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The Stochastic Finite Element in the Natural Frequency of Functionally Graded Material Beams

Nguyen Van Thuan and Noh Hyuk Chun

Abstract

In this study, the stochastic finite element solution is given to obtain the variability in the natural frequency of functionally graded material (FGM) beam. The elastic modulus is assumed to vary in the thickness direction and the width of the beam to vary as well in the longitudinal direction following the exponential law. The random material properties of elastic modulus and mass density of the FGM beam are assumed to be one-dimensional homogeneous stochastic processes. The stochastic finite element analysis of FGM beam is performed in conjunction with Monte Carlo simulation (MCS) employing the spectral representation method for 16, the description of random processes of the random material properties under consideration. The response variability of the natural frequency due to random elastic modulus is evaluated for various states of randomness. Furthermore, the investigation on the effect of the correlation between random elastic modulus and random mass density on the response variability is addressed in detail as well.

Keywords: functionally graded materials, finite element method, FGM beam, Monte Carlo simulation

1. Introduction

Functionally graded materials (FGM) have received considerable attention in many engineering applications, since the theory of FGM was firstly introduced in 1984. In general, FGM is made from the volume fractions of two or more material components that have continuous variation of material properties from one surface to another [1]. Nowadays, FGM suits the specific demand in different engineering applications, especially for high temperature environment applications of heat exchanger tubes, thermal barrier coating for turbine blades, thermoelectric generators, furnace linings, electrically insulated metal ceramic joints, space/aerospace industries, automotive applications, biomedical area, etc.

The manufacturing of FGM with fully specified profile of material gradation, however, is very difficult causing significant variability in their mechanical and structural properties. Therefore, proper handling of the randomness in the material properties is required for accurate prediction of structural response for safe and reliable design. The stochastic analysis is a useful analytical tool to predict the

response of structures with random material properties. In this direction, there is a reasonable body of recent research on the effect of uncertainties in material properties on the mechanical behavior of FGM. Investigators used stochastic simulation to study the effect of microstructural randomness on stress in FGM [2]. Ferrante et al. studied the effect of non-Gaussian porosity randomness on the response of functionally graded plate [3]. Yang et al. dealt with the stochastic bending response of moderately thick FGM plates [4, 5]. The effect of random material properties on post buckling response of FGM plate are presented in Lal et al. [6].

However, the above mentioned literatures are for the static analysis. To the best of author's knowledge, few limited works have been done on the eigen analysis of FGM structures involving randomness in system parameters. Certain efforts have been made in the past to predict the dynamic behavior of structures with randomness. In most of the studies conducted, investigators dealt with the free vibration of functionally graded laminates with random material properties using first-order perturbation technique (FOPT) incorporating mixed type and semi-analytical approach to derive the standard eigenvalue problem [7]. Some of these papers presented the stochastic finite element method (SFEM) to investigate the natural frequency of functionally graded plates based on the higher-order shear deformation theory (HSDT) utilizing first-order reliability method and second-order reliability method [8]. In most cases, Jagtap et al. [9] examined the stochastic nonlinear free vibration response of FGM plate using HSDT with von-Karman kinematic nonlinearity via direct iterative stochastic finite element method. Shegokar et al. investigated the stochastic finite element nonlinear free vibration analysis of FGM beam with random material properties due to thermo-piezoelectric loadings [10]. The above mentioned literatures investigated the free vibration and nonlinear behavior of FGM beam and plate. The material properties, such as Young's modulus, shear modulus, and Poisson's ratio of FGM, are modeled as independent random variables.

In this chapter, the stochastic finite element solution is suggested to obtain the variability in the natural frequency of functionally graded material (FGM) beam. The elastic modulus and width of the FGM beam are assumed to vary in thickness and longitudinal directions following the exponential law. The uncertain material properties, such as modulus of elasticity and mass density of the FGM beam, are considered to be a one-dimensional homogeneous stochastic process. The stochastic finite element analysis of FGM beam is performed using the spectral representation method for the description of randomness in conjunction with Monte Carlo simulation (MCS). The response variability of natural frequency due to random elastic modulus and mass density in FGM beam is given. Furthermore, the effect of correlation between the two random parameters is observed as well.

2. Theory formulation of FGM

An FGM is defined to be a material which has a continuous gradation through-the-thickness (h). One side of the material is typically ceramic and the other side is typically metal. A mixture of the two materials composes the through-the-thickness characteristics. Let us consider a functionally graded beam shown in **Figure 1**. The parameters of the model FGM beam are as follows: L is the length of the beam, h is the thickness of the beam, and b is the width of the beam.

The elastic material properties vary through-the-beam thickness according to the volume fractions of the constituents using power law distribution (as shown in **Figure 2**).

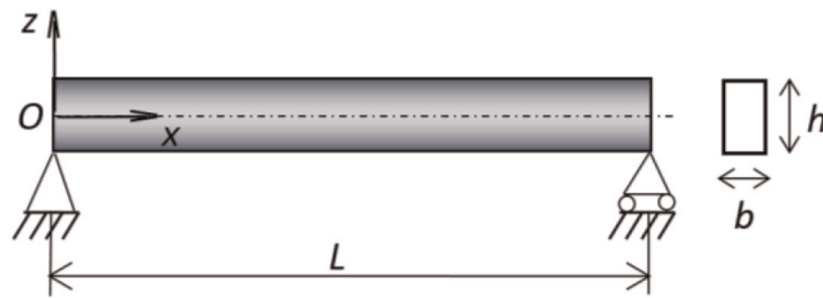


Figure 1.
 Model of FGM beams.

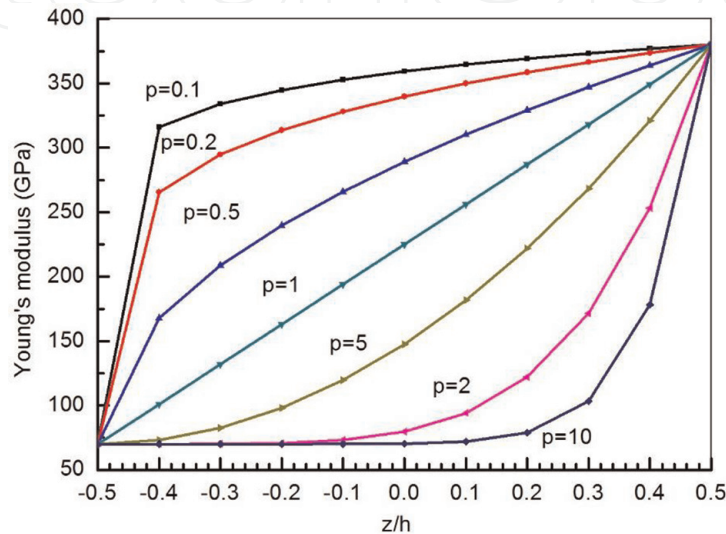


Figure 2.
 The variation of elasticity modulus along the corresponding directions.

$$E(z) = (E_c - E_m)g(z) + E_m$$

with

$$g(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^p, \quad -0.5h \leq z \leq 0.5h, \quad 0 \leq p \leq \infty$$

where E denotes the effective Young's modulus of elasticity, and E_m and E_c represent the Young's modulus of metal and ceramic, respectively. The parameters $g(z)$ and p represent the volume fraction of the metal and ceramic exponent, respectively.

In exponential law, for the material parameter of Young's modulus E , mass density ρ of the beam and the width of the beam b , with absolute values for z coordinate, which endows the symmetric characteristic to the beam with respect to mid-plane.

$$E(z) = E_0 e^{\beta|z|}; \rho(z) = \rho_0 e^{\beta|z|}; b(x) = b_0 e^{\psi x}$$

In Eq. (3), E_0 , ρ_0 are the values of the Young's modulus and mass density at the mid-plane ($z = 0$) of the beam. The parameter β in the exponent characterizes the material property variation along the thickness direction. The width of the beam b varies according to the non-uniformity parameter ψ along the axis of the beam. In order to give the insight into the varying characteristics of the FGM beam under

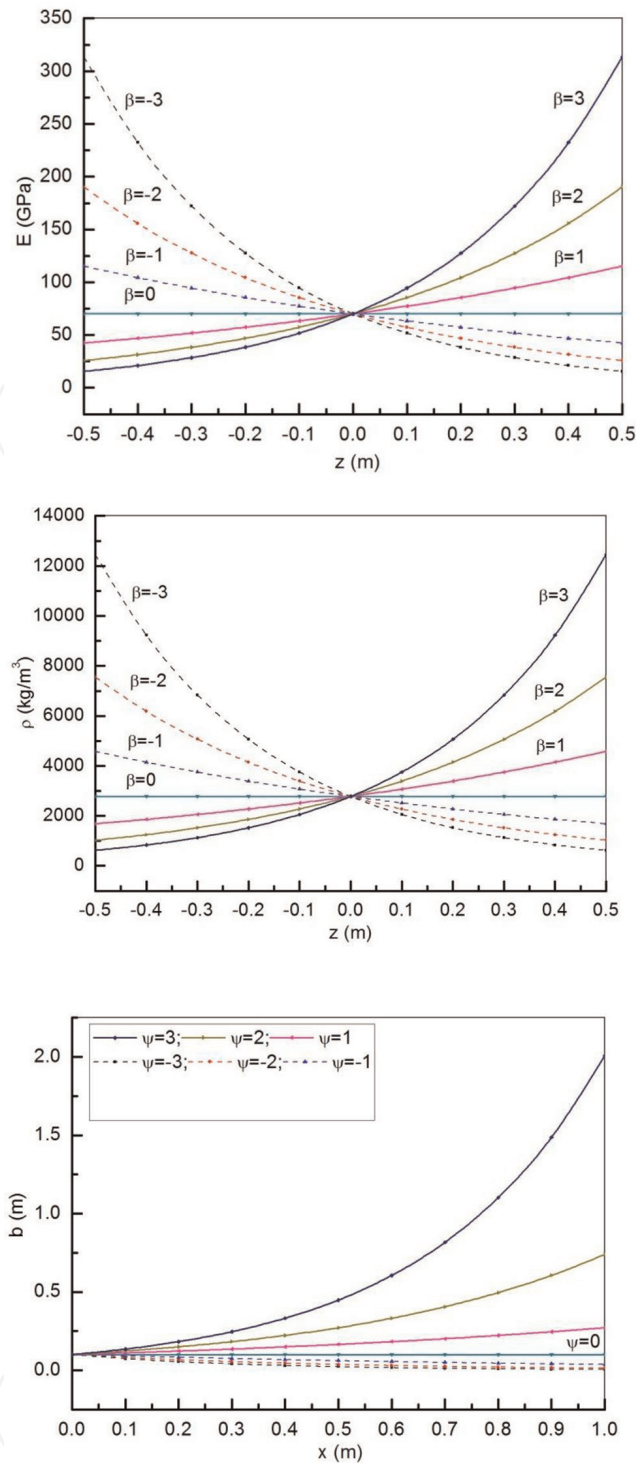


Figure 3. The variation of elasticity modulus, mass density, and width along the corresponding directions.

consideration, change of material constants and width depending on the corresponding parameters are shown in **Figure 3**.

3. Finite element formulation for FGM beam and frequency analysis

3.1 Finite elements

In case of four degree of freedom beam element, as shown in **Figure 4**, the transverse displacement function may be assumed as a cubic polynomial in x , and the corresponding shape functions are Hermite interpolation functions.

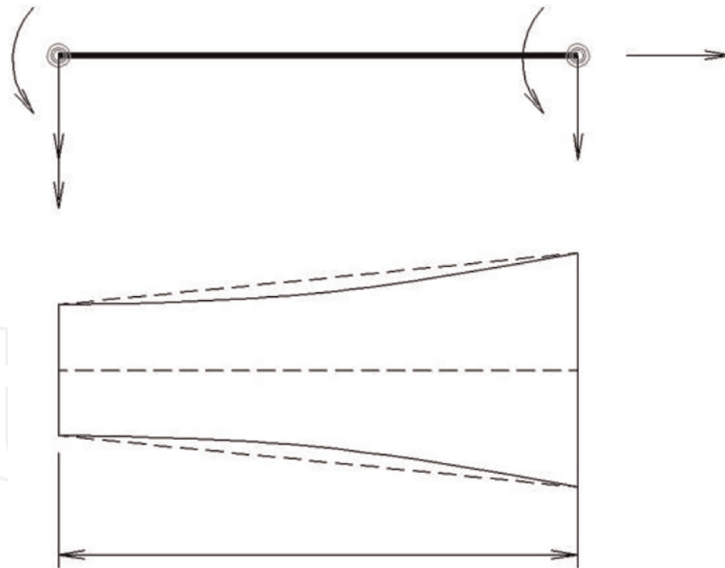


Figure 4.
 Beam element.

The width and mass per unit length of the element are

$$b_e(x) = b_1 + \frac{b_2 - b_1}{L_e}x,$$

$$m_e(x) = mb_e(x) = m \left(b_1 + \frac{b_2 - b_1}{L_e}x \right),$$

where $m = \int_{-h/2}^{h/2} \rho(z) dz$ and $\rho(z)$ denote the mass density at z .

The nodal displacement vector of the element is

$$\{q\}_e = \{q_1 \ q_2 \ q_3 \ q_4\}^T,$$

then, the displacement field is

$$w_e = \langle N \rangle \{q\}_e,$$

where $\langle N \rangle = \langle N_1 \ N_2 \ N_3 \ N_4 \rangle$, and N_i is Hermite shape function of i -th degree of freedom.

In this case, the stiffness of the beam $b_e D_{11}$ is similar to EI of the homogeneous beam

$$D_{11} = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \nu^2} z^2 dz,$$

where ν is Poisson's ratio.

3.2 Application of Hamilton's principle

Hamilton's principle may be a theoretical base for dynamical systems by its nature of integral form in time with Lagrangian density to account for continuous space. In this paper, the analysis of natural frequency of FGM beam is performed

using Hamilton's principle. The strain energy expression U_e for bending is given as following:

$$U_e = \int_0^{L_e} \frac{b_e D_{11}}{2} \left(\frac{\partial^2 w_e}{\partial x^2} \right)^2 dx.$$

The kinematic energy T_e for flexural vibration is

$$T_e = \int_0^{L_e} \frac{m b_e \dot{w}_e^2}{2} dx.$$

where

$$K_e = \int_0^{L_e} b_e D_{11} \langle N \rangle^T \langle N \rangle dx.$$

$$M_e = \int_0^{L_e} m_e b_e \langle N \rangle^T \langle N \rangle dx.$$

Substituting Eq. (7) into Eqs. (9) and (10), the following can be obtained:

$$U_e = \frac{1}{2} \{q\}_e^T K_e \{q\}_e,$$

$$T_e = \frac{1}{2} \{\dot{q}\}_e^T M_e \{\dot{q}\}_e.$$

The governing differential equations of motion and the related governing equation can be derived using Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(\sum_1^N U_e - \sum_1^N T_e \right) dt = 0,$$

where N denotes the number of finite elements.

Substituting Eqs. (13) and (14) into Eq. (15), the following can be obtained:

$$M\ddot{q} + Kq = 0,$$

here, q is the nodal displacement of the beam.

For simple harmonic vibration, we assume the displacements to be $q(x, t) = w(x)e^{i\omega t}$. Accordingly with Eq. (16), we can obtain the following:

$$(K - \omega^2 M)w = 0,$$

where K is the assembled global stiffness matrix of K_e , the element stiffness matrix, which is given in detail in the Appendix. For Eq. (17) to be valid, i.e., to have nontrivial solution, the following needs to be satisfied:

$$\det(K - \omega^2 M) = 0.$$

4. Modeling of randomness

4.1 Mathematical expression

In order to model the randomness in the material properties, the modulus of elasticity and mass density along the mid-plane are assumed to vary along its length of FGM beam in a random manner. We can model these variations as one-dimensional univariate (1D-1V) homogeneous stochastic processes. The simple mathematical expressions for the randomly varying modulus of elasticity and mass density can be written as

$$E_0(x) = \bar{E}_0 [1 + f_E(x)],$$

$$\rho_0(x) = \bar{\rho}_0 [1 + f_\rho(x)],$$

where x is the coordinate along the axis of the FGM beam, $\bar{E}_0, \bar{\rho}_0$ are the expected values of E_0 and ρ_0 , respectively, and $f_E(x), f_\rho(x)$ are one-dimensional stochastic process which are homogeneous with zero-mean values.

The numerical generation of sample functions of Gaussian zero-mean homogeneous stochastic processes, which describe the randomness in parameters of the structure, is accomplished using the spectral representation method. For a one-dimensional univariate (1D-1V) stochastic process, we have [11]

$$f(x) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \phi_n),$$

$$A_n = \sqrt{2S_{ff}(\omega_n)\Delta\omega},$$

$$\Delta\omega = \frac{\omega_u}{N},$$

$$\omega_n = n\Delta\omega, n = 0, 1, 2, \dots, N - 1.$$

In Eq. (21), ω_u denotes the upper cut-off frequency beyond which the power spectral density function $S_{ff}(\omega_n)$ may be assumed to be zero for either mathematical or physical reasons. The following criterion is usually used to estimate the value of ω_u :

$$\int_0^{\omega_u} S_{ff}(\omega_n) d\omega_n = (1 - \varepsilon) \int_0^{\infty} S_{ff}(\omega_n) d\omega_n,$$

where $\varepsilon \ll 1$.

The uniform random phase angle ϕ_n in Eq. lies in the range of $[0, 2\pi]$. The power spectral density function used in Eq. (20) is given as

$$S_{ff}(\omega_n) = \frac{1}{\sqrt{\pi}} \sigma_f^2 d e^{(-d^2 \omega_n^2)}, \quad -\infty < \omega_n < \infty,$$

here, σ_f denotes the standard deviation of the stochastic process $f(x)$ and d is the correlation distance of the stochastic process along the x axis.

In all examples, the coefficient of variation (COV) of natural frequency, which is defined as a ratio of the standard deviation of response to the absolute mean response, will be used to give the variability of the response.

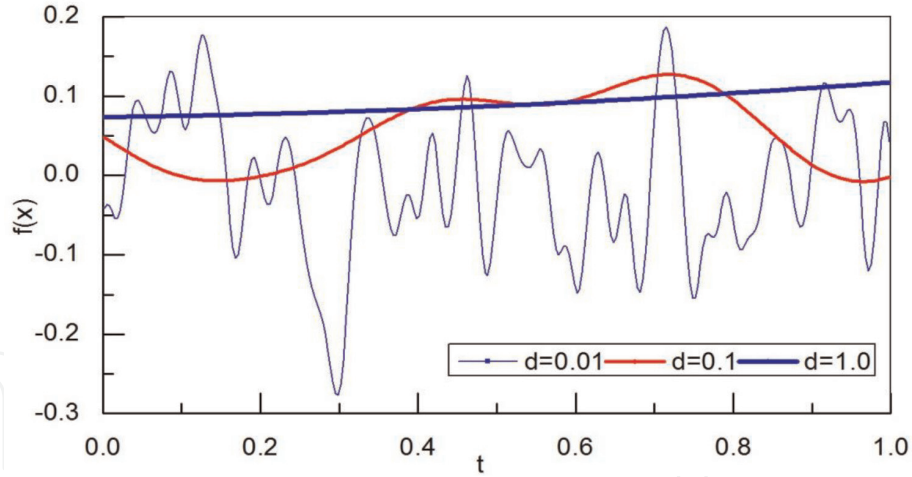


Figure 5.
 Example plot of random process used ($d = 0.01, 0.1, 1.0, \sigma_f = 0.1$).

$$COV = \frac{\text{Standard deviation}}{|\text{Mean}|},$$

here,

$$\left. \begin{aligned} \text{Standard deviation} &= \sqrt{\text{var}(\omega)} \\ \text{Var}(\omega) &= E[(\omega - \bar{\omega})^2] \end{aligned} \right\}$$

and ω is the natural frequency of the FGM beam, and $\bar{\omega}$ denotes the mean of the natural frequency.

4.2 Monte Carlo analysis

In order to obtain the response variability in the natural frequency of the FGM beam, we employed the scheme of Monte Carlo simulation (MCS). As a matter of fact, the MCS corresponds to the deterministic analyses on a set of heterogeneous models of the given structure, in which the material properties have different values depending on the position in the domain of the structure.

The generation of heterogeneous random samples is accomplished by the aforementioned spectral representation scheme, and we use 10,000 samples for respective analyses. In particular, we adopt the local average scheme other than the mid-point rule in applying the MCS, with which better results can be obtained especially for the processes with small correlation distance. **Figure 5** shows an example plot of the processes employed to model the randomness in the system parameters.

5. Numerical example

The geometric dimensions of the example FGM beam are: $h = 0.1$ m, $L = 1$ m, and $b_0 = 0.1$ m. It is assumed that the material properties are $E_0 = 70$ GPa, $\rho_0 = 2780$ kg/m³, and $\nu = 0.33$. E_0 denotes the Young's modulus at the mid-surface of the beam and E_1 at the top and bottom surfaces following Eq. (3) (**Figure 6**).

5.1 Deterministic analysis results

The results in **Figure 7** correspond to the prismatic homogeneous beam since the parameters in exponents, β and ψ , are all zero. The natural frequency is obtained in

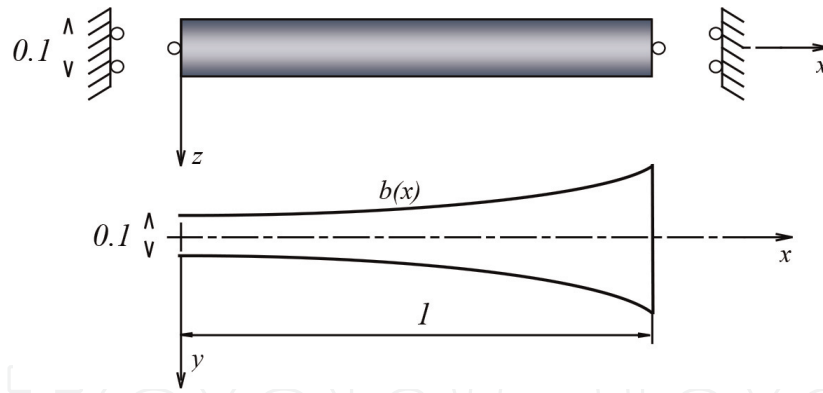


Figure 6.
 FGM beam model.

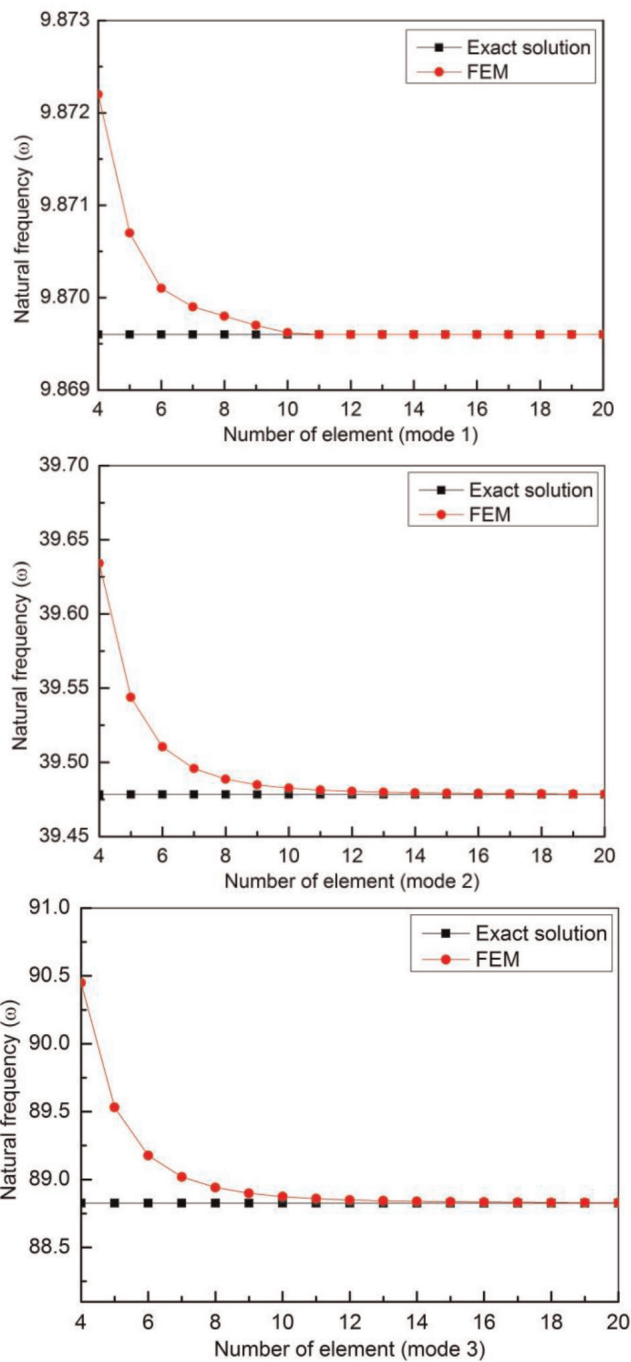


Figure 7.
 The convergence between exact solution and finite element method.

the present study to compare with the results of analytical solution given by Hassen et al. [12].

The discrepancies between exact and finite element solutions for the frequencies for the first three modes are shown in **Figure 8**. The differences given in percentile tends to zero as the number of finite elements is increased, meaning the results are converging to exact solutions.

Figure 9 shows the first three normalized natural frequencies of uniform FGM beams for three cases of modulus ratio (E_1/E_0). The natural frequency increases as the ratio of Young's modulus increases from $E_1 = 0.2E_0$ to $E_1 = 5E_0$.

5.2 Variability of natural frequency due to randomness in elastic modulus

Figure 10 shows the COV of response versus the correlation distance (d) when the elastic modulus E_0 is random. The standard deviation of the random elastic modulus is denoted by σ_f . In all cases, the COV of natural frequency shows similar trends, starting from small values for small correlation distance, up to large values

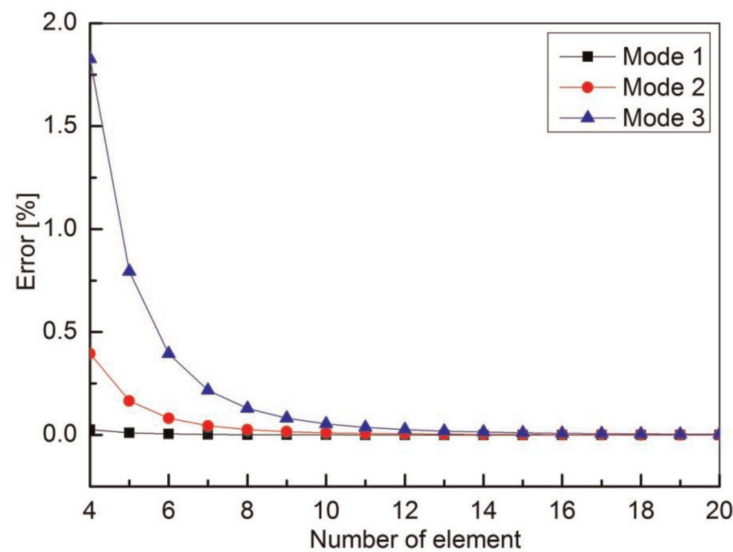


Figure 8.
Error depending on mesh refinement ($\beta = 0, \psi = 0$).

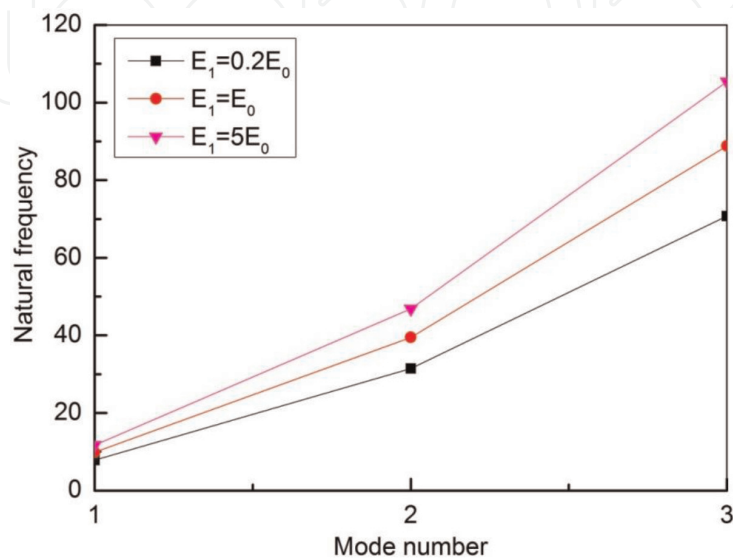


Figure 9.
First three normalized natural frequencies of FGM beams for different ratio of Young's modulus ($\psi = 0$).

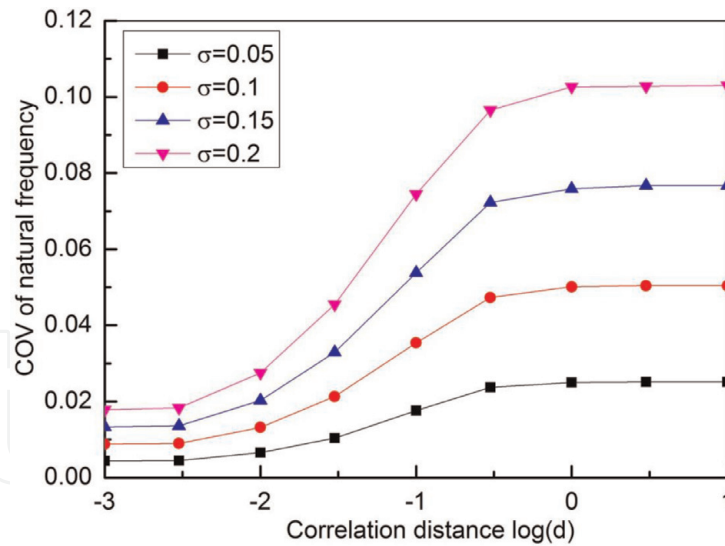


Figure 10.
 COV of natural frequency as a function of the correlation distance (d) for different standard deviation of stochastic process.

for large correlation distance. In the FGM beam under consideration, the response variability, when the correlation distance tends to infinity, is obtained to be about 50% of the input standard deviation of the stochastic process.

The relationship between COV of natural frequency and the COV of stochastic process is shown in **Figure 11**. The standard deviation of stochastic process is changed from 0.0 to 0.25. As seen in **Figure 11**, the COV of response shows a slightly nonlinear pattern in all the cases of $d = 0.01, 0.1,$ and 10 .

The effect of mesh refinement on the COV of natural frequency is shown in **Figure 12**. The correlation distance $\log(d)$ is assumed to be from -3 to 1 . As seen in **Figure 12**, the COV of natural frequency is not affected by the mesh refinement.

Figure 13 shows the variation of the coefficient of variation (COV) depending on the non-uniformity parameter (ψ) of the FGM beam. As seen in the figure, the COV of response is not slightly affected by the non-uniformity parameter in particular for large correlation distances.

The overall features of the effect of Young's modulus ratio on COV of natural frequency are shown in **Figure 14**. The COV of natural frequency is not affected by the parameter β . These results can easily be understood because the standard deviation and the mean of natural frequency in Eq. (27) increase in the same rate.

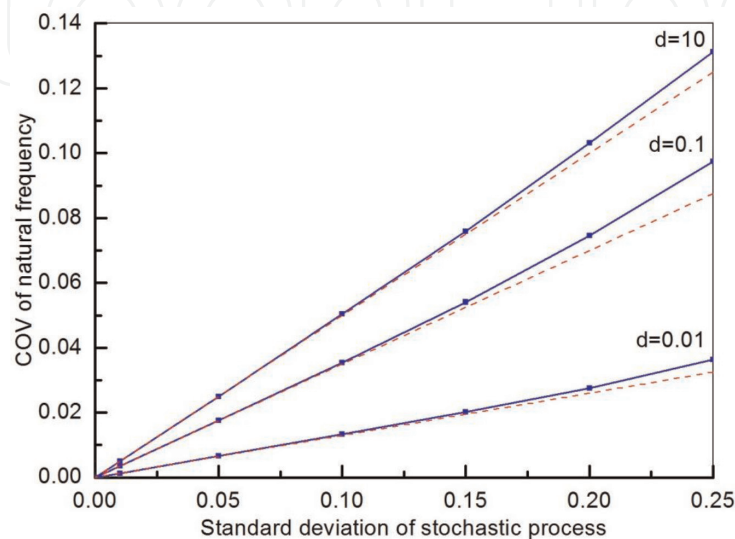


Figure 11.
 COV of natural frequency as a function of the standard deviation of stochastic process $f_E(x)$.

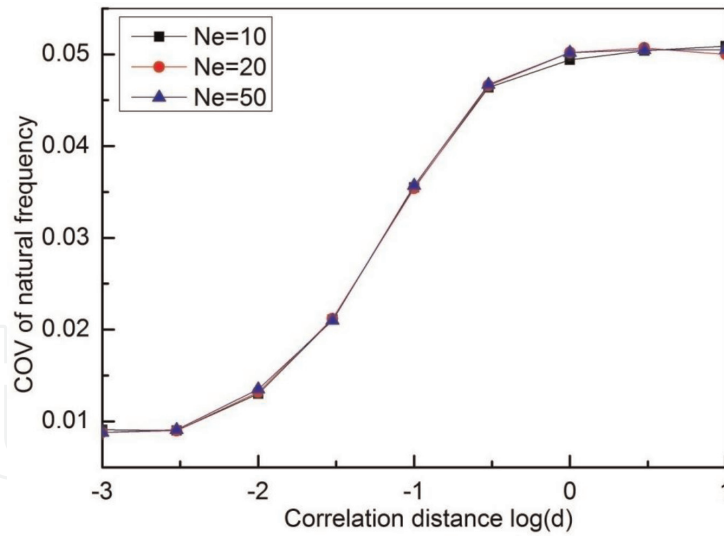


Figure 12.
Effect of mesh refinement on the COV of natural frequency ($\sigma_f = 0.1$).

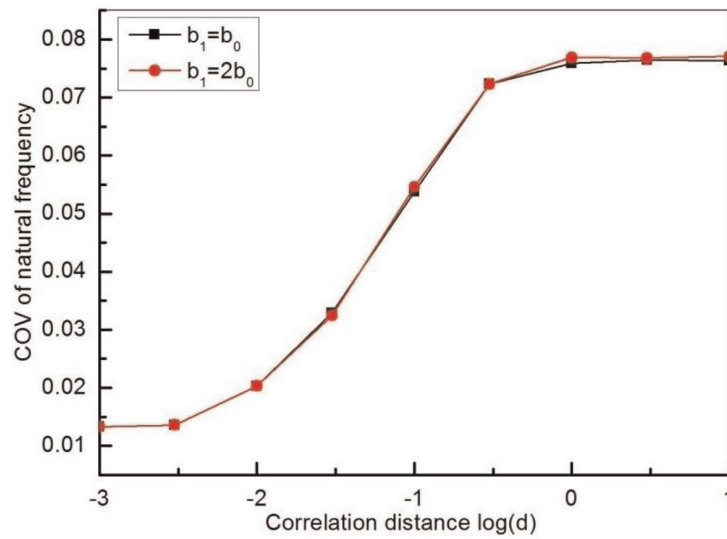


Figure 13.
Effect of non-uniformity parameter (ψ) on COV of natural frequency ($Ne = 20$, $\sigma_f = 0.15$).

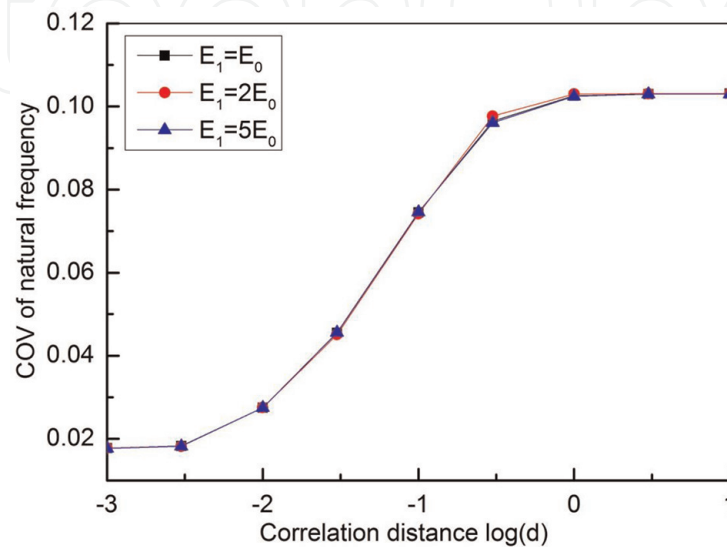


Figure 14.
Effect of parameter β on the COV of natural frequency ($Ne = 20$, $\sigma_f = 0.2$).

5.3 FGM beam having correlation multiple randomness

It is natural to have preposition that not only the elastic modulus, but also the mass density of the material can have randomness. Therefore, we need to consider

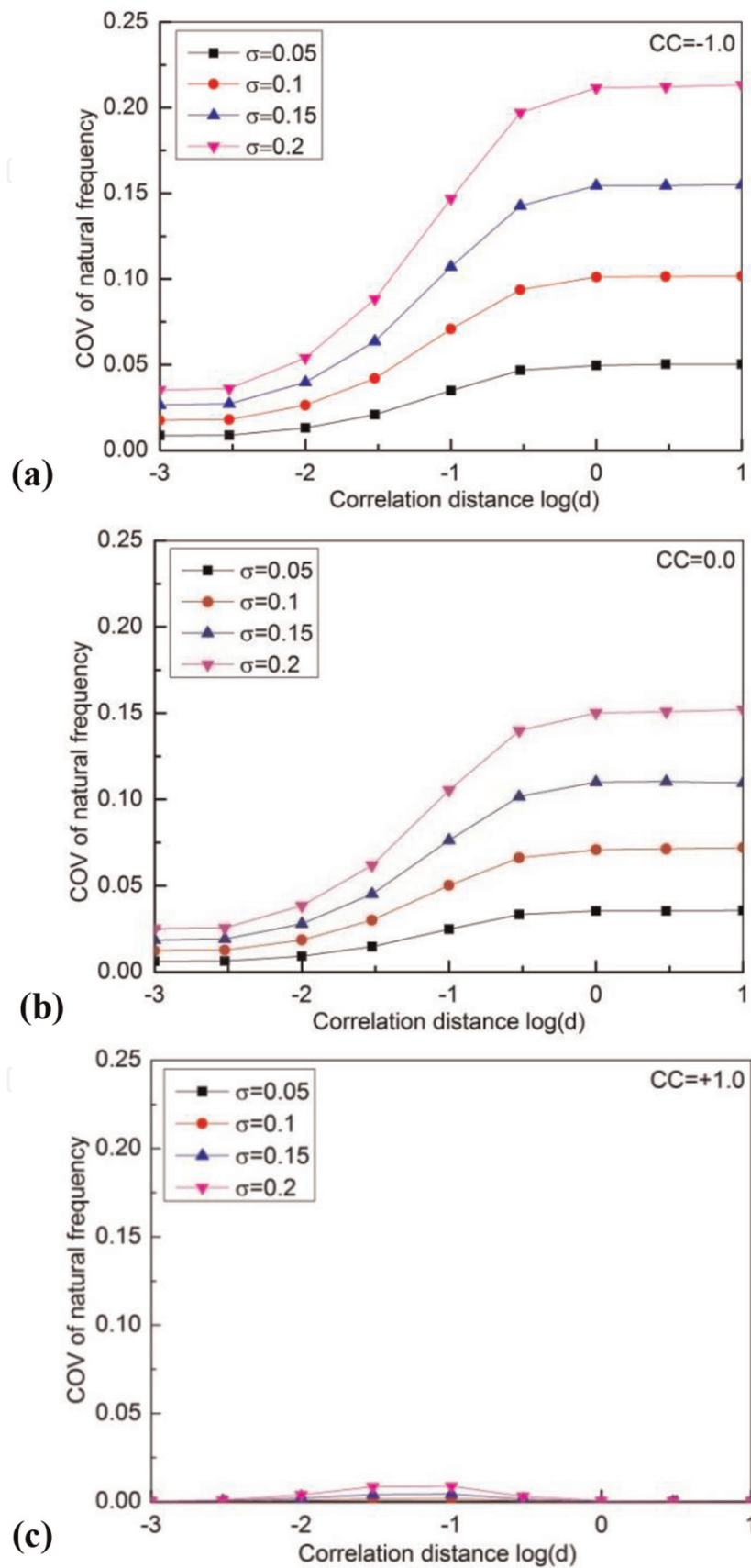


Figure 15. Effect of the correlation between two random parameters: (a) negative perfect correlation ($CC = -1.0$), (b) no correlation ($CC = 0.0$), and (c) positive perfect correlation ($CC = +1.0$).

the effect of correlation between two random parameters of elastic modulus and mass density. To this aim, we consider three correlation cases: +1.0, 0.0, and -1.0. When the random processes for elastic modulus, f_E , and mass density, f_ρ , are exactly the same, the correlation coefficient (CC) is +1.0. If the values have negative

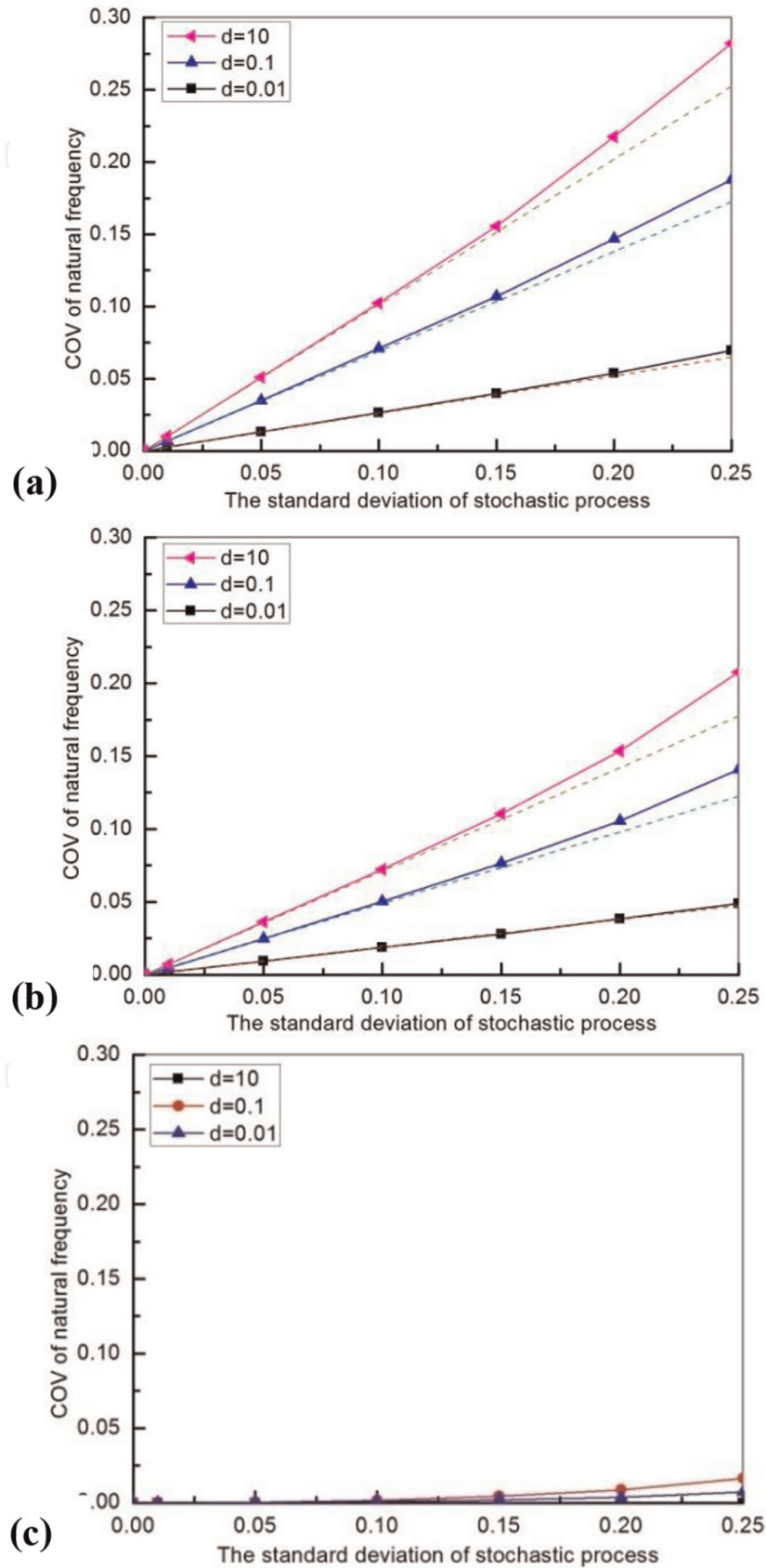


Figure 16. COV of natural frequency as a function of the standard deviation of stochastic process: (a) negative perfect correlation (CC = -1.0), (b) no correlation (CC = 0.0), and (c) positive perfect correlation (CC = +1.0).

amounts, then the correlation coefficient is -1.0 . The zero-correlation ($CC = 0.0$) means that stochastic processes of the two parameters are theoretically independent [13].

As shown in **Figure 15**, the maximum COV exceeds the input standard deviation of the stochastic process in the case of negative perfect correlation, while it is about 75% in the case of no correlation. However, in case of positive perfect correlation (**Figure 15c**), the response COV of natural frequency of FGM beam is small enough to be ignored. These results can easily be understood because the stiffness and mass matrix of each elements in Eq. (18) increase or decrease with the same rate in the case of positive perfect correlation. In case of negative perfect correlation, the ratio of random parts in the stiffness and mass are relatively large since the random parts have opposite sign, which makes the response variability large.

In the case, when we take the correlated multiple random material properties into account, we also obtained the slight nonlinear pattern of variation of COV in terms of COV of stochastic fields, as shown in **Figure 16**.

6. Conclusions

To evaluate response variability due to a single parameter of the random Young's modulus and multiple uncertain material properties, a formulation in the context of stochastic finite element solution is suggested for the natural frequency of FGM beam. In deriving the formula for the covariance of the response, modified power spectral density and correlation function are defined by using the general formula of random processes. The Monte Carlo simulation is performed employing the statistical preconditioning scheme as a random process generation technique. The local average method is employed instead of mid-point rule in Monte Carlo simulation.

In FGM beam natural frequency, the response COV for correlation between a random of single parameter and two uncertain parameters is observed. The coefficient of variation of natural frequency can only reach about 50% of the input standard deviation of the stochastic process in a single parameter of the random elastic modulus. However, the number of values is increased over 100% of the input standard deviation of the stochastic process in multiple uncertain material properties, when the correlation distance tends to infinity. The results showed that the COV of natural frequency of FGM beam in a single parameter of the random Young's modulus and multiple uncertain material properties achieved maximum variability for d about 1.0.

There is a very small difference between deterministic natural frequency and probabilistic natural frequency of FGM beam for the case of positive perfect correlation. Also, the COV of natural frequency does not depend on the number of elements, Young's modulus ratio, and the ratio of non-uniformity parameter of FGM beam. The importance of these parameters needs to be studied as a further work.

A. Appendix

Hermite shape functions of beam finite element:

$$N_1 = x \left(1 - 2\frac{x}{L} + \frac{x^2}{L^2} \right); N_2 = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}; N_3 = x \left(-\frac{x}{L} + \frac{x^2}{L^2} \right); N_4 = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}.$$

Stiffness matrix:

$$K_e = \frac{D_{11}}{L^3} \begin{bmatrix} 6(b_2 + b_1) & 2(b_2 + 2b_1)L & -6(b_2 + b_1) & 2(2b_2 + b_1)L \\ & (b_2 + 3b_1)L^2 & -2(b_2 + 2b_1)L & (b_2 + b_1)L^2 \\ \text{Sym.} & & 6(b_2 + b_1) & -2(2b_2 + b_1)L \\ & & & (3b_2 + b_1)L^2 \end{bmatrix}.$$

Mass matrix:

$$M_e = \frac{mL}{840} \begin{bmatrix} 24(3b_2 + 10b_1) & 2(b_2 + b_1)L & 54(b_2 + b_1) & -2(6b_2 + 7b_1)L \\ & (3b_2 + 5b_1)L^2 & 2(7b_2 + 6b_1)L & -3(b_2 + b_1)L^2 \\ \text{Sym.} & & 24(10b_2 + 3b_1) & -2(15b_2 + 7b_1)L \\ & & & (5b_2 + 3b_1)L^2 \end{bmatrix}.$$

Author details

Nguyen Van Thuan^{1*} and Noh Hyuk Chun²

1 Nha Trang University, Vietnam

2 Sejong University, South Korea

*Address all correspondence to: thuannv@ntu.edu.vn

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