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Supersymmetric Quantum Mechanics: Two Factorization Schemes and Quasi-Exactly Solvable Potentials

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Abstract

We present the general ideas on supersymmetric quantum mechanics (SUSY-QM) using different representations for the operators in question, which are defined by the corresponding bosonic Hamiltonian as part of SUSY Hamiltonian and its supercharges, which are defined as matrix or differential operators. We show that, although most of the SUSY partners of one-dimensional Schrödinger problems have already been found, there are still some unveiled aspects of the factorization procedure which may lead to richer insights of the problem involved.

Keywords: supersymmetric quantum mechanics, quasi-exactly solvable potentials

1. Introduction

We present the general ideas on supersymmetric quantum mechanics (SUSY-QM) using different representations for the operators in question, which are defined by the corresponding bosonic Hamiltonian as part of SUSY Hamiltonian and its supercharges, \hat{Q}^- and \bar{Q}^+ , which are defined as matrix or differential operators. We show that, although most of the SUSY partners of one-dimensional Schrödinger problems have already been found [1], there are still some unveiled aspects of the factorization procedure which may lead to richer insights of the problem involved. In particular, we refer to the factorization of the Hamiltonian in terms of two non-mutually adjoint operators [2, 3].

In this work, we try three main schemes; the first one consists on finding the eigenvalue Schrödinger equation in one dimension using the matrix representation via the appropriate factorization with ladder-like operators and finding the one parameter isospectral equation for this one. In this scheme, the wave function is written as a supermultiplet. Continuing with the Schrödinger model, we extend SUSY to include two-parameter factorizations, which include the SUSY factorization as particular case. As examples, we include the case of the harmonic oscillator and the Pöschl-Teller potentials. Also, we include the steps for the two-dimensional case and apply it to particular cases. The second scheme uses the differential representation in Grassmann numbers, where the wave function can be written as an

n-dimensional vector or as an expansion in Grassmann variables multiplied by bosonic functions. We apply the scheme in two bosonic variables a particular cosmological model and compare the corresponding solutions found. The third scheme tries on extensions to the SUSY factorization and to the case of quasi-exactly solvable potentials; we present a particular case which does not form part of the class of potentials found using Lie algebras.

To establish the different approaches presented here, we will briefly describe the different main formalisms applied to supersymmetric quantum mechanics, techniques that are now widely used in a rich spectrum of physical problems, covering such diverse fields as particle physics, quantum field theory, quantum gravity, quantum cosmology, and statistical mechanics, to mention some of them:

- In one dimension, SUSY-QM may be considered an equivalent formulation of the Darboux transformation method, which is well-known in mathematics from the original paper of Darboux [4], the book by Ince [5], and the book by Matveev and Salle [6], where the method is widely used in the context of the soliton theory. An essential ingredient of the method is the particular choice of a transformation operator in the form of a differential operator which intertwines two Hamiltonian and relates their eigenfunctions. When this approach is applied to quantum theory, it allows to generate a huge family of exactly solvable local potential starting with a given exactly solvable local potential [7]. This technique is also known in the literature as isospectral formalism [7–10].
- Those defined by means of the use of supersymmetry as a square root [11–14], in which the Grassmann variables are auxiliary variables and are not identified as the supersymmetric partners of the bosonic variables. In this formalism, a differential representation is used for the Grassmann variables. Also the supercharges for the n-dimensional case read as

$$\hat{Q}^- = \psi^\mu \left[-\hbar \partial_{q^\mu} + \frac{\partial S}{\partial q^\mu} \right], \quad \hat{Q}^+ = \bar{\psi}^\nu \left[-\hbar \partial_{q^\nu} - \frac{\partial S}{\partial q^\nu} \right], \quad (1)$$

where S is known as the superpotential function which are related to the physical potential under consideration, when the Hamiltonian density is written as the Hamilton-Jacobi equation, and the algebra for the variables ψ^μ and $\bar{\psi}^\nu$ is

$$\{\psi^\mu, \bar{\psi}^\nu\} = \eta^{\mu\nu}, \quad \{\psi^\mu, \psi^\nu\} = 0, \quad \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0. \quad (2)$$

There are two forms where the equations in 1D are satisfied: in the literature we find either the matrix representation or the differential operator scheme. However for more than one dimensions, there exist many applications to cosmological models, where the differential representation for the Grassmann variables is widely applied [14–18]. There are few works in more dimensions in the first scheme [19]; we present in this work the main ideas to build the 2D case, where the supercharge operators become 4×4 matrices.

2. Factorization method in one dimension: matrix approach

We begin by introducing the main ideas for the one-dimensional quantum harmonic oscillator. The corresponding Hamiltonian is written in operator form as

$$\hat{H}_B = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega_B^2\hat{q}^2 \quad (3)$$

where \hat{q} is the generalized coordinate and \hat{p} is the associated momentum, the canonical commutation relation between these quantities being $[\hat{q}, \hat{p}] = i$. We introduce two new operators, known as the creation and annihilation operators \hat{a}^+ , \hat{a}^- , respectively, defined as

$$\hat{a}^- = \frac{1}{\sqrt{2\omega_B}}(\hat{p} - i\omega_B\hat{q}), \quad \hat{a}^+ = \frac{1}{\sqrt{2\omega_B}}(\hat{p} + i\omega_B\hat{q}). \quad (4)$$

This Hamiltonian can be written in terms of the anti-commutation relation between these operators as

$$\hat{H}_B = \frac{\omega_B}{2} \{\hat{a}^+, \hat{a}^-\}. \quad (5)$$

The symmetric nature of \hat{H}_B under the interchange of \hat{a}^- and \hat{a}^+ suggests that these operators satisfy Bose-Einstein statistics, and it is therefore called bosonic.

Now, we build the operators \hat{b}^- and \hat{b}^+ that obey similar rules to operators \hat{a}^- , \hat{a}^+ changing $[\cdot, \cdot] \Leftrightarrow \{\cdot, \cdot\}$, that is,

$$\{\hat{b}^-, \hat{b}^+\} = 1; \quad \{\hat{b}^-, \hat{b}^-\} = \{\hat{b}^+, \hat{b}^+\} = 0, \quad (6)$$

and in analogy to (5), we define the corresponding new Hamiltonian as

$$\hat{H}_F = \frac{\omega_F}{2} [\hat{b}^+, \hat{b}^-]. \quad (7)$$

The antisymmetric nature of \hat{H}_F under the interchange of \hat{b}^- and \hat{b}^+ suggests that these operators satisfy the Fermi-Dirac statistics, and it is called fermionic.

These operators \hat{b}^- and \hat{b}^+ admit a matrix representations in terms of Pauli matrices that satisfy all rules defined above, that is,

$$\hat{b}^- = \sigma_-, \quad \hat{b}^+ = \sigma_+, \quad \sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \quad (8)$$

$$\text{with } [\sigma_+, \sigma_-] = \sigma_3, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now, consider both Hamiltonians as a composite system, that is, we consider the superposition of two oscillators, one being bosonic and one fermionic, with energy $E_T = E_B + E_F$

$$E_T = \omega_B \left(n_B + \frac{1}{2} \right) + \omega_F \left(n_F - \frac{1}{2} \right) = \omega_B n_B + \omega_F n_F + \frac{1}{2}(\omega_B - \omega_F). \quad (9)$$

When we demand that both frequencies are the same, $\omega_B = \omega_F = \omega$, we introduce a new symmetry, called supersymmetry (SUSY); we can see that the simultaneous creation of a quantum fermion ($n_F \rightarrow n_F + 1$) causes the destruction of quantum boson ($n_B \rightarrow n_B - 1$) and vice versa, in the sense that the total energy is

unaltered. The ground energy state is exact and no degenerate. The degeneration appears from $n = 1$, where it is double degenerate.

In this way, we have the super-Hamiltonian \hat{H}_{susy} , written as

$$\begin{aligned} \hat{H}_{\text{susy}} &= \frac{\omega}{2} \{\hat{a}^+, \hat{a}^-\} + \frac{\omega}{2} [\hat{b}^+, \hat{b}^-] = \frac{\omega}{2} \{\hat{a}^+, \hat{a}^-\} I + \frac{\omega}{2} \sigma_3 = \omega \begin{pmatrix} \hat{a} - \hat{a}^+ & 0 \\ 0 & \hat{a}^+ \hat{a}^- \end{pmatrix} \\ &= \begin{pmatrix} \hat{H}_- & 0 \\ 0 & \hat{H}_+ \end{pmatrix}, \end{aligned} \quad (10)$$

where I is a 2×2 unit matrix and where the two components of \hat{H}_{susy} in (10) can be written independently as

$$\hat{H}_+ = \frac{1}{2} \hat{p}^2 + \frac{1}{2} (\omega^2 q^2 - \omega) \equiv \omega \hat{a}^+ \hat{a}^- \quad (11)$$

$$\hat{H}_- = \frac{1}{2} \hat{p}^2 + \frac{1}{2} (\omega^2 q^2 + \omega) \equiv \omega \hat{a}^- \hat{a}^+. \quad (12)$$

From Eqs. (18) and (19), we can see that \hat{H}_+ and \hat{H}_- are the same representation of one Hamiltonian with a constant shifting ω in the energy spectrum.

The question is, what are the generators for this SUSY Hamiltonian? The answer is, considering that the degeneration is the result of the simultaneous destruction (creation) of quantum boson and the creation (destruction) of quantum fermion, the corresponding generators for this symmetry must be written as $\hat{a}^- \hat{b}^+$ (or $\hat{a}^+ \hat{b}^-$). Therefore we introduce the following generators, called supercharges \hat{Q}^- and \hat{Q}^+ , defined as

$$\hat{Q}^- = \sqrt{2\omega} \hat{a}^- \hat{b}^+ = \sqrt{2\omega} \begin{pmatrix} 0 & \hat{a}^- \\ 0 & 0 \end{pmatrix}, \quad \hat{Q}^+ = \sqrt{2\omega} \hat{a}^+ \hat{b}^- = \sqrt{2\omega} \begin{pmatrix} 0 & 0 \\ \hat{a}^+ & 0 \end{pmatrix}, \quad (13)$$

implying that

$$\hat{H}_{\text{susy}} = \frac{1}{2} \{\hat{Q}^+, \hat{Q}^-\} \quad (14)$$

and satisfying the following relations

$$\{\hat{Q}^-, \hat{Q}^-\} = \{\hat{Q}^+, \hat{Q}^+\} = 0; \quad [\hat{Q}^-, \hat{H}_{\text{susy}}] = [\hat{Q}^+, \hat{H}_{\text{susy}}] = 0. \quad (15)$$

We can generalize this procedure for a certain function $W(q)$, and at this point, we can define two new operators \hat{A}^- and \hat{A}^+ with a property similar to (4),

$$\hat{A}^- = \frac{1}{\sqrt{2\omega}} (\hat{p} - i\omega W(q)), \quad \hat{A}^+ = \frac{1}{\sqrt{2\omega}} (\hat{p} + i\omega W(q)). \quad (16)$$

In order to obtain the general solutions, we can use an arbitrary potential in Eq. (3), that is,

$$\hat{H}_B = \frac{1}{2} \hat{p}^2 + V(q). \quad (17)$$

The Hamiltonians \hat{H}^+ and \hat{H}^- determine two new potentials,

$$\hat{H}_+ = \frac{1}{2}\hat{p}^2 + V_+ = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\left(W^2 - \frac{dW}{dq}\right) \quad (18)$$

$$\hat{H}_- = \frac{1}{2}\hat{p}^2 + V_- = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\left(W^2 + \frac{dW}{dq}\right), \quad (19)$$

where the potential term $V_+(q)$ is related to the superpotential function $W(q)$ via the Riccati equation

$$V_+ = \frac{1}{2}\left(W^2 - \frac{dW}{dq}\right), \quad (20)$$

(modulo constant ϵ , which is related to some energy eigenvalue) and $V_- = \frac{1}{2}\left(W^2 + \frac{dW}{dq}\right) = V_+ + \frac{dW}{dq}$, with the same spectrum, except for the ground state, which is related to the energy potential V_+ .

In a general way, let us now find the general form of the function W . The quantum equation (17) applied to stationary wave function u_i becomes

$$-\frac{1}{2}\frac{d^2u_i}{dq^2} + V(q)u_i = E_i u_i, \quad (21)$$

where E_i are the energy eigenvalues. Considering the transformation $W(q) = -\frac{d\ln[u_i(q)]}{dq}$ and introducing it into (18), we have that

$$V(q) - E_i = \frac{1}{2}\left(W^2 - \frac{dW}{dq}\right) = \left(\frac{1}{2u_i}\frac{du_i}{dq}\right)^2 - \frac{\left(\frac{du_i}{dq}\right)^2 - u_i\frac{d^2u_i}{dq^2}}{2u_i^2} = \frac{1}{2u_i}\frac{d^2u_i}{dq^2}.$$

Then, this equation is the same as the original one, Eq. (21), that is, W is related to an initial solution of the bosonic Hamiltonian. What happens to the isopotential $V_-(q) = \frac{1}{2}\left(W^2 + \frac{dW}{dq}\right)$? Considering that

$$2V_- = W^2 + \frac{dW}{dq} \equiv \hat{W}^2 + \frac{d\hat{W}}{dq} = 2\hat{V}_-,$$

the question is, what is \hat{W} if we know the function W ? Finding it we can build a family of potentials \hat{V}_- depending on a free parameter λ , the supersymmetric parameter that, to some extent, plays the role of internal time. Following the procedure $\hat{W} = W + \frac{1}{y(q)}$, where the function $y(q)$ satisfy the linear differential equation $\frac{dy}{dq} - 2Wy = 1$, the solution implies

$$y(q) = \frac{\lambda + \int u_i^2 dq}{u_i^2}, \quad \rightarrow \quad \hat{W} = W + \frac{u_i^2}{\lambda + \int u_i^2 dq}. \quad (22)$$

The family of potentials \hat{V}_+ can be built now as

$$\hat{V}_+ - E_i = \frac{1}{2}\left(\hat{W}^2 - \frac{d\hat{W}}{dq}\right) = V_- + \frac{d\hat{W}}{dq}. \quad (23)$$

Finally

$$\hat{u} = g(\lambda) \frac{u_i}{\lambda + \int u_i^2 dq} \quad (24)$$

is the isospectral solution of the Schrödinger-like equation for the new family potential (23), with the condition $g(\lambda) = \sqrt{\lambda(\lambda + 1)}$, which in the limit

$$\lambda \rightarrow \pm\infty, \quad g(\lambda) = \lambda, \quad \hat{u}_i \rightarrow u_i.$$

This λ parameter is included not for factorization reasons; in particular, in quantum cosmology the wave functions are still nonnormalizable, and λ is used as a decoherence parameter embodying a sort of quantum cosmological dissipation (or damping) distance.

2.1 Two-dimensional case

We use Witten's idea [20] to find the supersymmetric supercharge operators Q^- and Q^+ that generate the super-Hamiltonian H_{susy} . Using Eqs. (13)–(15), we can generalize the one-dimensional factorization scheme. We define the two-dimensional Hamiltonian as

$$\hat{H}_B(x, y) = \frac{1}{2} \hat{p}_x^2 + \frac{1}{2} \hat{p}_y^2 + V(x) + V(y), \quad (25)$$

where the Schrödinger-like equation can be obtained as the bosonic sector of this super-Hamiltonian in the superspace, i.e., when all fermionic fields are set equal to zero (classical limit).

In two dimensions, the supercharges are defined by the tensorial products

$$Q^- = \sqrt{2} d|^- \otimes \sigma_+, \quad Q^+ = \sqrt{2} d|^+ \otimes \sigma_- \quad (26)$$

with

$$d|^- = \begin{pmatrix} a^- & 0 \\ 0 & b^- \end{pmatrix}, \quad d|^+ = \begin{pmatrix} a^+ & 0 \\ 0 & b^+ \end{pmatrix}, \quad (27)$$

where σ_{\pm} are the same as in (8). From Eq. (26) we have that the supercharges are 4×4 matrices

$$\hat{Q}^+ = \sqrt{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a^+ & 0 & 0 & 0 \\ 0 & b^+ & 0 & 0 \end{bmatrix} \quad \hat{Q}^- = \sqrt{2} \begin{bmatrix} 0 & 0 & a^- & 0 \\ 0 & 0 & 0 & b^- \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (28)$$

where the super-Hamiltonian, (14), can be written as

$$H_{\text{susy}} = \begin{pmatrix} a^- a^+ & 0 & 0 & 0 \\ 0 & b^- b^+ & 0 & 0 \\ 0 & 0 & a^+ a^- & 0 \\ 0 & 0 & 0 & b^+ b^- \end{pmatrix} = \begin{pmatrix} H_-^1(x) & 0 & 0 & 0 \\ 0 & H_-^1(y) & 0 & 0 \\ 0 & 0 & H_+^2(x) & 0 \\ 0 & 0 & 0 & H_+^2(y) \end{pmatrix}, \quad (29)$$

where

$$a^- = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + W(x) \right), \quad a^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W(x) \right) \quad (30)$$

$$b^- = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + Z(y) \right), \quad b^+ = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + Z(y) \right) \quad (31)$$

and $V(x, y) = W(x) + Z(y)$.

The Riccati equation (20) is written in 2D as

$$V_+(x, y) = V_{+1}(x) + V_{+2}(y) = \frac{1}{2} \left(W^2 - \frac{dW}{dx} \right) + \frac{1}{2} \left(Z^2 - \frac{dZ}{dy} \right), \quad (32)$$

and, using separation variables, we get

$$V_1(x) - \frac{1}{2} \left(W^2(x) - \frac{dW}{dx} \right) = C_0 \quad (33)$$

$$V_2(y) - \frac{1}{2} \left(Z^2(y) - \frac{dZ}{dy} \right) = -C_0. \quad (34)$$

In general, we find that each potential V_{+i} satisfies

$$\frac{1}{2} \frac{d^2}{dx^2} u_i(x) + V_{+i} u_i(x) = E_i u_i(x), \quad i = 1, 2, \quad (35)$$

and we can find the isopotential as $W = -\frac{1}{u_1} \frac{du_1}{dx}$, when u_1 is known.

Following the same steps as in the 1D case, we find that the solutions (22) are the same in this case. So, the general solution for \hat{W} is $\hat{W} = W + \frac{1}{y(x)}$, with $y = u_1^{-2}(x) [E_1 + \int u_1^2(x) dx]$. The general solution for the superpotential $\hat{W}(x)$ is

$$\hat{W} = -\frac{1}{u_1} \frac{du_1}{dx} + \frac{u_1^2}{\lambda_1 + \int u_1^2 dx} = W_p + \frac{d}{dx} [\text{Ln}(\lambda_1 + I_1)] \quad (36)$$

where $W_p = -\frac{1}{u_1} \frac{du_1}{dx}$ and $I_1 = \int u_1^2 dx$.

In the same manner, we have that

$$\hat{Z} = -\frac{1}{u_2} \frac{du_2}{dy} + \frac{u_2^2}{\lambda_2 + \int u_2^2 dy} = Z_p + \frac{d}{dy} [\text{Ln}(\lambda_2 + I_2)] \quad (37)$$

with $Z_p = -\frac{1}{u_2} \frac{du_2}{dy}$ and $I_2 = \int u_2^2 dy$.

On the other hand, using the Riccati equation, we can build a generalization for the isopotential, using the new potential \hat{W} , as

$$\hat{V}_{+1}(x, \lambda_1) = \frac{1}{2} \left(\hat{W}^2 - \hat{W}' \right) = V_+(x) - \frac{2u_1 \frac{du_1}{dx}}{\lambda_1 + I_1} + \frac{u_1^4}{(\lambda_1 + I_1)^2}. \quad (38)$$

For the other coordinate, we have

$$\hat{V}_{+2}(y, \lambda_2) = \frac{1}{2} \left(\hat{Z}^2 - \frac{d\hat{Z}}{dy} \right) = V_+(y) - \frac{2u_2 \frac{du_2}{dy}}{\lambda_2 + I_2} + \frac{u_2^4}{(\lambda_2 + I_2)^2}. \quad (39)$$

The general solutions for \hat{u}_i depend on the initial solutions to the original Schrödinger equations in the variables (x, y) , that is, $u_1 = u_1(x)$, $u_2 = u_2(y)$, being

$$\hat{u}_1(x, \lambda_1) = C_1(\lambda_1) \frac{u_1}{\lambda_1 + I_1}, \quad \hat{u}_2(y, \lambda_2) = C_2(\lambda_2) \frac{u_2}{\lambda_2 + I_2}, \quad (40)$$

where the variables $C_i(\lambda_i)$ have the same properties that $g(\lambda)$ obtained in the 1D case.

2.2 Application to cosmological Taub model

The Wheeler-DeWitt equation for the cosmological Taub model is given by

$$\frac{\partial^2 \Psi}{\partial \alpha^2} - \frac{\partial^2 \Psi}{\partial \beta^2} + e^{4\alpha} V(\beta) \Psi = 0 \quad (41)$$

where $V(\beta) = \frac{1}{3}(e^{-8\beta} - 4e^{-2\beta})$. These equations can be separated using $x_1 = 4\alpha - 8\beta$ and $x_2 = 4\alpha - 2\beta$, rendering

$$-\frac{\partial^2 f_1(x_1)}{\partial x_1^2} + \frac{1}{144} e^{x_1} f_1(x_1) = \frac{\omega^2}{4} f_1(x_1), \quad -\frac{\partial^2 f_2(x_2)}{\partial x_2^2} + \frac{1}{9} e^{x_2} f_2(x_2) = \omega^2 f_2(x_2), \quad (42)$$

where the parameter ω is the separation constant. These equations possess the solutions

$$f_1 = K_{i\omega} \left(\frac{1}{6} e^{\frac{x_1}{2}} \right), \quad f_2 = L_{2i\omega} \left(\frac{2}{3} e^{\frac{x_2}{2}} \right) + K_{2i\omega} \left(\frac{2}{3} e^{\frac{x_2}{2}} \right) \quad (43)$$

where K (or I) is the modified Bessel function of imaginary order and the function L is defined as

$$L_{2i\omega} = \frac{\pi i}{2 \sinh(2\omega\pi)} (I_{2i\omega} + I_{-2i\omega}).$$

Using Eqs. (38) and (39), we obtain the isopotential for this model

$$\begin{aligned} \hat{V}(x_1) &= V_+(x_1) - \frac{2K_{i\omega} K'_{i\omega}}{\lambda_1 + I_1} + \frac{K_{i\omega}^4}{(\lambda_1 + I_1)^2}, \\ \hat{V}(x_2) &= V_+(x_2) - \frac{2(L_{2i\omega} + K_{2i\omega})(L_{2i\omega} + K_{2i\omega})'}{\lambda_2 + I_2} + \frac{(L_{2i\omega} + K_{2i\omega})^4}{(\lambda_2 - I_2)^2}. \end{aligned} \quad (44)$$

Using Eq. (40) we can obtain general solutions for the functions f_1 and f_2 in the following way

$$\hat{f}_1 = \frac{C_1 K_{i\omega} \left(\frac{1}{6} e^{\frac{x_1}{2}} \right)}{\lambda_1 + I_1}, \quad \hat{f}_2 = \frac{C_2 \left[L_{2i\omega} \left(\frac{2}{3} e^{\frac{x_2}{2}} \right) + K_{2i\omega} \left(\frac{2}{3} e^{\frac{x_2}{2}} \right) \right]}{\lambda_2 + I_2}. \quad (45)$$

3. Differential approach: Grassmann variables

The supersymmetric scheme has the particularity of being very restrictive, because there are many constraint equations applied to the wave function. So, in this work and in others, we found that there exist a tendency for supersymmetric vacua to remain close to their semiclassical limits, because the exact solutions found are also the lowest-order WKB-like approximations and do not correspond to the full quantum solutions found previously for particular models [14–18].

Maintaining the structure of Eqs. (13)–(16), taking the differential representation for the fermionic operator $\hat{b} \leftrightarrow \psi^\mu$ for convenience in the calculations, and changing the function $W \rightarrow \frac{\partial S}{\partial q^\mu}$, the supercharges for the n-dimensional case read as

$$\hat{Q}^- = \psi^\mu \left[P_\mu + i \frac{\partial S}{\partial q^\mu} \right], \quad \hat{Q}^+ = \bar{\psi}^\nu \left[P_\nu - i \frac{\partial S}{\partial q^\nu} \right], \quad (46)$$

where S is known as the superpotential functions which are related to the physical potential under consideration, when the Hamiltonian density is written as the Hamilton-Jacobi equation, and the following algebra for the variables ψ^μ and $\bar{\psi}^\nu$ (similar to Eq. (6))

$$\{\psi^\mu, \bar{\psi}^\nu\} = \eta^{\mu\nu}, \quad \{\psi^\mu, \psi^\nu\} = 0, \quad \{\bar{\psi}^\mu, \bar{\psi}^\nu\} = 0. \quad (47)$$

These rules are satisfied when we use a differential representation for these $\psi^\mu, \bar{\psi}^\nu$ variables in terms of the Grassmann numbers, as

$$\psi^\mu = \eta^{\mu\nu} \frac{\partial}{\partial \theta^\nu}, \quad \bar{\psi}^\nu = \theta^\nu, \quad (48)$$

where $\eta^{\mu\nu}$ is a diagonal constant matrix, its dimensions depending on the independent bosonic variables that appear in the bosonic Hamiltonian. Now the super-Hamiltonian is written as

$$H_S = \frac{1}{2} \{ \hat{Q}^+, \hat{Q}^- \} = \mathcal{H}_0 + \frac{\hbar}{2} \frac{\partial^2 S}{\partial q^\mu \partial q^\nu} [\bar{\psi}^\mu, \psi^\nu], \quad (49)$$

where $\mathcal{H}_0 = \square + U(q^\mu)$ is the quantum version of the classical bosonic Hamiltonian, \square is the d'Alembertian in three dimension when we have three bosonic independent coordinates, and $U(q^\mu)$ is the potential energy in consideration.

The superspace for three-dimensional model becomes $(q_1, q_2, q_3, \theta^0, \theta^1, \theta^2)$, where the variables θ^i are the coordinate in the fermionic space, as the Grassmann numbers, which have the property of $\theta^i \theta^j = -\theta^j \theta^i$, and the wave function has the representation

$$\Psi = \mathcal{A}_+ + \mathcal{B}_0 \theta^0, \quad 1 \text{ dimension} \quad (50)$$

$$\Psi = \mathcal{A}_+ + \mathcal{B}_0 \theta^0 + \mathcal{B}_1 \theta^1 + \mathcal{A}_- \theta^0 \theta^1, \quad 2 \text{ dimensions} \quad (51)$$

$$\Psi = \mathcal{A}_+ + \mathcal{B}_\nu \theta^\nu + \frac{1}{2} \epsilon_{\mu\nu\lambda} \mathcal{C}^\lambda \theta^\mu \theta^\nu + \mathcal{A}_- \theta^0 \theta^1 \theta^2, \quad 3 \text{ dimensions} \quad (52)$$

where the indices μ, ν, λ values are 0, 1, and 2 and $\mathcal{A}_\pm, \mathcal{B}_\nu$ and \mathcal{C}^λ are bosonic functions which depend on the bosonic coordinates q^μ and not on the Grassmann numbers. Here, the wave function representation structure is set in terms of 2^n

components, for n independent bosonic coordinates, with half of the terms coming from the bosonic (fermionic) contribution into the wave function.

It is well-known that the physical states are determined by the applications of the supercharges \hat{Q}^- and \hat{Q}^+ on the wave functions, that is,

$$\hat{Q}^- \Psi = 0, \quad \hat{Q}^+ \Psi = 0, \quad (53)$$

where we use the usual representation for the momentum $P_\mu = -i \frac{\partial}{\partial q^\mu}$. Considering the 2D case, the last second equation gives

$$\theta^0 : \left[\frac{\partial A_+}{\partial q^0} - A_+ \frac{\partial S}{\partial q^0} \right] = 0, \quad (54)$$

$$\theta^1 : \left[\frac{\partial A_+}{\partial q^1} - A_+ \frac{\partial S}{\partial q^1} \right] = 0, \quad (55)$$

$$\theta^0 \theta^1 : \left[\frac{\partial B_1}{\partial q^0} - B_1 \frac{\partial S}{\partial q^0} \right] - \left[\frac{\partial B_0}{\partial q^1} - B_0 \frac{\partial S}{\partial q^1} \right] = 0. \quad (56)$$

From (54)–(55), we obtain the relation $\frac{\partial A_+}{\partial q^\mu} - A_+ \frac{\partial S}{\partial q^\mu} = 0$ with the solution $A_+ = a_+ e^S$.

On the other hand, the first equation in (53) gives

$$\theta^0 : \left[\frac{\partial A_-}{\partial q^1} + A_- \frac{\partial S}{\partial q^1} \right] = 0, \quad (57)$$

$$\theta^1 : \left[\frac{\partial A_-}{\partial q^0} + A_- \frac{\partial S}{\partial q^0} \right] = 0, \quad (58)$$

$$\text{free term} : - \left[\frac{\partial B_0}{\partial q^0} + B_0 \frac{\partial S}{\partial q^0} \right] + \left[\frac{\partial B_1}{\partial q^1} + B_1 \frac{\partial S}{\partial q^1} \right] = 0. \quad (59)$$

The free term equation is written as $\eta^{\mu\nu} (\partial_\mu B_\nu + B_\nu \partial_\mu S) = 0$, and taking the ansatz $B_\mu = e^{-S} \partial_\nu f_+(q^\mu)$, Eq. (56) is fulfilled, so we obtain for the free term,

$$\square f_+ + 2\eta^{\mu\nu} \nabla_\mu S \nabla_\nu f_+ = 0, \quad (60)$$

with the solution to $f_+ = h(q^1 - q^2)$, with h an arbitrary function depending of its argument. However, this function f must depend on the potential under consideration.

Also, Eqs. (57) and (58) are written as

$$\frac{\partial A_-}{\partial q^\mu} + A_- \frac{\partial S}{\partial q^\mu} = 0, \quad \frac{1}{A_-} \frac{\partial A_-}{\partial q^\mu} = - \frac{\partial S}{\partial q^\mu} \rightarrow \frac{\partial \text{Ln} A_-}{\partial q^\mu} = - \frac{\partial S}{\partial q^\mu} \quad (61)$$

whose solution is $A_- = a_- e^{-S}$. In this way, all functions entering the wave function are

$$A_\pm = a_\pm e^{\pm S}, \quad B_0 = e^{-S} \partial_0(f_+), \quad B_1 = e^{-S} \partial_1(f_+).$$

3.1 The unnormalized probability density

To obtain the wave function probability density $|\Psi|^2$ in this supersymmetric fashion, we need first to integrate over the Grassmann variables θ^i . This procedure

is well-known [21], and here we present the main ideas. Let Ψ_1 and Ψ_2 be two functions that depend on Grassmann numbers; the product $\langle \Psi_1, \Psi_2 \rangle$ is defined as

$$\langle \Psi_1, \Psi_2 \rangle = \int (\Psi_1(\theta^*))^* \Psi_2(\theta^*) e^{-\sum_i \theta_i^* \theta_i} \Pi_i d\theta_i^* d\theta_i, \quad (C\theta_i \dots \theta_r)^* = \theta_r^* \dots \theta_i^* C^*,$$

and the integral over the Grassmann numbers is $\int \theta_i^* \theta_i \dots \theta_m \theta_m^* d\theta_m^* d\theta_m \dots d\theta_i^* d\theta_i = 1$. In 2D, the main contributions to the term $e^{-\sum_i \theta_i^* \theta_i}$ come from

$$e^{-\sum_i \theta_i^* \theta_i} = e^{-\sum_i \theta_i \theta_i^*} = 1 + \theta^0 \theta^{*0} + \theta^1 \theta^{*1} + \theta^0 \theta^{*0} \theta^1 \theta^{*1},$$

and using that $\int \theta d\theta = 1$, and $\int d\theta = 0$, which act as a filter, we obtain that

$$|\Psi|^2 = \mathcal{A}_+^* \mathcal{A}_+ + \mathcal{B}_0^* \mathcal{B}_0 + \mathcal{B}_1^* \mathcal{B}_1 + \mathcal{A}_-^* \mathcal{A}_-.$$

By demanding that $|\Psi|^2$ does not diverge when $|q^0|, |q^1| \rightarrow \infty$, only the contribution with the exponential e^{-2S} will remain.

4. Beyond SUSY factorization

Although most of the SUSY partners of 1D Schrödinger problems have been found [1], there are still some unveiled aspects of the factorization procedure. We have shown this for the simple harmonic oscillator in previous works [2, 3] and will proceed here in the same way for the problem of the modified Pöschl-Teller potential. The factorization operators depend on two supersymmetric type parameters, which when the operator product is inverted, allow us to define a new SL operator, which includes the original QM problem.

The Hamiltonian of a particle in a modified Pöschl-Teller potential is [1, 22]

$$H_{m+1} \Psi = \left(-\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} - \frac{\alpha^2 m(m+1)}{\cosh^2 \alpha x} \right) \Psi = E \Psi, \quad (62)$$

where $\alpha > 0$ and the integer m is greater than 0. To shorten the algebraic equations, we shall set $\frac{\hbar^2}{2\mu} = 1$.

The eigenvalue problem may be solved using the Infeld and Hull's (IH) factorizations [23],

$$A_{m+1}^+ A_{m+1}^- \psi_{m-n}^m = (H_{m+1} + \epsilon_{m+1}) \psi_{m-n}^m, \quad (63)$$

$$A_m^- A_m^+ \psi_{m-n}^m = (H_{m+1} + \epsilon_m) \psi_{m-n}^m, \quad (64)$$

where the IH raising/lowering operators are given by

$$A_m^\mp = k(x, m) \mp \frac{d}{dx} \quad (65)$$

and where $k(x, m) = \alpha m \tanh \alpha x$; also $\epsilon_m = \alpha^2 m^2$, and n is the eigenvalue index,

$$\Psi_n = \psi_{m-n}^m, \quad E_n = -\epsilon_{m-n} = -\alpha^2 (m-n)^2, \quad n = 0, 1, 2, \dots < m. \quad (66)$$

Beginning with the zeroth-order eigenfunctions, the eigenfunctions can be found by successive applications of the raising operator, which only increases the value of the upper index. That is,

$$\psi_\ell^\ell(x) = \sqrt{\frac{\alpha\Gamma(\ell + \frac{1}{2})}{\sqrt{\pi}\Gamma(\ell)}} \cosh^{-\ell} \alpha x. \quad (67)$$

We repeatedly apply the creation operator $A_{s+1}^- \psi_\ell^s = \psi_\ell^{s+1}$. Note that from (63) and (64), $A_m^- A_m^+$ and $A_m^+ A_m^-$ give different Hamiltonian operators.

4.1 Two-parameter factorization of the Pöschl-Teller Hamiltonian

Following our previous work [2, 3], we define two non-mutually adjoint first-order operators,

$$B_m = \eta_m^{-1} \frac{d}{dx} + \beta_m, \quad B_m^* = -\eta_m \frac{d}{dx} + \beta_m, \quad (68)$$

where β_m and η_m are functions of x , and we require that $B_{m+1} B_{m+1}^* = H_{m+1} + \epsilon_{m+1}$. Then β_{m+1} and η_{m+1} are the solutions of

$$-\frac{\eta'}{\eta} + \frac{\beta}{\eta} - \beta\eta = 0, \quad \frac{\beta'}{\eta} + \beta^2 = -\frac{\alpha^2 m(m+1)}{\cosh^2 \alpha x} + \epsilon. \quad (69)$$

By multiplying the first equation by β/η and adding, we have that

$$\left(\frac{\beta_{m+1}}{\eta_{m+1}}\right) + \left(\frac{\beta_{m+1}}{\eta_{m+1}}\right)^2 = -\frac{\alpha^2 m(m+1)}{\cosh^2 \alpha x} + \epsilon_{m+1}. \quad (70)$$

This Riccati equation was found in [24]; it has the solution $\beta/\eta = D \tanh \alpha x$, with $\epsilon = D^2$, and two possible values for D , $D = \alpha(m+1)$, $-\alpha m$. If we simply set $\eta_m \rightarrow 1$, we recover the factorization (63).

The constant ϵ is usually related to the lowest energy eigenvalue, but here the two different values come from the index asymmetry in the factorizations (63) and (64). Following Ref. [24], we solve for $D = \alpha(m+1)$.

The general solution to the pair of coupled equations (69) is

$$\eta_{m+1}(x) = \left[1 + \frac{\gamma_2 \operatorname{sech}^{2(m+1)} \alpha x}{\left(1 + \gamma_1 \int_0^x \operatorname{sech}^{2(m+1)} \alpha y dy\right)^2} \right]^{-1/2}, \quad (71)$$

and

$$\beta_{m+1}(x) = \left[\alpha(m+1) \tanh \alpha x + \frac{\gamma_1 \operatorname{sech}^{2(m+1)} \alpha x}{1 + \gamma_1 \int_0^x \operatorname{sech}^{2(m+1)} \alpha y dy} \right] \times \eta_{m+1}(x), \quad (72)$$

where γ_1 has to satisfy $|\gamma_1| < 2\alpha\Gamma(m+3/2)/(\sqrt{\pi}\Gamma(m+1))$. The corresponding condition on γ_2 involves transcendental functions, but one may use $\gamma_2 > -1 + \gamma_1^2$ to

determine the (γ_1, γ_2) parameter space. When $\gamma_1 = \gamma_2 = 0$ we recover the original IH raising/lowering operators.

4.2 Reversing the operator product: new Sturm-Liouville operator

Now we invert the first-order operators' product, keeping in mind Eq. (64),

$$B_m^* B_m = -\frac{d^2}{dx^2} + 2\frac{\eta'_m d}{\eta_m dx} + \left(V_0 + \epsilon_m - \eta_m \beta'_m - \frac{\beta'_m}{\eta_m} \right). \quad (73)$$

Then we can define a new Sturm-Liouville (SL) eigenvalue problem $\mathcal{L}\Phi_n + \omega(x)E_n\Phi_n = 0$, where

$$\mathcal{L} = \frac{d}{dx} \left[\eta_m^{-2} \frac{d}{dx} \right] + (\epsilon_m - \beta_m^2)(1 + \eta_m^{-2}) - \alpha^2 m(m+1) \operatorname{sech}^2(\alpha x) \quad (74)$$

$$\Phi_n = \phi_{m-n}^m \equiv B_m^* \psi_{m-n}^{m-1}, \quad (75)$$

with the weight function $\omega(x) = \eta_m^{-2}(x)$.

This new SL operator is isospectral to the original PT problem. The zeroth-order eigenfunction is easily found by solving $B\phi_0 = \left[\frac{d}{dx} + \beta_m \eta_m \right] \Phi_0 = 0$ which gives

$$\Phi_0 = \eta_m(x) \times \frac{\operatorname{sech}^{m+1}(\alpha x)}{1 + \gamma_1 \int_0^x \operatorname{sech}^{2(m+1)}(\alpha y) dy}. \quad (76)$$

4.3 Regions in the two-parameter space

We may recover the original QM problem when $\gamma_1 = \gamma_2 = 0$, the origin of the two-parameter space. Moreover, the SUSY partner of the PT problem arises when one sets $\gamma_2 = 0$, moving along the horizontal axis. In this case, \mathcal{L} becomes

$$\mathcal{L} = \frac{d^2}{dx^2} + \alpha^2 \lambda(\lambda+1) \operatorname{sech}^2(\alpha x) - 2S_1^2(\alpha x) - 4\alpha \lambda \tanh(\alpha x) S_1(\alpha x) \quad (77)$$

where $\lambda = m + 1$, with $S_1(\alpha x) = \frac{\gamma_1 \operatorname{sech}^{2\lambda}(\alpha x)}{1 + \gamma_1 \int_0^x \operatorname{sech}^{2\lambda}(\alpha y) dy}$, and $\omega(x) = 1$. These in turn define a SUSY PT problem

$$\left[-\frac{d^2}{dx^2} + \tilde{V}(x) \right] \Phi_n = E_n \Phi_n(x) \quad (78)$$

where the partner SUSY potentials are given by

$$\tilde{V} = -\alpha^2 \lambda(\lambda+1) \operatorname{sech}^2(\alpha x) + 2S_1^2(\alpha x) + 4\alpha \lambda \tanh(\alpha x) S_1(\alpha x). \quad (79)$$

The zeroth-order eigenfunction is defined by $B^- \phi_0 = 0$, that is,

$$\phi_0 = \frac{\operatorname{sech}^\lambda(\alpha x)}{1 + \gamma_1 \int_0^x \operatorname{sech}^{2\lambda}(\alpha y) dy}. \quad (80)$$

5. Quasi-exactly solvable potentials

In exactly solvable problems, the whole spectrum is found analytically, but the vast majority of problems have to be solved numerically. A new possibility arises with the class of QES potentials, where a subset of the spectrum may be found analytically [25–27]. QES potentials have been studied using the Lie algebraic method [25]: Manning [28], Razavy [29], and Ushveridze [30] potentials belong to this class (see also [31]). These are double-well potentials, which received much attention due to their applications in theoretical and experimental problems. Furthermore, hyperbolic-type potentials are found in many physical applications, like the Rosen-Morse potential [32], Dirac-type hyperbolic potentials [33], bidimensional quantum dot [34], Scarf-type entangled states [35], etc. QES potentials' classification has been given by Turbiner [25] and Ushveridze [30].

Here we show that the Lie algebraic procedure may impose strict restrictions on the solutions: we shall construct here analytical solutions for the Razavy-type potential $V(x) = V_0(\sin h^4(x) - k \sin h^2(x))$ based on the polynomial solutions of the related confluent Heun equation (CHE) and show that in that case the energy eigenvalues diverge when $k \rightarrow -1$, a feature solely of the procedure. We shall also show that other QES potentials may be found that do not belong to any of the potentials found using the Lie algebraic method.

5.1 A Razavy-type QES potential

Let us consider Schrödinger's problem for the Razavy-type potential $V(x) = V_0(\sin h^4(x) - k \sin h^2(x))$,

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi(x)}{dx^2} + V_0(\sinh^4(\lambda x) - k \sin h^2(\lambda x))\psi(x) = E\psi(x). \quad (81)$$

For simplicity, we set $\mu = \hbar = \lambda = 1$ [35, 36].

Here the potential function is the hyperbolic Razavy potential $V(x) = \frac{1}{2}(\zeta \cosh(2x) - M)^2$, with $V_0 = 2\zeta^2$, where M energy levels are found if M is a positive integer [29]. It may also be viewed as the Ushveridze potential $V(x) = 2\xi^2 \sinh^4(x) + 2\xi[\xi - 2(\gamma + \delta) - 2\ell]\sinh^2(x) + 2(\delta - \frac{1}{4})(\delta - \frac{3}{4})\text{csch}^2(x) - 2(\gamma - \frac{1}{4})(\gamma - \frac{3}{4})\text{sech}^2(x)$, when $\gamma = \frac{1}{4}$ and $\delta = \frac{3}{4}$, or vice versa [30], which is QES if $\ell = 0, 1, 2, \dots$ (with $\delta \geq \frac{1}{4}$). El-Jaick et al. showed that it is also QES if $\ell = \text{half-integer}$ and $\gamma, \delta = \frac{1}{4}, \frac{3}{4}$ [37].

In the case of the Razavy potential, the solutions obtained by Finkel et al. are

$$\psi_{\sigma\eta}(x, E_R) \propto (\sinh x)^{\frac{1}{2}(1-\sigma-\eta)} (\cosh x)^{\frac{1}{2}(1-\sigma+\eta)} e^{-\frac{\zeta}{2}\cosh(2x)} \sum_{j=0}^n \frac{\hat{P}_j^{\sigma\eta}(E_R)}{(2j + \frac{\eta-\sigma+1}{2})!} \cosh^{2j}(x) \quad (82)$$

with the parameters $(\sigma, \eta) = (\pm 1, 0)$ or $(0, \pm 1)$, the energy eigenvalues being the roots of the polynomials $P_{j+1}^{\sigma\eta}(E_R)$, satisfying the three-term recursive relations

$$\hat{P}_{j+1}^{\sigma\eta} = (E_R - b_j)\hat{P}_j^{\sigma\eta}(E_R) - a_j\hat{P}_{j-1}^{\sigma\eta}(E_R), \quad j \geq 0 \quad (83)$$

with $E_R = 2E$, and

$$\begin{aligned} a_j &= 16\zeta j(2j - \sigma + \eta)(j - n - 1) \\ b_j &= -4j(j + 1 - \sigma + 2\zeta) + (2n + 1)(2(n - \sigma) + 3) + \zeta(\zeta - 2\eta + 4n). \end{aligned} \quad (84)$$

5.2 Symmetric solutions for $V(x) = V_0 \sinh^4(x)$

To find the even solutions to Eq. (81) with $k = 0$, let us set $\beta(x) = \cosh^2(x)$, to get

$$\beta(\beta - 1) \frac{d^2\psi}{d\beta^2} + \left(\beta - \frac{1}{2}\right) \frac{d\psi}{d\beta} + \frac{1}{4} [2E - 2V_0\beta^2 + 4V_0\beta - 2V_0] = 0 \quad (85)$$

and to ensure that $\psi(x)$ vanishes as $x \rightarrow \pm\infty$, let $\psi(x) = e^{-\frac{\alpha}{2}\beta} f(\beta)$. Previous works may not include square-integrable solutions to the Razavy potential [38–40]. By requiring $\alpha^2 = 2V_0$, we obtain [41]

$$\beta(\beta - 1) \frac{d^2f}{d\beta^2} + \left[-\alpha\beta(\beta - 1) + \left(\beta - \frac{1}{2}\right)\right] \frac{df}{d\beta} + \left[\frac{\alpha^2\beta}{4} - \frac{\alpha\beta}{2} + \frac{\alpha}{4} + \frac{E}{2} - \frac{\alpha^2}{4}\right] f = 0. \quad (86)$$

We shall look for rank N polynomial solutions: $f(\beta) = f_0$ for $N = 0$, or $f(\beta) = f_0 \prod_{i=1}^N (\beta - \beta_i)$ for $N > 0$, the β_i being the roots of the resulting polynomial in Eq. (86). Sometimes the $N = 0$ solution is not even considered [35].

The highest power of β in Eq. (86) fix α to $\alpha = 4N + 2$. The energy eigenvalues and the roots satisfy

$$E = \frac{1}{2} \left[\alpha^2 + \alpha \left(4 \sum_{i=1}^N \beta_i - 1 - 4N \right) - 4N^2 \right] \quad (87)$$

$$\sum_{i \neq j}^N \frac{2}{\beta_i - \beta_j} + \frac{-\alpha\beta_i^2 + (\alpha + 1)\beta_i - \frac{1}{2}}{\beta_i^2 - \beta_i} = 0, \quad i = 1, 2, \dots, n. \quad (88)$$

V_0 is found to depend on the order of the polynomial, $V_0 = 2(2N + 1)^2$ for even solutions, and solutions with different N cannot be scaled one into the other due to the $\sinh^4(x)$ dependence of the potential function. The highest solution order is $n = 2N$, and we use subindexes $\{N, n\}$ to label eigenvalues/eigenfunctions.

For $N = 0$, $f(\beta) = 1$, we get $V_0 = 2$, $E_{0,0} = 1$, and the (unnormalized) ground-state eigenfunction $\psi_{0,0}(x) = e^{-\cosh^2(x)}$. For $N = 2$, $f(\beta) = f_0(\beta - \beta_1)(\beta - \beta_2)$, equating to zero the coefficients of the polynomial $P(\beta)$, we get the coupled equations

$$\begin{aligned} \frac{\alpha^2}{4} - \frac{5\alpha}{2} &= 0 \\ 3 + (\beta_1 + \beta_2) \left(-\frac{\alpha^2}{4} + \frac{3\alpha}{2} \right) + \left(-\frac{\alpha^2}{4} + \frac{9\alpha}{2} + \frac{E}{2} \right) &= 0 \\ -3 - (\beta_1 + \beta_2) \left(-\frac{\alpha^2}{4} + \frac{5\alpha}{4} + \frac{E}{2} + 1 \right) + \beta_1\beta_2 \left(\frac{\alpha^2}{4} - \frac{\alpha}{2} \right) &= 0 \\ \frac{1}{2}(\beta_1 + \beta_2) + \beta_1\beta_2 \left(-\frac{\alpha^2}{4} + \frac{\alpha}{4} + \frac{E}{2} \right) &= 0. \end{aligned} \quad (89)$$

Solving these, we find that $V_0 = 50$, and the three possible eigenvalues, $E_{2,0} = 2.6301$, $E_{2,2} = 19.0121$, and $E_{2,4} = 43.2490$.

5.3 Antisymmetric solutions

In order to find antisymmetric solutions to Eq. (86), we set $f(\beta) = \sinh(x)g(\beta)$, to obtain

$$\beta[\beta - 1] \frac{d^2g}{dx^2} + \left[-\alpha\beta^2 + (\alpha + 2)\beta - \frac{1}{2} \right] \frac{dg}{dx} + \left[\left(-\alpha + \frac{\alpha^2}{4} \right) \beta + \left(-\frac{\alpha^2}{4} + \frac{\alpha}{4} + \frac{E}{2} + \frac{1}{4} \right) \right] g = 0. \quad (90)$$

This CHE can be solved in power series: $g(\beta) = g_0$ if $N = 0$, or $g(\beta) = g_0 \prod_{i=1}^N (\beta - \beta_i)$ for $N > 0$. Then, $\alpha = 4(N + 1)$, and

$$E = \frac{1}{2} \left[\alpha^2 + \alpha \left(4 \sum_{i=1}^N \beta_i - 1 - 4N \right) - 4N^2 - 4N - 1 \right]. \quad (91)$$

Here, $V_0 = 8(N + 1)^2$, and all even and odd solutions have different V_0 . The maximum solutions order is $n = 2N + 1$. For example, for $N = 3$ we get $\alpha = 16$, $V_0 = 128$, and

$$\begin{aligned} (\beta_1 + \beta_2 + \beta_3) \left(3\alpha - \frac{\alpha^2}{4} \right) + \left(-\frac{\alpha^2}{4} + \frac{13\alpha}{4} + \frac{E}{2} - \frac{49}{4} \right) &= 0 \\ (\beta_1 + \beta_2 + \beta_3) \left(\frac{\alpha^2}{4} - \frac{9\alpha}{4} - \frac{E}{2} - \frac{25}{4} \right) + (\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1) \left(\frac{\alpha^2}{4} - 2\alpha \right) - \frac{15}{2} &= 0 \\ 3(\beta_1 + \beta_2 + \beta_3) + (\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1) \left(-\frac{\alpha^2}{4} + \frac{5\alpha}{4} + \frac{9}{4} + \frac{E}{2} \right) + \beta_1\beta_2\beta_3 \left(-\frac{\alpha^2}{4} + \alpha \right) &= 0 \\ -\frac{1}{2} (\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1) - \beta_1\beta_2\beta_3 \left(\frac{\alpha^2}{4} - \frac{\alpha}{4} - \frac{E}{2} - \frac{1}{4} \right) &= 0. \end{aligned} \quad (92)$$

We find four eigenvalues, $E_{3,1} = 12.8152$, $E_{3,3} = 40.4568$, $E_{3,5} = 75.7246$, and $E_{3,7} = 117.003$.

6. The potential function $V(x) = V_0(\sinh^4(x) - k \sinh^2(x))$

Now we apply our analysis to the problem with $V(x) = V_0(\sinh^4(x) - k \sinh^2(x))$, which is a symmetric double well if $k > 0$. To find even solutions, we set again $\beta(x) = \cosh^2(x)$ and $\psi(\beta) = e^{-\frac{\alpha}{2}\beta}f(\beta)$, with $\alpha^2 = 2V_0$,

$$\beta(\beta - 1) \frac{d^2f}{d\beta^2} + \left[-\alpha\beta(\beta - 1) + \left(\beta - \frac{1}{2} \right) \right] \frac{df}{d\beta}$$

$$+ \left[\frac{\alpha^2 \beta}{4} (1+k) - \frac{\alpha \beta}{2} + \frac{\alpha}{4} + \frac{E}{2} - \frac{\alpha^2}{4} (1+k) \right] f = 0. \quad (93)$$

We now find that $V_0 = \frac{2(2N+1)^2}{1+k}$, k varying freely. For example, if $N = 0$, $E_{0,0} = 1/(1+k)$, and no negative energy eigenvalues may exist. For $N = 1$ the two energy eigenvalues found are

$$E = \frac{9 - (1+k) \pm \sqrt{(1+k)^2 + 36}}{1+k} \quad (94)$$

meaning that for $k > 3/2$ we will have negative eigenvalues. Note that for $N > 0$, it is always possible to find a zero-energy ground state, a feature that may have cosmological implications [18].

For the case with $N = 2$, choosing $k = 4$, the energy eigenvalues are $E_{2,0} = -3.74456$, $E_{2,2} = 1.00000$, and $E_{2,4} = 7.74456$. The corresponding eigenfunctions are plotted in **Figure 1**.

Now, to find the antisymmetric eigenfunctions, we set $f(\beta) = \sinh(x) g(\beta)$, to get the CHE

$$\beta(\beta - 1) \frac{d^2 g}{d\beta^2} + \left[-\alpha\beta^2 + (\alpha + 2)\beta - \frac{1}{2} \right] \frac{dg}{d\beta} + \left[\beta \left(\frac{\alpha^2}{4} (1+c) - \alpha \right) + \left(\frac{\alpha}{4} + \frac{E}{2} - \frac{\alpha^2}{4} (1+c) + \frac{1}{4} \right) \right] g = 0. \quad (95)$$

For $N = 0$ we get that $\alpha = 4/(1+k)$ and $E_1 = 6/(1+k) - 1/2$, such that if $k > 11$, we may find negative energy eigenvalues. For $N = 2$, $\alpha = 12/(1+k)$, if we set $k = 5$, the energy eigenvalues found are $E_{2,1} = -7.11693$, $E_{2,3} = 1.08119$, and $E_{2,5} = 9.53574$. The eigenfunctions are plotted in **Figure 1**.

Note that in this case $(E_1 - E_0)/E_0 = 0.0052$, and it is not possible to distinguish these eigenvalue's lines from each other in **Figure 1** for antisymmetric

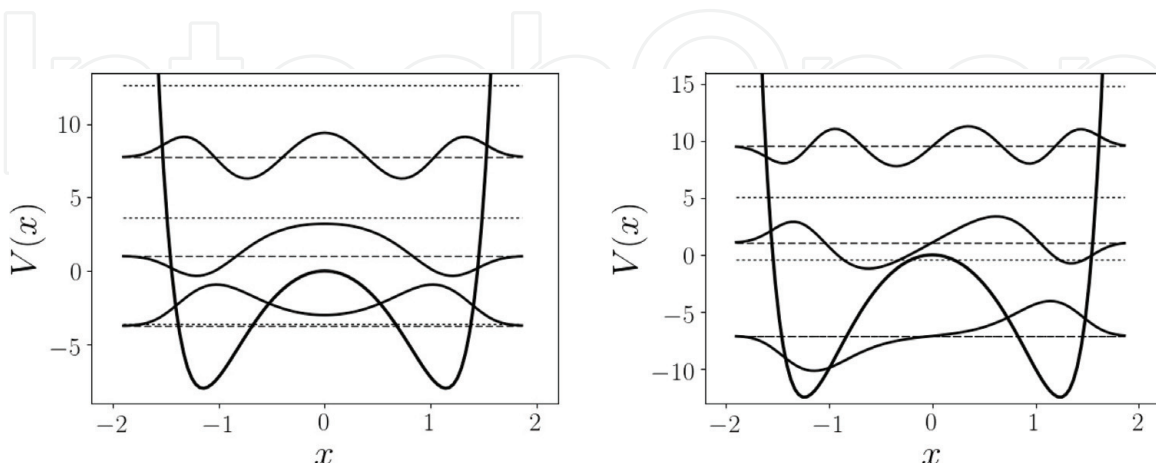


Figure 1. Left: the three even eigenfunctions (narrow solid lines) found analytically for $k = 4$ and $N = 2$, together with the corresponding eigenvalues (dashed lines). Right: the three odd eigenfunctions (narrow solid lines) found analytically for $k = 5$ and $N = 2$, together with the corresponding eigenvalues (dashed lines). The unsolved eigenvalues are shown in dotted lines.

eigenvalues, implying quasi-degenerate eigenstates. A similar effect is seen in the symmetric case.

6.1 The case with $k = -1$

As was seen in Section VI, the ground-state energy diverges as $1/(1+k)$ and as $k \rightarrow -1$, and this also happens to all higher-order even eigenvalues (see Eq. (94)). This is a strange behavior, since it is clear that the potential function has a rather simple functional form for any value of k : a single or double well with infinite barriers. We can see that this is only a characteristic due to the analytical solution procedure, coming from the fact that the potential strength V_0 is also divergent when $k \rightarrow -1$.

6.2 Unclassified QES potentials

Finally, we would like to emphasize that there should be other potential functions which may not be classified from the Lie algebraic method [25].

Indeed, let us consider Schrödinger's problem with the potential function

$$V(x) = \frac{\alpha^2}{2} \cosh^2(x) - \frac{3\alpha}{2} \cosh(x) + \frac{\alpha}{\cosh(x)}. \quad (96)$$

For this problem, the ground-state eigenfunction and eigenvalue are given by

$$\psi = \psi_0 e^{-\alpha \cosh(x)} \cosh(x), \quad E = \frac{\alpha^2 - 1}{2} \quad (97)$$

while this particular problem does not belong to the class of potentials found using the Lie algebraic method. Similar potentials may be found which do not belong to that class, leaving space for further developments.

Acknowledgements

This work was partially supported by CONACYT 179881 grants and PROMEP grants UGTO-CA-3. This work is part of the collaboration within the Instituto Avanzado de Cosmología. E. Condori-Pozo is supported by a CONACYT graduate fellowship.

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