_n \leq \frac{1}{n^{1/r}}$ , for all  $n > n_1$ . Next, for  $m > n \geq n_1$  we have

$$d(z_n, z_m) \leq \sum_{i=n}^{m-1} d(z_i, z_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}},$$

since  $0 < k < 1$ ,  $\sum_{i=n}^{m-1} \frac{1}{i^{1/k}}$  converges. Therefore,  $d(z_n, z_m) \rightarrow 0$  as  $m, n \rightarrow \infty$ . Thus,  $\{z_n\}$  is a Cauchy sequence. Since  $\mathcal{X}$  is complete, there exists  $z^* \in \mathcal{X}$  such that  $z_n \rightarrow z^*$  as  $n \rightarrow \infty$ . From Eqs. (44) and  $(F_2)$ , we have  $\lim_{n \rightarrow \infty} D(z_n, Tz_n) = 0$ . Since  $z \rightarrow D(z, Tz)$  is lower semi-continuous, then

$$0 \leq D(z, Tz) \leq \liminf_{n \rightarrow \infty} D(z_n, Tz_n) = 0.$$

Thus,  $T$  has a fixed point.  $\square$

In the following theorem we take  $C(\mathcal{X})$  instead of  $K(\mathcal{X})$ , then we need to take  $\mathcal{F} \in \Delta(F_*)$  in Definition 3.10.

**Theorem 3.13** *Let  $(\mathcal{X}, d)$  be a complete metric space and  $T : \mathcal{X} \rightarrow C(\mathcal{X})$  be an multivalued  $\alpha$ -F-weak-contraction with  $\mathcal{F} \in \Delta(F_*)$  satisfying all the assertions of Theorem 3.12. Then  $T$  has a fixed point in  $\mathcal{X}$ .*

*Proof.* Let  $z_0 \in \mathcal{X}$ , since  $Tz \in C(\mathcal{X})$  for every  $z \in \mathcal{X}$  and  $\mathcal{F} \in \Delta(F_*)$ , the set  $F_\sigma^z$  is non-empty for any  $\sigma > 0$ , then there exists  $z_1 \in F_\sigma^{z_0}$  and by hypothesis  $\alpha(z_0, z_1) \geq 1$ . Assume that  $z_1 \notin Tz_1$ , otherwise  $z_1$  is the fixed point of  $T$ . Then, since  $Tz_1$  is closed,  $D(z_1, Tz_1) > 0$ , so, from (28), we have

$$\tau(d(z_0, z_1)) + \alpha(z_0, z_1)\mathcal{F}(D(z_1, Tz_1)) \leq \mathcal{F}(M(z_0, z_1)), \quad (46)$$

where

$$M(z_0, z_1) = \max \left\{ d(z_0, z_1), D(z_0, Tz_0), D(z_1, Tz_1), \frac{D(z_1, Tz_0) + D(z_0, Tz_1)}{2}, \frac{D(z_1, Tz_1)[1 + D(z_0, Tz_0)]}{1 + d(z_0, z_1)}, \frac{D(z_1, Tz_0)[1 + D(z_0, Tz_1)]}{1 + d(z_0, z_1)} \right\}. \quad (47)$$

The rest of the proof can be completed as in the proof of Theorem 3.12 by considering the closedness of  $Tz$ , for all  $z \in \mathcal{X}$ .  $\square$

**Theorem 3.14** *Let  $(\mathcal{X}, d)$  be a complete metric space,  $T : \mathcal{X} \rightarrow K(\mathcal{X})$  be a continuous mapping and  $\mathcal{F} \in \Delta(F)$ . Assume that the following assertions hold:*

1.  $T$  is generalized  $\alpha_*$ -admissible mapping;
2. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in Tz_0$  such that  $\alpha(z_0, z_1) \geq 1$ ;
3. there exists  $\tau : (0, \infty) \rightarrow (0, \infty)$  such that

$$\liminf_{t \rightarrow s^+} \tau(t) > 0 \quad \text{for all } s \geq 0$$

and for all  $z \in \mathcal{X}$  with  $H(Tz, Ty) > 0$ , there exist a function  $\alpha : \mathcal{X} \times \mathcal{X} \rightarrow \{-\infty\} \cup (0, +\infty)$  satisfying

$$\tau(d(z, y)) + \alpha(z, y)\mathcal{F}(H(Tz, Ty)) \leq \mathcal{F}(M(z, y)), \quad (48)$$

where  $M(z, y)$  is defined in (29).

Then  $T$  has a fixed point in  $\mathcal{X}$ .

*Proof.* By following the steps in the proof of Theorem 3.12, we get the required result.  $\square$

Note that Theorem 3.14 cannot be obtained from Theorem 3.12, because in Theorem 3.12,  $\sigma$  cannot be equal to zero.

**Theorem 3.15** *Let  $(\mathcal{X}, d)$  be a complete metric space,  $T : \mathcal{X} \rightarrow C(\mathcal{X})$  be a continuous mapping and  $\mathcal{F} \in \Delta(F_*)$  satisfying all assertions of Theorem 3.14. Then  $T$  has a fixed point in  $\mathcal{X}$ .*

From Theorems 3.14 and 3.15, we get the following fixed point result for single valued mappings:

**Theorem 3.16** *Let  $(\mathcal{X}, d)$  be a complete metric space,  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a continuous mapping and  $\mathcal{F} \in \Delta(F)$ . Assume that the following assertions hold:*

1.  $T$  is  $\alpha$ -admissible mapping;

2. there exists  $z_0, z_1 \in \mathcal{X}$  such that  $\alpha(z_0, z_1) \geq 1$ ;

3. there exists  $\tau : (0, \infty) \rightarrow (0, \infty)$  such that

$$\liminf_{t \rightarrow s^+} \tau(t) > 0 \quad \text{for all } s \geq 0$$

and for all  $z \in \mathcal{X}$  with  $d(Tz, Ty) > 0$ , there exist a function  $\alpha : \mathcal{X} \times \mathcal{X} \rightarrow \{-\infty\} \cup (0, +\infty)$  satisfying

$$\tau(d(z, y)) + \alpha(z, y)\mathcal{F}(d(Tz, Ty)) \leq \mathcal{F}(m(z, y)), \quad (49)$$

where

$$m(z, y) = \max \left\{ d(z, y), d(z, Tz), d(y, Ty), \frac{d(y, Tz) + d(z, Ty)}{2}, \frac{d(y, Ty)[1 + d(z, Tz)]}{1 + d(z, y)}, \frac{d(y, Tz)[1 + d(z, Ty)]}{1 + d(z, y)} \right\}. \quad (50)$$

Then  $T$  has a fixed point in  $\mathcal{X}$ .

Now, let  $(\mathcal{X}, d, \leq)$  be a partially ordered metric space. Recall that  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  is monotone increasing if  $Ty \leq Tz$  for all  $y, z \in \mathcal{X}$ , for which  $y \leq z$  (see [33]). There are many applications in differential and integral equations of monotone mappings in ordered metric spaces (see [34–36] and references therein).

**Theorem 3.17** Let  $(\mathcal{X}, d, \leq)$  be a complete partially ordered metric space and let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a closed valued mapping satisfying the following assertions for all  $y, z \in \mathcal{X}$  with  $y \leq z$ :

1.  $T$  is monotone increasing;

2.  $\vartheta(H(Ty, Tz), d(y, z)) \geq 0$ ;

3. there exists  $z_0 \in X$  and  $z_1 \in Tz_0$  such that  $z_0 \leq z_1$ ;

4. for a sequence  $\{z_n\} \subset \mathcal{X}$ ,  $\lim_{n \rightarrow \infty} \{z_n\} = z$  and  $z_n \leq z_{n+1}$  for all  $n \in \mathbb{N}$ , we have  $z_n \leq z$  for all  $n \in \mathbb{N}$ .

Then  $\text{Fix}\{T\} \neq \emptyset$ .

*Proof.* Define  $\alpha, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  by

$$\alpha(y, z) = \begin{cases} 1 & y \leq z \\ 0 & \text{otherwise} \end{cases} \quad \eta(y, z) = \begin{cases} \frac{1}{2} & y \leq z \\ 0 & \text{otherwise,} \end{cases}$$

then for  $y, z \in \mathcal{X}$  with  $y \leq z$ ,  $\alpha(y, z) \geq \eta(y, z)$  implies  $\alpha_*(Ty, Tz) = 1 > \frac{1}{2} = \eta_*(Ty, Tz)$  and  $\alpha_*(Ty, Tz) = \eta_*(Ty, Tz) = 0$  otherwise. Thus, all the conditions of Theorem 3.4 are satisfied and hence  $T$  has a fixed point.  $\square$

In case of single valued mapping Theorem 3.17 reduced to the following:

**Theorem 3.18** Let  $(\mathcal{X}, d, \leq)$  be a complete partially ordered metric space and let  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a self-map fulfilling the following assertions:

1.  $T$  is monotone increasing;

2.  $\vartheta(d(Ty, Tz), d(y, z)) \geq 0$ ;
3. there exists  $z_0 \in \mathcal{X}$  and  $z_1 = Tz_0$  such that  $z_0 \preceq z_1$ ;
4. for a sequence  $\{z_n\} \subset \mathcal{X}$ ,  $\lim_{n \rightarrow \infty} \{z_n\} = z$  and  $z_n \preceq z_{n+1}$  for all  $n \in \mathbb{N}$ , we have  $z_n \preceq z$  for all  $n \in \mathbb{N}$ .

for all  $y, z \in \mathcal{X}$  with  $y \preceq z$  and  $\vartheta \in \widehat{Man}(\mathbb{R})$ . Then  $Fix\{T\} \neq \emptyset$ .

**Definition 3.19** Let  $T : \mathcal{X} \rightarrow 2^{\mathcal{X}}$  be a multivalued mapping on a partially ordered metric space  $(\mathcal{X}, d, \preceq)$ , then  $T$  is said to be an ordered  $\mathcal{F}$ - $\tau$ -contraction on  $\mathcal{X}$ , if there exists  $\sigma > 0$  and  $\tau : (0, \infty) \rightarrow (\sigma, \infty)$ ,  $\mathcal{F} \in \Delta(F)$  such that for all  $z \in \mathcal{X}$ ,  $y \in \mathcal{F}_\sigma^z$  with  $z \preceq y$  and  $D(z, Tz) > 0$  satisfying

$$\tau(d(z, y)) + \mathcal{F}(D(y, Ty)) \leq \mathcal{F}(M(z, y)), \quad (51)$$

where,

$$M(z, y) = \max \left\{ d(z, y), D(z, Tz), D(y, Ty), \frac{D(y, Tz) + D(z, Ty)}{2}, \frac{D(y, Ty)[1 + D(z, Tz)]}{1 + d(z, y)}, \frac{D(y, Tz)[1 + D(z, Ty)]}{1 + d(z, y)} \right\}. \quad (52)$$

**Theorem 3.20** Let  $(\mathcal{X}, d, \preceq)$  be a complete partially ordered metric space and  $T : \mathcal{X} \rightarrow K(\mathcal{X})$  be an ordered  $\mathcal{F}$ - $\tau$ -contraction satisfying the following assertions:

1.  $T$  is monotone increasing;
2. the map  $z \rightarrow D(z, Tz)$  is lower semi-continuous;
3. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in Tz_0$  such that  $z_0 \preceq z_1$ ;
4.  $\tau$  satisfies

$$\liminf_{t \rightarrow s^+} \tau(t) > \sigma \quad \text{for all } s \geq 0$$

Then  $T$  has a fixed point in  $\mathcal{X}$ .

*Proof.* By using the similar arguments as in the proof of Theorem 3.17 and using Theorem 3.12, we get the result.  $\square$

**Theorem 3.21** Let  $(\mathcal{X}, d, \preceq)$  be a complete partially ordered metric space and  $T : \mathcal{X} \rightarrow C(\mathcal{X})$  be an ordered  $\mathcal{F}$ - $\tau$ -contraction with  $\mathcal{F} \in \Delta(F_*)$  satisfying all the assertions of Theorem 3.20. Then  $T$  has a fixed point in  $\mathcal{X}$ .

**Theorem 3.22** Let  $(\mathcal{X}, d, \preceq)$  be a complete partially ordered metric space,  $T : \mathcal{X} \rightarrow K(\mathcal{X})$  be a continuous mapping and  $\mathcal{F} \in \Delta(F)$ . Assume that the following assertions hold:

1.  $T$  is monotone increasing;
2. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in Tz_0$  such that  $z_0 \preceq z_1$ ;
3. there exists  $\tau : (0, \infty) \rightarrow (0, \infty)$  such that

$$\liminf_{t \rightarrow s^+} \tau(t) > 0 \quad \text{for all } s \geq 0$$

and for all  $z, y \in \mathcal{X}$  with  $z \preceq y$  and  $H(Tz, Ty) > 0$  satisfying

$$\tau(d(z, y)) + \mathcal{F}(H(Tz, Ty)) \leq \mathcal{F}(M(z, y)), \quad (53)$$

where  $M(z, y)$  is defined in (52).

Then  $T$  has a fixed point in  $\mathcal{X}$ .

*Proof.* By defining  $\alpha : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$  as in the proof of Theorem 3.17 and by using Theorem (3.14), we get the required result.  $\square$

**Theorem 3.23** Let  $(\mathcal{X}, d, \preceq)$  be a complete partially ordered metric space,  $T : \mathcal{X} \rightarrow C(\mathcal{X})$  be a continuous mapping and  $\mathcal{F} \in \Delta(F_*)$  satisfying all assertions of Theorem 3.22. Then  $T$  has a fixed point in  $\mathcal{X}$ .

From Theorems 3.22 and 3.23, we get the following fixed point result for single valued mapping.

**Theorem 3.24** Let  $(\mathcal{X}, d, \preceq)$  be a complete partially ordered metric space,  $T : \mathcal{X} \rightarrow \mathcal{X}$  be a continuous mapping and  $\mathcal{F} \in \Delta(F)$ . Assume that the following assertions hold:

1.  $T$  is monotone increasing;
2. there exists  $z_0, z_1 \in \mathcal{X}$  such that  $z_0 \preceq z_1$ ;
3. there exists  $\tau : (0, \infty) \rightarrow (0, \infty)$  such that

$$\liminf_{t \rightarrow s^+} \tau(t) > 0 \quad \text{for all } s \geq 0$$

and for all  $z, y \in \mathcal{X}$  with  $z \preceq y$  and  $d(Tz, Ty) > 0$  satisfying

$$\tau(d(z, y)) + \mathcal{F}(d(Tz, Ty)) \leq \mathcal{F}(m(z, y)), \quad (54)$$

where

$$m(z, y) = \max \left\{ d(z, y), d(z, Tz), d(y, Ty), \frac{d(y, Tz) + d(z, Ty)}{2}, \frac{d(y, Ty)[1 + d(z, Tz)]}{1 + d(z, y)}, \frac{d(y, Tz)[1 + d(z, Ty)]}{1 + d(z, y)} \right\}. \quad (55)$$

Then  $T$  has a fixed point in  $\mathcal{X}$ .

## 4. Existence of solution

In this section, by using the fixed point results proved in the previous section, we obtain the existence of the solution of integral Eq. (2) and matrix Eq. (1).

### 4.1 Solution of Fredholm integral equation of second kind

Let  $\ll$  be a partial order relation on  $\mathbb{R}^n$ . Define  $T : \mathcal{X} \rightarrow \mathcal{X}$  by

$$Tz(r) = \int_b^c B(r, s, z(s)) ds + g(r), \quad r \in [a, b]. \quad (56)$$

**Theorem 4.1** Let  $\mathcal{X} = C([b, c], \mathbb{R}^n)$  with the usual spermium norm. Suppose that

1.  $B : [b, c] \times [b, c] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous;



2. there exists a continuous function  $p : [b, c] \times [b, c] \rightarrow [b, c]$  such that

$$|\mathcal{B}(r, s, u) - \mathcal{B}(r, s, v)| \leq p(r, s)|u - v|, \quad (57)$$

for each  $r, s \in [b, c]$  and  $u, v \in \mathbb{R}^n$  with  $u \ll v$ .

3.  $\sup_{r \in [b, c]} \int_b^c p(r, s) ds = q \leq \frac{1}{4}$ ;

4. there exists  $z_0 \in \mathcal{X}$  and  $z_1 \in \mathcal{T}z_0$  such that  $z_0 \leq z_1$ ;

5. for a sequence  $\{z_n\} \subset \mathcal{X}$ ,  $\lim_{n \rightarrow \infty} \{z_n\} = z$  and  $z_n \leq z_{n+1}$  for all  $n \in \mathbb{N}$ , we have  $z_n \leq z$  for all  $n \in \mathbb{N}$ .

Then the integral Eq. (2) has a solution in  $\mathcal{X}$ .

*Proof.* Let  $\mathcal{X} = C([b, c], \mathbb{R}^n)$  and  $\|z\| = \max_{r \in [b, c]} |z(r)|$ , for  $z \in C([a, b])$ . Consider a partial order defined on  $\mathcal{X}$  by

$$y, z \in C([b, c], \mathbb{R}^n), \quad y \leq z \quad \text{if and only if} \quad y(r) \ll z(r), \quad \text{for } r \in [b, c]. \quad (58)$$

Then  $(\mathcal{X}, \|\cdot\|, \leq)$  is a complete partial ordered metric space and for any increasing sequence  $\{z_n\}$  in  $\mathcal{X}$  converging to  $z \in \mathcal{X}$ , we have  $z_n(r) \ll z(r)$  for any  $r \in [b, c]$  (see [36]). By using Eq. (56), conditions (2, 3) and taking  $\vartheta(r, s) = \frac{1}{2}s - r$  for all  $y, z \in \mathcal{X}$  with  $y \leq z$ , we obtain

$$\begin{aligned} |\mathcal{T}y(r) - \mathcal{T}z(r)| &= \left| \int_b^c \mathcal{B}(r, s, y(s)) ds - \int_b^c \mathcal{B}(r, s, z(s)) ds \right| \\ &\leq \int_b^c |\mathcal{B}(r, s, y(s)) - \mathcal{B}(r, s, z(s))| ds \\ &\leq \int_b^c p(r, s) |y(s) - z(s)| ds \\ &\leq \frac{1}{4} \|y - z\|. \end{aligned}$$

This implies that

$$\frac{1}{2} \|y - z\| - \|\mathcal{T}y - \mathcal{T}z\| \geq \frac{1}{2} \|y - z\| - \frac{1}{4} \|y - z\| = \frac{1}{4} \|y - z\|.$$

So  $\vartheta(d(\mathcal{T}y, \mathcal{T}z), d(y, z)) \geq 0$  for all  $y, z \in \mathcal{X}$  with  $y \leq z$ . Hence all the conditions of Theorem 3.18 are satisfied. Therefore  $\mathcal{T}$  has a fixed point, consequently, integral Eq. (2) has a solution in  $\mathcal{X}$ .  $\square$

#### 4.2 Solution of non-linear matrix equation

**Theorem 4.2** Let  $\gamma : H(n) \rightarrow H(n)$  be an order-preserving mapping which maps  $P(n)$  into  $P(n)$  and  $Q \in P(n)$ . Assume that there exists a positive number  $N$  for which  $\sum_{i=1}^m A_i A_i^* < N I_n$  and  $\sum_{i=1}^m A_i^* \gamma(Q) A_i > 0$  such that for all  $X \leq Y$  we have

$$d(\gamma(X), \gamma(Y)) \leq \frac{1}{N} m(Y, X) e^{-\left(\frac{2+d(X, Y)}{2d(X, Y)}\right)}, \quad (59)$$

where

$$m(X, Y) = \max \left\{ d(X, Y), d(X, \mathcal{T}X), d(Y, \mathcal{T}Y), \frac{d(Y, \mathcal{T}Y) + d(X, \mathcal{T}X)}{2}, \frac{d(Y, \mathcal{T}Y)[1 + d(X, \mathcal{T}X)]}{1 + d(X, Y)}, \frac{d(Y, \mathcal{T}X)[1 + d(X, \mathcal{T}Y)]}{1 + d(X, Y)} \right\}.$$

Then (1) has a solution in  $P(n)$ .

*Proof.* Define  $T : H(n) \rightarrow H(n)$  and  $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}$  by

$$T(X) = Q + \sum_{i=1}^m A_i^* \gamma(X) A_i \quad (60)$$

and  $F(r) = \ln r$  respectively. Then a fixed point of  $T$  is a solution of (1). Let  $X, Y \in H(n)$  with  $X \leq Y$ , then  $\gamma(X) \leq \gamma(Y)$ . So, for  $d(X, Y) > 0$  and  $\tau(t) = \frac{1}{t} + \frac{1}{2}$ , we have

$$\begin{aligned} d(TX, TY) &= \|TY - TX\|_1 \\ &= \text{tr}(TY - TX) \\ &= \sum_{i=1}^m \text{tr}(A_i A_i^* (\gamma(Y) - \gamma(X))) \\ &= \text{tr} \left( \left( \sum_{i=1}^m A_i A_i^* \right) (\gamma(Y) - \gamma(X)) \right) \\ &\leq \left\| \sum_{i=1}^m A_i A_i^* \right\| \|\gamma(Y) - \gamma(X)\|_1 \\ &\leq \frac{\left\| \sum_{i=1}^m A_i A_i^* \right\|}{N} m(Y, X) e^{-\left( \frac{2 + \|Y - X\|_1}{2\|Y - X\|_1} \right)} \\ &< m(Y, X) e^{-\left( \frac{2 + \|Y - X\|_1}{2\|Y - X\|_1} \right)}, \end{aligned}$$

and so,

$$\ln(\|TY - TX\|_1) < \ln \left( m(Y, X) e^{-\left( \frac{2 + \|Y - X\|_1}{2\|Y - X\|_1} \right)} \right) = \ln(m(X, Y)) - \frac{2 + \|Y - X\|_1}{2\|Y - X\|_1}.$$

This implies that

$$\frac{1}{\|Y - X\|_1} + \frac{1}{2} + \ln(\|TY - TX\|_1) < \ln(m(X, Y)).$$

Consequently,

$$\tau(d(X, Y)) + \mathcal{F}(d(TX, TY)) < \mathcal{F}(m(X, Y)).$$

Also, from  $\sum_{i=1}^m A_i^* \gamma(Q) A_i > 0$ , we have  $Q \leq T(Q)$ . Thus, by using Theorem 3.24, we conclude that  $T$  has a fixed point and hence (1) has a solution in  $P(n)$ .  $\square$

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