Chapter

# An Overview of Stress-Strain Analysis for Elasticity Equations

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## Abstract

The present chapter contains the analysis of stress, analysis of strain and stress-strain relationship through particular sections. The theory of elasticity contains equilibrium equations relating to stresses, kinematic equations relating to the strains and displacements and the constitutive equations relating to the stresses and strains. Concept of normal and shear stresses, principal stress, plane stress, Mohr's circle, stress invariants and stress equilibrium relations are discussed in analysis of stress section while strain-displacement relationship for normal and shear strain, compatibility of strains are discussed in analysis of strain section through geometrical representations. Linear elasticity, generalized Hooke's law and stress-strain relations for triclinic, monoclinic, orthotropic, transversely isotropic, fiber-reinforced and isotropic materials with some important relations for elasticity are discussed.

**Keywords:** analysis of stress, analysis of strain, Mohr's circle, compatibility of strain, stress-strain relation, generalized Hooke's law

## 1. Introduction

If the external forces producing deformation do not exceed a certain limit, the deformation disappears with the removal of the forces. Thus the elastic behavior implies the absence of any permanent deformation. Every engineering material/ composite possesses a certain extent of elasticity. The common materials of construction would remain elastic only for very small strains before exhibiting either plastic straining or brittle failure. However, natural polymeric composites show elasticity over a wider range and the widespread use of natural rubber and similar composites motivated the development of finite elasticity. The mathematical theory of elasticity is possessed with an endeavor to decrease the computation for condition of strain, or relative displacement inside a solid body which is liable to the activity of an equilibrating arrangement of forces, or is in a condition of little inward relative motion and with tries to obtain results which might have been basically essential applications to design, building, and all other helpful expressions in which the material of development is solid.

The elastic properties of continuous materials are determined by the underlying molecular structure, but the relation between material properties and the molecular structure and arrangement in materials is complicated. There are wide classes of materials that might be portrayed by a couple of material constants which can be determined by macroscopic experiments. The quantity of such constants relies upon the nature of the crystalline structure of the material. In this section, we give a short but then entire composition of the basic highlights of applied elasticity having pertinence to our topics. This praiseworthy theory, likely the most successful and best surely understood theory of elasticity, has been given numerous excellent and comprehensive compositions. Among the textbooks including an ample coverage of the problems, we deal with in this chapter which are discussed earlier by Love [1], Sokolnikoff [2], Malvern [3], Gladwell [4], Gurtin [5], Brillouin [6], Pujol [7], Ewing, Jardetsky and Press [8], Achenbach [9], Eringen and Suhubi [10], Jeffreys and Jeffreys [11], Capriz and Podio-Guidugli [12], Truesdell and Noll [13] whose use of direct notation and we find appropriate to avoid encumbering conceptual developments with component-wise expressions. Meriam and Kraige [14] gave an overview of engineering mechanics in theirs book and Podio-Guidugli [15, 16] discussed the strain and examples of concentrated contact interactions in simple bodies in the primer of elasticity. Interestingly, no matter how early in the history of elasticity the consequences of concentrated loads were studied, some of those went overlooked until recently [17–22]. The problem of the determination of stress and strain fields in the elastic solids are discussed by many researchers [23–33]. Belfield et al. [34] discussed the stresses in elastic plates reinforced by fibers lying in concentric circles. Biot [35–38] gave the theory for the propagation of elastic waves in an initially stressed and fluid saturated transversely isotropic media. Borcherdt and Brekhovskikh [39–41] studied the propagation of surface waves in viscoelastic layered media. The fundamental study of seismic surface waves due to the theory of linear viscoelasticity and stress-strain relationship is elaborated by some notable researchers [42–46]. The stress intensity factor is computed due to diffraction of plane dilatational waves by a finite crack by Chang [47], magnetoelastic shear waves in an infinite self-reinforced plate by Chattopadhyay and Choudhury [48]. The propagation of edge wave under initial stress is discussed by Das and Dey [49] and existence and uniqueness of edge waves in a generally anisotropic laminated elastic plates by Fu and Brookes [50, 51]. The basic and historical literature about the stress-strain relationship for propagation of elastic waves in kinds of medium is given by some eminent researchers [52–57]. Kaplunov, Pichugin and Rogersion [58–60] have discussed the propagation of extensional edge waves in in semi-infinite isotropic plates, shells and incompressible plates under the influence of initial stresses. The theory of boundary layers in highly anisotropic and/or reinforced elasticity is studied by Hool, Kinne and Spencer [61, 62].

This chapter addresses the analysis of stress, analysis of strain and stress-strain relationship through particular sections. Concept of normal and shear stress, principal stress, plane stress, Mohr's circle, stress invariants and stress equilibrium relations are discussed in analysis of stress section while strain-displacement relationship for normal and shear strain, compatibility of strains are discussed in analysis of strain section through geometrical representations too. Linear elasticity generalized Hooke's law and stress-strain relation for triclinic, monoclinic, orthotropic, transversely isotropic and isotropic materials are discussed and some important relations for elasticity are deliberated.

## 2. Analysis of stress

A body consists of huge number of grains or molecules. The internal forces act within a body, representing the interaction between the grains or molecules of the body. In general, if a body is in statically equilibrium, then the internal forces are

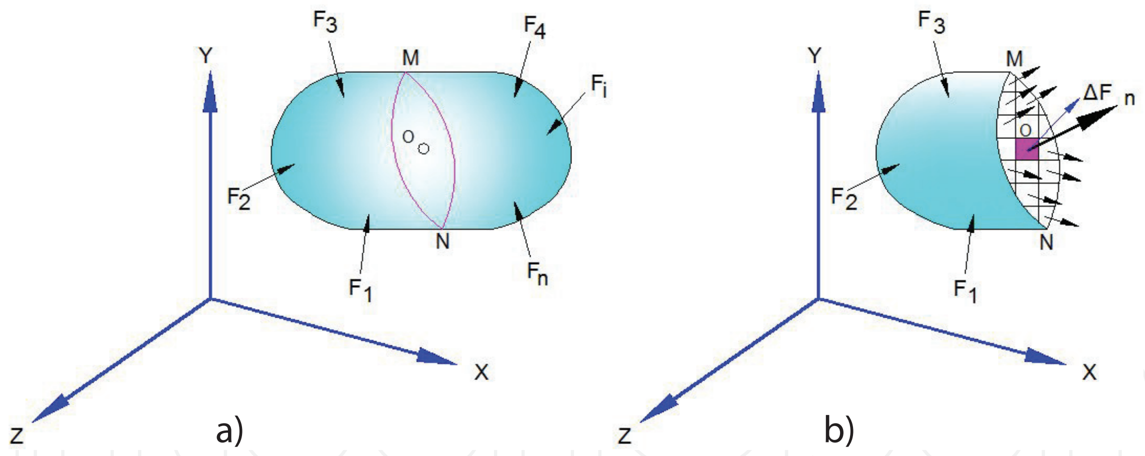


**Figure 1.**
*Forces acting on a (a) body, (b) cross-section of the body.*

equilibrated on the basis of Newton's third law. The internal forces are always present even though the external forces are not active.

To examine these internal forces at a point $O$ in **Figure 1(a)**, inside the body, consider a plane $MN$ passing through the point $O$. If the plane is divided into a number of small areas, as in the **Figure 1(b)**, and the forces acting on each of these are measured, it will be observed that these forces vary from one small area to the next. On the small area $\Delta A$ at point $O$, a force $\Delta F$ will be acting as shown in **Figure 1(b)**. From this the concept of stress as the internal force per unit area can be understood. Assuming that the material is continuous, the term "stress" at any point across a small area $\Delta A$ can be defined by the limiting equation as below.

$$\textbf{Stress}\ (\sigma) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1}$$

where $\Delta F$ is the internal force on the area $\Delta A$ surrounding the given point.
Forces which act on an element of material may be of two types:

i. body forces and

ii. surface forces.

Body forces always act on every molecule of a body and are proportional to the volume whereas surface force acts over the surface of the body and is measure in terms of force per unit area. The force acting on a surface may resolve into normal stress and shear stress. Normal stress may be tensile or compressive in nature. Positive side of normal stress is for tensile stress whilst negative side is for compressive.

### 2.1 Concept of normal stress and shear stress

**Figure 2(a)** shows the rectangular components of the force vector $\Delta F$ referred to corresponding axes. Taking the ratios $\Delta F_x/\Delta A_x$, $\Delta F_y/\Delta A_x$, $\Delta F_z/\Delta A_x$, three quantities that set up the average intensity of the force on the area $\Delta A_x$ When the limit $\Delta A \to 0$, the above ratios are characterized as the force intensity acting on $X$-face at point $O$. These values associated with three intensities are defined as the "Stress components" related with the $X$-face at point $O$. The stress component parallel to the surface are called "Shear stress component," is indicated by $\tau$. The

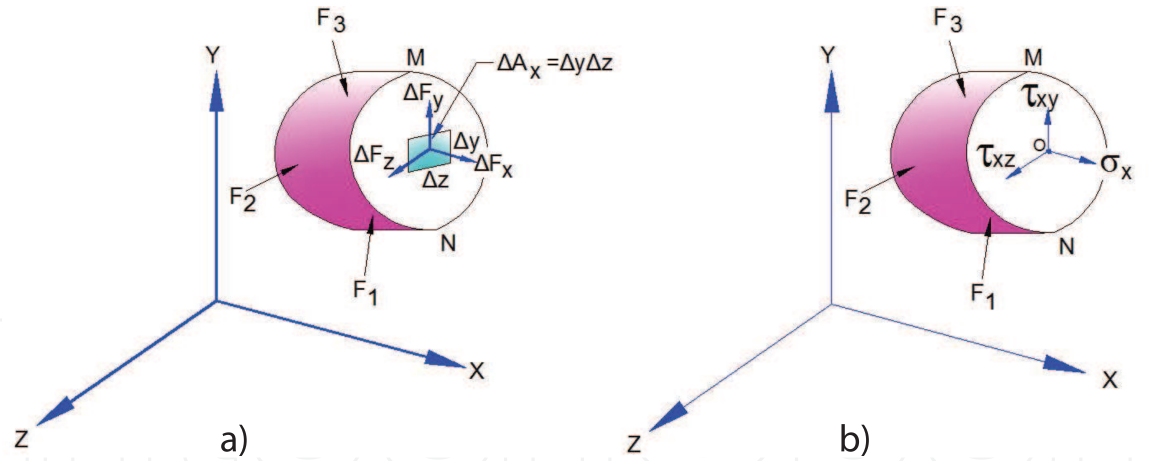


**Figure 2.**
*(a) Force components of ΔF acting on small area centered at point O and (b) stress components at point O.*

shear stress component acting on the $X$-face in the $Y$-direction is identified as $\tau_{xy}$. The stress component perpendicular to the face is called "Normal Stress" or "Direct stress" component and is denoted by $\sigma$.

From the above discussions, the stress components on the $X$-face at point $O$ are defined as follows in terms of force intensity ratios

$$\left.\begin{aligned} \sigma_x &= \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x} \\ \tau_{xy} &= \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x} \\ \tau_{xz} &= \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x} \end{aligned}\right\} \tag{2}$$

and the above stress components are illustrated in **Figure 2(b)**.

### 2.2 Stress components

Three mutually perpendicular coordinate axes $x, y, z$ are taken. We consider the stresses act on the surface of the cubic element of the substance. When a force is applied, as mean that the state of stress is perfectly homogeneous throughout the element and that the body is in equilibrium as shown in **Figure 3**. There are nine quantities which are acting on the faces of the cubic and are known as the stress components.

In matrix notation, the stress components can be written as

$$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \tag{3}$$

which completely define the state of stress in the elemental cube. The first suffix of the shear stress refers to the normal to the plane on which the stress acts and the second suffix refer to the direction of shear stress on this plane. The nine stress components which are derived in matrix form are not all independent quantities.

### 2.3 Principal stress and stress invariants

Let us consider three mutually perpendicular planes in which shear stress is zero and on these planes the normal stresses have maximum or minimum values. These

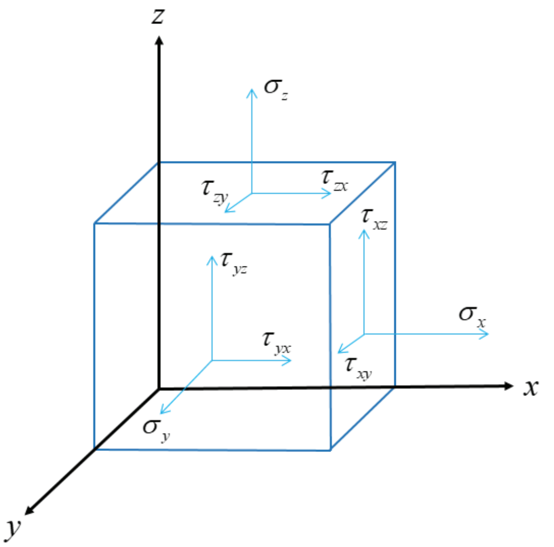


**Figure 3.**
*Stress components acting on cube.*

normal stresses are referred to as principal stresses and the plane in which these normal stresses act is called principal plane.

Invariants mean those amounts that are unexchangeable and do not differ under various conditions. With regards to stress components, invariants are such quantities that don't change with rotation of axes or which stay unaffected under transformation, from one set of axes to another. Subsequently, the combination of stresses at a point that don't change with the introduction of co-ordinate axis is called stress invariants.

### 2.4 Plane stress

Numerous metal shaping procedures include biaxial condition of stress. On the off chance that one of the three normal and shear stresses acting on a body is zero, the state of stress is called plane stress condition. All stresses act parallel to $x$ and $y$ axes. Plane pressure condition is gone over in numerous engineering and forming applications. Regularly, slip can be simple if the shear stress following up on the slip planes is adequately high and acts along favored slip direction. Slip planes may be inclined with respect to the external stress acting on solids. It becomes necessary to transform the stresses acting along the original axes into the inclined planes. Stress change ends up essential in such cases.

#### *2.4.1 Stress transformation in plane stress*

Consider the plane stress condition acting on a plane as shown in **Figure 4**. Let us investigate the state of stresses onto a transformed plane which is inclined at an angle $\theta$ with respect to $x$, $y$ axes.

Let by rotating of the $x$ and $y$ axes through the angle $\theta$, a new set of axes $X'$ and $Y'$ will be formed. The stresses acting on the plane along the new axes are obtained when the plane has been rotated about the $z$ axis. In order to obtain these transformed stresses, we take equilibrium of forces on the inclined plane both perpendicular to and parallel to the inclined plane.

Thus, the expression for transformed stress using the direction cosines can be written as

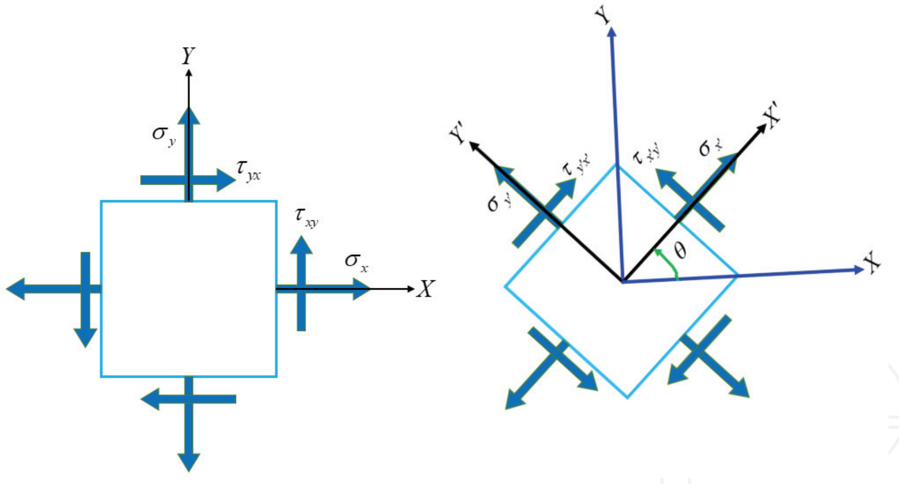


**Figure 4.**
*Representation of stresses on inclined plane.*

$$\begin{aligned}\sigma_{x'} &= l_{x'x}^2\sigma_x + l_{x'y}^2\sigma_y + 2l_{x'x}l_{x'y}\tau_{xy} \\ &= 2\cos^2\theta\sigma_x + 2\sin^2\theta\sigma_y + 2\cos\theta\sin\theta\tau_{xy}\end{aligned} \tag{4}$$

Similarly, write for the $y$' normal stress and shear stress.
The transformed stresses are given as

$$\begin{aligned}\sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta\end{aligned} \tag{5}$$

and

$$\tau_{x'y'} = \frac{\sigma_y - \sigma_x}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

where $\sigma_{x'}$ and $\tau_{x'y'}$ are respectively the normal and shear stress acting on the inclined plane. The above three equations are known as transformation equations for plane stress.

In order to design components against failure the maximum and minimum normal and shear stresses acting on the inclined plane must be derived. The maximum normal stress and shear stress can be found when we differentiate the stress transformation equations with respect to $\theta$ and equate to zero. The maximum and minimum stresses are known as principal stresses and the plane of acting is named as principal planes.

Maximum normal stress is given by

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{6}$$

and maximum shear stress is

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{7}$$

$$\text{with } \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}. \tag{8}$$

*DOI: http://dx.doi.org/10.5772/intechopen.82066*

The plane on which the principal normal stress acts, the shear stress is zero and vice versa. The angle corresponding to the principal planes can be obtained from $\tan 2\theta = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$ for the principal normal planes and $\tan 2\theta = \frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}$ is for the principal shear plane.

### *2.4.2 Mohr's circle for plane stress*

The transformation equations of plane stress which are given by Eq. (5) can be represented in a graphical form (**Figure 5**) by *Mohr's circle*. The transformation equations are sufficient to get the normal and shear stresses on any plane at a point, with Mohr's circle one can easily visualize their variation with respect to plane orientation $\theta$.

#### *2.4.2.1 Equations of Mohr's circle*

Rearranging the terms of Eq. (5), we get

$$\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \tag{9.1}$$

and

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta \tag{9.2}$$

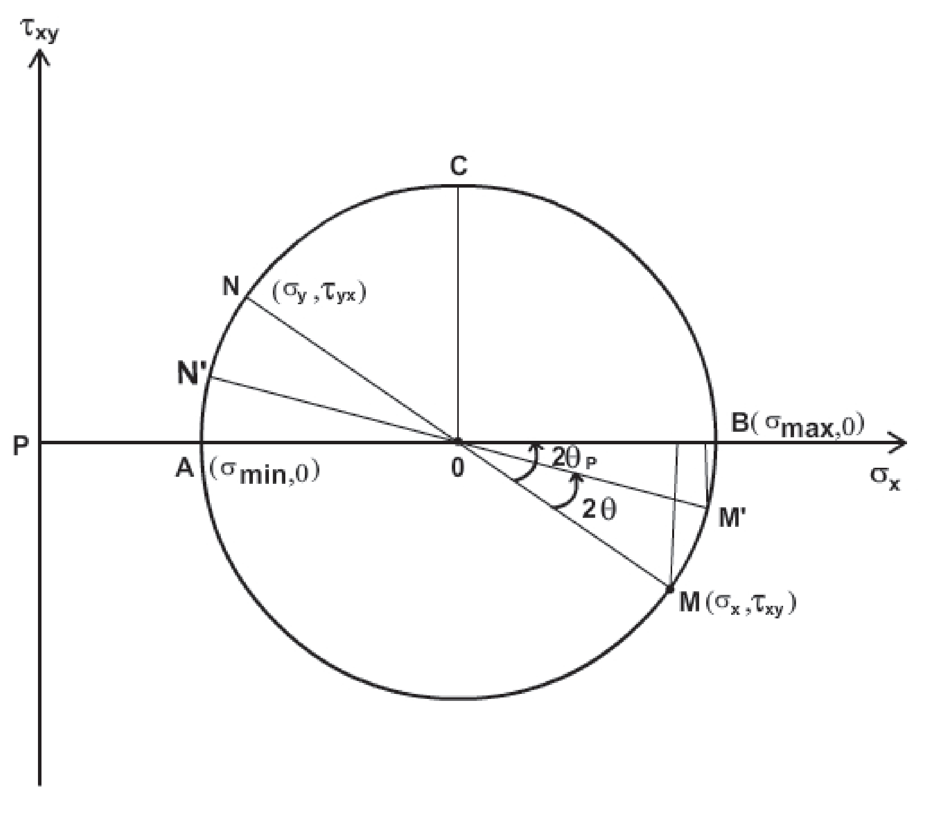


**Figure 5.**
*Mohr's circle diagram.*

Squaring and adding the Eqs. (9.1) and (9.2), result in

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{10}$$

For simple representation of Eq. (10), the following notations are used

$$\sigma_{\text{av}} = \frac{\sigma_x + \sigma_y}{2}, \quad r = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2} \tag{11}$$

Thus, the simplified form of Eq. (10) can be written as

$$(\sigma_{x'} - \sigma_{\text{av}})^2 + \tau_{x'y'}^2 = r^2 \tag{12}$$

Eq. (12) represents the equation of a circle in a standard form. This circle has $\sigma_{x'}$ as its abscissa and $\tau_{x'y'}$ as its ordinate with radius $r$. The coordinate for the center of the circle is $(\sigma_{\text{av}}, 0)$.

Mohr's circle is drawn by considering the stress coordinates $\sigma_x$ as its abscissa and $\tau_{xy}$ as its ordinate, and this plane is known as the stress plane. The plane on the element bounded with $xy$ coordinates in the material is named as physical plane. Stresses on the physical plane $M$ is represented by the point $M$ on the stress plane with $\sigma_x$ and $\tau_{xy}$ coordinates.

Stresses on the physical plane which is normal to i.e. $N$, is given by the point $N$ on the stress plane with $\sigma_y$ and $\tau_{yx}$. $O$ is the intersecting point of line $MN$ and which is at the center of the circle and radius of the circle is $OM$. Now, the stresses on a plane, making $\theta$ inclination with $x$ axis in physical plane can be determined as follows.

An important point to be noted here is that a plane which has a $\theta$ inclination in physical plane will make $2\theta$ inclination in stress plane $M$. Hence, rotate the line $OM$ in stress plane by $2\theta$ counter clockwise to obtain the plane $M'$. The coordinates of $M'$ in stress plane define the stresses acting on plane $M'$ in physical plane and it can be easily verified.

$$\sigma_{x'} = PO + r\cos\left(2\theta_p - 2\theta\right) \tag{13}$$

where $PO = \frac{\sigma_x + \sigma_y}{2}$, $r = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$, $\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2r}$, $\sin 2\theta_p = \frac{\tau_{xy}}{2r}$.

On simplifying Eq. (13)

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{14}$$

Eq. (14) is same as the first equation of Eq. (5).

This way it can be proved for shear stress $\tau_{x'y'}$ on plane $M'$ (do yourself).

### *2.4.3 Stress equilibrium relation*

Let $\sigma_x$, $\tau_{yx}$, $\tau_{zx}$ are the stress components acting along the $x$-direction, $\tau_{xy}$, $\sigma_y$, $\tau_{zy}$ are the stress components acting along the $y$-direction and $\tau_{xz}$, $\tau_{yz}$, $\sigma_z$ are the stress components acting along the $z$-direction. The body forces $F_x$, $F_y$, $F_z$ acting along $x$,

DOI: http://dx.doi.org/10.5772/intechopen.82066

$y$, $z$ direction respectively. Then the stress equilibrium relation or equation of motion in terms of stress components are given by

$$\left.\begin{array}{l}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{yx}}{\partial y}+\dfrac{\partial\tau_{zx}}{\partial z}+F_x=0,\\ \dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{zy}}{\partial z}+F_y=0,\\ \dfrac{\partial\tau_{xz}}{\partial x}+\dfrac{\partial\tau_{yz}}{\partial y}+\dfrac{\partial\sigma_z}{\partial z}+F_z=0.\end{array}\right\} \tag{15}$$

## 3. Analysis of strain

While defining a stress it was pointed out that stress is an abstract quantity which cannot be seen and is generally measured indirectly. Strain differs in this respect from stress. It is a complete quantity that can be seen and generally measured directly as a relative change of length or shape. In generally, stress is the ratio of change in original dimension and the original dimension. It is the dimensionless constant quantity.

### 3.1 Types of strain

Strain may be classified into three types; normal strain, shear strain and volumetric strain.

The normal strain is the relative change in length whether shearing strain relative change in shape. The volumetric strain is defined by the relative change in volume.

### 3.2 Strain-displacement relationship

#### *3.2.1 Normal strain*

Consider a line element of length $\Delta x$ emanating from position $(x, y)$ and lying in the $x$-direction, denoted by $AB$ in **Figure 6**. After deformation the line element occupies $A'B'$, having undergone a translation, extension and rotation.

The particle that was originally at $x$ has undergone a displacement $u_x(x,y)$ and the other end of the line element has undergone a displacement $u_x(x+\Delta x, y)$. By the definition of normal strain

$$\varepsilon_{xx}=\frac{A'B^{*}-AB}{AB}=\frac{u_x(x+\Delta x,y)-u_x(x,y)}{\Delta x}. \tag{16}$$

In the limit $\Delta x\to 0$, Eq. (16) becomes

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x} \tag{17}$$

This partial derivative is a **displacement gradient**, a measure of how rapid the displacement changes through the material, and is the strain at $(x, y)$. Physically, it represents the (approximate) unit change in length of a line element.

Similarly, by considering a line element initially lying in the $y$-direction, the strain in the $y$-direction can be expressed as

$$\varepsilon_{yy}=\frac{\partial u_y}{\partial y}. \tag{18}$$

*3.2.2 Shear strain*

The particles $A$ and $B$ in **Figure 6** also undergo displacements in the $y$-direction and this is shown in **Figure 7(a)**. In this case, we have

$$B^*B' = \frac{\partial u_y}{\partial x} \Delta x. \tag{19}$$

A similar relation can be derived by considering a line element initially lying in the $y$-direction. From the **Figure 7(b)**, we have

$$\theta \approx \tan\theta = \frac{\partial u_y/\partial x}{1 + \partial u_x/\partial x} \approx \frac{\partial u_y}{\partial x} \tag{20}$$

provided that (i) $\theta$ is small and (ii) the displacement gradient $\partial u_x/\partial x$ is small. A similar expression for the angle $\lambda$ can be derived as

$$\lambda \approx \frac{\partial u_x}{\partial y} \tag{21}$$

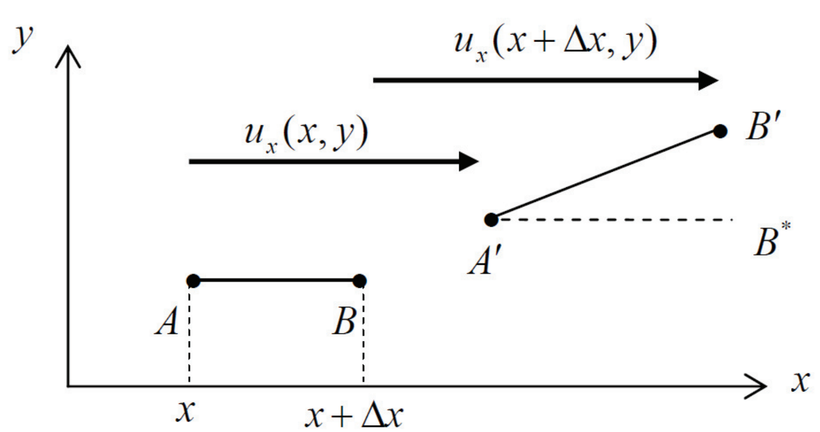


**Figure 6.**
*Deformation of a line element.*

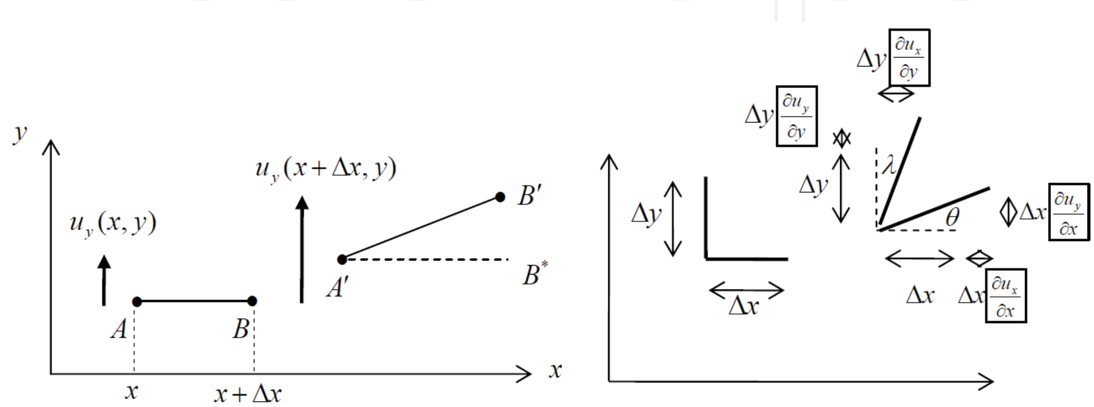


**Figure 7.**
*(a) Deformation of a line element and (b) strains in terms of displacement gradients.*

and hence the shear strain can be written in terms of displacement gradients as

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right). \tag{22}$$

In similar manner, the strain-displacement relation for three dimensional body is given by

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \\ &\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right). \end{aligned} \tag{23}$$

### 3.3 Compatibility of strain

As seen in the previous section, there are three strain-displacement relations Eqs. (17), (18) and (22) but only two displacement components. This implies that the strains are not independent but are related in some way. The relations between the strains are called compatibility conditions.

#### *3.3.1 Compatibility relations*

Let us suppose that the point $P$ which is act *(x,y)* before straining and it will be at $P'$ after straining on the co-ordinate plane $Oxy$ as depicted in **Figure 8**. Then *(u,v)* is a displacement corresponding to the point $P$. The variable $u$ and $v$ are the functions of $x$ and $y$.

Using the fundamental notation

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{24}$$

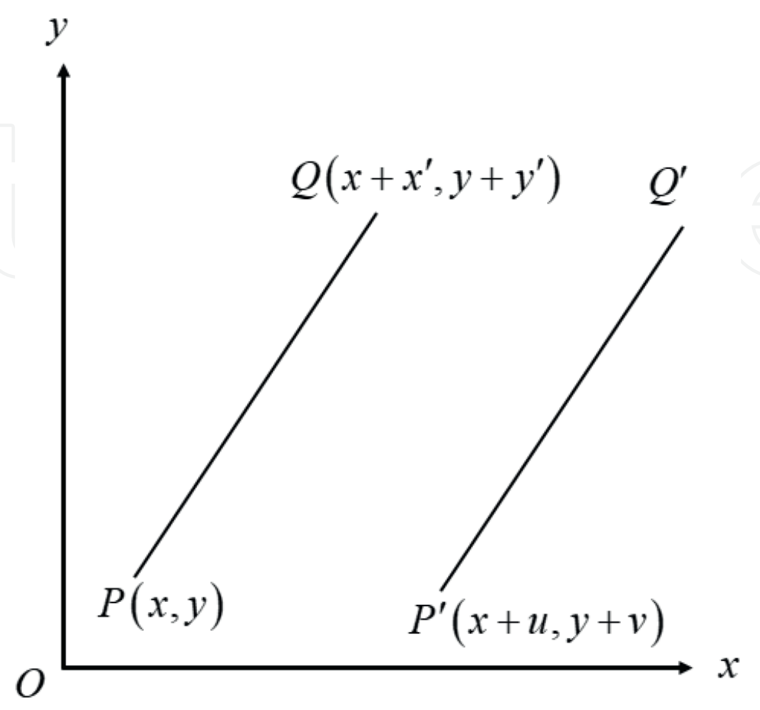


**Figure 8.**
*Deformation of line element.*

we get

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2}, \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial x^2 \partial y} \tag{25}$$

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^3 u_y}{\partial x^2 \partial y} + \frac{\partial^3 u_x}{\partial x \partial y^2}\right). \tag{26}$$

Eqs. (25) and (26) result in

$$\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}\right) \tag{27}$$

which is the compatibility condition in two dimension.

## 4. Stress-strain relation

In the previous section, the state of stress at a point was characterized by six components of stress, and the internal stresses and the applied forces are accompanied with the three equilibrium equation. These equations are applicable to all types of materials as the relationships are independent of the deformations (strains) and the material behavior.

Also, the state of strain at a point was defined in terms of six components of strain. The strains and the displacements are related uniquely by the derivation of six strain-displacement relations and compatibility equations. These equations are also applicable to all materials as they are independent of the stresses and the material behavior and hence.

Irrespective of the independent nature of the equilibrium equations and strain-displacement relations, usually, it is essential to study the general behavior of materials under applied loads including these relations. Strains will be developed in a body due to the application of a load, stresses and deformations and hence it is become necessary to study the behavior of different types of materials. In a general three-dimensional system, there will be 15 unknowns namely 3 displacements, 6 strains and 6 stresses. But we have only 9 equations such as 3 equilibrium equations and 6 strain-displacement equations to achieve these 15 unknowns. It is important to note that the compatibility conditions are not useful for the determination of either the displacements or strains. Hence the additional six equations relating six stresses and six strains will be developed. These equations are known as "Constitutive equations" because they describe the macroscopic behavior of a material based on its internal constitution.

### 4.1 Linear elasticity generalized Hooke's law

Hooke's law provides the unique relationship between stress and strain, which is independent of time and loading history. The law can be used to predict the deformations used in a given material by a combination of stresses.

The linear relationship between stress and strain is given by

$$\sigma_x = E\varepsilon_{xx} \tag{28}$$

where $E$ is known as Young's modulus.

*DOI: http://dx.doi.org/10.5772/intechopen.82066*

In general, each strain is dependent on each stress. For example, the strain $\varepsilon_{xx}$ written as a function of each stress as

$$\varepsilon_{xx} = C_{11}\sigma_x + C_{12}\sigma_y + C_{13}\sigma_z + C_{14}\tau_{xy} + C_{15}\tau_{yz} + C_{16}\tau_{zx} + C_{17}\tau_{xz} + C_{18}\tau_{zy} + C_{19}\tau_{yx}. \tag{29}$$

Similarly, stresses can be expressed in terms of strains which state that at each point in a material, each stress component is linearly related to all the strain components. This is known as **generalized Hook's law**.

For the most general case of three-dimensional state of stress, Eq. (28) can be written as

$$\left(\sigma_{ij}\right)_{9\times1} = \left(D_{ijkl}\right)_{9\times9}\left(\varepsilon_{kl}\right)_{9\times1} \tag{30}$$

where $\left(D_{ijkl}\right)$ is elasticity matrix, $\left(\sigma_{ij}\right)$ is stress components, $\left(\varepsilon_{kl}\right)$ is strain components.

Since both stress $\sigma_{ij}$ and strain $\varepsilon_{ij}$ are second-order tensors, it follows that $D_{ijkl}$ is a fourth order tensor, which consists of $3^4 = 81$ material constants if symmetry is not assumed.

Now, from $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$, the number of 81 material constants is reduced to 36 under symmetric conditions of $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{jilk}$ which provides stress-strain relation for most general form of anisotropic material.

#### *4.1.1 Stress-strain relation for triclinic material*

The stress-strain relation for triclinic material will consist 21 elastic constants which is given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}. \tag{31}$$

#### *4.1.2 Stress-strain relation for monoclinic material*

The stress-strain relation for monoclinic material will consist 13 elastic constants which is given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & D_{15} & 0 \\ D_{12} & D_{22} & D_{23} & 0 & D_{25} & 0 \\ D_{13} & D_{23} & D_{33} & 0 & D_{35} & 0 \\ 0 & 0 & 0 & D_{44} & 0 & D_{46} \\ D_{15} & D_{25} & D_{35} & 0 & D_{55} & 0 \\ 0 & 0 & 0 & D_{46} & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}. \tag{32}$$

#### *4.1.3 Stress-strain relation for orthotropic material*

A material that exhibits symmetry with respect to three mutually orthogonal planes is called an orthotropic material. The stress-strain relation for orthotropic material will consist 9 elastic constants which is given by

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}. \tag{33}$$

#### *4.1.4 Stress-strain relation for transversely isotropic material*

Transversely isotropic material exhibits a rationally elastic symmetry about one of the coordinate axes $x$, $y$ and $z$. In such case, the material constants reduce to 5 as shown below

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{11} & D_{13} & 0 & 0 & 0 \\ D_{13} & D_{13} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & (D_{11}-D_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}. \tag{34}$$

#### *4.1.5 Stress-strain relation for fiber-reinforced material*

The constitutive equation for a fiber-reinforced material whose preferred direction is that of a unit vector $\vec{a}$ is

$$\begin{aligned} \tau_{ij} = {} & \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha\left(a_k a_m e_{km}\delta_{ij} + e_{kk}a_i a_j\right) + 2(\mu_L - \mu_T)\left(a_i a_k e_{kj} + a_j a_k e_{ki}\right) \\ & + \beta a_k a_m e_{km} a_i a_j; \quad i, j, k, m = 1, 2, 3 \end{aligned} \tag{35}$$

where $\tau_{ij}$ are components of stress, $e_{ij}$ are components of infinitesimal strain, and $a_i$ the components of $\vec{a}$ , which are referred to rectangular Cartesian co-ordinates $x_i$. The vector $\vec{a}$ may be a function of position. Indices take the value 1, 2 and 3, and the repeated suffix summation convention is adopted. The coefficients $\lambda$, $\mu_L$, $\mu_T$, $\alpha$ and $\beta$ are all elastic constant with the dimension of stress.

#### *4.1.6 Stress-strain relation for isotropic material*

For a material whose elastic properties are not a function of direction at all, only two independent elastic material constants are sufficient to describe its behavior completely. This material is called isotropic linear elastic. The stress-strain relationship for this material is written as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{12} & D_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (D_{11}-D_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (D_{11}-D_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (D_{11}-D_{12})/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} \tag{36}$$

which consists only two independent elastic constants. Replacing $D_{12}$ and $D_{12}\,(D_{11} - D_{12})/2$ by $\lambda$ and $\mu$ which are called Lame's constants and in particular $\mu$ is also called shear modulus of elasticity, we get

$$\left.\begin{array}{l} \sigma_x = (2\mu + \lambda)\varepsilon_{xx} + \lambda(\varepsilon_{yy} + \varepsilon_{zz}), \\ \sigma_y = (2\mu + \lambda)\varepsilon_{yy} + \lambda(\varepsilon_{xx} + \varepsilon_{zz}), \\ \sigma_z = (2\mu + \lambda)\varepsilon_{zz} + \lambda(\varepsilon_{yy} + \varepsilon_{xx}), \\ \tau_{xy} = \mu\varepsilon_{xy}, \tau_{yz} = \mu\varepsilon_{yz}, \tau_{zx} = \mu\varepsilon_{zx}. \end{array}\right\} \tag{37}$$

Also, from the above relation some important terms are induced which are as follow

(1) **Bulk modulus:** Bulk modulus is the relative change in the volume of a body produced by a unit compressive or tensile stress acting uniformly over its surface. Symbolically

$$K = \lambda + \frac{2}{3}\mu. \tag{38}$$

(2) **Young's modulus:** Young's modulus is a measure of the ability of a material to withstand changes in length when under lengthwise tension or compression. Symbolically

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{39}$$

(3) **Poisson's ratio:** The ratio of transverse strain and longitudinal strain is called Poisson's ratio. Symbolically

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{40}$$

## 5. Conclusions

This chapter dealt the analysis of stress, analysis of strain and stress-strain relationship through particular sections. Concept of normal and shear stress, principal stress, plane stress, Mohr's circle, stress invariants and stress equilibrium relations are discussed in analysis of stress section while strain-displacement relationship for normal and shear strain, compatibility of strains are discussed in analysis of strain section through geometrical representations. Linear elasticity, generalized Hooke's law and stress-strain relation for triclinic, monoclinic, orthotropic, transversely-isotropic, fiber-reinforced and isotropic materials with some important relations for elasticity are discussed mathematically.

## Acknowledgements

The authors convey their sincere thanks to Indian Institute of Technology (ISM), Dhanbad, India for facilitating us with best research facility and provide a Senior Research Fellowship to Mr. Pulkit Kumar and also thanks to DST Inspire India to provide Senior Research Fellowship to Ms. Moumita Mahanty.

## Conflict of interest

There is no conflict of interest to declare.

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## References

[1] Love AEH. A Treatise on the Mathematical Theory of Elasticity. Cambridge: Cambridge University Press; 1927

[2] Sokolnikoff IS. Mathematical Theory of Elasticity. New York: McGraw-Hill; 1956

[3] Malvern LE. Introduction to the Mechanics of a Continuous Medium. Englewood Cliffs: Prentice-Hall; 1969

[4] Gladwell GM. Contact Problems in the Classical Theory of Elasticity. Netherland: Springer Science & Business Media; 1980

[5] Gurtin ME. The linear theory of elasticity. In: Linear Theories of Elasticity and Thermoelasticity. Berlin, Heidelberg: Springer; 1973. pp. 1-295

[6] Brillouin L. Tensors in Mechanics and Elasticity. New York: Academic; 1964

[7] Pujol J. Elastic Wave Propagation and Generation in Seismology. Cambridge, New York: Cambridge University Press; 2003

[8] Ewing WM, Jardetsky WS, Press F. Elastic Waves in Layered Media. New York: McGraw-Hill; 1957

[9] Achenbach J. Wave Propagation in Elastic Solids. Amsterdam: North-Holland; 1973

[10] Eringen A, Suhubi E. Elastodynamics. Vol. II. New York: Academic; 1975

[11] Jeffreys H, Jeffreys B. Methods of Mathematical Physics. Cambridge: Cambridge University Press; 1956

[12] Capriz G, Podio-Guidugli P. Whence the boundary conditions in modern continuum physics? In: Capriz G, Podio-Guidugli P, editors. Proceedings of the Symposium. Rome: Accademia dei Lincei; 2004

[13] Truesdell C, Noll W. The non-linear field theories of mechanics. In: The Non-Linear Field Theories of Mechanics. Berlin, Heidelberg: Springer; 2004. pp. 1-579

[14] Meriam JL, Kraige LG. Engineering Mechanics. Vol. 1. New York: Wiley; 2003

[15] Podio-Guidugli P. Strain. In: A Primer in Elasticity. Dordrecht: Springer; 2000

[16] Podio-Guidugli P. Examples of concentrated contact interactions in simple bodies. Journal of Elasticity. 2004;**75**:167-186

[17] Schuricht F. A new mathematical foundation for contact interactions in continuum physics. Archive for Rational Mechanics and Analysis. 2007;**184**: 495-551

[18] Aki K, Richards PG. Quantitative Seismology. Vol. 2. San Francisco: W. H. Freeman & Co; 1980

[19] Mase G. Theory and Problems of Continuum Mechanics, Schaum's outline series. New York: McGraw-Hill; 1970

[20] Morse P, Feshbach H. Methods of Theoretical Physics. Vol. 2. New York: McGraw-Hill; 1953

[21] Mahanty M, Chattopadhyay A, Dhua S, Chatterjee M. Propagation of shear waves in homogeneous and inhomogeneous fibre-reinforced media on a cylindrical Earth model. Applied Mathematical Modelling. 2017;**52**: 493-511

[22] Kumar P, Chattopadhyay A, Singh AK. Shear wave propagation due to a point source. Procedia Engineering. 2017;**173**:1544-1551

[23] Chattopadhyay A, Singh P, Kumar P, Singh AK. Study of Love-type wave propagation in an isotropic tri layers elastic medium overlying a semi-infinite elastic medium structure. Waves in Random and Complex Media. 2017;**28** (4):643-669

[24] Chattopadhyay A, Saha S, Chakraborty M. Reflection and transmission of shear waves in monoclinic media. International Journal for Numerical and Analytical Methods in Geomechanics. 1997;**21**(7):495-504

[25] Chattopadhyay A. Wave reflection and refraction in triclinic crystalline media. Archive of Applied Mechanics. 2004;**73**(8):568-579

[26] Chattopadhyay A, Singh AK. Propagation of magnetoelastic shear waves in an irregular self-reinforced layer. Journal of Engineering Mathematics. 2012;**75**(1):139-155

[27] Udias A, Buforn E. Principles of Seismology. Cambridge, New York: Cambridge University Press; 2017

[28] Novotny O. Seismic Surface Waves. Bahia, Salvador: Instituto de Geociencias; 1999

[29] Brillouin L. Wave Propagation and Group Velocity. Academic Press, New York; 1960

[30] Badriev IB, Banderov VV, Makarov MV, Paimushin VN. Determination of stress-strain state of geometrically nonlinear sandwich plate. Applied Mathematical Sciences. 2015;**9**(77–80): 3887-3895

[31] Jaeger JC, Cook NG, Zimmerman R. Fundamentals of Rock Mechanics. Malden, U.S.A: Blackwell Publishing. 2009

[32] Press F. Seismic wave attenuation in the crust. Journal of Geophysical Research. 1964;**69**(20):4417-4418

[33] Ben-Menahem A, Singh SJ. Seismic Waves and Sources. New York: Springer Verlag; 1981

[34] Belfield AJ, Rogers TG, Spencer AJM. Stress in elastic plates reinforced by fibres lying in concentric circles. Journal of the Mechanics and Physics of Solids. 1983;**31**(1):25-54

[35] Biot MA. The influence of initial stress on elastic waves. Journal of Applied Physics. 1940;**11**(8):522-530

[36] Biot MA. Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low frequency range. The Journal of the Acoustical Society of America. 1956;**28**(2):168-178

[37] Biot MA. Theory of propagation of elastic waves in a fluid-saturated porous solid. II. Higher frequency range. The Journal of the Acoustical Society of America. 1956;**28**(2):179-191

[38] Biot MA, Drucker DC. Mechanics of incremental deformation. Journal of Applied Mechanics. 1965;**32**(1):957.

[39] Borcherdt RD. Rayleigh-type surface wave on a linear viscoelastic half-space. The Journal of the Acoustical Society of America. 1974;**55**(1):13-15

[40] Borcherdt RD. Viscoelastic Waves in Layered Media. Cambridge, New York: Cambridge University Press; 2009

[41] Brekhovskikh L. Waves in Layered Media. New York: Academic Press; 1980

[42] Bullen KE. Compressibility-pressure hypothesis and the Earth's interior. Geophysical Journal International. 1949; **5**:335-368

[43] Bullen KE, Bullen KE, Bolt BA. An Introduction to the Theory of Seismology. Cambridge, New York: Cambridge University Press; 1985

[44] Chapman C. Fundamentals of Seismic Wave Propagation. Cambridge, UK: Cambridge University Press; 2004

[45] Carcione JM. Wave propagation in anisotropic linear viscoelastic media: Theory and simulated wavefields. Geophysical Journal International. 1990; **101**(3):739-750

[46] Chadwick P. Wave propagation in transversely isotropic elastic media—I. Homogeneous plane waves. Proceedings of the Royal Society of London A. 1989; **422**(1862):23-66

[47] Chang SJ. Diffraction of plane dilatational waves by a finite crack. The Quarterly Journal of Mechanics and Applied Mathematics. 1971;**24**(4): 423-443

[48] Chattopadhyay A, Choudhury S. Magnetoelastic shear waves in an infinite self-reinforced plate. International Journal for Numerical and Analytical Methods in Geomechanics. 1995;**19**(4):289-304

[49] Das SC, Dey S. Edge waves under initial stress. Applied Scientific Research. 1970;**22**(1):382-389

[50] Fu YB. Existence and uniqueness of edge waves in a generally anisotropic elastic plate. Quarterly Journal of Mechanics and Applied Mathematics. 2003;**56**(4):605-616

[51] Fu YB, Brookes DW. Edge waves in asymmetrically laminated plates. Journal of the Mechanics and Physics of Solids. 2006;**54**(1):1-21

[52] Sneddon IN. The distribution of stress in the neighbourhood of a crack in an elastic solid. Proceedings of the Royal Society A. 1946;**187**(1009):229-260

[53] Sengupta PR, Nath S. Surface waves in fibre-reinforced anisotropic elastic media. Sadhana. 2001;**26**(4):363-370

[54] Shearer PM. Introduction to Seismology. 2nd ed. Cambridge: Cambridge University Press; 2009

[55] Sheriff RE, Geldart LP. Exploration Seismology. Cambridge, New York: Cambridge University Press; 1995

[56] Kolsky H. Stress Waves in Solids. New York: Dover Publication; 1963

[57] Gubbins D. Seismology and Plate Tectonics. Cambridge, New York: Cambridge University Press; 1990

[58] Kaplunov J, Prikazchikov DA, Rogerson GA. On three-dimensional edge waves in semi-infinite isotropic plates subject to mixed face boundary conditions. The Journal of the Acoustical Society of America. 2005;**118**(5): 2975-2983

[59] Kaplunov J, Pichugin AV, Zernov V. Extensional edge modes in elastic plates and shells. The Journal of the Acoustical Society of America. 2009;**125**(2): 621-623

[60] Pichugin AV, Rogerson GA. Extensional edge waves in pre-stressed incompressible plates. Mathematics and Mechanics of Solids. 2012;**17**(1):27-42

[61] Hool GA, Kinne WS, Zipprodt RR. Reinforced Concrete and Masonry Structures. New York: McGraw-Hill; 1924

[62] Spencer AJM. Boundary layers in highly anisotropic plane elasticity. International Journal of Solids and Structures. 1974;**10**(10):1103-1123