Chapter

Fourier Transforms for Generalized Fredholm Equations

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Abstract

In this chapter we take the conventional Fredholm integral equations as a guideline to define a broad class of equations we name generalized Fredholm equations with a larger scope of applications. We show first that these new kind of equations are really vector-integral equations with the same properties but with redefined and also enlarged elements in its structure replacing the old traditional concepts like in the case of the source or inhomogeneous term with the generalized source useful for describing the electromagnetic wave propagation. Then we can apply a Fourier transform to the new equations in order to obtain matrix equations to both types, inhomogeneous and homogeneous generalized Fredholm equations. Meanwhile, we discover new properties of the field we can describe with this new technology, that is, mean; we recognize that the old concept of nuclear resonances is present in the new equations and reinterpreted as the brake of the confinement of the electromagnetic field. It is important to say that some segments involving mathematical details of our present work were published somewhere by us, as part of independent researches with different specific goals, and we recall them as a tool to give a sound support of the Fourier transforms.

Keywords: Fredholm equations, electromagnetic resonances, electromagnetic confinement, evanescent waves, left-hand materials, Fourier transforms, vector-matrix equations

1. Introduction

There is a very broad class of problems on physics that requires a tool that not only serves to handle the mathematical problem related to the solution of some differential equation describing the behavior of a system but that gives us an alternative description of them from a distinct point of view in a manner that allows us to discover some hidden physical properties, that is, we need to generalize the application of the Fourier transform from the conventional task to achieve a set of algebraic equations to a complete alternative formulation in terms of the Fourier transform of the integral Fredholm equations [1–5, 13, 17]. Many of the problems we want to consider are those related with vector fields like the electromagnetic. For this situation we dedicate the present chapter first to the integral equation formulation of the electromagnetic traveling waves, and then, by the application of

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the Fourier transform, we obtain finally a matrix-vector formulation [9, 10, 12, 14, 18]. To this end we go from the conventional Fredholm equations to new vector-integral equations we name generalized Fredholm equations proving that really they have the same properties of the conventional scalar Fredholm equations. In the meantime we discover that the new formulation brings a resonant behavior solution when some specific conditions are accomplished. The resonant behavior can be associated with the physical phenomenon of a brake of confinement of the so-called evanescent waves [6–8, 10–12, 19, 20] which leaves the region known as the near-field zone and is strongly related to the condition we name a left-hand material condition of the propagation media. The name left-hand material conditions describes the fact that are related with a negative refraction index observed in artificial materials created by man and we have used for describe the propagation media property in which in some embedded region the electromagnetic waves are diffracted like in a left-hand material. We find in the first part of the present chapter a brief discussion about the relation between the inhomogeneous generalized Fredholm equations or GIFE [9, 10, 12, 18] and the homogeneous generalized Fredholm equations or GHFE. The GHFE are behind the presence of the resonant behavior, and we show how a sudden change in a little set of physical parameters related to propagation properties triggers the brake of the confinement of the evanescent waves. Then we incorporate to our description the plasma sandwich model or PSM and their own parameters in order to propose that the change in these last parameters changes drastically the wave propagation properties of media. It is important to advise that our procedures are applied to continuous systems and therefore are strictly original, and only the topics related to the funds of the PSM were taken from previous works that involved discrete systems.

2. Beginning of the generalized Fredholm equations

In this section we will build the generalized Fredholm equations mentioned in the introduction of this chapter. To this end, we suppose that both electric and magnetic fields have the linearity property, and for this reason we can relate their values represented with the symbol $F^m(\mathbf{r},t)$ at different times and places $\mathbf{r},t$ and $\mathbf{r}',t'$. Due to the mentioned linearity of the wave equation, we can write (bearing in mind that we can have more general conditions different to empty space)

$$F^m(\mathbf{r},t) = F^{m(\circ)}(\mathbf{r},t) + \int_V \sum_{n=1}^{3} \int_{-\infty}^{\infty} G^{mn(\circ)}\left(\mathbf{r},t;\mathbf{r}',t'\right) U^{mn}\left(\mathbf{r}'\right) F^n\left(\mathbf{r}',t'\right) dt'\,dV' \qquad (1)$$

Here

$$G^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t') \qquad (2)$$

is the free Green's function, and the complex dispersion coefficients are $U^{mn}(\mathbf{r}')$ which contain the complete linear or nonlinear space-dependent interaction, but only time-independent ones are considered. By interchanging the volume and time differentials on integrands in Eq. (1), we obtain

$$F^m(\mathbf{r},t) = F^{m(\circ)}(\mathbf{r},t) + \int_{-\infty}^{\infty} \sum_{n=1}^{3} \int_V G^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t') U^{mn}(\mathbf{r}') F^n(\mathbf{r}',t') dV'\,dt' \qquad (3)$$

or

$$F^m(\mathbf{r},t) = F^{m(\circ)}(\mathbf{r},t) + \int_{-\infty}^{\infty}\int_V K^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t')F^n(\mathbf{r}',t')dV'dt' \tag{4}$$

This equation resembles inhomogeneous Fredholm's integral equation (IFE) but not as defined in scalar conventional form, and we will prove below that is strictly the case, so we call it generalized inhomogeneous Fredholm's integral equation or GIFE and the homogeneous version generalized homogeneous Fredholm's equation or GHFE.

Also, we have used summation convention over n and defined the kernel:

$$K^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t') = G^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t')U^{mn}(\mathbf{r}') \tag{5}$$

The signal $F^n(\mathbf{r}',t')$ can be written in terms of a well-behaved non-null function $Z^n(\mathbf{r}',t')$ defined by

$$F^n(\mathbf{r}',t') = \begin{cases} 0 \ \text{if} \ t' \in (-\infty,0)\cup(T,\infty) \\ Z^n(\mathbf{r}',t') \ \text{if} \ t' \in [0,T] \end{cases} \tag{6}$$

For convenience, we return to Eq. (2), which can be written as

$$Z^m(\mathbf{r},t) = Z^{m(\circ)}(\mathbf{r},t) + \sum_{n=1}^{3}\int_V\int_0^T G^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t')U^{mn}(\mathbf{r}')Z^m(\mathbf{r}',t')dt'dV' \tag{7}$$

On the other hand, we can express the Green's function in terms of its Fourier transform associated with frequency ω

$$G^{mn(\circ)}(\mathbf{r},t;\mathbf{r}',t') = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} G_\omega^{mn(\circ)}(\mathbf{r};\mathbf{r}')e^{i\omega(t-t')}d\omega \tag{8}$$

so that Eq. (7) becomes

$$Z^m(\mathbf{r},t) = Z^{m(\circ)}(\mathbf{r},t) + \tfrac{1}{2\pi}\sum_{n=1}^{3}\int_V U^{mn}(\mathbf{r}')\int_{-\infty}^{\infty} e^{i\omega t}G_\omega^{mn(\circ)}(\mathbf{r};\mathbf{r}')g^n(\mathbf{r}',\omega)d\omega dV' \tag{9}$$

where we have defined the function

$$g^m(\mathbf{r}',\omega) = \int_0^T e^{i\omega t'}Z^m(\mathbf{r}',t')dt' \tag{10}$$

That is, $g^m(\mathbf{r}',\omega)$ is the Fourier transform of $Z^n(\mathbf{r}',t')$

We also have

$$Z^m(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} g^m(\mathbf{r},\omega)d\omega \quad (11)$$

Substituting in Eq. (9) and performing some algebra, we obtain

$$g^m(\mathbf{r},\omega) = g^{m(\circ)}(\mathbf{r},\omega) + \sum_{n=1}^{3}\int_V U^{mn}(\mathbf{r}')G_\omega^{mn(\circ)}(\mathbf{r};\mathbf{r}')g^n(\mathbf{r}',\omega)d\omega dV' \quad (12)$$

Now we introduce a very useful and powerful notation we call vector-matrix form for Eq. (12) (vectors have another vectors as components, and also matrices have matrices as components):

$$\mathbf{g}^{m(\circ)}(\omega) = \left[\mathbf{1} - \mathbf{K}^{(\circ)}(\omega)\right]_n^m \mathbf{g}^n(\omega) \quad (13)$$

(Einstein summation convention was used here)
where

$$\mathbf{K}^{(\circ)}\left(\mathbf{r};\mathbf{r}';\omega\right) \equiv U^{mn}(\mathbf{r})G_\omega^{mn(\circ)}\left(\mathbf{r};\mathbf{r}'\right) \quad (14)$$

and also define

$$\int_V U^{mn}\left(\mathbf{r}'\right)G_\omega^{mn(\circ)}\left(\mathbf{r};\mathbf{r}'\right)g^n dV' \equiv \mathbf{K}^{mn(\circ)}(\omega)g^n(\mathbf{r})$$

3. The vector-matrix forward equation

Eq. (13) can be inverted formally as

$$\mathbf{g}^n(\omega) = [\left[\mathbf{1} - \mathbf{K}^{(\circ)}(\omega)\right]^{-1}]_m^n \mathbf{g}^{m(\circ)}(\omega) \quad (15)$$

By means of the development of this equation, we find the generalized Neumann series [12] and obtain the Fourier transform of complete Green's function $G_\omega^{mn}(\mathbf{r}_j,\mathbf{r}_k)$.. The result is [1]

$$\mathbf{g}^n(\omega) = \left[\mathbf{1} + \mathbf{K}(\omega)\right]_m^n \mathbf{g}^{m(\circ)}(\omega) \quad (16)$$

Here we have defined

$$\mathbf{K}_n^m\left(\mathbf{r};\mathbf{r}';\omega\right) = U^{mn}\left(\mathbf{r}'\right)G_\omega^{mn}\left(\mathbf{r};\mathbf{r}'\right) \quad (17)$$

and the integral

$$\int_V U^{mn}\left(\mathbf{r}'\right)G_\omega^{mn}\left(\mathbf{r};\mathbf{r}'\right)g^{m(\circ)}\left(\mathbf{r}'\right)dV' \equiv \mathbf{K}^{mn(\circ)}(\omega)g^{m(\circ)}(\mathbf{r})$$

Eqs. (16) and (17) comprise the basic tools needed to describe the forward transmission of information but, as we will see in the next chapter, an incomplete description for time reversal. We can use Eq. (16) to get experimental data on the **components of** $\mathbf{K}^n_m(\mathbf{r};\mathbf{r}';\omega)$ since the Fourier transforms of the original signals $g^{m(\circ)}(\mathbf{r}',\omega)$ are known, we can measure the arriving signals $g^n(\mathbf{r},\omega)$. In practice, we may consider Eqs. (16) and (17) as our starting point instead of assuming that there is no signal for t < 0.

4. The role of the Fourier transforms assisting time reverse

Nowadays, there is not any device capable to manipulate electromagnetic signals in the easy way; we can manipulate sound waves mostly when we make a time reverse on them. Nevertheless, we have proposed in another work a recipe to handle this problem, so we are convinced that the treatment of the time reversal process that we now describe corresponds to a completely possible fact. Suppose that we have recorded a signal during a time T and now the reversed signal returns to site **r**. Then we can write

$$F^n(\mathbf{r},T-t)=F^{n(\circ)}(\mathbf{r},T-t)$$
$$+\sum_{m=1}^{n}\int_V\int_{-\infty}^{\infty}U^{mn*}(\mathbf{r})G^{mn(\circ)*}(\mathbf{r}',T-t';\mathbf{r},t)F^m(\mathbf{r}',T-t')dt'dV' \quad (18)$$

This Eq. (18) can be written in terms of the function $Z^m(\mathbf{r}',t)$ as

$$Z^n(\mathbf{r},T-t)=Z^{n(\circ)}(\mathbf{r},T-t)$$
$$+\int_V\int_{-\infty}^{\infty}U^{mn*}(\mathbf{r}')G^{mn(\circ)*}(\mathbf{r}',T-t';\mathbf{r},t)\sum_{m=1}^{3}Z^m(\mathbf{r}',T-t')dt'dV' \quad (19)$$

We can express Eq. (19) in terms of the Fourier transform $G^{nm(\circ)*}_{-\omega}(\mathbf{r}';\mathbf{r})$

$$G^{nm(\circ)*}(\mathbf{r}',T-t';\mathbf{r},t)=\tfrac{1}{2\pi}\int_{-\infty}^{\infty}G^{nm(\circ)*}_{-\omega}(\mathbf{r}',\mathbf{r})e^{i\omega(T-t'-t)}d\omega \quad (20)$$

in the form

$$Z^n(\mathbf{r},T-t)=Z^{n(\circ)}(\mathbf{r},T-t)$$
$$+\sum_{m=1}^{3}\int_V\int_0^T\tfrac{1}{2\pi}U^{mn*}(\mathbf{r}')\int_{-\infty}^{\infty}G^{mn(\circ)*}_{-\omega}(\mathbf{r}';\mathbf{r})e^{i\omega(T-t'-t)}d\omega Z^m(\mathbf{r}',T-t')dt'dV' \quad (21)$$

And recalling the Fourier transform for $Z^n(\mathbf{r},t)$, this can be written as

$$g^n(\mathbf{r},\omega)=g^{n(\circ)}(\mathbf{r},\omega)+\sum_{m=1}^{3}\int_V g^m(\mathbf{r}',\omega)U^{mn*}(\mathbf{r}')G^{mn(\circ)*}_{\omega}(\mathbf{r}';\mathbf{r})e^{-i\omega T}dV' \quad (22)$$

At this point it is important to distinguish between functions related to forward phenomena and those related to backward direction when necessary. So we will use a different notation for both cases, and also, we introduce a quantum mechanics resembling notation for the product between matrices and vectors; in this manner, Eq. (22) can be written in vector (row vector) form like

$$\mathcal{g}^{n(\circ)}(\mathbf{r},\omega) = \mathcal{g}^{n}(\mathbf{r},\omega) - \left\langle \mathcal{g}^{m}(\mathbf{r}',\omega)\mathbf{M}^{mn(\circ)*}(\mathbf{r};\mathbf{r}';\omega) \right\rangle \quad (23)$$

where we introduced the quantum mechanics resembling notation:

$$\left\langle \mathcal{g}^{m}(\mathbf{r}',\omega)\mathbf{M}^{mn(\circ)*}(\mathbf{r};\mathbf{r}';\omega) \right\rangle = \int_V \mathcal{g}^{m}(\mathbf{r}',\omega)\mathbf{M}^{mn(\circ)*}(\mathbf{r};\mathbf{r}';\omega)\,dV' \quad (24)$$

Also we define

$$\mathbf{M}^{mn(\circ)*}(\mathbf{r}';\mathbf{r};\omega) \equiv U^{mn*}(\mathbf{r}')\mathcal{G}_{\omega}^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) \equiv U^{mn*}(\mathbf{r}')G_{\omega}^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) \quad (25)$$

and

$$\int_V \mathcal{g}^{n}(\mathbf{r}')U^{mn*}(\mathbf{r}')\mathcal{G}_{\omega}^{mn(\circ)*}(\mathbf{r}';\mathbf{r})dV' \equiv \mathcal{g}^{n}\mathbf{M}^{mn(\circ)*}(\omega) \quad (26)$$

Factorizing in Eq. (28) and using definition Eq. (29)

$$\mathcal{g}^{n(\circ)}(\mathbf{r},\omega) \equiv \mathcal{g}^{m}(\mathbf{r}',\omega)[1-\mathbf{M}^{(\circ)*}(\mathbf{r}';\mathbf{r};\omega)]_n^m \equiv \left\langle \mathcal{g}^{m}(\mathbf{r}',\omega)[1\delta(\mathbf{r}'-\mathbf{r})-\mathbf{M}^{(\circ)*}(\mathbf{r}';\mathbf{r};\omega)]_n^m \right\rangle \quad (27)$$

In the following we will use systematically Eqs. (23), (25), and (27).

5. Fourier transforms and Neumann series make up a powerful tool

It is possible to invert formally Eq. (27)

$$\mathcal{g}^{n}(\mathbf{r},\omega) \equiv \mathcal{g}^{m(\circ)}(\mathbf{r}',\omega)[\left[\mathbf{1}-\mathbf{M}^{(\circ)*}(\mathbf{r}';\mathbf{r};\omega)\right]^{-1}]_n^m \quad (28)$$

Formally

$$g^{n}(\mathbf{r},\omega) = \sum_{m=1}^{3} [\delta_m^n g^{m(\circ)}(\mathbf{r},\omega) + \mathcal{g}^{m(\circ)}(\mathbf{r}',\omega)\mathbf{M}^{mn(\circ)*}(\mathbf{r}';\mathbf{r};\omega) + \mathcal{g}^{m(\circ)}(\mathbf{r}',\omega)\left[\mathbf{M}^{mn(\circ)*}(\mathbf{r}';\mathbf{r};\omega)\right]^2 + \mathcal{g}^{m(\circ)}(\mathbf{r}',\omega)\left[\mathbf{M}^{mn(\circ)*}(\mathbf{r}';\mathbf{r};\omega)\right]^3 + \cdots\} \quad (29)$$

or

$$g^n(\mathbf{r},\omega) = \sum_{m=1}^{3} \{\delta_m^n g^{n(\circ)}(\mathbf{r},\omega) + \int_V g^{m(\circ)}(\mathbf{r}',\omega) U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) e^{-i\omega T} dV'$$

$$+ \int_V \int_V g^{m(\circ)}(\mathbf{r}',\omega) U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}'',\mathbf{r}) e^{-i\omega T} U^{mn*}(\mathbf{r}'') G_\omega^{mn(\circ)*}(\mathbf{r}',\mathbf{r}'') e^{-i\omega T} dV' dV'' + \cdots\} \tag{30}$$

Now we substitute in Eq. (29) this last expression for $g^n(\mathbf{r},\omega)$:

$$g^n(\mathbf{r},\omega) = \sum_{m=1}^{3} [\delta_m^n g^{m(\circ)}(\mathbf{r},\omega)$$

$$+ \int_V U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) e^{-i\omega T} \{ g^{m(\circ)}(\mathbf{r}',\omega) dV'$$

$$+ \int_V U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) e^{-i\omega T} U^{mn*}(\mathbf{r}'') G_\omega^{mn(\circ)*}(\mathbf{r}'';\mathbf{r}') e^{-i\omega T} g^{m(\circ)}(\mathbf{r}'',\omega) dV'' dV' + \cdots \}] \tag{31}$$

Canceling parentheses we obtain

$$g^n(\mathbf{r},\omega) = \sum_{m=1}^{3} [\delta_m^n g^{m(\circ)}(\mathbf{r},\omega)$$

$$+ \int_V U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) e^{-i\omega T} \{ g^{m(\circ)}(\mathbf{r}',\omega) dV'$$

$$+ \int_V \int_V U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)}(\mathbf{r}';\mathbf{r}) e^{-i\omega T} U^{mn*}(\mathbf{r}'') G_\omega^{mn(\circ)}(\mathbf{r}'';\mathbf{r}') e^{-i\omega T} g^{m(\circ)}(\mathbf{r}'',\omega) dV'' dV' + \cdots] \tag{32}$$

We then obtain the Neumann series [12] for the Fourier transform of the integral equation solution for time reversal (for reference see Eq. (18)):

$$g^n(\mathbf{r},\omega) = \sum_{m=1}^{3} [\delta_m^n g^{m(\circ)}(\mathbf{r},\omega)$$

$$+ \int_V U^{mn*}(\mathbf{r}') G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}) e^{-i\omega T}$$

$$+ \int_V G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}'') e^{-i\omega T} U^{mn*}(\mathbf{r}'') G_\omega^{mn(\circ)*}(\mathbf{r}'';\mathbf{r}) e^{-i\omega T} dV''$$

$$+ \int_V \int_V G_\omega^{mn(\circ)*}(\mathbf{r}';\mathbf{r}'') e^{-i\omega T} U^{mn*}(\mathbf{r}'') G_\omega^{mn(\circ)*}(\mathbf{r}'';\mathbf{r}''') e^{-i\omega T} U^{mn*}(\mathbf{r}''') G_\omega^{mn(\circ)}(\mathbf{r}''';\mathbf{r}) e^{-i\omega T} dV'' dV''' + \cdots\} dV'] \tag{33}$$

6. An algebraic equation for time reverse

Because the bracketed expression in Eq. (36) is convergent, then it must equal the Fourier transform of complete Green's function $\mathcal{G}_\omega^{mn*}(\mathbf{r}';\mathbf{r})$, so that we can write

$$g^{\,n}(\mathbf{r},\omega) = \sum_{m=1}^{3}[\delta_m^n g^{\,m(\circ)}(\mathbf{r},\omega)] + \int_V U^{mn*}(\mathbf{r}')\mathcal{G}_\omega^{mn*}(\mathbf{r}';\mathbf{r})\, g^{\,m(\circ)}(\mathbf{r}',\omega)dV'] \tag{34}$$

Equation (34) can be written in a compact row vector form:

$$g^{\,m}(\omega) = g^{\,n(\circ)}(\omega)[\mathbf{1}+\mathbf{M}^*(\omega)]_m^n \tag{35}$$

In this equation, we define the kernel

$$\mathbf{M}^*(\mathbf{r}';\mathbf{r};\omega) \equiv U^{mn*}(\mathbf{r}')\mathcal{G}_\omega^{mn*}(\mathbf{r}';\mathbf{r}) \tag{36}$$

and also define

$$\int_V g^{\,n(\circ)}(\mathbf{r}')U^{mn*}(\mathbf{r}')\mathcal{G}_\omega^{mn*}(\mathbf{r}';\mathbf{r})dV' \equiv g^{\,n(\circ)}\mathbf{M}^{mn*}(\omega) \tag{37}$$

Transposing Eq. (35) we obtain finally the column vector form (for real interactions):

$$\mathbf{g}^n(\omega) = [\mathbf{1}+\mathbf{M}(\omega)]_m^n \mathbf{g}^{m(\circ)}(\omega) \tag{38}$$

Obviously, Eq. (38) is identical with Eq. (16) but with $\mathbf{M}(\omega)$ instead of $\mathbf{K}(\omega)$.

7. Operators and resonances on continuum formulation

Eqs. (16) and (38) are algebraic representations of integral equations, that is, they are strongly dependent on the Fourier transform of the Green function; indeed the behavior of the late referred function determines the solution whether or not the regime was resonant. For this reason it is convenient to analyze how the Green function changes in the neighborhood of a resonance. With this purpose in mind, we recall Eqs. (13) and (16):

$$\mathbf{g}^{m(\circ)}(\omega) = \left[\mathbf{1}-\mathbf{K}^{(\circ)}(\omega)\right]_n^m \mathbf{g}^n(\omega) \tag{39}$$

$$\mathbf{g}^n(\omega) = \left[\mathbf{1}+\mathbf{K}(\omega)\right]_m^n \mathbf{g}^{m(\circ)}(\omega) \tag{40}$$

By applying the operator $\left[\mathbf{1}+\mathbf{K}(\omega)\right]_m^s$ from the left to Eq. (13) and summing over m, we have

$$\left[\mathbf{1}+\mathbf{K}(\omega)\right]_m^s \mathbf{g}^{m(\circ)}(\omega)=\left[\mathbf{1}+\mathbf{K}(\omega)\right]_m^s\left[\mathbf{1}-\mathbf{K}^{(\circ)}(\omega)\right]_n^m \mathbf{g}^n(\omega) \tag{41}$$

Then using Eq. (16), we obtain

$$\mathbf{g}^s(\omega)=\left[\mathbf{1}+\mathbf{K}(\omega)\right]_m^s\left[\mathbf{1}-\mathbf{K}^{(\circ)}(\omega)\right]_n^m \mathbf{g}^n(\omega) \tag{42}$$

or

$$\mathbf{g}(\omega)=\left[\mathbf{1}+\mathbf{K}(\omega)\right]\left[\mathbf{1}-\mathbf{K}^{(\circ)}(\omega)\right]\mathbf{g}(\omega) \tag{43}$$

In this expression $\mathbf{g}(\omega)$ is also a short notation for a "vector" whose components are $\mathbf{g}^m(\omega)$ or as we have seen $\mathcal{M}^{\ m}\left(\mathbf{r},\mathbf{r}_0;\omega\right)$

Now, by spanning Eq. (41)

$$\mathbf{g}(\omega)=\left[\mathbf{1}-\mathbf{K}^{(\circ)}(\omega)+\mathbf{K}(\omega)-\mathbf{K}(\omega)\mathbf{K}^{(\circ)}(\omega)\right]\mathbf{g}(\omega) \tag{44}$$

This can be expressed as

$$\mathbf{g}(\omega)=\mathbf{1}\mathbf{g}(\omega)+\left[-\mathbf{K}^{(\circ)}(\omega)+\mathbf{K}(\omega)-\mathbf{K}(\omega)\mathbf{K}^{(\circ)}(\omega)\right]\mathbf{g}(\omega) \tag{45}$$

Then we can write

$$\left[-\mathbf{K}^{(\circ)}(\omega)+\mathbf{K}(\omega)-\mathbf{K}(\omega)\mathbf{K}^{(\circ)}(\omega)\right]\mathbf{g}(\omega)=0 \tag{46}$$

and by rearrangement of terms and writing only the operators

$$\mathbf{K}(\omega)=\mathbf{K}^{(\circ)}(\omega)+\mathbf{K}(\omega)\mathbf{K}^{(\circ)}(\omega) \tag{47}$$

But we can now explicitly write Eq. (45) in terms of Green's function:

$$\mathbf{G}(\omega)\mathbf{U}=\mathbf{G}^{(\circ)}(\omega)\mathbf{U}+\mathbf{G}(\omega)\mathbf{U}\mathbf{G}^{(\circ)}(\omega)\mathbf{U} \tag{48}$$

Here we have defined the product

$$U^{nm}(\mathbf{r})G_\omega^{mn}\left(\mathbf{r};\mathbf{r}'\right)\equiv[\mathbf{UG}(\omega)]_m^n \tag{49}$$

That is, the Fourier transform of Green's function satisfies the equation

$$\mathbf{G}(\omega)=\mathbf{G}^{(\circ)}(\omega)+\mathbf{K}(\omega)\mathbf{G}^{(\circ)}(\omega) \tag{50}$$

And if we start with Eq. (39) (time reversal), we obtain by a similar procedure

$$\mathbf{G}(\omega)=\mathbf{G}^{(\circ)}(\omega)+\mathbf{M}(\omega)\mathbf{G}^{(\circ)}(\omega) \tag{51}$$

Now, if we are near a resonance, Eqs. (48) and (49) are transformed in homogeneous equations with solutions we will denote as $\mathbf{w}_e(\omega)$, and if we denote the interaction as $\mathbf{U}$ and the kernel $\mathbf{K}^{(\circ)}(\omega)$, then from Eqs. (48) or (49) without the source term, we have the following relation:

$$\mathbf{w}_l^\dagger(\omega)\mathbf{U}\mathbf{w}_u(\omega)\left[\eta_u^{-1} - \eta_l^{-1}\right] = 0 \tag{52}$$

This relation establishes that the resonant solutions are mutually orthogonal and the functions $\eta(\omega)$ are known as the Fredholm eigenvalues.

8. The homogeneous Fredholm equation and Fredholm's eigenvalue

As we saw in Section 8, the resonant solutions are orthogonal and in Eq. (50) the Fredholm eigenvalues appear, but these last functions emerge when the inhomogeneous Fredholm equations are transformed in a homogeneous equation near a resonance. The resulting homogeneous equation is

$$w_e^m(\mathbf{r};\omega) = \eta_e(\omega)\int_0^\infty \mathbb{K}_n^{m(\circ)}(\omega;\mathbf{r},\mathbf{r}')w_e^m(\mathbf{r}';\omega)dr' \tag{53}$$

According to the theory of homogeneous Fredholm equations [1, 2, 3, 5, 9, 15, 16], one of the conditions for the existence of solutions is that first Fredholm's minor $\mathcal{M}^{\ m}\left(\mathbf{r},\mathbf{r}_0;\omega\right)$ complies

$$\mathcal{M}^{\ m}(\mathbf{r},\mathbf{r}_0;\omega) = \eta(\omega)\Delta(\eta,\omega)$$

$$+\eta(\omega)\int_0^\infty \mathbf{K}_n^{m(\circ)}(\omega;\mathbf{r},\mathbf{s})\mathcal{M}^{\ n}(\mathbf{s},\mathbf{r}_0;\omega)ds \tag{54}$$

From Eqs. (51) and (52) and after a little algebra, we arrive to the following equation:

$$\mathcal{M}^{\ m}(\mathbf{r},\mathbf{r}_0;\omega) - \Delta(\eta,\omega)g^m(\mathbf{r},\omega) =$$

$$\Delta(\eta,\omega)[\eta(\omega) - g^{m(\circ)}(\mathbf{r},\omega)]$$

$$+\eta\int_0^\infty \mathbf{K}_n^{m(\circ)}(\omega;\mathbf{r},\mathbf{s})\left[\mathcal{M}^{\ n}(\mathbf{s},\mathbf{r}_0;\omega) - \Delta(\eta,\omega)g^n(\mathbf{r}',\omega)\right]ds$$

$$+\Delta(\eta,\omega)\left[\eta(\omega) - \upsilon(\omega)\right]\int_0^\infty \mathbf{K}_n^{m(\circ)}(\omega;\mathbf{r},\mathbf{r}')g^n(\mathbf{r}',\omega)dr' \tag{55}$$

At this point, it is convenient to make the following definitions:

$$\Phi(\mathbf{r},\omega) = \mathcal{M}^{\ m}(\mathbf{r},\mathbf{r}_0;\omega) - \Delta(\eta,\omega)g^m(\mathbf{r},\omega) \tag{56}$$

with also

$$\Phi^{(\circ)}(\mathbf{r};\omega)=$$

$$\Delta(\eta,\omega)[\eta(\omega)-g^{m(\circ)}(\mathbf{r},\omega)]$$

$$+\Delta(\eta,\omega)[\eta(\omega)-\upsilon(\omega)]\int_0^\infty \mathbf{K}_n^{m(\circ)}(\omega,\mathbf{r},\mathbf{r}')g^n(\mathbf{r}',\omega)dr' \quad (57)$$

We can reduce the last equations to a compact one:

$$\Phi(\mathbf{r},\omega)=$$

$$\Phi^{(\circ)}(\mathbf{r},\omega)+\eta(\omega)\int_0^\infty \mathbf{K}_n^{m(\circ)}(\omega;\mathbf{r},\mathbf{s})\Phi(\mathbf{s};\omega)ds \quad (58)$$

It is clear that our procedure leads to an inhomogeneous Fredholm equation in which it is possible to observe that the transit from a non-resonant regime to a resonant regime is described by the generalized source term $\Phi^{(\circ)}(\mathbf{r},\omega)$.

9. The role of resonances on broadcasting applications

In precedent sections we have seen how we can go from inhomogeneous to homogeneous Fredholm equations, that is, from non-resonant or conventional

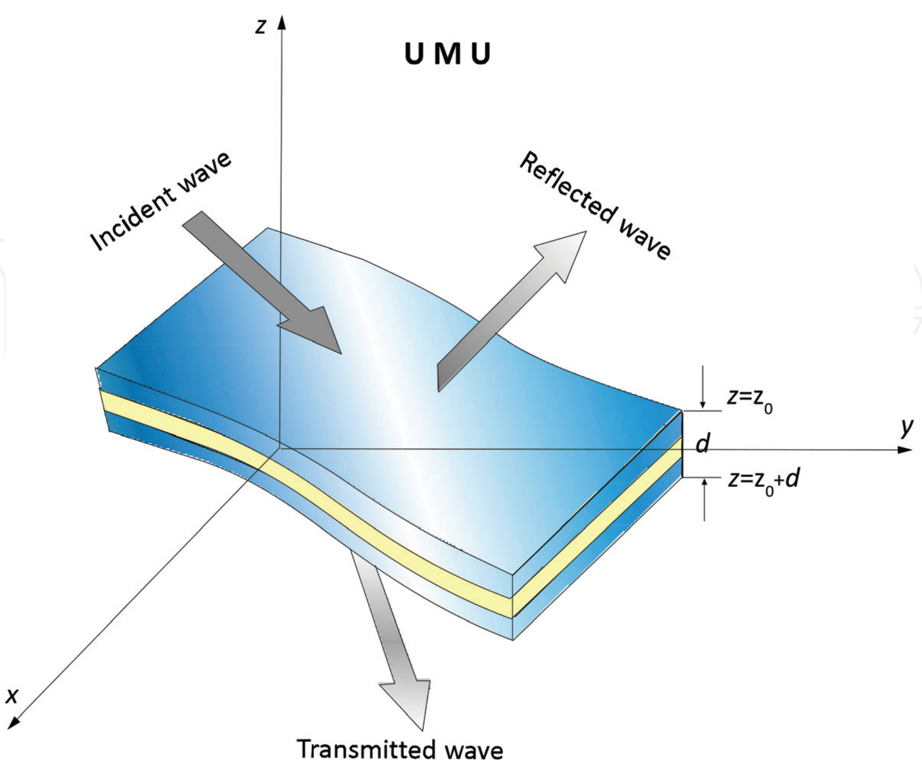

Figure 1.
We show the superposition of three plasma layers subjected to local high electromagnetic potential creating resonances and releasing evanescent waves: Layer M is composed of magnetic plasma, and the U layers are composed of unmagnetized plasma.

solutions to resonant ones. But we know that the resonant solutions are related with a left-hand behavior of the transmitting media, that is, with negative refraction index. On the other hand, Xiang-kun Kong et al. [7, 11] have studied the sign change of the refraction index on devices with superposed layers of magnetized an unmagnetized plasma. This experiment suggested us to propose the plasma sandwich model for transmitting media illustrated in **Figure 1** that consists in itinerant and random appearing of superposed magnetized and unmagnetized plasma layers in high atmosphere that creates localized zones with negative refraction index. According to the precedent results, the change to negative refraction index must establish completely different conditions for the crossing of electromagnetic signals, and we have the appropriate tool to handle these very important phenomena. That is we can observe the transition from evanescent waves (non-traveling waves) to traveling waves like an increase in the polarization effect. In **Figure 1** three plasma regions appear named U (unmagnetized), M (magnetized), and U (again unmagnetized) representing a region on the atmosphere. When some local electromagnetic potential values occur, it is possible to reach left-hand material conditions.

10. Conclusions

In this chapter we have expanded the scope of Fourier transforms by application to a relatively new class (really a vector generalization) of integral equations we named generalized Fredholm equations (GFE). We think that the very relevant subjects we discussed, not only because they are far-reaching implications but also for they are not presented nowadays by other authors, are the properties we have discovered about both the GFE and its own solutions. We have shown a strong relation between the resonant solutions of the generalized homogeneous Fredholm equations for the electromagnetic field and the resonances observed in scattering in nuclear physics. The physical interpretation of the new class of resonances allows us to discern completely new applications in different subjects like electromagnetic wave propagation or the understanding of meta-materials. We give the mathematical proofs for properties of the integral equations, the relation between homogeneous and inhomogeneous equations, and the mechanism for release of the evanescent waves converting them in traveling ones.

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