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Chapter

Computation of Two-Dimensional Fourier Transforms for Noisy Band-Limited Signals

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Abstract

The computation of the two-dimensional Fourier transform by the sampling points creates an ill-posed problem. In this chapter, we will cover this problem for the band-limited signals in the noisy case. We will present a regularized algorithm based on the two-dimensional Shannon Sampling Theorem, the two-dimensional Fourier series, and the regularization method. First, we prove the convergence property of the regularized solution according to the maximum norm. Then an error estimation is given according to the L^2-norm. The convergence property of the regularized Fourier series is given in theory, and some examples are given to compare the numerical results of the regularized Fourier series with the numerical results of the Fourier series.

Keywords: Fourier transform, band-limited signal, ill-posedness, regularization

AMS subject classifications: 65T40, 65R20, 65R30, 65R32

1. Introduction

The two-dimensional Fourier transform is widely applied in many fields [1–9]. In this chapter, the ill-posedness of the problem for computing two-dimensional Fourier transform is analyzed on a pair of spaces by the theory and examples in detail. A two-dimensional regularized Fourier series is presented with the proof of the convergence property and some experimental results.

First, we describe the band-limited signals.

Definition. For two positive $\Omega_1, \Omega_2 \in \mathbb{R}$, a function $f \in L^2(\mathbb{R}^2)$ is said to be band-limited if

$$\hat{f}(\omega_1, \omega_2) = 0, \forall(\omega_1, \omega_2) \in \mathbb{R}^2 \backslash [-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2].$$

Here $\hat{f}$ is the Fourier transform of:

$$F(f)(\omega_1, \omega_2) = \hat{f}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2) e^{it_1\omega_1 + it_2\omega_2} dt_1 dt_2, \ (\omega_1, \omega_2) \in \mathbb{R}^2. \qquad (1)$$

We will consider the problem of computing $\hat{f}(\omega_1, \omega_2)$ from $f(t_1, t_2)$.

For band-limited signals, we have the following sampling theorem [4, 10, 11]. For the two-dimensional band-limited function above, we have

$$f(t_1,t_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} f(n_1H_1,n_2H_2)\frac{\sin\Omega_1(t_1-n_1H_1)}{\Omega_1(t_1-n_1H_1)}\frac{\sin\Omega_2(t_2-n_2H_2)}{\Omega_2(t_2-n_2H_2)}, \tag{2}$$

where $H_1 := \pi/\Omega_1$ and $H_2 := \pi/\Omega_2$.

Calculating the Fourier transform of $f(t_1,t_2)$ by the formula (2), we have the formula which is same as the Fourier series

$$\hat{f}(\omega_1,\omega_2) = H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} f(n_1H_1,n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2), \tag{3}$$

where $P_\Omega(\omega_1,\omega_2) = 1_{[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]}(\omega_1,\omega_2)$ is the characteristic function of $[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$.

In many practical problems, the samples $\{f(n_1H_1,n_2H_2)\}$ are noisy:

$$f(n_1H_1,n_2H_2) = f_T(n_1H_1,n_2H_2) + \eta(n_1H_1,n_2H_2), \tag{4}$$

where $\{\eta(n_1H_1,n_2H_2)\}$ is the noise

$$|\eta(n_1H_1,n_2H_2)| \le \delta, \tag{5}$$

and $f_T \in L^2$ is the exact band-limited signal.

The noise in the two-dimensional case is discussed in [5, 6], and the Tikhonov regularization method is used. However, there is too much computation in the Tikhonov regularization method since the solution of an Euler equation is required.

The ill-posedness in the one-dimensional case is considered in [12, 13]. The regularized Fourier series

$$\hat{f}_\alpha(\omega) = H\sum_{n=-\infty}^{\infty}\frac{f(nH)e^{inH\omega}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}P_\Omega(\omega)$$

in [12] is given based on the regularized Fourier transform

$$F_\alpha[f] = \int_{-\infty}^{\infty}\frac{f(t)e^{i\omega t}dt}{1+2\pi\alpha+2\pi\alpha t^2}$$

in [14]. The regularized Fourier transform was found by finding the minimizer of the Tikhonov's smoothing functional.

In this chapter, we will find a reliable algorithm for this ill-posed problem using a two-dimensional regularized Fourier series. In Section 2, the ill-posedness is discussed in the two-dimensional case. In Section 3, the regularized Fourier series and the proof of the convergence property are given. The bias and variance of regularized Fourier series are given in Section 4. The algorithm and the experimental results of numerical examples are given in Section 5. Finally, the conclusion is given in Section 6.

2. The ill-posedness

We will first study the ill-posedness of the problem (3) in the noisy case (4). The concept of ill-posed problems was introduced in [15]. Here we borrow the following definition from it.

Definition 2.1 Assume A: $D \to U$ is an operator in which D and U are metric spaces with distances $\rho_D(*, *)$ and $\rho_U(*, *)$, respectively. The problem

$$Az = u. \tag{6}$$

of determining a solution z in the space D from the "initial data" u in the space U is said to be well-posed on the pair of metric spaces (D, U) in the sense of Hadamard if the following three conditions are satisfied:

i. For every element $u \in U$, there exists a solution z in the space D; in other words, the mapping A is surjective.

ii. The solution is unique; in other words, the mapping A is injective.

iii. The problem is stable in the spaces (D, U): $\forall \in >0$, $\exists \delta > 0$, such that

$$\rho_U(u_1, u_2) < \delta \Rightarrow \rho_D(z_1, z_2) < \in.$$

In other words, the inverse mapping A^{-1} is uniformly continuous. Problems that violate any of the three conditions are said to be ill-posed.

In this section, we discuss the ill-posedness of $A\hat{f} = f$ on the pair of Banach spaces $(L^2[-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2], l^\infty(\mathbb{Z}^2))$, where $\hat{f}(\omega_1, \omega_2)$ is given by the Fourier series in Eq. (3).

The operator A in Eq. (6) is defined by the following formula:

$$A\hat{f} = f, \tag{7}$$

where $= \{f(n_1 H_1, n_2 H_2)$: $\mathrm{n}_1 \in \mathbb{Z}, \mathrm{n}_2 \in \mathbb{Z}\}$.

As usual, l^∞ is the space $\{a(n) : \mathrm{n} \in \mathbb{Z}^2\}$ of bounded sequences. The norm of l^∞ is defined by

$$\|\boldsymbol{a}\|_{l^\infty} = \sup_{n \in \mathbb{Z}^2} |a(n)|,$$

where

i. The existence condition is not satisfied.

ii. The uniqueness condition is satisfied.

iii. The stability condition is not satisfied. The proof is similar to the proof in [10].

3. The regularized Fourier series

Based on the one-dimensional regularized Fourier series in [12], we construct the two-dimensional regularized Fourier series:

$$\hat{f}_\alpha(\omega_1, \omega_2) = H_1 H_2 \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \frac{f(n_1 H_1, n_2 H_2) e^{i n_1 H_1 \omega_1 + i n_2 H_2 \omega_2}}{\left[1 + 2\pi\alpha + 2\pi\alpha (n_1 H_1)^2\right]\left[1 + 2\pi\alpha + 2\pi\alpha (n_2 H_2)^2\right]} P_\Omega(\omega_1, \omega_2), \tag{8}$$

where $f(n_1H_1, n_2H_2)$ is given in (4). We will give the convergence property of the regularized Fourier series in this section.

Lemma 3.1

$$F\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\frac{\sin\Omega(t-nH)}{\Omega(t-nH)}\right] = \frac{H}{1+2\pi\alpha+2\pi\alpha(nH)^2}e^{inH\omega} - \frac{H}{4\pi a\alpha}(-1)^n\left[\frac{e^{a(\omega-\Omega)}}{a-inH}+\frac{e^{-a(\omega+\Omega)}}{a+inH}\right], \tag{9}$$

where $a := \sqrt{\frac{1+2\pi\alpha}{2\pi\alpha}}$.

Proof.

$$F\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\frac{\sin\Omega(t-nH)}{\Omega(t-nH)}\right] = \frac{1}{2\pi}F\left[\frac{1}{1+2\pi\alpha+2\pi\alpha t^2}\right]*F\left[\frac{\sin\Omega(t-nH)}{\Omega(t-nH)}\right]$$

$$= \frac{1}{2\pi}\frac{1}{2a\alpha}e^{-a|\omega|}*\left[He^{i\omega nH}P_\Omega(\omega)\right] = H\frac{1}{4\pi a\alpha}\int_{-\infty}^{\infty} e^{-a|u|}e^{inH(\omega-u)}1_{[\omega-\Omega,\omega+\Omega]}(u)du$$

$$= H\frac{1}{4\pi a\alpha}e^{inH\omega}\int_{\omega-\Omega}^{\omega+\Omega} e^{-a|u|-inHu}du = H\frac{1}{4\pi a\alpha}e^{inH\omega}\left(\int_{\omega-\Omega}^{0} e^{au-inHu}du + \int_{0}^{\omega+\Omega} e^{-au-inHu}du\right)$$

$$= H\frac{1}{4\pi a\alpha}e^{inH\omega}\left[\frac{1}{a-inH}-\frac{e^{(a-inH)(\omega-\Omega)}}{a-inH}+\frac{1}{a+inH}-\frac{e^{-(a+inH)(\omega+\Omega)}}{a+inH}\right]$$

$$= H\frac{1}{4\pi a\alpha}e^{inH\omega}\left(\frac{1}{a-inH}+\frac{1}{a+inH}\right) - H\frac{1}{4\pi a\alpha}e^{inH\omega}\left[\frac{e^{(a-inH)(\omega-\Omega)}}{a-inH}+\frac{e^{-(a+inH)(\omega+\Omega)}}{a+inH}\right]$$

$$= H\frac{1}{2\pi\alpha}\frac{e^{inH\omega}}{a^2+(nH)^2} - H\frac{1}{4\pi a\alpha}e^{inH\omega}(-1)^n\left[\frac{e^{a(\omega-\Omega)-inH\omega}}{a-inH}+\frac{e^{-a(\omega+\Omega)-inH\omega}}{a+inH}\right]$$

$$= H\frac{e^{inH\omega}}{1+2\pi\alpha+2\pi\alpha(nH)^2} - H\frac{1}{4\pi a\alpha}(-1)^n\left[\frac{e^{a(\omega-\Omega)}}{a-inH}+\frac{e^{-a(\omega+\Omega)}}{a+inH}\right].$$

Lemma 3.2 For any band-limited function $g(t_1, t_2)$ and $(\omega_1, \omega_2) \in [-\Omega_1, \Omega_1] \times [-\Omega_2, \Omega_2]$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{g(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}$$

$$= H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\frac{e^{in_1H_1\omega_1+in_2H_2\omega_2}}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}$$

$$-H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\left[\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\frac{(-1)^{n_2}}{4\pi a\alpha}\left(\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right)\right]$$

$$-H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\left[\frac{e^{in_2H_2\omega_2}}{1+2\pi\alpha+2\pi\alpha(n_2H_2)^2}\frac{(-1)^{n_1}}{4\pi a\alpha}\left(\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right)\right]$$

$$+H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\frac{(-1)^{n_1+n_2}}{(4\pi a\alpha)^2}\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\cdot\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]. \tag{10}$$

Proof. By the sampling theorem

$$
\begin{aligned}
I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{g(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{it_1\omega_1+it_2\omega_2}}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)} \\
&\quad \cdot \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\frac{\sin\Omega_1(t_1-n_1H_1)}{\Omega_1(t_1-n_1H_1)}\frac{\sin\Omega_2(t_2-n_2H_2)}{\Omega_2(t_2-n_2H_2)}dt_1dt_2 \\
&= \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\int_{-\infty}^{\infty}\frac{1}{1+2\pi\alpha+2\pi\alpha t_1^2}\frac{\sin\Omega_1(t_1-n_1H_1)}{\Omega_1(t_1-n_1H_1)}e^{it_1\omega_1}dt_1 \\
&\qquad \cdot \int_{-\infty}^{\infty}\frac{1}{1+2\pi\alpha+2\pi\alpha t_2^2}\frac{\sin\Omega_2(t_2-n_2H_2)}{\Omega_2(t_2-n_2H_2)}e^{it_2\omega_2}dt_2
\end{aligned}
$$

By Lemma 3.1 and the FOIL method, Eq. (10) is true.

Lemma 3.3 For each arbitrarily small $c>0$ and $\omega\in[-\Omega+c,\Omega-c]$,

$$
\sum_{n=-\infty}^{\infty}\left|\frac{e^{a(\omega-\Omega)}}{a-inH}+\frac{e^{-a(\omega+\Omega)}}{a+inH}\right|^2 = O\left(\frac{e^{-2ac}}{a}\right). \tag{11}
$$

Proof. By the inequality $|a+b|^2\le 2\left(|a|^2+|b|^2\right)$,

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\left|\frac{e^{a(\omega-\Omega)}}{a-inH}+\frac{e^{-a(\omega+\Omega)}}{a+inH}\right|^2 \le 2\sum_{n=-\infty}^{\infty}\left[\left|\frac{e^{a(\omega-\Omega)}}{a-inH}\right|^2+\left|\frac{e^{-a(\omega+\Omega)}}{a+inH}\right|^2\right] \\
&\le 4\sum_{n=-\infty}^{\infty}\frac{e^{-2ac}}{a^2+(nH)^2} \le \frac{4}{H}e^{-2ac}\int_{-\infty}^{\infty}\frac{dx}{a^2+x^2}+\frac{4}{a^2}e^{-2ac} = \frac{4\pi e^{-2ac}}{Ha}+\frac{4}{a^2}e^{-2ac}.
\end{aligned}
$$

Lemma 3.4 For each arbitrarily small $c>0$ and $(\omega_1,\omega_2)\in[-\Omega_1+c,\Omega_1-c]\times[-\Omega_2+c,\Omega_2-c]$,

$$
\begin{aligned}
&\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\frac{(-1)^{n_1+n_2}}{(4\pi a\alpha)^2}\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right] \\
&\qquad \left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right], = O(ae^{-2ac}).
\end{aligned} \tag{12}
$$

for $\alpha\to+0$ and g that is Ω-band-limited.

Proof. By the Cauchy inequality,

$$
\begin{aligned}
&\left|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} g(n_1H_1,n_2H_2)\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2 \\
&\qquad \le \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|g(n_1H_1,n_2H_2)|^2 \\
&\qquad \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2,
\end{aligned}
$$

where $\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|g(n_1H_1,n_2H_2)|^2$ is bounded by Parseval equality, and

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2$$

$$=\sum_{n_1=-\infty}^{\infty}\left|\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\right|^2\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{a(\omega_1-\Omega_1)}}{a-in_1H_1}+\frac{e^{-a(\omega_1+\Omega_1)}}{a+in_1H_1}\right]\right|^2.$$

By Lemma 3.3, Eq. (12) is true.

Lemma 3.5

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH)^2}\right|^2=O\left(\frac{1}{\sqrt{\alpha}}\right). \tag{13}$$

Proof.

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH)^2}\right|^2\le\left|\frac{1}{1+2\pi\alpha}\right|^2+\sum_{n\neq 0}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH)^2}\right|^2,$$

where

$$\sum_{n\neq 0}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH)^2}\right|^2\le 2\sum_{n=1}^{\infty}\frac{1}{\left[1+2\pi\alpha+2\pi\alpha(nH)^2\right]^2}\le$$

$$\frac{2}{H}\int_0^{\infty}\frac{dx}{(1+2\pi\alpha+2\pi\alpha x^2)^2}=O\left(\frac{1}{\sqrt{\alpha}}\right).$$

So

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH)^2}\right|^2=O\left(\frac{1}{\sqrt{\alpha}}\right).$$

Lemma 3.6 For each arbitrarily small $c>0$ and $(\omega_1,\omega_2)\in[-\Omega_1+c,\Omega_1-c]\times[-\Omega_2+c,\Omega_2-c]$,

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}g(n_1H_1,n_2H_2)\left[\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\frac{(-1)^{n_2}}{4\pi\alpha a}\left(\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right)\right]$$
$$=O\left(a^{\frac{1}{2}}e^{-ac}\right), \tag{14}$$

for $\alpha\to+0$ and g that is Ω-band-limited.

Proof. By Cauchy inequality,

$$\left|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}g(n_1H_1,n_2H_2)\left[\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2$$

$$\le\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|g(n_1H_1,n_2H_2)|^2$$

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2,$$

where $\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|g(n_1H_1, n_2H_2)|^2$ is bounded by the Parseval equality, and

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\right]\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2$$
$$=\sum_{n_1=-\infty}^{\infty}\left|\frac{e^{in_1H_1\omega_1}}{1+2\pi\alpha+2\pi\alpha(n_1H_1)^2}\right|^2\sum_{n_2=-\infty}^{\infty}\left|\left[\frac{e^{a(\omega_2-\Omega_2)}}{a-in_2H_2}+\frac{e^{-a(\omega_2+\Omega_2)}}{a+in_2H_2}\right]\right|^2.$$

By Lemma 3.3 and Lemma 3.5 Eq. (14) is true.
Lemma 3.7

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\frac{\eta(n_1H_1, n_2H_2)}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}\right| = O\left(\frac{\delta}{\alpha}\right) \quad (15)$$

for $\delta\to+0$ and $\alpha\to+0$, where η and δ are given in (4) and (5) in Section 1.
Proof.

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_1)^2}\right| \le \left|\frac{1}{1+2\pi\alpha}\right| + \sum_{n\neq 0}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_1)^2}\right|,$$

where

$$\sum_{n\neq 0}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_1)^2}\right| \le 2\sum_{n=1}^{\infty}\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_1)^2}$$
$$\le\frac{2}{H_1}\int_0^{\infty}\frac{dx}{1+2\pi\alpha+2\pi\alpha x^2}=O\left(\frac{1}{\sqrt{\alpha}}\right).$$

So

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_1)^2}\right| = O\left(1/\sqrt{\alpha}\right).$$

For the same reason,

$$\sum_{n=-\infty}^{\infty}\left|\frac{1}{1+2\pi\alpha+2\pi\alpha(nH_2)^2}\right| = O\left(\frac{1}{\sqrt{\alpha}}\right).$$

So Eq. (15) is true.

Theorem 3.1 Suppose $f_T \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$ is band-limited. For each arbitrarily small $c>0$, if we choose $\alpha=\alpha(\delta)$ such that $\alpha(\delta)\to 0$ and $\delta/\alpha(\delta)\to 0$ as $\delta\to 0$, then $\hat{f}_\alpha(\omega_1,\omega_2)\to\hat{f}_T(\omega_1,\omega_2)$ uniformly in $(\omega_1,\omega_2)\in[-\Omega_1+c, \Omega_1-c]\times[-\Omega_2+c, \Omega_2-c]$ as $\delta\to 0$.
Proof. By Lemma 3.2, Lemma 3.4 and Lemma 3.6, we have

$$H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{f_T(n_1H_1, n_2H_2)}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}e^{in_1H_1\omega_1+in_2H_2\omega_2}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}+O\left(a^{\frac{1}{2}}e^{-ac}\right).$$

Therefore,

$$\hat{f_\alpha}(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)=H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{[f_T(n_1H_1,n_2H_2)+\eta(n_1H_1,n_2H_2)]e^{in_1H_1\omega_1+in_2H_2\omega_2}}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}P_\Omega(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)$$

$$=H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{f_T(n_1H_1,n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}P_\Omega(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)$$

$$+H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1,n_2H_2)}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2)$$

$$=\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right]P_\Omega(\omega_1,\omega_2)$$

$$+H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1,n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}P_\Omega(\omega_1,\omega_2)+O\left(a^{\frac{1}{2}}e^{-ac}\right).$$

This implies

$$\left|\hat{f}_\alpha(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)\right|$$
$$\le\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right|$$
$$+H_1H_2\left|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1,n_2H_2)}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}e^{in_1H_1\omega_1+in_2H_2\omega_2}\right|$$
$$+O\left(a^{\frac{1}{2}}e^{-ac}\right)$$

where

$$\left|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1,n_2H_2)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}e^{in_1H_1\omega_1+in_2H_2\omega_2}\right|=O\left(\frac{\delta}{\alpha}\right).$$

For any $\varepsilon>0$, there exists $M>0$ such that

$$\iint_{|t_1|\ge M\ or\ |t_2|\ge M}\left|f_T(t_1,t_2)\right|dt_1dt_2<\varepsilon.$$

Then

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right|$$
$$\le\left|\iint_{|t_1|\le M\ and\ |t_2|\le M}\frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right|$$
$$+\left|\iint_{|t_1|\ge M\ or\ |t_2|\ge M}\frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)}f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right|,$$

where

$$\left|\iint\limits_{|t_1|\geq M\ or\ |t_2|\geq M} \frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)} f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right|$$

$$\leq \iint\limits_{|t_1|\geq M\ or\ |t_2|\geq M} \left|f_T(t_1,t_2)\right|dt_1dt_2 < \varepsilon$$

and

$$\left|\iint\limits_{|t_1|\leq M\ and\ |t_2|\leq M} \frac{4\pi\alpha+2\pi\alpha t_1^2+2\pi\alpha t_2^2+\left(2\pi\alpha+2\pi\alpha t_1^2\right)\left(2\pi\alpha+2\pi\alpha t_2^2\right)}{\left(1+2\pi\alpha+2\pi\alpha t_1^2\right)\left(1+2\pi\alpha+2\pi\alpha t_2^2\right)} f_T(t_1,t_2)e^{it_1\omega_1+it_2\omega_2}dt_1dt_2\right| \to 0$$

as $\alpha \to 0$.

4. Error analysis

In last section we have proved the convergence property of the regularized Fourier series under the condition $f_T \in L^1(\mathbb{R}^2)$. In this section, we give the error analysis of the regularized Fourier series according to the L^2-norm for the functions $f_T \in L^2(\mathbb{R}^2)$. The bound of the variance of the regularized Fourier series is presented.

By Lemma 3.5, we have next lemma.

Lemma 4.1

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\frac{\eta(n_1H_1,n_2H_2)}{[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2][1+2\pi\alpha+2\pi\alpha(n_2H_2)^2]}\right|^2 = O\left(\delta^2\right)+O\left(\frac{\delta^2}{\alpha}\right)$$

for $\delta \to +0$ and $\alpha \to +0$, where η and δ are given in Eq. (4) and Eq. (5) in Section 1.

Theorem 4.1 Suppose $f_T \in L^2(\mathbb{R}^2)$ is band-limited. If we choose $\alpha = \alpha(\delta)$ such that $\alpha(\delta) \to 0$ and $\delta^2/\alpha(\delta) \to 0$ as $\delta \to 0$, then $\hat{f}_\alpha(\omega_1,\omega_2) \to \hat{f}_T(\omega_1,\omega_2)$ in $L^2[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$ as $\delta \to 0$.

Proof.

$$\hat{f}_\alpha(\omega_1,\omega_2) - \hat{f}_T(\omega_1,\omega_2)$$

$$= H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\left[f_T(n_1H_1,n_2H_2)+\eta(n_1H_1,n_2H_2)\right]e^{in_1H_1\omega_1+in_2H_2\omega_2}}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}P_\Omega(\omega_1,\omega_2) - \hat{f}_T(\omega_1,\omega_2)$$

$$= -H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$

$$\cdot f_T(n_1H_1, n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2)$$
$$+H_1H_2\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1, n_2H_2)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2).$$

Let

$$\mathrm{S}(\omega_1,\omega_2) := \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}$$
$$\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot f_T(n_1H_1, n_2H_2)e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2).$$

Then

$$\left\|\hat{f}_\alpha(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)\right\|_{L^2}^2 \le 2H_1^2H_2^2\|\mathrm{S}(\omega_1,\omega_2)\|^2+2H_1^2H_2^2$$
$$\left\|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1, n_2H_2)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2)\right\|^2,$$

where

$$\left\|\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\eta(n_1H_1, n_2H_2)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}e^{in_1H_1\omega_1+in_2H_2\omega_2}P_\Omega(\omega_1,\omega_2)\right\|^2$$
$$=\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\left|\frac{\eta(n_1H_1, n_2H_2)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}\right|^2 = O\left(\frac{\delta^2}{\alpha}\right)$$

by Lemma 4.1 and

$$\|\mathrm{S}(\omega_1,\omega_2)\|^2 = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}$$
$$\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot\left|f_T(n_1H_1, n_2H_2)\right|^2.$$

For every $\varepsilon>0$, there exists $N>0$ such that

$$\sum_{|n_1|\ge N}\ \sum_{or\ |n_2|\ge N}\left|f_T(n_1H_1, n_2H_2)\right|^2 < \epsilon,$$

since

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$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot\left|f_T(n_1H_1,n_2H_2)\right|^2=\sum_{|n_1|\le N}\sum_{and\ |n_2|\le N}$$
$$\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot\left|f_T(n_1H_1,n_2H_2)\right|^2+\sum_{|n_1|>N}\sum_{or\ |n_2|>N}$$
$$\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot\left|f_T(n_1H_1,n_2H_2)\right|^2,$$

where

$$\sum_{|n_1|>N}\sum_{or\ |n_2|>N}\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}$$
$$\cdot\left|f_T(n_1H_1,n_2H_2)\right|^2$$
$$\le\sum_{|n_1|\ge N}\sum_{or\ |n_2|\ge N}\left|f_T(n_1H_1,n_2H_2)\right|^2<\varepsilon$$

and

$$\sum_{|n_1|\le N}\sum_{and\ |n_2|\le N}\frac{4\pi\alpha+2\pi\alpha(n_1H_1)^2+2\pi\alpha(n_2H_2)^2+\left(2\pi\alpha+2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha+2\pi\alpha(n_2H_2)^2\right)}{\left[1+2\pi\alpha+2\pi\alpha(n_1H_1)^2\right]\left[1+2\pi\alpha+2\pi\alpha(n_2H_2)^2\right]}\left|f_T(n_1H_1,n_2H_2)\right|^2\to 0$$

as $\alpha\to 0$.

Therefore, $\left\|\hat{f}_\alpha(\omega_1,\omega_2)-\hat{f}_T(\omega_1,\omega_2)\right\|_{L^2}^2\to 0.$

Theorem 4.2 Suppose $f_T\in L^2(\mathbb{R}^2)$ is band-limited. If the noise in Eq. (4) is white noise such that $E[\eta(n_1H_1,n_2H_2)]=0$ and $Var[\eta(n_1H_1,n_2H_2)]=\sigma^2$, then the bias $\hat{f}_T(\omega_1,\omega_2)-E\left[\hat{f}_\alpha(\omega_1,\omega_2)\right]\to 0$ in $L^2[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$ as $\alpha\to 0$ and

$$Var\left[\hat{f}_\alpha(\omega_1,\omega_2)\right]=O\left(\sigma^2\right)+O\left(\sigma^2/\alpha\right)$$

if $\alpha(\sigma) \to 0$ and $\sigma^2/\alpha(\sigma) \to 0$ as $\sigma \to 0$.

Proof. We can calculate

$$\left\|\hat{f}_T(\omega_1,\omega_2) - E\left[\hat{f}_\alpha(\omega_1,\omega_2)\right]\right\|_{L^2}^2 = H_1^2H_2^2 \cdot \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}$$

$$\frac{4\pi\alpha + 2\pi\alpha(n_1H_1)^2 + 2\pi\alpha(n_2H_2)^2 + \left(2\pi\alpha + 2\pi\alpha(n_1H_1)^2\right)\left(2\pi\alpha + 2\pi\alpha(n_2H_2)^2\right)}{\left[1 + 2\pi\alpha + 2\pi\alpha(n_1H_1)^2\right]\left[1 + 2\pi\alpha + 2\pi\alpha(n_2H_2)^2\right]}$$

$$\cdot\left|f_T(n_1H_1, n_2H_2)\right|^2$$

and

$$Var\left[\hat{f}_\alpha(\omega_1,\omega_2)\right] = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\frac{\sigma^2}{\left[1 + 2\pi\alpha + 2\pi\alpha(n_1H_1)^2\right]^2\left[1 + 2\pi\alpha + 2\pi\alpha(n_2H_2)^2\right]^2}.$$

By the proof of Theorem 4.1, we can see that $\hat{f}_T(\omega_1,\omega_2) - E\left[\hat{f}_\alpha(\omega_1,\omega_2)\right] \to 0$ in $L^2[-\Omega_1,\Omega_1]\times[-\Omega_2,\Omega_2]$ as $\alpha \to 0$ and $Var\left[\hat{f}_\alpha([-\omega_1,\omega_1])\right] = O(\sigma^2) + O(\sigma^2/\alpha)$.

5. The algorithm and experimental results

In this section, we give the algorithm and an example to show that the regularized Fourier series is more effective in controlling noise than the Fourier series.

In practical computation, we choose a large integer N and use the next formula in computation:

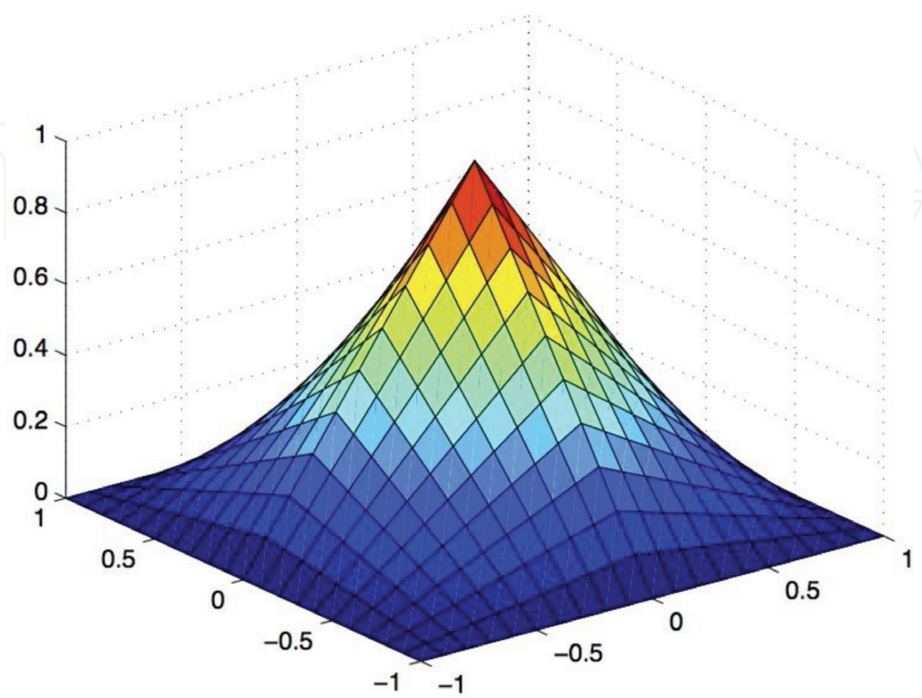

Figure 1.
The exact Fourier transform in Example 2.

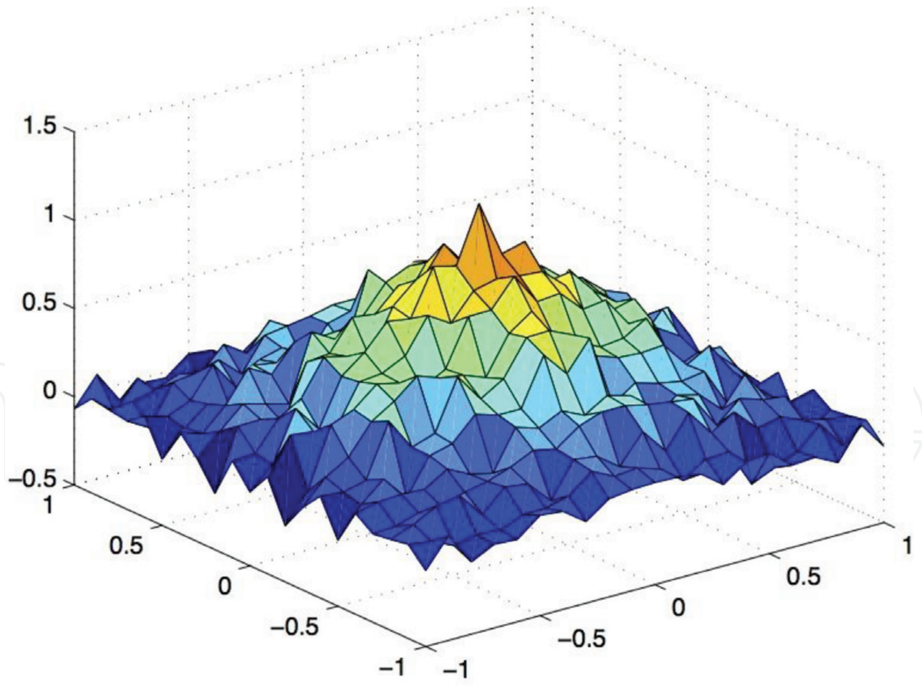

Figure 2.
The numerical results by Fourier series in Example 2.

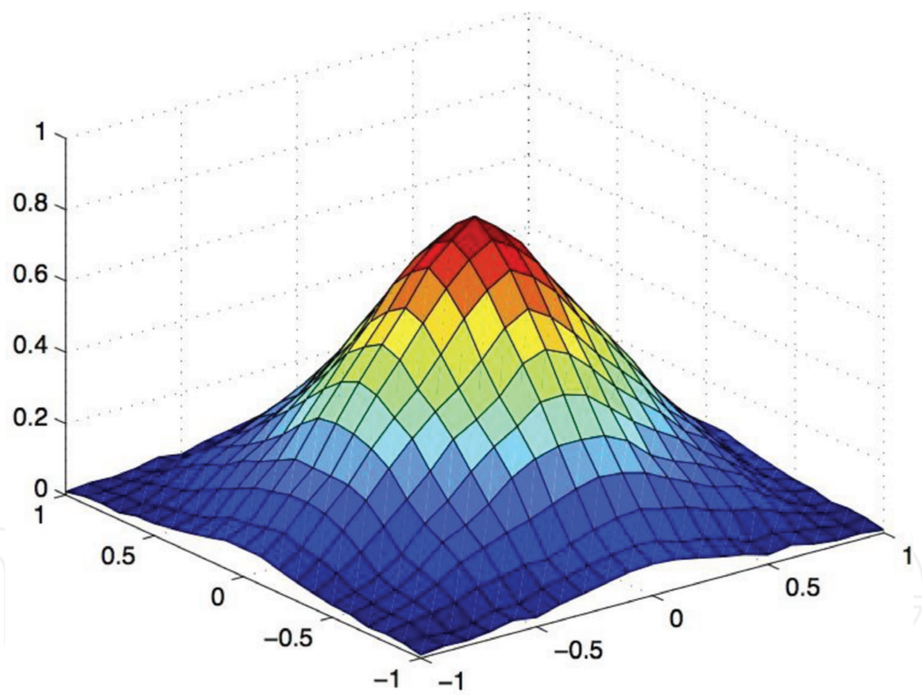

Figure 3.
The numerical results by the regularized Fourier series in Example 2.

$$\hat{f}_\alpha(\omega_1, \omega_2) = H_1 H_2 \sum_{n_1=-N}^{N} \sum_{n_2=-N}^{N} \frac{f(n_1 H_1, n_2 H_2) e^{i n_1 H_1 \omega_1 + i n_2 H_2 \omega_2}}{\left[1 + 2\pi\alpha + 2\pi\alpha (n_1 H_1)^2\right]\left[1 + 2\pi\alpha + 2\pi\alpha (n_2 H_2)^2\right]} P_\Omega(\omega_1, \omega_2).$$

Example 1. Suppose

$$f_T(t_1, t_2) = \frac{1 - \cos t_1}{\pi t_1^2} \frac{1 - \cos t_2}{\pi t_2^2}.$$

Then

$$\hat{f}_T(\omega_1, \omega_2) = (1 - |\omega_1|)(1 - |\omega_2|)P_\Omega(\omega_1, \omega_2),$$

where $\Omega_1 = 1$ and $\Omega_2 = 1$.

We add the white noise that is uniformly distributed in $[-0.0005, 0.0005]$ and choose $N = 20$. The exact Fourier transform is in **Figure 1**. The result of the Fourier series is in **Figure 2**. The result of the regularized Fourier series with $\alpha = 0.001$ is in **Figure 3**.

6. Conclusion

The problem of computing the two-dimensional Fourier transform is highly ill-posed. Noise can give rise to large errors if the Fourier series formula is used. The regularized two-dimensional Fourier series is presented. The convergence property is proved and tested by some examples. The convergence property and numerical results show that the regularized two-dimensional Fourier series is excellent in computation in noisy cases. The algorithm will be useful in image processing and multi-dimensional signal processing. The method will be of interest to: engineers who want higher precision in the gauging and design of function generators and analyzers; the electronic or electrical rectification industry; and also to the mathematics community for computing methods and the improvement of mathematics programs on signals and systems, for example, Simulink; and others since many problems in engineering involve noise.

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