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Chapter

# Oscillation Criteria of Two-Dimensional Time-Scale Systems

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## Abstract

Oscillation and nonoscillation theories have recently gotten too much attention and play a very important role in the theory of time-scale systems to have enough information about the long-time behavior of nonlinear systems. Some applications of such systems in discrete and continuous cases arise in control and stability theories for the unmanned aerial and ground vehicles (UAVs and UGVs). We deal with a two-dimensional nonlinear system to investigate the oscillatory behaviors of solutions. This helps us understand the limiting behavior of such solutions and contributes several theoretical results to the literature.

**Keywords:** oscillation, nonoscillation, two-dimensional systems, time scale, nonlinear system, fixed point theorems

## 1. Introduction

This chapter analyses the oscillatory behavior of solutions of two-dimensional (2D) nonlinear time-scale systems of first-order dynamic equations. We also investigate the existence and asymptotic properties of such solutions. The tools that we use are the most well-known fixed point theorems to consider the sign of the component functions of solutions of our system. A *time scale*, denoted by $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, which is introduced by a German mathematician, Stefan Hilger, in his PhD thesis in 1988 [1]. His primary purpose was to unify continuous and discrete analysis and extend the results to one comprehensive theory. For example, the results hold for differential equations when $\mathbb{T} = \mathbb{R}$, while the results hold for difference equations when $\mathbb{T} = \mathbb{Z}$. Therefore, there might happen to be two different proofs and maybe similar in most cases. In other words, our essential desire is to combine continuous and discrete cases in one comprehensive theory and remove the obscurity from both. For more details in the theory of differential and difference equations, we refer the books [2–4] to interested readers. As for the time-scale theory, we assume most of the readers are not familiar with the time-scale calculus, and thus we give a concise introduction to the theory of time scales from the books [5, 6] written by Bohner and Peterson in 2001 and 2003, respectively.

Two-dimensional dynamical systems have recently gotten too much attention because of their potential in applications in engineering, biology, and physics (see, e.g., [7–11]). For example, Bartolini and Pvdvnowski [12] consider a nonlinear system and propose a new method for the asymptotic linearization by means of continuous control law. Also Bartolini et al. [13, 14] consider an uncertain second-order nonlinear system and propose a new approximate linearization and sliding mode to control such systems. In addition to the nonoscillation for two-dimensional systems of first-order equations, periodic and subharmonic solutions are also investigated in [15–17], and significant contributions have been made. Another type of two-dimensional systems of dynamic equations is the Emden-Fowler type equation, named after E. Fowler after he did the mathematical foundation of a second-order differential equation in a series of four papers during 1914–1931 (see [18–21]). This system has several fascinating applications such as in gas dynamics and fluid mechanics, astrophysics, nuclear physics, relativistic mechanics, and chemically reacting systems (see [9, 22–24]).

This chapter is organized as follows: In Section 2, we give the calculus of the time-scale theory for those who are not familiar with the time scale (see [5]). In Section 3, referred to [25, 26], we show the existence and asymptotic behaviors of nonoscillatory solutions of a two-dimensional homogeneous dynamical system on time scales by using improper integrals and some inequalities. We also give enough examples for readers to see our results work nicely. Section 4, referred to [27], provides us oscillation criteria for two-dimensional nonhomogeneous time-scale systems by using famous inequalities and rules such as comparison theorem and chain rules on time scales. Finally, we give a conclusion and provide some exercises to the readers to have them comprehend the main results in the last two sections.

## 2. Preliminaries

The examples of the time scales are not restricted with the set of real numbers $\mathbb{R}$ and the set of integers $\mathbb{Z}$. There are several other time scales which are used in many application areas such as $q^{\mathbb{N}_0} = \{1, q, q^2, \cdots,\}, q > 1$ (called $q$-difference equations [28]), $\mathbb{T} = h\mathbb{Z}, h > 0$, $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$, etc. On the other hand, the set of rational numbers $\mathbb{Q}$, the set of irrational numbers $\mathbb{R}\backslash\mathbb{Q}$, and the open interval $(a, b)$ are not time scales since they are not closed subsets of $\mathbb{R}$. For the following definitions and theorems in this section, we refer [5], (Chapter 1), and [29] to the readers.

**Definition 2.1** Let $\mathbb{T}$ be a time scale. Then, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) \coloneqq \inf\{s \in \mathbb{T} : s > t\} \quad \text{for all} \quad t \in \mathbb{T}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is given by

$$\rho(t) \coloneqq \sup\{s \in \mathbb{T} : s < t\} \quad \text{for all} \quad t \in \mathbb{T}.$$

Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) \coloneqq \sigma(t) - t \quad \text{for all} \quad t \in \mathbb{T}$.

For a better explanation, the operator $\sigma$ is the first next point, while the operator $\rho$ is the first back point on a time scale. And $\mu$ is the length between the next point and the current point. So it is always nonnegative. **Table 1** shows some examples of the forward/backward jump operators and the graininess function for most known time scales.

If $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then $t$ is said to be *right-dense*, and if $t > \inf\mathbb{T}$ and $\rho(t) = t$, we say $t$ is *left-dense*. Also, if $t$ is right- and left-dense at the same time, then $t$ is said to be *dense*. In addition to left and right-dense points, it is said to be *right-scattered* when $\sigma(t) > t$, and t is called *left-scattered* when $\rho(t) < t$. Also, if $t$ is right-and left-scattered at the same time, then $t$ is called *isolated*. **Figure 1** shows the classification of points on time scales, clarifying the operators $\sigma$, $\rho$ and $\mu$ (see [5]).

Next, we introduce the definition of derivative on any time scale. Note that if $\sup\mathbb{T} < \infty$, then $\mathbb{T}^\kappa = \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}]$, and $\mathbb{T}^\kappa = \mathbb{T}$ if $\sup\mathbb{T} = \infty$. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function. Then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t)) \quad for\ \ all \quad t \in \mathbb{T}$.

**Definition 2.2** If there does exist a $\delta > 0$ such that

$$|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad for\ \ all \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

for any $\varepsilon$, then $g$ is called *delta differentiable* on $\mathbb{T}^\kappa$ and $g^\Delta$ is said to be *delta derivative* of $g$. Sometimes, delta derivative is referred as Hilger derivative in the literature (see [5]).

**Theorem 2.3** *Suppose that $f, g : \mathbb{T} \to \mathbb{R}$ is a function with $t \in \mathbb{T}^\kappa$. Then.*

*i. g is said to be continuous at t if g is differentiable at t.*

*ii. g is differentiable at t and*

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)},$$

provided $g$ is continuous at $t$ and $t$ is right-scattered.

*iii. Let t be right-dense, then g is differentiable at t if and only if*

$$g^\Delta(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}$$

is equal to a finite number.

| $\mathbb{T}$ | $\sigma(t)$ | $\rho(t)$ | $\mu(t)$ |
|---|---|---|---|
| $\mathbb{R}$ | $t$ | $t$ | $0$ |
| $h\mathbb{Z}$ | $t+h$ | $t-h$ | $h$ |
| $\mathbb{N}_0^2$ | $(\sqrt{t}+1)^2$ | | $1+2\sqrt{t}$ |
| $q^{\mathbb{N}_0}$ | $tq$ | $\frac{t}{q}$ | $(q-1)t$ |

**Table 1.**
*Examples of most known time scales.*

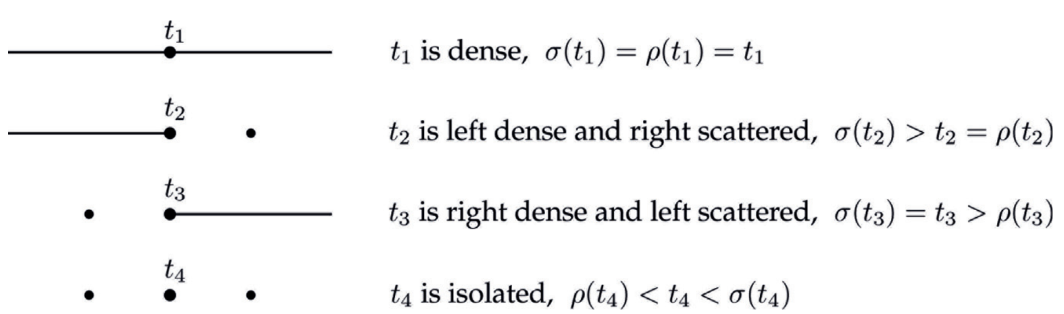


**Figure 1.**
*Classification of points.*

*iv*. If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at $t$ with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}$ turns out to be the usual derivative $f'$ on continuous case, while $f^{\Delta}$ is reduced to forward difference operator $\Delta f$, defined by $\Delta f(t) = f(t+1) - f(t)$ if $\mathbb{T} = \mathbb{Z}$. The following example is a good example of time scale applications in electrical engineering (see [5], Example 1.39–1.40).

**Example 2.4** Consider a simple electric circuit, shown in **Figure 2** with resistor $R$, inductor $L$, capacitor $C$ and the current $I$.

Suppose, we discharge the capacitor periodically every time unit and assume that the discharging small $\delta > 0$ time units. Then we can model it as

$$\mathbb{P}_{1-\delta,\delta} = \bigcup_{k \in \mathbb{N}_0} [k, k+1-\delta]$$

by using the time scale. Suppose that $Q(t)$ is the total charge on the capacitor at time $t$ and $I(t)$ is the current with respect to time $t$. Then the total charge $Q$ can be defined by

$$Q^{\Delta}(t) = \begin{cases} bQ(t) & \textit{if} \quad t \in \bigcup_{k \in \mathbb{N}} \{k - \delta\} \\ I & \textit{otherwise} \end{cases}$$

and

$$I^{\Delta}(t) = \begin{cases} 0 & \textit{if} \quad t \in \bigcup_{k \in \mathbb{N}} \{k - \delta\} \\ -\frac{1}{LC}Q(T) - \frac{R}{L}I(t) & \textit{otherwise,} \end{cases}$$

*where* $-1 < b\delta < 0$.

Finally, we introduce the integrals on time scales, but before that, we must give the following definition to define delta integrable functions (see [5]).

**Definition 2.5** $g : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (rd-continuous) if its left-sided limits exist at left-dense points in $\mathbb{T}$ and it is continuous at right-dense points in $\mathbb{T}$. We denote rd-continuous functions by $C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $g$ that are differentiable and whose derivative is *rd-continuous* is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$. Finally, we denote continuous functions by $C$ throughout this chapter.

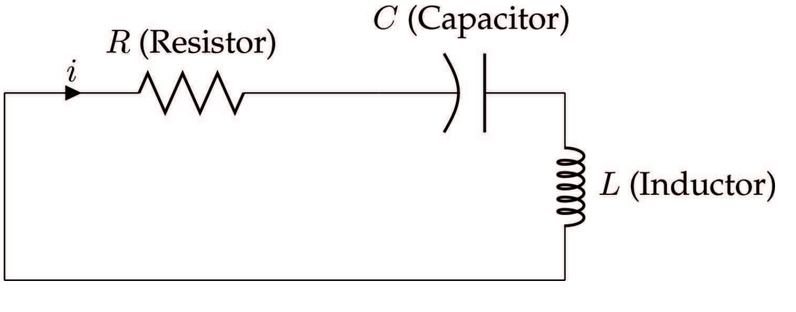


**Figure 2.**
*Electric circuit.*

**Theorem 2.6** ([5], Theorem 1.60) *For $\sigma : \mathbb{T} \to \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$, we have the following:*

i. *The jump operator $\sigma$ is rd-continuous.*

ii. *If $f$ is continuous, then it is rd-continuous.*

The Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

The following theorem presents the existence of antiderivatives.

**Theorem 2.7** *Every rd-continuous function has an antiderivative. Moreover, F given by*

$$F(t) = \int_{t_0}^t f(s)\Delta s \quad for \quad t \in \mathbb{T}$$

is an antiderivative of $f$.

Similar to the continuous analysis, we have integral properties and some of them are presented as follows ([5] or [29]):

**Theorem 2.8** *Suppose that $h_1$ and $h_2$ are rd-continuous functions, $c, d, e \in \mathbb{T}$ and $\beta \in \mathbb{R}$.*

i. $h_1$ is nondecreasing if $h_1^\Delta \geq 0$.

ii. If $h_1(t) \geq 0$ for all $c \leq t \leq d$, then $\int_c^d h_1(t)\Delta t \geq 0$.

iii. $\int_c^d [(\beta h_1(t)) + (\beta h_2(t))] = \beta \int_c^d h_1(t)\Delta t + \beta \int_a^b h_2(t)\Delta t$.

iv. $\int_c^e h_1(t)\Delta t = \int_c^d h_1(t)\Delta t + \int_d^e h_1(t)\Delta t$.

v. $\int_c^d h_1(t)h_2^\Delta(t)\Delta t = (h_1h_2)(d) - (h_1h_2)(c) - \int_c^d h_1^\Delta(t)h_2(\sigma(t))\Delta t$

vi. $\int_a^a f(t)\Delta t = 0$.

**Table 2** shows how the derivative and integral are defined for some time scales for $a, b \in \mathbb{T}$.

| $\mathbb{T}$ | $f^\Delta(t)$ | $\int_a^b f(t)\Delta t$ |
|---|---|---|
| $\mathbb{R}$ | $f'(t)$ | $\int_a^b f(t)dt$ |
| $\mathbb{Z}$ | $\Delta f(t)$ | $\sum_{t=a}^{b-1} f(t)$ |
| $q^{\mathbb{N}_0}$ | $\Delta_q f(t)$ | $\sum_{t \in [a,b)_{q^{\mathbb{N}_0}}} f(t)\mu(t)$ |

**Table 2.**
*Derivative and integrals for most common time scales.*

We finish the section by Schauder's fixed point theorem, proved by Juliusz Schauder in 1930, and Knaster fixed point theorem, proved by Knaster in 1928 (see [30], Theorem 2.A and [31], respectively).

**Theorem 2.9** Schauder's fixed point theorem. *Suppose that $S$ is a nonempty, bounded, closed, and convex subset of a Banach space $Y$ and that $F : S \to S$ is a compact operator. Then, we conclude that $F$ has a fixed point such that $y = Fy$.*

**Theorem 2.10** The Knaster fixed point theorem. *Suppose that $(S, \leq)$ is a complete lattice and that $F : S \to S$ is order preserving, then $F$ has a fixed point such that $y = Fy$. In fact, we say that the set of fixed points of $F$ is a complete lattice.*

Finally, we note that throughout this paper, we assume that $\mathbb{T}$ is unbounded above and whenever we write $t \geq t_1$, we mean $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$.

## 3. Nonoscillation on a two-dimensional time-scale systems

This section focuses on the nonoscillatory solutions of a two-dimensional dynamical system on time scales. To do this, we consider the system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = r(t)g(x(t)), \end{cases} \tag{1}$$

where $p, r \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and $f$ and $g$ are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$.

By a solution of (1), we mean a collection of functions, where $x, y \in \mathrm{C}^1_{\mathrm{rd}}([t_0, \infty)_T; R)$, $T \geq t_0$ and $(x, y)$ satisfies system (1) for all large $t \geq T$.

Note that system (1) is reduced to the system of differential equations when the time scale is the set of real numbers $\mathbb{R}$, i.e., $f^{\Delta} = f'$ (see [32]). And when $\mathbb{T} = \mathbb{Z}$, system (1) turns out to be a system of difference equations, i.e., $f^{\Delta} = \Delta f$ (see [33]). Other versions of system (1), the case $\mathbb{T} = \mathbb{Z}$, are investigated by Li et al. [34], Cheng et al. [35], and Marini et al. [36]. More details about the continuous and discrete versions of system (1) are given in the conclusion section.

**Definition 3.1** A solution $(x, y)$ of system (1) is said to be proper if

$$\sup\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$$

holds for $t \geq t_0$.

**Definition 3.2** A proper solution $(x, y)$ of (1) is said to be nonoscillatory if the component functions $x$ and $y$ are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise it is said to be oscillatory.

Suppose that $N$ is the set of all nonoscillatory solutions of system (1). It can easily be shown that any nonoscillatory solution $(x, y)$ of system (1) belongs to one of the following classes:

$$N^+ := \{(x, y) \in N : xy > 0 \text{ eventually}\}$$

$$N^- := \{(x, y) \in N : xy < 0 \text{ eventually}\}.$$

Let $(x, y)$ be a solution of system (1). Then one can show that the component functions $x$ and $y$ are themselves nonoscillatory (see, e.g., [37]). Throughout this section, we assume that the first component function $x$ of the nonoscillatory solution $(x,y)$ is eventually positive. The results can be obtained similarly for the case $x < 0$ eventually.

We obtain the existence criteria for nonoscillatory solutions of system (1) in $N^+$ and $N^-$ by using the fixed point theorems and the following improper integrals:

$$I_1 = \int_{t_0}^{\infty} p(t)f\left(k_1 \int_{t_0}^{t} r(s)\Delta s\right)\Delta t, \qquad I_2 = \int_{t_0}^{\infty} r(t)g\left(k_2 \int_{t_0}^{t} p(s)\Delta s\right)\Delta t,$$

$$I_3 = \int_{t_0}^{\infty} p(t)f\left(k_3 - k_4 \int_{t}^{\infty} r(s)\Delta s\right)\Delta t, \quad I_4 = \int_{t_0}^{\infty} r(t)g\left(k_5 \int_{t}^{\infty} p(s)\Delta s\right)\Delta t,$$

$$P(t_0,t) = \int_{t_0}^{t} p(s)\Delta s, \qquad R(t_0,t) = \int_{t_0}^{t} r(s)\Delta s,$$

where $k_i$, $i = 1-5$ are some constants.

### 3.1 Existence of nonoscillatory solutions of (1) in $N^+$

Suppose that $(x,y)$ is a nonoscillatory solution of (1) such that $x > 0$. Then system (1) implies that $x^\Delta > 0$ and $y^\Delta > 0$ eventually. Therefore, as a result of this, we have that $x$ converges to a positive finite number or $x \to \infty$ and similarly $y$ tends to a positive finite number or $y \to \infty$. One can have very similar asymptotic behaviors when $x < 0$. Hence, as a result of this information, the following subclasses of $N^+$ are obtained:

$$N_{F,F}^+ = \left\{(x,y) \in N^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\},$$

$$N_{F,\infty}^+ = \left\{(x,y) \in N^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = \infty\right\},$$

$$N_{\infty,F}^+ = \left\{(x,y) \in N^+ : \lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |y(t)| = d\right\},$$

$$N_{\infty,\infty}^+ = \left\{(x,y) \in N^+ : \lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |y(t)| = \infty\right\}.$$

To focus on $N^+$, first consider the following four cases for $t_0 \in \mathbb{T}$ :

1. $P(t_0,\infty) = \infty$ and $R(t_0,\infty) = \infty$

2. $P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$

3. $P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$

4. $P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$

Suppose $P(t_0,\infty) = \infty$ and $R(t_0,\infty) = \infty$ and that $(x,y)$ is a nonoscillatory solution in $N^+$. Integrating the equations of system (1) from $t_0$ to $t$ separately gives us

$$x(t) \geq x(t_0) + f(y(t_0)) \int_{t_0}^{t} p(s)\Delta s$$

and

$$y(t) \geq y(t_0) + g(x(t_0)) \int_{t_0}^{t} r(s)\Delta s, \quad t \geq t_0.$$

Thus, we get $x(t) \to \infty$ and $y(t) \to \infty$ as $t \to \infty$. In view of this information, the following theorem is given without any proof.

**Theorem 3.3** *Let $P(t_0, \infty) = \infty$ and $R(t_0, \infty) = \infty$. Then any nonoscillatory solution of system (1) belongs to $N^+_{\infty,\infty}$.*

Next, we consider the other three cases to obtain the nonoscillation criteria for system (1).

### *3.1.1 The case $P(t_0, \infty) = \infty$ and $R(t_0, \infty) < \infty$*

Suppose that $(x, y)$ is a nonoscillatory solution of system (1) such that $x > 0$ and $y > 0$ eventually. Then by the integration of the first equation of system (1) from $t_0$ to $t$, we have that there exists $k > 0$

$$x(t) \geq x(t_0) + k \int_{t_0}^{t} p(s)\Delta s, \quad t_0 \in \mathbb{T}. \tag{2}$$

Then by taking the limit of (2) as $t \to \infty$, we have that $x$ diverges. Therefore, we have the following lemma in the light of this information.

**Lemma 3.4** *Any nonoscillatory solution in $N^+$ belongs to $N^+_{\infty,F}$, or $N^+_{\infty,\infty}$ for $0 < c, d < \infty$.*

It is not easy to give the sufficient conditions for the existence of nonoscillatory solutions in $N^+_{\infty,\infty}$. So, we only provide the existence of nonoscillatory solutions in $N^+_{\infty,F}$.

**Theorem 3.5** *There exists a nonoscillatory solution in $N^+_{\infty,F}$ if and only if $I_2 < \infty$ for all $k_2 > 0$.*

*Proof.* Suppose that there exists a solution in $N^+_{\infty,F}$ such that $x(t) > 0$, $y(t) > 0$ for $t \geq t_0$, $x(t) \to \infty$ and $y(t) \to d$ as $t \to \infty$ for $d > 0$. Since $y$ is eventually increasing, there exist $k_2 > 0$ and $t_1 \geq t_0$ such that $f(y(t)) \geq k_2$ for $t \geq t_1$. Integrating the first equation from $t_1$ to $t$, the monotonicity of $f$ yields us

$$x(t) = x(t_1) + \int_{t_1}^{t} p(s) f(y(s))\Delta s \geq k_2 \int_{t_1}^{t} p(s)\Delta s, \quad t \geq t_1. \tag{3}$$

Integrating the second equation from $t_1$ to $t$, the monotonicity of $g$ and (3) gives us

$$y(t) = y(t_1) + \int_{t_1}^{t} r(s) g(x(s))\Delta s \geq \int_{t_1}^{t} r(s) g\left(k_2 \int_{t_1}^{s} p(u)\Delta u\right)\Delta s, \quad t \geq t_1. \tag{4}$$

So as $t \to \infty$, we have that $I_2 < \infty$ holds.

Conversely, suppose that $I_2 < \infty$ for all $k_2 > 0$. Then, there exists a large $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} r(t) g\left(k_2 \int_{t_1}^{t} p(s)\Delta s\right)\Delta t < \frac{c}{2}, \tag{5}$$

where $k_2 = f(c)$. Let $Y$ be the set of all bounded and continuous real-valued functions $y(t)$ on $[t_1, \infty)_{\mathbb{T}}$ with the supremum norm $\sup_{t \geq t_1} |y(t)|$. Then $Y$ is a Banach space (see [38]). Let us define a subset $\Omega$ of $Y$ such that

$$\Omega := \left\{ y(t) \in Y : \frac{c}{2} \leq y(t) \leq c, \quad t \geq t_1 \right\}.$$

One can prove that $\Omega$ is bounded, closed, and also convex subset of $Y$. Suppose that $T : \Omega \to Y$ is an operator given by

$$(Ty)(t) = c - \int_t^{\infty} r(s) g\left( \int_{t_1}^{s} p(u) f(y(u)) \Delta u \right) \Delta s. \tag{6}$$

The very first thing we do is to show that $T$ is mapping into itself, i.e., $T : \Omega \to \Omega$.

$$\frac{c}{2} \leq c - \int_t^{\infty} r(s) g\left( \int_{t_1}^{s} p(u) f(c) \Delta u \right) \Delta s \leq (Ty)(t) \leq c$$

by using (5) for $y \in \Omega$. The second thing we show that $T$ must be continuous on $\Omega$. Hence, for $y \in \Omega$, suppose that $y_n$ is a sequence in $\Omega$ so that $\|y_n - y\| \to 0$. Then

$$\begin{aligned} &|(Ty_n)(t) - (Ty)(t)| \\ &\leq \int_t^{\infty} r(s) \left| g\left( \int_{t_1}^{s} p(u) f(y_n(u)) \Delta u \right) - g\left( \int_{t_1}^{s} p(u) f(y(u)) \Delta u \right) \right| \Delta s \end{aligned}.$$

Then by the Lebesgue dominated convergence theorem and by the continuity of $f$ and $g$, we have that $\|Ty_n - Ty\| \to 0$ as $n \to \infty$, i.e., $T$, is continuous. Finally, we show that $T\Omega$ is relatively compact, i.e., equibounded and equicontinuous. Since

$$0 < (Ty)^{\Delta}(t) = r(t) g\left( \int_{t_1}^{t} p(u) f(y(u)) \Delta u \right) \leq r(t) g\left( k_2 \int_{t_1}^{t} p(u) \Delta u \right) < \infty,$$

we have that $Ty$ is relatively compact by the Arzelá-Ascoli and mean value theorems. Therefore, Theorem 2.9 implies that there exists $\bar{y} \in \Omega$ such that $\bar{y} = T\bar{y}$. Then we have

$$\bar{y}^{\Delta}(t) = (T\bar{y})^{\Delta}(t) = r(t) g\left( \int_{t_1}^{t} p(u) f(\bar{y}(u)) \Delta u \right) \quad t \geq t_1. \tag{7}$$

Setting $\bar{x}(t) = \int_{t_1}^{t} p(u) f(\bar{y}(u)) \Delta u$ gives us $x^{\Delta}(t) = p(t) f(\bar{y}(t))$. Hence, we have that $(\bar{x}, \bar{y})$ is a nonoscillatory solution of system (1) such that $\bar{x}(t) \to \infty$ and $\bar{y}(t) \to c$ as $t \to \infty$, i.e., $N^{+}_{\infty, F} \neq \varnothing$.

### *3.1.2 The case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$*

In this subsection, we show that the existence of nonoscillatory solutions of (1) is only possible in $N^{+}_{F,F}$ and $N^{+}_{\infty,\infty}$ for $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$, i.e., $N^{+}_{F,\infty} = N^{+}_{\infty,F} = \varnothing$.

**Lemma 3.6** *Suppose $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$ and that $(x, y)$ is a nonoscillatory solution of system* (1). *Then $x(t)$ tends to a finite nonzero number $c$ if and only if $y(t)$ tends to a finite nonzero number $d$ as $t \to \infty$.*

*Proof.* We prove the theorem by assuming $x > 0$ without loss of generality. Therefore by the definition of $N^+$, $y$ is also a positive component function of the solution $(x, y)$. By taking the integral of the second equation of system (1) from $t_0$ to $t$ and by the monotonicity of $g$ and $x$, we have that there exists a positive constant $k$ such that

$$y(t) \leq y(t_0) + k \int_{t_0}^{t} r(s)\Delta s,$$

where $k = g(c)$. Then we have that $y$ is convergent because $P(t_0, \infty) < \infty$ as $t \to \infty$. The sufficiency can be shown similarly.

**Theorem 3.7** $N^+_{F,F} \neq \varnothing$ *if and only if* $I_1 < \infty$ *for all* $k_1 > 0$.

*Proof.* The necessity part can be shown similar to Theorem 3.5. So for sufficiency, suppose $I_1 < \infty$ holds for all $k_1 > 0$. Then choose $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(t) f\left(k_1 \int_{t_1}^{t} r(s)\Delta s\right)\Delta t < \frac{c}{2}, \tag{8}$$

where $k_1 = g(c)$ and $t \geq t_1$. Let $X$ be the Banach space of all bounded real-valued and continuous functions on $[t_0, \infty)_{\mathbb{T}}$ with usual pointwise ordering $\leq$ and the norm $\sup_{t \geq t_1} |x(t)|$. Let $Y$ be a subset of $X$ such that

$$Y := \left\{x \in X : \frac{c}{2} \leq x(t) \leq c \quad t \geq t_1\right\}$$

and $F : \Omega \to X$ be an operator such that

$$(Fx)(t) = \frac{c}{2} + \int_{t_1}^{t} p(s) f\left(\int_{t_1}^{s} r(u) g(x(u))\Delta u\right)\Delta t, \quad t \geq t_1.$$

One can easily have that $\inf B \in Y$ and $\sup B \in Y$ for any subset $B$ of $Y$, which implies that $(Y, \leq)$ is a complete lattice. First, let us show that $F : Y \to Y$ is an increasing mapping.

$$\frac{c}{2} \leq (Fx)(t) \leq \frac{c}{2} + \int_{t_1}^{t} p(s) f\left(g(c) \int_{t_1}^{s} r(u)\Delta u\right)\Delta t \leq c, \quad t \geq t_1,$$

that is $F : Y \to Y$. Note also that for $x_1 \leq x_2$, $x_1, x_2 \in Y$, we have $Fx_1 \leq Fx_2$, i.e., $F$, which is an increasing mapping. Then by Theorem 2.10, there exists a function $\overline{x} \in Y$ such that $\overline{x} = F\overline{x}$. By taking the derivative of $F\overline{x}$, we have

$$(F\overline{x})^{\Delta}(t) = p(t) f\left(\int_{t_1}^{t} r(u) g(\overline{x}(u))\Delta u\right), \quad t \geq t_1.$$

By letting

$$\overline{y}(t) = \int_{t_1}^{t} r(u) g(\overline{x}(u))\Delta u,$$

we have $\overline{y}^{\Delta}(t) = r(t) g(\overline{x}(t))$, and $(\overline{x}, \overline{y})$ is a nonoscillatory solution of system (1) such that $\overline{x}$ and $\overline{y}$ have finite limits as $t \to \infty$. This completes the assertion.

**Remark 3.8** Suppose that $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$. Then, as a result of this, we have $I_1 < \infty$. So Theorem 3.7 also holds for $P(t_0, \infty) < \infty$ and $R(t_0, \infty) < \infty$.

**Exercise 3.9** Prove Remark 3.8.

### *3.1.3 The case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$*

We present the nonoscillation criteria in $N^+$ under the case $P(t_0, \infty) < \infty$ and $R(t_0, \infty) = \infty$ in this subsection. Therefore, we have the following lemma.

**Lemma 3.10** *Suppose that $R(t_0, \infty) = \infty$. Then any nonoscillatory solution in $N^+$ belongs to $N^+_{F,\infty}$ or $N^+_{\infty,\infty}$, i.e., $N^+_{F,F} = N^+_{\infty,F} = \varnothing$.*

**Exercise 3.11** Prove Lemma 3.10.

The following theorem shows us the nonexistence of nonoscillatory solutions in $N^+_{F,\infty}$. We skip the proof of the following theorem, since it is very similar to the proof of Theorem 3.5.

**Theorem 3.12** $N^+_{F,\infty} \neq \varnothing$ *if and only if $I_1 < \infty$ for all $k_1 > 0$.*

### *3.1.4 Examples*

Examples are great ways to see that theoretical claims actually work. Therefore, we provide two examples about the existence of nonoscillatory solutions of system (1). But before the examples, we need the following proposition because our examples consist of scattered points.

**Proposition 1** ([5], Theorem 1.79) *Let $a, b \in \mathbb{T}$ and $h \in \mathrm{C}_{rd}$. If $[a, b]$ consists of only isolated points, then*

$$\int_a^b h(t)\Delta t = \sum_{t \in [a,b)_{\mathbb{T}}} \mu(t)h(t).$$

**Example 3.13** *Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider*

$$\begin{cases} \Delta_q x(t) = \left(\dfrac{t}{2t-1}\right)^{\frac{1}{61}} (y(t))^{\frac{1}{61}} \\ \Delta_q y(t) = \dfrac{1}{2t^{\frac{13}{5}}} (x(t))^{\frac{3}{5}}, \end{cases} \tag{9}$$

where $\Delta_q$ is known as a $q$-derivative and defined as $\Delta_q h(t) = \frac{h(\sigma(t)) - h(t)}{\mu(t)}$, where $\mu(t) = t$, $\sigma(t) = 2t$, and $t = 2^n$, (see [5]). In this example, it is shown that we have a nonoscillatory solution in $N^+_{\infty,F}$ to highlight Theorem 3.5. Therefore, we need that $P(t_0, \infty)$ is divergent and $R(t_0, \infty)$ is convergent. Indeed, by Proposition 1, we have

$$P(1, T) = \int_1^T p(t)\Delta t = \sum_{t \in [1,T)_{2^{\mathbb{N}_0}}} \left(\frac{t}{2t-1}\right)^{\frac{1}{61}} \cdot t.$$

*Hence, we have $P(1, \infty) = \infty$ as $T$ tends to infinity. Note that we use the limit divergence test to show the divergence of $P(1, \infty)$. Next, we continue with the convergence of $R(1, \infty)$. To do that, we note*

$$R(1, T) = \int_1^T r(t)\Delta t = \sum_{t \in [1,T)_{2^{\mathbb{N}_0}}} \frac{1}{2t^{\frac{13}{5}}} \cdot t.$$

*As* $T \to \infty$, *we have*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2\cdot 2^n}\right)^{\frac{8}{5}} < \infty$$

by the geometric series, i.e., $R(1,\infty) < \infty$. Finally, we have to show $I_2 < \infty$. Let $k_2 = 1$. Then we get

$$\int_1^T r(t) g\left(k_2 \int_1^t p(s)\Delta s\right)\Delta t = \int_1^T \frac{1}{2t^{\frac{13}{5}}}\left(\sum_{s\in[1,t)_{2^{\mathbb{N}_0}}}\left(\frac{s}{2s-1}\right)^{\frac{1}{61}}\cdot s\right)^{\frac{3}{5}}\Delta t$$

$$\leq \int_1^T \frac{1}{2t^{\frac{13}{5}}}\left(\sum_{s\in[1,t)_{2^{\mathbb{N}_0}}} s^{\frac{62}{61}}\right)^{\frac{3}{5}}\Delta t \leq \int_1^T \frac{1}{2t^{\frac{13}{5}}}\cdot t^{\frac{62}{105}}\Delta t = \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{t^{\frac{208}{105}}}.$$

*So as* $t \to \infty$, *we have*

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right)^{\frac{208}{105}} < \infty$$

by the ratio test. Therefore, $I_2 < \infty$ by the comparison test. One can also show that $\left(t, 2-\frac{1}{t}\right)$ is a solution of system (9) such that $x(t) \to \infty$ and $y(t) \to 2$ as $t \to \infty$, i.e., $N^+_{\infty,F} \neq \emptyset$ by Theorem 3.5

**Example 3.14** *Let* $\mathbb{T} = \left\{\frac{n}{2} : n \in \mathbb{N}_0\right\}$, $f(z) = z^{\frac{1}{3}}$, $g(z) = z^{\frac{1}{5}}$, $p(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t - 1)^{\frac{1}{3}}}$, $r(t) = \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t - 1)^{\frac{1}{5}}}$, *and* $t = \frac{n}{2}$ *in system* (1). *We show that there exists a nonoscillatory solution in* $N^+_{F,F}$. *So by Theorem* 3.7, *we need to show* $P(t_0,\infty) < \infty$ *and* $R(t_0,\infty) < \infty$ *and* $I_1 < \infty$. *Proposition 1 gives us*

$$\int_0^T p(t)\Delta t = \sum_{t\in[0,T)_{\mathbb{T}}} \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t - 1)^{\frac{1}{3}}}\cdot\frac{1}{2} \leq \sum_{t\in[0,T)_{\mathbb{T}}} \frac{1}{2^{\frac{2t}{3}}}.$$

*So as* $T \to \infty$, *we have*

$$\sum_{n=0}^{\infty} \frac{1}{2^{\frac{n}{3}}} < \infty$$

by the geometric series, i.e., $P(t_0,\infty) < \infty$. Also

$$\int_0^T r(t)\Delta t = \sum_{t\in[0,T)_{\mathbb{T}}} \frac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t - 1)^{\frac{1}{5}}}\cdot\frac{1}{2} \leq \sum_{t\in[0,T)_{\mathbb{T}}} \frac{1}{2^{\frac{4t}{5}}}.$$

*Hence, we have*

$$\sum_{n=0}^{\infty} \frac{1}{2^{\frac{2n}{5}}} < \infty$$

*as* $T \to \infty$. *Note also that* $I_1 < \infty$ *if* $P(t_0,\infty) < \infty$ *and* $R(t_0,\infty) < \infty$ *(see Remark* (8)). *It can be confirmed that* $\left(2-\frac{1}{2^t}, 3-\frac{1}{2^t}\right)$ *is a nonoscillatory solution of*

$$\begin{cases} x^{\Delta}(t) = \dfrac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{2t}{3}}(3\cdot 2^t-1)^{\frac{1}{3}}}(y(t))^{\frac{1}{3}} \\ \\ y^{\Delta}(t) = \dfrac{\sqrt{2}(\sqrt{2}-1)}{2^{\frac{4t}{5}}(2\cdot 2^t-1)^{\frac{1}{5}}}(x(t))^{\frac{1}{5}} \end{cases}$$

such that $x(t)\to 2$ and $y(t)\to 3$ as $t\to\infty$, i.e., $N^{+}_{F,F}\neq \varnothing$ by Theorem 3.7.

### 3.2 Existence of nonoscillatory solutions of (1) in $N^{-}$

Suppose that $(x,y)$ is a nonoscillatory solution of system (1) such that $x>0$ eventually. Then by the first and second equations of system (1) and the similar discussion as in Section 3.1, we obtain the following subclasses of $N^{-}$.

$$N^{-}_{F,F} = \left\{(x,y)\in N^{-} : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = -d\right\},$$

$$N^{-}_{F,0} = \left\{(x,y)\in N^{-} : \lim_{t\to\infty} x(t) = c, \quad \lim_{t\to\infty} y(t) = 0\right\},$$

$$N^{-}_{0,F} = \left\{(x,y)\in N^{-} : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = -d\right\},$$

$$N^{-}_{0,0} = \left\{(x,y)\in N^{-} : \lim_{t\to 0} x(t) = 0, \quad \lim_{t\to 0} y(t) = 0\right\}.$$

This section presents us the existence and nonexistence of nonoscillatory solutions of system (1) under the monotonicity condition on $f$ and $g$.

**Theorem 3.15** *Let $R(t_0,\infty)<\infty$. Then there exists a nonoscillatory solution in $N^{-}_{F,F}\neq\varnothing$ if and only if $I_3<\infty$ for all $k_3<0$ and $k_4>0$.*

*Proof.* Suppose $N^{-}_{F,F}\neq\varnothing$. Then there exists a solution $(x,y)\in N^{-}_{F,F}$ such that $x>0$, $y<0$, $x(t)\to c_1$, and $y(t)\to -d_1$ as $t\to\infty$ for $0<c_1<\infty$ and $0<d_1<\infty$. By integrating the second equation of system (1) from $t$ to $\infty$, we obtain

$$\begin{aligned} y(t) &= y(\infty) - \int_t^{\infty} r(s)g(x(s))\Delta s \\ &\leq -d_1 - k_4\int_t^{\infty} r(s)\Delta s, \quad \text{where} \quad k_4 = g(c_1). \end{aligned} \tag{10}$$

Integrating the first equation from $t_1$ to $t$, using (10) and the fact that $x$ is bounded yield us

$$\begin{aligned} c_1 \leq x(t) &= x(t_1) + \textstyle\int_{t_1}^{t} p(s)f(y(s))\Delta s \\ &\leq x(t_1) + \textstyle\int_{t_1}^{t} p(s)f\left(-d_1 - k_4\int_s^{\infty} r(u)\Delta u\right)\Delta s \leq x(t_1), \quad t\geq t_1. \end{aligned}$$

Therefore, it implies $I_3<\infty$ as $t\to\infty$, where $-d_1=k_3$.

Conversely, suppose that $I_3<\infty$. Then there exist $t_1\geq t_0$ and $k_3<0$, $k_4>0$ such that

$$\int_{t_0}^{\infty} p(t)f\left(k_3 - k_4\int_t^{\infty} r(s)\Delta s\right)\Delta t > \frac{-1}{2} \tag{11}$$

where $k_4 = g\left(\frac{3}{2}\right)$. Let $C_B$ be the set of all continuous and bounded real-valued functions $x(t)$ on $[t_1, \infty)_{\mathbb{T}}$ with the supremum norm $\sup_{t \geq t_1} |x(t)|$. Observe that $C_B$ is a Banach space (see [38]). Suppose that $B$ is a subset of $C_B$ such that

$$B := \left\{ x(t) \in C_B : 1 \leq x(t) \leq \frac{3}{2}, \quad t \geq t_1 \right\}.$$

We have that $B$ meets the assumptions of Theorem 2.9. Suppose also that $F : B \to B$ is an operator such that

$$(Fx)(t) = 1 - \int_t^\infty a(s) f\left(k_3 - \int_s^\infty b(u) g(x(u)) \Delta u\right) \Delta s \tag{12}$$

First, we need to show $F$ is a mapping into itself, i.e., $F : B \to B$. Indeed,

$$1 \leq (Fx)(t) \leq 1 - \int_t^\infty a(s) f\left(k_3 - g\left(\frac{3}{2}\right) \int_{t_1}^s b(u) \Delta u\right) \Delta s \leq \frac{3}{2}$$

because $x \in B$ and (5) hold. Next, let us verify that $F$ is continuous on $B$. In order to do that, let $x_n$ be a sequence in $B$ such that $x_n \to x$, where $x \in B = \overline{B}$. Then

$$\begin{aligned} &|(Fx_n)(t) - (Fx)(t)| \\ &\leq \int_t^\infty p(s) \left| f\left(k_3 - \int_s^\infty r(u) g(x_n(u)) \Delta u\right) - f\left(k_3 - \int_s^\infty r(u) g(x(u)) \Delta u\right) \right| \Delta s. \end{aligned}$$

Therefore, the continuity of $f$ and $g$ and the Lebesgue dominated convergence theorem gives us $Fx_n \to Fx$ as $n \to \infty$, which implies $F$ is continuous on $B$. Finally, we prove that $FY$ is equibounded and equicontinuous, i.e., relatively compact. Because

$$\begin{aligned} 0 < -(Fx)^\Delta(t) &= -p(t) f\left(k_3 - \int_t^\infty r(u) g(x(u)) \Delta u\right) \\ &\leq -p(t) f\left(k_3 - k_4 \int_{t_1}^t r(u) \Delta u\right) < \infty, \end{aligned}$$

we have that $Fx$ is relatively compact. Hence, Theorem 2.9 implies that there exists $\overline{x} \in B$ such that $\overline{x} = F\overline{x}$. Thus, we have $\overline{x} > 0$ eventually and $\overline{x}(t) \to 1$ as $t \to \infty$. Also

$$\overline{x}^\Delta(t) = (F\overline{x})^\Delta(t) = p(t) f\left(k_3 - \int_t^\infty r(u) g(\overline{x}(u)) \Delta u\right) \quad t \geq t_1.$$

Letting

$$\overline{y}(t) = k_3 - \int_t^\infty r(u) g(\overline{x}(u)) \Delta u < 0, \quad t \geq t_1 \tag{13}$$

and taking the derivative of (13) give $\overline{y}^\Delta(t) = b(t) g(\overline{x}(t))$. So, we conclude that $(\overline{x}, \overline{y})$ is a nonoscillatory solution of system (1). Finally, taking the limit of Eq. (13) results in $\overline{y}(t) \to k_3 < 0$. Therefore, we get $N_{F,F}^- \neq ø$.

**Theorem 3.16** *Suppose* $P(t_0, \infty) < \infty$. $N^-_{0,F} \neq ø$ *if and only if* $I_4 < \infty$ *for* $k_5 > 0$.

**Exercise 3.17** Prove Theorem 3.16.

**Theorem 3.18** *Suppose* $P(t_0, \infty) < \infty$. $N^-_{0,0} \neq ø$ *if* $I_3 < \infty$ *and* $I_4 = \infty$ *for all* $k_3 = 0$, $k_4 < 0$ *and* $k_5 > 0$, *provided* $f$ *is odd.*

*Proof.* Suppose that $I_3 < \infty$, and $I_4 = \infty$. Then there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} p(s)f\left(-k_4 \int_s^{\infty} r(u)\Delta u\right)\Delta s < 1$$

and

$$\int_{t_1}^{\infty} r(s)g\left(k_5 \int_s^{\infty} p(u)\Delta u\right)\Delta s > \frac{1}{2}$$

for $t \geq t_1$, $k_4 = -g(1)$. Let $X$ be the space that is claimed as in the proof of Theorem 3.7. Let $Y$ be a subset of $X$ and given by

$$Y := \left\{x \in X : c_1 \int_t^{\infty} a(s)\Delta s \leq x(t) \leq 1 \quad t \geq t_1\right\},$$

where $c_1 = f\left(\frac{1}{2}\right)$. Define an operator $T : Y \to X$ such that

$$(Tx)(t) = \int_t^{\infty} p(s)f\left(\int_s^{\infty} r(u)g(x(u))\Delta u\right)\Delta t, \quad t \geq t_1.$$

One can show that $(Y, \leq)$ is a complete lattice and $T$ is an increasing mapping such that $T : Y \to Y$. As a matter of fact,

$$(Tx)(t) \leq \int_t^{\infty} p(s)f\left(g(1) \int_s^{\infty} r(u)\Delta u\right)\Delta t \leq 1, \quad t \geq t_1$$

and

$$\begin{aligned}(Tx)(t) &\geq \int_t^{\infty} p(s)f\left(\int_s^{\infty} r(u)g\left(c_1 \int_u^{\infty} p(v)\Delta v\right)\Delta u\right)\Delta s \\ &\geq f\left(\frac{1}{2}\right) \int_t^{\infty} p(s)\Delta s,\end{aligned}$$

where $c_1 = k_5$, i.e., $T : Y \to Y$. Then by Theorem 2.10, there exists a function $\overline{x} \in Y$ such that $\overline{x} = T\overline{x}$. By taking the derivative of $T\overline{x}$ and using the fact that $f$ is odd, we have

$$(T\overline{x})^{\Delta}(t) = p(t)f\left(-\int_t^{\infty} r(u)g(\overline{x}(u))\Delta u\right), \quad t \geq t_1.$$

Setting

$$\overline{y}(t) = -\int_t^{\infty} r(u)g(\overline{x}(u))\Delta u$$

yields $\bar{y}^{\Delta}(t) = b(t)g(\bar{x}(t))$, and $(\bar{x}, \bar{y})$ is a solution of system (1) in $N^-_{0,0}$, i.e., $\bar{x}$ and $\bar{y}$ both tend to zero.

**Theorem 3.19** *Suppose $R(t_0, \infty) < \infty$. $N^-_{F,0} \neq \varnothing$ if and only if $I_3 < \infty$, where $k_3 = 0$ and $k_4 > 0$.*

**Exercise 3.20** Prove Theorem 3.19. Hint: Use Theorem 2.10 with the operator

$$(Fx)(t) = \frac{1}{2} - \int_t^{\infty} a(s) f\left( - \int_s^{\infty} b(u) g(x(u)) \Delta u \right) \Delta t, \quad t \geq t_1.$$

Examples make results clearer and give more information to readers. Therefore, we give the following example to validate our claims. The beauty of our example is that we do not only show the theorem holds but also find the explicit solutions, which might be very hard for some nonlinear systems.

**Example 3.21** *Consider $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$ with the system*

$$\begin{cases} x^{\Delta}(t) = \dfrac{1}{t^{\frac{1}{3}}\left(\sqrt{t}+1\right)^2 (t^2+1)^{\frac{1}{3}}} (y(t))^{\frac{1}{3}} \\ \\ y^{\Delta}(t) = \dfrac{\left(\sqrt{t}+1\right)^4 - t^2}{t^{\frac{9}{5}}\left(\sqrt{t}+1\right)^4 \left(1+2\sqrt{t}\right)} (x(t))^{\frac{1}{5}}, \end{cases} \tag{14}$$

*where $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$ for $\sigma(t) = \left(\sqrt{t}+1\right)^2$ and $\mu(t) = 1 + 2\sqrt{t}$* (see [5]). *First, let us show $P(t_0, \infty) < \infty$, where $t_0 \geq 1$.*

$$\int_1^T p(t)\Delta t = \sum_{t \in [1,T)_{\mathbb{N}_0{}^2}} \frac{1}{t^{\frac{1}{3}}\left(\sqrt{t}+1\right)^2 (t^2+1)^{\frac{1}{3}}} \cdot \left(1+2\sqrt{t}\right) \leq \sum_{t \in [1,T)_{\mathbb{N}_0{}^2}} \frac{1+2\sqrt{t}}{t^2}.$$

*Since $t = n^2$, as $T \to \infty$, we have*

$$\sum_{n=1}^{\infty} \frac{1+2n}{n^4} < \infty$$

by the geometric series. Therefore, $P(1, \infty) < \infty$ by the comparison test. Next, we show $I_4 < \infty$. Since $P(1, \infty) < \infty$, we have $\int_t^{\infty} p(s)\Delta s < \alpha$ for $t \geq 1$ and $0 < \alpha < \infty$. Hence,

$$\begin{aligned} \int_1^T r(t) g\left( \int_t^{\infty} p(s)\Delta s \right) \Delta t \leq \alpha \int_1^T r(t)\Delta t \\ = \alpha \sum_{t \in [1,T)_{\mathbb{N}_0{}^2}} \frac{\left(\sqrt{t}+1\right)^4 - t^2}{t^{\frac{9}{5}}\left(\sqrt{t}+1\right)^4 \left(1+2\sqrt{t}\right)} \cdot \left(1+2\sqrt{t}\right) \\ \leq \alpha \sum_{t \in [1,T)_{\mathbb{N}_0{}^2}} \frac{1}{t^{\frac{9}{5}}}. \end{aligned}$$

*So as $T$ tends to infinity, we get*

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{18}{5}}} < \infty,$$

*i.e.,* $I_2 < \infty$. *Also, note that* $\left(\frac{1}{t}, -1 - \frac{1}{t^2}\right)$ *is a solution of system* (14) *in* $N^-$ *such that* $x$ *tends to zero, while* $y$ *tends to* $-1$, *i.e.,* $N^-_{0,F} \neq \varnothing$.

## 4. Oscillation of a two-dimensional time-scale systems

Motivated by [39], this section deals with the system

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = -b(t)g(x(t)) + c(t), \end{cases} \tag{15}$$

where $a, b \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, $c \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}\right)$ and functions $f$ $g$ have the same characteristics as in system (1) and $g$ is continuously differentiable. Note that we can rewrite system (15) as a non-homogenous dynamic equations on time scales and putting $\sigma$ on $x$ inside the function $g$. Therefore, we have the following dynamic equation

$$\left(a(t)x^\Delta(t)\right)^\Delta + b(t)g(x^\sigma(t)) = c(t) \tag{16}$$

and systems of dynamical equations

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = -b(t)g(x^\sigma(t)) + c(t). \end{cases} \tag{17}$$

Oscillation criteria for Eq. (16), system (17), and other similar versions of (15) and (17) are investigated in [39–42]. A solution $(x, y)$ of system (15) is called oscillatory if $x$ and $y$ have arbitrarily large zeros. System (15) is called oscillatory if all solutions are oscillatory.

Before giving the main results, we present some propositions so that we can use them in our theoretical claims (see [43], Theorem 4.2 (comparison theorem) and [5], Theorem 1.90).

**Proposition 2** *Let* $z_1$ *be a function from* $\mathbb{T}$ *to* $\mathbb{R}$ *and* $v$ *be a nondecreasing function from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that* $v \circ z_1$ *is rd-continuous. Suppose also that* $p \geq 0$ *is rd-continuous and* $\alpha \in \mathbb{R}$. *Then*

$$z_1(t) \leq \alpha + \int_{t_0}^{t} p(\tau)v(z_1(\tau))\Delta\tau, \quad t \geq t_0$$

implies $z_1(t) \leq z_2(t)$, where $z_2$ solves the initial value problem

$$z_2^\Delta(t) = p(t)v(z_2(t)), \quad z_2(t_0) = z_{20} > \alpha.$$

**Proposition 3 (chain rule).** ([5], Theorem 1.90) *Let* $h_1 : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $h_2 : \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $h_1 \circ h_2 : \mathbb{T} \to \mathbb{R}$ *is delta differentiable, and the formula*

$$(h_1 \circ h_2)^\Delta(t) = \left\{ \int_0^1 h_{1'}\left(h_2(t) + h\mu(t)h_2^\Delta(t)\right)dh \right\} h_2^\Delta(t)$$

holds.

For simplicity, set

$$A(t,s)=\int_t^s a(u)\Delta u,\quad B(t,s)=\int_t^s b(u)\Delta u,$$

$$C(t,s)=\int_t^s |c(u)|\Delta u,\quad D(t,s)=\int_t^s \left(b(u)-\frac{c(u)}{g(x(u))}\right)\Delta u,$$

$$Y(t,s)=\int_t^s \frac{y^\sigma(u)x^\Delta(u)\int_0^1 [g'(x(u)+h\mu(u)x^\Delta(u))dh]}{g(x(u))g(x^\sigma(u))}\Delta u.$$

Next, note that if $(x,y)$ is a nonoscillatory solution of system (15), then one can easily prove that $x$ is also nonoscillatory. This result was shown by Anderson in [37] when $c(t)\equiv 0$. Because the proof when $c(t)\not\equiv 0$ is very similar to the proof of the case $c(t)\equiv 0$, we leave it to the readers.

**Lemma 4.1** *Suppose that* $(x,y)$ *is a nonoscillatory solution of system* (15) *and* $t_1, t_2\in\mathbb{T}$. *If there exists a constant* $K>0$ *such that*

$$H(t)\geq K,\quad t\geq t_2, \tag{18}$$

where $H$ is defined as

$$H(t)=-\frac{y(t_1)}{g(x(t_1))}+D(t_1,t)+Y(t_1,t_2), \tag{19}$$

*then* $y(t)\leq -Kg(x(t_2)),\quad t\geq t_2$.

*Proof*. Suppose that $(x,y)$ is a nonoscillatory solution of system (15). Then, we have that $x$ is also nonoscillatory. Without loss of generality, assume that $x(t)>0$ for $t\geq t_1\geq t_0$, where $t_1, t_0\in\mathbb{T}$. Integrating the second equation of system (15) from $t_1$ to $t$ and Theorem 2.8 (v.) gives us

$$\int_{t_1}^t b(s)\Delta s=\frac{y(t_1)}{g(x(t_1))}-\frac{y(t)}{g(x(t))}+\int_{t_1}^t\left(\frac{1}{g(x(s))}\right)^\Delta y^\sigma(s)\Delta s+\int_{t_1}^t\frac{c(s)}{g(x(s))}\Delta s. \tag{20}$$

By applying Theorem 2.3 (iv) and Proposition 3 to Eq. (20), we have

$$\int_{t_1}^t b(s)\Delta s=\frac{y(t_1)}{g(x(t_1))}-\frac{y(t)}{g(x(t))}+\int_{t_1}^t\frac{c(s)}{g(x(s))}\Delta s-Y(t_1,t),\quad t\geq t_1. \tag{21}$$

Rewriting Eq. (21) gives us

$$-\frac{y(t)}{g(x(t))}=D(t_1,t)-\frac{y(t_1)}{g(x(t_1))}+Y(t_1,t),\quad t\geq t_1. \tag{22}$$

Now by using (18) and (19), we get

$$-\frac{y(t)}{g(x(t))}\geq K+Y(t_2,t),\quad t\geq t_2\geq t_1. \tag{23}$$

Note that $y(t)<0$ and $x^\Delta(t)<0$ for $t\geq t_2$ since $y(t)x^\Delta(t)=a(s)y(s)f(y(s))>0$. Otherwise, we would have $\frac{-y(t)}{g(x(t))}>0$, which is a contradiction. Let

$$\frac{-v(t)}{g(x(t))}=K+Y(t_2,t),\quad t\geq t_2. \tag{24}$$

So one can obtain

$$\left(\frac{-v(t)}{g(x(t))}\right)^{\Delta} = \frac{y^{\sigma}(t)x^{\Delta}(t)\int_0^1 [g'(x(t)+h\mu(t)x^{\Delta}(t))dh]}{g(x(t))g(x^{\sigma}(t))} > 0, \quad t \geq t_2. \tag{25}$$

Because $x(t)$ is a positive and $v(t)$ is a negative function for $t \geq t_2$, we have $\frac{-y(t)}{g(x(t))} \geq \frac{-v(t)}{g(x(t))}$, i.e., $y(t) \leq v(t) < 0$ for $t \geq t_2$. Therefore, we have by (25) that

$$\left(\frac{-v(t)}{g(x(t))}\right)^{\Delta} \geq \frac{v^{\sigma}(t)x^{\Delta}(t)\int_0^1 [g'(x(t)+h\mu(t)x^{\Delta}(t))dh]}{g(x(t))g(x^{\sigma}(t))} > 0, \quad t \geq t_2$$

since $v(t) < 0$ and $x^{\Delta}(t) < 0$ for $t \geq t_2$. By setting

$$\frac{w(t)}{g(x(t))} = K - \int_{t_2}^{t} \frac{w^{\sigma}(s)x^{\Delta}(s)\int_0^1 [g'(x(s)+h\mu(s)x^{\Delta}(s))dh]}{g(x(s))g(x^{\sigma}(s))}\Delta s \tag{26}$$

and using (24), we have $\frac{-v(t_2)}{g(x(t_2))} = K = \frac{w(t_2)}{g(x(t_2))}$. Then, setting $z_1 = \frac{v(t)}{g(x(t))}, z_2 = \frac{-w(t)}{g(x(t))}, h(u) = \frac{u^{\sigma}(t)}{g(x(t))}$ in Proposition 2, it follows $v(t) \leq -w(t)$, which implies $y(t) \leq -w(t)$, $t \geq t_2$. Note also by Theorem 2.3 (iv) and Proposition 3 that

$$\left(\frac{w(t)}{g(x(t))}\right)^{\Delta} = \frac{w^{\Delta}(t)}{g(x^{\sigma}(t))} - \frac{w^{\sigma}(t)x^{\Delta}(t)\int_0^1 g'(x(t)+h\mu(t)x^{\Delta}(t))dh}{g(x(t))g(x^{\sigma}(t))}, \quad t \geq t_2. \tag{27}$$

Taking the derivative of (26) and comparing the resulting equation with (27) yield us

$$\frac{w^{\Delta}(t)}{g(x^{\sigma}(t))} = 0, \quad i.e., w^{\Delta}(t) = 0, \quad t \geq t_2.$$

Therefore, we have

$$w(t_2) = K \cdot g(x(t_2)) = w(t), \quad i.e., y(t) \leq -w(t) = -K \cdot g(x(t_2)).$$

So the proof is completed.

### 4.1 Results for oscillation

After giving the preliminaries in the previous section, it is presented the conditions for oscillatory solutions in this section.

**Theorem 4.2** *Let* $A(t_0,\infty) = \infty, B(t_0,\infty) < \infty,$ *and* $C(t_0,\infty) < \infty$. *Assume*

$$f(u)f(v) \leq f(uv) \leq -f(u)f(-v) \tag{28}$$

and

$$\int_{t_0}^{\infty} \frac{x^{\Delta}(s)}{f(g(x(s)))}\Delta s < \infty. \tag{29}$$

*Then system* (15) *is oscillatory if*

$$\int_{t_0}^{\infty} a(t)f(B(t,\infty) - k \cdot C(t,\infty))\Delta t = \infty \tag{30}$$

*for* $k \neq 0$.

*Proof.* Suppose that system (15) has a nonoscillatory solution $(x, y)$ such that $x > 0$ eventually. Then there exist $t_1 \geq t_0$ and a constant $k_6$ such that $g(x(t)) \geq k_6$ for $t \geq t_1$ by the monotonicity of $g$. Then by Eq. (22), we have

$$\frac{y(t)}{g(x(t))} = \frac{y(t_1)}{g(x(t_1))} - D(t_1, t) - Y(t_1, t), \quad t \geq t_1. \tag{31}$$

Note that $Y(t_1, t) < \infty$. Otherwise, we have a contradiction to the fact that $x(t) > 0$ for $t \geq t_1$ since $A(t_0, \infty) = \infty$. Equality (31) can be rewritten as

$$\frac{y(t)}{g(x(t))} = \gamma + D(t, \infty) + Y(t, \infty), \tag{32}$$

where $\gamma = \frac{y(t_1)}{g(x(t_1))} - D(t_1, \infty) - Y(t_1, \infty), \quad t \geq t_1$. It can be shown that $\gamma \geq 0$. Otherwise, we can choose a large $t_2$ such that $B(t, \infty) \leq -\gamma$, $Y(t_2, \infty) \leq \frac{-\gamma}{4}$, and $\left|\int_t^\infty \frac{c(s)}{g(x(s))} \Delta s\right| \leq \frac{-\gamma}{4}$ for $t \geq t_2$. Then $H(t) \geq \frac{-\gamma}{4} > 0$ for $t \geq t_2$. Then by setting $K = \frac{-\gamma}{4}$ in Lemma 4.1 found, we have $y(t) \leq -Kg(x(t_2))$ for $t \geq t_2$. Integrating the first equation of system (15) from $t_2$ to $\infty$ and the monotonicity of $f$ yields us

$$x(t) \leq x(t_2) + f(-Kg(x(t_2))) \int_{t_2}^t a(s) \Delta s, \quad t \geq t_2.$$

So as $t \to \infty$, we have a contradiction to $x > 0$ eventually. Therefore $\gamma \geq 0$. Then by Eq. (32), we have

$$y(t) \geq g(x(t)) \left[\int_t^\infty b(s) \Delta s - \frac{1}{k_6} \int_t^\infty |c(s)| \Delta s\right], \quad t \geq t_2.$$

By the first equation of system (15), the monotonicity of $f$ and Eq. (28), we have

$$x^\Delta(t) \geq a(t) f(g(x(t))) f\left(\int_t^\infty b(s) \Delta s - \frac{1}{k_6} \int_t^\infty |c(s)| \Delta s\right), \quad t \geq t_2. \tag{33}$$

Then by Eqs. (33) and (29), we have

$$\int_{t_2}^t a(s) f\left(\int_s^\infty b(u) \Delta u - k \int_s^\infty |c(u)| \Delta u\right) \leq \int_{t_2}^t \frac{x^\Delta(s)}{f(g(x(s)))} \Delta s < \infty$$

where $k = \frac{1}{k_6}$. But as $t \to \infty$, this contradicts to Eq. (30). The proof is completed.

**Theorem 4.3** *System (15) is oscillatory if* $A(t_0, \infty) = B(t_0, \infty) = \infty$ *and* $C(t_0, \infty) < \infty$.

*Proof.* We use the method of contradiction to prove the theorem. Thus, assume there is a nonoscillatory solution $(x, y)$ of system (15) such that the component function $x$ is eventually positive. Because $g$ is nondecreasing, we have that there exist $t_1 \geq t_0$ and $k_7 > 0$ such that $g(x(t)) \geq k_7$ for $t \geq t_1$. Then since $C(t_0, \infty) < \infty$, we have that there exists $0 < k_8 < \infty$ such that

$$\left|\int_{t_1}^t \frac{c(s)}{g(x(s))} \Delta s\right| \leq \frac{1}{k_7} \int_{t_1}^t |c(s)| \Delta s < k_8, \quad t \geq t_1. \tag{34}$$

The first equation of system (15), and the monotonicity of $g$ give us that there exist $K > 0$ and $t_2 \geq t_1$ so large that

$$x^{\Delta}(t) \leq a(t)f(-Kg(x(t_2))), \quad t \geq t_2. \tag{35}$$

Integrating (35) from $t_2$ to $t$ yields

$$x(t) \leq x(t_2) + k_9 \int_{t_2}^{t} a(s)\Delta s, \quad \text{where} \quad k_9 = f(-Kg(x(t_2))) < 0, \quad t \geq t_2.$$

As $t \to \infty$, we have a contradiction to $x(t) > 0$ for $t \geq t_2$. This proves the assertion.

Finally, an example is provided to highlight Theorem 4.3 by finding the explicit solution of the dynamical system.

**Example 4.4** *Consider the time scale* $\mathbb{T} = 5\mathbb{Z}^+$ *with* $a(t) = \frac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)}$, $b(t) = \frac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)}$, $f(z) = z^{\frac{1}{3}}$, $g(z) = z^3$, $c(t) = \frac{(-1)^{3t}\left(-3t^5-27t^4-125t^3-237t^2-195t-59\right)}{5(t+1)^4(t+4)(t+6)(t+9)}$, *and* $t = 5n$, *where* $n \in \mathbb{N}$ *in system* (15). *We show that* $A(t_0, \infty) = \infty$, $B(t_0, \infty) = \infty$, *and* $C(t_0, \infty) < \infty$. *Indeed,*

$$A(5, T) = \int_5^T \frac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)}\Delta t = \sum_{t \in [5,T)_{5\mathbb{Z}^+}} \frac{(t+4)^{\frac{1}{3}}(2t+7)}{(t+1)^{\frac{2}{3}}(t+6)}.$$

*So as* $T \to \infty$, *we have*

$$\sum_{n=1}^{\infty} \frac{(5n+4)^{\frac{1}{3}}(10n+7)}{(5n+1)^{\frac{2}{3}}(5n+6)} = \infty \quad \text{by the limit comparison test. Therefore,} \quad A(5, \infty) = \infty.$$

*Similarly,*

$$\begin{aligned} B(5,T) &= \int_5^T \frac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)}\Delta t = \sum_{t \in [5,T)_{5\mathbb{Z}^+}} \frac{t^5+t^4+t^3+t^2+t+1}{(t+1)(t+4)(t+6)(t+9)} \\ &\geq \sum_{t \in [5,T)_{5\mathbb{Z}^+}} \frac{t^5}{(t+1)(t+4)(t+6)(t+9)}. \end{aligned}$$

*Taking the limit as* $T \to \infty$ *gives us*

$$B(5, \infty) \geq 625 \cdot \sum_{n=1}^{\infty} \frac{n^5}{(5n+1)(5n+4)(5n+6)(5n+9)} = \infty$$

*by the limit divergence test. Therefore,* $B(5, \infty) = \infty$ *by the comparison test. Finally, we show* $C(t_0, \infty) < \infty$.

$$\begin{aligned} C(5,T) &= \sum_{t \in [5,T)_{5\mathbb{Z}^+}} \frac{3t^5 + 27t^4 + 125t^3 + 237t^2 + 195t + 59}{(t+1)^4(t+4)(t+6)(t+9)} \\ &\leq \sum_{t \in [5,T)_{5\mathbb{Z}^+}} \frac{3}{t^2} + \frac{27}{t^3} + \frac{125}{t^5} + \frac{195}{t^6} + \frac{59}{t^7}. \end{aligned}$$

*So as* $T \to \infty$, *we have*

$$C(5,\infty) \leq \sum_{n=1}^{\infty} \frac{3}{n^2} + \frac{27}{n^3} + \frac{125}{n^5} + \frac{195}{n^6} + \frac{59}{n^7} < \infty$$

by the geometric series. One can also show that $\left(\frac{(-1)^{t+1}}{t+1}, \frac{(-1)^{3t}}{(t+1)(t+4)}\right)$ is an oscillatory solution of system

$$\begin{cases} x^{\Delta}(t) = \dfrac{(t+4)^{\frac{1}{3}}(2t+7)}{5(t+1)^{\frac{2}{3}}(t+6)} y^{\frac{1}{3}}(t) \\ y^{\Delta}(t) = -\dfrac{t^5+t^4+t^3+t^2+t+1}{5(t+1)(t+4)(t+6)(t+9)} x^3(t) + \dfrac{(-1)^{3t}(-3t^5-27t^4-125t^3-237t^2-195t-59)}{5(t+1)^4(t+4)(t+6)(t+9)}, \end{cases}$$

where we define $h^{\Delta}(t) = \frac{h(\sigma(t))-h(t)}{\mu(t)}$ for $\sigma(t) = t+5$ and $\mu(t) = 5$ (see [5]).

## 5. Conclusion

This chapter focuses on the oscillation/nonoscillation criteria of two-dimensional dynamical systems on time scales. We do not only show the oscillatory behaviors of such solutions but also guarantee the existence of such solutions, which might be challenging most of the time for nonlinear systems. In the first and second sections, we present some introductory parts to dynamical systems and basic calculus of the time-scale theory for the readers to comprehend the idea behind the time scales. In Section 3, we consider

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = r(t)g(x(t)) \end{cases}$$

and investigate the nonoscillatory behavior of solutions under some certain circumstances. Recall that system (1) turns out to be a differential equation system

$$\begin{cases} x'(t) = p(t)f(y(t)) \\ y'(t) = r(t)g(x(t)) \end{cases}$$

when $\mathbb{T} = \mathbb{R}$. And the asymptotic behaviors of nonoscillatory solutions were presented by Li in [32]. Also when $\mathbb{T} = \mathbb{Z}$, system (1) is reduced to the difference equation system,

$$\begin{cases} \Delta x_n = p_n f(y_n) \\ \Delta y_n = r_n g(x_n), \end{cases}$$

and the existence of nonoscillatory solutions were investigated in [33]. Therefore, we unify the results for oscillation and nonoscillation theory, which was shown in $\mathbb{R}$ and $\mathbb{Z}$ and extends them in one comprehensive theory, which is called time-scale theory. These results were inspired from the book chapter written by Elvan Akın and Özkan Öztürk (see [29]). In that book chapter, it was considered a second-order dynamical system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = -r(t)g(x(t)) \end{cases} \tag{36}$$

and delay system

$$\begin{cases} x^{\Delta}(t) = p(t)f(y(t)) \\ y^{\Delta}(t) = -r(t)g(x((\tau(t))), \end{cases} \tag{37}$$

where $\tau$ is rd-continuous function such that $\tau(t) \leq t$ and $\tau(t) \to \infty$ as $t \to \infty$. When the latter systems were considered, because of the negative sign of the second equation of systems, the subclasses for $N^+$ an $N^-$ would be totally different. So in [29], the existence of nonoscillatory solutions in different subclasses was shown. Another crucial thing on the results is that it is assumed that $f$ must be an odd function for some main results. However, we do not have these strict conditions on our results. Another interesting observation for system (37) is that we lose some subclasses when we consider the delay in system (37). It is because of the setup fixed point theorem and the delay function $\tau$. Therefore, this is a big disadvantage of delayed systems on time scales.

Akın and Öztürk also considered the system

$$\begin{cases} x^{\Delta}(t) = p(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) \\ y^{\Delta}(t) = -r(t)|x(\sigma(t))|^{\beta} \operatorname{sgn} x^{\sigma}(t), \end{cases} \tag{38}$$

where $\alpha, \beta > 0$. System (38) is known as *Emden-Fowler* dynamical systems on time scales in the literature that has been mentioned in Section 1 with applications. Akın et al. [44, 45] showed the asymptotic behavior of nonoscillatory solutions by using $\alpha$ and $\beta$ relations.

For example, system (38) turns out to be a system of first-order differential equation

$$\begin{cases} x'(t) = p(t)|y(t)|^{\alpha} \operatorname{sgn} y(t) \\ y'(t) = -r(t)|x(t)|^{\beta} \operatorname{sgn} x(t), \end{cases}$$

when the time scale $\mathbb{T} = \mathbb{R}$. On the other hand, system (38) ends up with the system of difference equations

$$\begin{cases} \Delta x_n = p_n|y_n|^{\alpha} \operatorname{sgn} y_n \\ \Delta y_n = -r_n|x_{n+1}|^{\beta} \operatorname{sgn} x_{n+1}, \end{cases}$$

when the time scale $\mathbb{T} = \mathbb{Z}$. For both cases, several contributions have been made by Zuzana et al. in [46] and [47], respectively.

Finally, we finish this section with the following tables, showing summaries about the existence of nonoscillatory solutions of system (1) in $N^+$ and $N^-$ (**Tables 3** and **4**).

| | | | |
|---|---|---|---|
| $N^+_{\infty,F}$ | $\neq \emptyset$ | $P(t_0,\infty) = \infty$ and $R(t_0,\infty) < \infty$ | $I_2 < \infty$ |
| $N^+_{F,F}$ | $\neq \emptyset$ | $P(t_0,\infty) < \infty$ and $R(t_0,\infty) < \infty$ | $I_1 < \infty$ |
| $N^+_{F,\infty}$ | $\neq \emptyset$ | $P(t_0,\infty) < \infty$ and $R(t_0,\infty) = \infty$ | $I_1 < \infty$ |

**Table 3.**
*Existence for (1) in $N^+$.*

| | | | |
|---|---|---|---|
| $N^-_{F,F}$ | $\neq \emptyset$ | $R(t_0, \infty) < \infty$ | $I_3 < \infty$ |
| $N^-_{0,F}$ | $\neq \emptyset$ | $P(t_0, \infty) < \infty$ | $I_4 < \infty$ |
| $N^-_{0,0}$ | $\neq \emptyset$ | $P(t_0, \infty) < \infty$ | $I_3 < \infty$ and $I_4 = \infty$ |
| $N^-_{F,0}$ | $\neq \emptyset$ | $R(t_0, \infty) < \infty$ | $I_3 < \infty$ |

**Table 4.**
*Existence for (1) in $N^-$.*

# A. Appendix

We give the following exercises to the interested readers that help them practicing the theoretical results. The examples are in q-calculus which takes too much attention recently. Recall from Example 3.13 that $\Delta_q$ is defined as

$$\Delta_q f(t) = \frac{f(tq) - f(t)}{(q-1)t}. \tag{39}$$

With the help of Eq. (39), we provide the following exercises.

**Exercise 6.1** *Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Consider the following system:*

$$\begin{cases} \Delta_q x(t) = \dfrac{1}{4t^2(1+t)^{\frac{1}{7}}}(y(t))^{\frac{1}{7}} \\ \Delta_q y(t) = \dfrac{2t}{4t-1}x(t) \end{cases} \tag{40}$$

and show that $\left(2 - \frac{1}{2t}, t+1\right)$ is a nonoscillatory solution of Eq. (40) in $N^+_{F,\infty} \neq \emptyset$ by checking the conditions given in Theorem 3.12 for $k_1 = 1$.

**Exercise 6.2** *Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. Consider the following system:*

$$\begin{cases} \Delta x_q(t) = \dfrac{1}{qt^{\frac{8}{5}}(2t^2+1)^{\frac{1}{5}}}(y(t))^{\frac{1}{5}} \\ \Delta y_q(t) = \dfrac{q+1}{q^2t^2(t+1)}x(t), \end{cases} \tag{41}$$

where $\Delta h_q(t) = \frac{h(\sigma(t)) - h(t)}{\mu(t)}$ and show that there exists a nonoscillatory solution of system (41), given by $\left(1 + \frac{1}{t}, -2 - \frac{1}{t^2}\right)$, in $N^-_{F,F} \neq \emptyset$ by Theorem 3.15 for $k_3 = -1$ and $k_4 = 1$.

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