Cauchy-Kovalevskaya extension theorem in fractional Clifford analysis*

N. Vieira

CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mail: nloureirovieira@gmail.com

Abstract

In this paper, we establish the fractional Cauchy-Kovalevskaya extension (FCK-extension) theorem for fractional monogenic functions defined on $\mathbb{R}^d$. Based on this extension principle, fractional Fueter polynomials, forming a basis of the space of fractional spherical monogenics, i.e. fractional homogeneous polynomials, are introduced. We studied the connection between the FCK-extension of functions of the form $x^\alpha P_l$ and the classical Gegenbauer polynomials. Finally we present two examples of FCK-extension.

Keywords: Cauchy-Kovalevskaya extension theorem; Fractional Clifford analysis; Fractional monogenic polynomials; Fractional Dirac operator; Caputo derivatives.

MSC2010: 30G35; 35A10; 26A33; 30A05; 31B05.

1 Introduction

In the last years, there is an increasing interest in finding fractional correspondences to various structures in classical mathematics. This popularity arises naturally because on the one hand different problems can be considered in the framework of fractional derivatives like, for example, in optics and quantum mechanics, and on the other hand fractional calculus gives us a new degree of freedom which can be used for more complete characterization of an object or as an additional encoding parameter. An important issue is the construction of a fractional function theory, not only as a counterpart of the theory of holomorphic functions in the complex plane, which nowadays is well established, but also of its higher-dimensional version. These higher-dimensional analogues were developed in two major directions, the first one being several complex variable analysis and the second one being Clifford analysis, i.e., the theory of null solutions of a Dirac operator, called monogenic functions [3, 5].

The major problem with most of the fractional approaches is the presence of non-local fractional differential operators. Furthermore, the adjoint of a fractional differential used to describe the dynamics is non-negative itself. Other complicated problems arise during the mathematical manipulations, as the appearance of a very complicated rule which replaces the Leibniz rule for product of functions in the case of the classic derivative. Also we have a lack of any sufficiently good analogue of the chain rule. It is important to remark that there are several definitions for fractional derivatives (Riemann-Liouville, Caputo, Riesz, Feller, ...), however not many of those allow our approach. For the purposes of this work, the definition of fractional derivatives in the sense of Caputo is the most appropriate and applicable.

Recently in [11], a framework for a fractional counterpart of Euclidian Clifford analysis was set up and developed, based on the introduction of fractional Weyl relations. Definitions were give for a fractional Dirac operator via Caputo derivatives, fractional monogenic functions and fractional spherical monogenics, i.e., fractional homogeneous polynomials, defined as the eigenfunctions of a fractional Dirac operator. Moreover, some basic results of fractional function theory, such a fractional Fischer decomposition, were obtained.

*The final version is published in Complex Analysis and Operator Theory, **9**-No.5, (2015), 1089-1109. It as available via the website http://link.springer.com/article/10.1007%2Fs11785-014-0395-x

The Cauchy-Kovalevskaya extension theorem, which we will denote simply as CK-extension, (see [4, 13]) is very well tool in Clifford analysis. In its most simple presentation, it reads as follows:

Theorem 1.1 *If the functions $F, f_0, \ldots, f_{k-1}$ are analytic in a neighborhood of the origin, then the initial value problem*

$$\begin{aligned} \partial_t^k h(\overline{x}, t) &= F(\overline{x}, t, \partial_t^s \ \partial_{\overline{x}}^\alpha h) \\ \partial_t^j h(\overline{x}, 0) &= f_j(\overline{x}), \qquad\qquad j = 0, \ldots, k-1 \end{aligned}$$

has a unique solution which is analytic in a neighborhood of the origin, provided that $|\alpha| + s \leq k$.

When the differential operator involved is the Cauchy-Riemann operator, i.e., when the differential equation reduces to $\partial_t h = -i\partial_x h$ (with $k = 1$, $|\alpha| = 1$, $s = 0$), the theorem states that a holomorphic function in an appropriate region of the complex plane is completely determined by its restrictions to the real axis. When we are dealing with harmonic functions, we have $\partial_t^2 h = -\partial_x^2 h$ (with $k = 2$, $|\alpha| = 2$, $s = 0$), which means that additionally the values of its normal derivative on the real axis should be given in order to determine it uniquely. In fact, the necessity of these restrictions as initial values becomes clear in the following construction formula for holomorphic and harmonic CK-extensions:

Proposition 1.2 *If the function $f_0(x)$ is real-analytic in $|x| < a$, then*

$$F(z) = \exp\left(iy \ \frac{d}{dx}\right)[f_0(x)] = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} \ f_0^{(k)}(x)$$

is holomorphic in $|z| < a$ and $F(z)\big|_{\mathbb{R}} = f_0(x)$. If moreover $f_1(x)$ is real-analytic in $|x| < a$, then

$$G(z) = \sum_{j=0}^{\infty} \frac{(-1)^j \ y^{2j}}{(2j)!} \left(\frac{d}{dx}\right)^{2j} [f_0(x)] + \sum_{j=0}^{\infty} \frac{(-1)^j \ y^{2j+1}}{(2j+1)!} \left(\frac{d}{dx}\right)^{2j} [f_1(x)]$$

is harmonic in $|z| < a$ and $G(z)\big|_{\mathbb{R}} = f_0(x)$, while $\frac{\partial}{\partial_y} \ G(z)\big|_{\mathbb{R}} = f_1(x)$.

The CK-extension in Euclidian Clifford analysis is a direct generalization to higher dimension of the complex plane case, and can be founded in [5]. Generalizations the CK-extension to another Clifford algebra settings can be found for instance in [2, 6, 7, 8, 9, 15].

The aim of this paper is to establish a FCK-extension theorem for fractional monogenic functions, and, in particular, to apply it for the construction of bases for the spaces of fractional spherical monogenics. The author would like to point that in spite of some similarities between the formulation of this fractional approach and the classical case, the proofs are very different because we can not apply polar or spherical coordinates when we are dealing with fractional derivatives. This impossibility comes from the fact that an explicit and complete derivation of fractional operators in polar or spherical coordinates is still an open task, despite the fact, that there have been several attempts in the past (Goldfain [10], Tarasov [17], Roberts [16], Li [14]). The idea to overcome this problem is to adapt the approach presented for the discrete case (see [7]).

The outline of the paper reads as follows. In the Preliminaries we recall some basic facts about fractional Clifford analysis, fractional Caputo derivatives, fractional Dirac operators and fractional Fischer decomposition. In Section 3 we establish a FCK-extension theorem for fractional monogenic functions defined on $\mathbb{R}^d$. In the following section, based on this extension principle, fractional Fueter polynomials, forming a basis of the space of fractional spherical monogenics, i.e., fractional homogeneous monogenic polynomials, are introduced. In Section 5 we go into detail about the connection between the FCK-extension of functions of the form $x^\alpha P_l$ and the classical Gegenbauer polynomials. In the last section we present two examples of FCK-extension.

2 Preliminaries

We consider the d-dimensional vector space $\mathbb{R}^d$ endowed with an orthonormal basis $\{e_1, \cdots, e_d\}$. We define the universal real Clifford algebra $\mathbb{R}_{0,d}$ as the 2^d-dimensional associative algebra which obeys the multiplication rules $e_i e_j + e_j e_i = -2\delta_{i,j}$. A vector space basis for $\mathbb{R}_{0,d}$ is generated by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, d\}$, for $1 \leq h_1 < \cdots < h_k \leq d$. The Clifford conjugation is defined by

$\overline{1} = 1$, $\overline{e_j} = -e_j$ for all $j = 1, \ldots, d$, and we have $\overline{ab} = \overline{b}\overline{a}$. An important property of algebra $\mathbb{R}_{0,d}$ is that each non-zero vector $x \in \mathbb{R}_1^d$ has a multiplicative inverse given by $\frac{\overline{x}}{||x||^2}$. Now, we introduce the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes \mathbb{R}_{0,d} = \left\{ w = \sum_A w_A e_A,\ w_A \in \mathbb{C}, A \subset M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, d$. We have a pseudonorm on $\mathbb{C}_d$ viz $|w| := \sum_A |w_A|$ where $w = \sum_A w_A e_A$, as usual. Notice also that for $a, b \in \mathbb{C}_d$ we only have $|ab| \leq 2^d |a||b|$. The other norm criteria are fulfilled.

An important subspace of the real Clifford algebra $\mathbb{R}_{0,d}$ is the so-called space of paravectors $\mathbb{R}_1^d = \mathbb{R} \bigoplus \mathbb{R}^d$, being the sum of scalars and vectors. An element $\underline{x} = (x_0, x_1, \ldots, x_d)$ of $\mathbb{R}^d$ will be identified by $\underline{x} = x_0 + x$, with $x = \sum_{i=1}^d e_i x_i$. From now until the end of the paper, we will consider paravectors of the form $\underline{x}^\alpha = x_0^\alpha + x^\alpha$, where

$$x_j^\alpha = \begin{cases} \exp(\alpha \ \ln |x_j|); & x_j > 0 \\ 0; & x_j = 0 \\ \exp(\alpha \ \ln |x_j| + i\alpha\pi); & x_j < 0 \end{cases},$$

with $0 < \alpha < 1$, and $j = 0, 1, \ldots, d$.

An $\mathbb{C}_d-$valued function f over $\Omega \subset \mathbb{R}_1^d$ has representation $f = \sum_A e_A f_A$, with components $f_A : \Omega \to \mathbb{C}$. Properties such as continuity will be understood component-wisely. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^d e_j \ \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplacian, i.e., $D^2 = -\Delta = -\sum_{j=1}^d \partial x_j^2$. A $\mathbb{C}_d$-valued function f is called *left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *right-monogenic* if it satisfies $uD = 0$ on Ω). For more details about Clifford algebras and monogenic function we refer [5].

The fractional Dirac operator will correspond to the fractional differential operator $D^\alpha = \sum_{j=1}^d e_j \ {}_+^C\partial_j^\alpha$, where ${}_+^C\partial_j^\alpha$ is the fractional Caputo derivative with respect to x_j^α defined as (see [12])

$$\left({}_+^C\partial_j^\alpha \ f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x_j^\alpha} \frac{1}{(x_j^\alpha - u)^\alpha} \ f'_u(x_1^\alpha, \ldots, x_{j-1}^\alpha, u, x_{j+1}^\alpha, \ldots, x_n^\alpha) \ du. \tag{1}$$

A $\mathbb{C}_n$-valued function f is called *fractional left-monogenic* if it satisfies $D^\alpha u = 0$ on Ω (resp. *fractional right-monogenic* if it satisfies $uD^\alpha = 0$ on Ω). The fractional Dirac operator verifies the following identity $\Delta^{2\alpha} = -D^\alpha D^\alpha$, i.e., factorizes the fractional Laplace operator.

Now we recall some facts proved in [11]. There the authors introduced the following fractional Weyl relation

$$\left[\ {}_+^C\partial_i^\alpha, x_i^\alpha \ \right] = {}_+^C\partial_i^\alpha \ x_i^\alpha - x_i^\alpha \ {}_+^C\partial_i^\alpha = \frac{\alpha\pi}{\sin(\alpha\pi) \ \Gamma(1-\alpha)} =: K_\alpha, \tag{2}$$

with $i = 1, \ldots, d$, $0 < \alpha < 1$ and ${}_+^C\partial_i^\alpha$ the fractional Caputo derivative with respect to x_j^α defined in (1). In the same paper the authors showed that

$$\{D^\alpha, x^\alpha\} = D^\alpha x^\alpha + x^\alpha D^\alpha = -2\mathbb{E}^\alpha - K_\alpha d, \qquad [x^\alpha, D^\alpha] = x^\alpha D^\alpha - D^\alpha x^\alpha = -2\Gamma^\alpha + K_\alpha d, \tag{3}$$

where $\mathbb{E}^\alpha$, Γ^α are, respectively, the fractional Euler and Gamma operators. They also presented the following expressions for $\mathbb{E}^\alpha$ and Γ^α

$$\mathbb{E}^\alpha = \sum_{i=1}^d x_i^\alpha \ {}_+^C\partial_i^\alpha, \qquad \Gamma^\alpha = \sum_{i<j} e_i e_j (x_i^\alpha \ {}_+^C\partial_j^\alpha - {}_+^C\partial_j^\alpha \ x_i^\alpha). \tag{4}$$

Furthermore, in [11] the authors deduced the following relations

$$\mathbb{E}^\alpha + \Gamma^\alpha = -x^\alpha D^\alpha, \qquad [\mathbb{E}^\alpha, \Gamma^\alpha] = 0, \qquad [x^\alpha, \mathbb{E}^\alpha] = -K_\alpha x^\alpha, \qquad [D^\alpha, \mathbb{E}^\alpha] = K_\alpha D^\alpha, \tag{5}$$

In [11] was introduced the following definition of fractional homogeneity of a polynomial by means of the fractional Euler operator.

Definition 2.1 *A polynomial P is called fractional homogeneous of degree $l \in \mathbb{N}_0$, if and only if* $\mathbb{E}^\alpha P = K_{l,\alpha} \ l \ P$, *where* $K_{l,\alpha} = \frac{\alpha \ \Gamma(\alpha l)}{\Gamma(1+\alpha(l-1))}$.

From the previous definition the basic homogeneous powers are given by $(x^\alpha)^\beta = (x_1^\alpha)^{\beta_1}\dots(x_d^\alpha)^{\beta_d}$, with $l = \beta_1+\dots+\beta_d$. In combination with the third relation in (5), this definition also implies that the multiplication of a fractional homogeneous polynomial of degree l by x^α, will result in a homogeneous polynomial of degree $l+1$, and thus may be seen as a raising operator. Moreover, we can also ensure that for a fractional homogeneous polynomial P_l of degree l, $D^\alpha P_l$ is a fractional homogeneous polynomial of degree $l-1$.

From the fact recalled previously we have that $(x_j^\alpha)^k$ are the basic fractional homogeneous polynomials of degree k in the variable x_j. In the following result their fundamental properties are listed

Theorem 2.2 *For all $k\in\mathbb{N}$ and $i,j=1,\dots,d$ we have*

$$\begin{array}{ll} {}_+^C\partial_j^\alpha (x_j^\alpha)^k = k\ K_\alpha\ (x_j^\alpha)^{k-1}, & {}_+^C\partial_i^\alpha (x_j^\alpha)^k = 0, \quad i\neq j, \\ {}_+^C\partial_j^\alpha (x_j^\alpha)^{k_1}(x_i^\alpha)^{k_2} = k_1\ K_\alpha\ (x_j^\alpha)^{k_1-1}\ (x_i^\alpha)^{k_2}, \quad i\neq j. & \end{array} \tag{6}$$

Moreover, for any two multi-index $\underline{\gamma}=(\gamma_1,\dots,\gamma_d)$ and $\underline{\beta}=(\beta_1,\dots,\beta_d)$ with $|\underline{\gamma}|=|\underline{\beta}|$, it holds that

$${}_+^C\partial_1^{\gamma_1}\dots{}_+^C\partial_1^{\gamma_d}((x_1^\alpha)^{\beta_1}\dots(x_d^\alpha)^{\beta_d}) = \begin{cases} (K_\alpha)^{\gamma!}\ \gamma!, & \text{if } \underline{\gamma}=\underline{\beta} \\ 0, & \text{if } \underline{\gamma}\neq\underline{\beta} \end{cases}$$

where we have put $\underline{\gamma}! = \gamma_1!\dots\gamma_d!$.

The proof of this result is immediate and therefore we will omit it from the text. Furthermore, from the previous theorem we conclude that a closed form for the fractional homogeneous polynomials are given by

$$(e_j x_j^\alpha)^{2n+1} = (-1)^n e_j (x_j^\alpha)^{2n+1}, \qquad (e_j x_j^\alpha)^{2n} = (-1)^n (x_j^\alpha)^{2n}, \tag{7}$$

for $n=1,2,\dots$ and $j=1,\dots,d$.

3 FCK-extension

Due to the formal similarities with the classical setting, we propose the following form for the FCK-extension:

$$F(x_1^\alpha, x_2^\alpha,\dots,x_d^\alpha) = \sum_{k=0}^{\infty} \frac{(e_1 x_1^\alpha)^k}{k!}\ f_k(x_2^\alpha,\dots,x_d^\alpha),$$

with $f_0=f$. Taking into account (7), we conclude that the function F takes the correct values and satisfies $F|_{x_1=0}=f$. For F to be fractional monogenic it must be vanish under the action of the fractional Dirac operator D^α, which can be rewrite as

$$D^\alpha = {}_+^C\partial_1^\alpha + \sum_{j=2}^{d} e_j\ {}_+^C\partial_j^\alpha = {}_+^C\partial_1^\alpha + D_*^\alpha.$$

In order to determine the coefficient functions f_k, $k=1,2,\dots,d$ in such that $D^\alpha F=0$, we proceed by direct calculation. Taking into account Theorem 2.2 for the action of ${}_+^C\partial_j^\alpha$ over x_1^α, and from the facts that ${}_+^C\partial_1^\alpha$ only acts on $(x_1^\alpha)^k$ and D_*^α anticommutes with x_1^α, we obtain

$$0 = D^\alpha F = \left({}_+^C\partial_1^\alpha + D_*^\alpha\right)\left(\sum_{k=0}^{\infty}\frac{(e_1x_1^\alpha)^k}{k!}\ f_k\right) = \sum_{k=0}^{\infty}\frac{(e_1x_1^\alpha)^k}{k!}\ f_{k+1} + \sum_{k=0}^{\infty}(-1)^k\ \frac{(e_1x_1^\alpha)^k}{k!}\ D_*^\alpha f_k,$$

resulting into the recurrence relation

$$f_{k+1} = (-1)^{k+1}\ D_*^\alpha f_k.$$

Hence we obtain the following definition for the FCK-extension:

Definition 3.1 *The FCK-extension of a function $f=f(x_2^\alpha,\dots,x_d^\alpha)$ is the fractional monogenic function*

$$FCK[f](x_1^\alpha,x_2^\alpha,\dots,x_d^\alpha) = \sum_{k=0}^{\infty}\frac{(e_1x_1^\alpha)^k}{k!}\ f_k(x_2^\alpha,\dots,x_d^\alpha), \tag{8}$$

where $f_0=f$ and $f_{k+1}=(-1)^{k+1}D_*^\alpha f_k$.

Let us observe that the previous definition does not impose any conditions to the original function f. From (7) follows

$$(e_1 x_1^\alpha)^{2n+1} = 0 \qquad \text{for } n \leq |x_1^\alpha|,$$
$$(e_1 x_1^\alpha)^{2n} = 0, \qquad \text{for } n \leq |x_1^\alpha| + 1,$$

which implies that for every point $(x_1^\alpha, \ldots, x_d^\alpha) \in \mathbb{R}^d$, there exists $N \in \mathbb{N}$ such that all but the first N terms in the series (8) vanish, and therefore the series reduces to a finite sum in every point of $\mathbb{R}^d$. This fact implies that function $f(x_2^\alpha, \ldots, x_d^\alpha)$ its FCK-extension is well-defined on $\mathbb{R}^d$. The uniqueness of the extension is a corollary of the following result.

Theorem 3.2 *Let F be a fractional monogenic function defined on $\mathbb{R}^d$, with $F|_{x_1^\alpha=0} \equiv 0$. Then F is the null function.*

Proof: The fractional monogenicity of F explicitly reads as $\left({}_{+}^{C}\partial_1^\alpha + D_*^\alpha\right) F = 0$. Now take $(x_1^\alpha, x_2^\alpha, \ldots, x_d^\alpha) \in \mathbb{R}^d$ with $x_1^\alpha = 0$. Since $F|_{x_1^\alpha=0} \equiv 0$ the above expression reduces to ${}_{+}^{C}\partial_i^\alpha\, F = 0$. Furthermore $-\Delta^{2\alpha} F = D^\alpha D^\alpha F = 0$, from which we obtain, for $(0, x_2^\alpha, \ldots, x_d^\alpha) \in \mathbb{R}^d$ with $x_1^\alpha = 0$, that $F \equiv 0$. Repeating this procedure, we find $F \equiv 0$ on $\mathbb{R}^d$.

■

Corollary 3.3 (***Uniqueness of the* FCK*-extension***) *Let F_1 and F_2 be two fractional monogenic functions such that $F_1|_{x_1^\alpha=0} = f$ and $F_2|_{x_1^\alpha=0} = f$. Then F_1 and F_2 coincide.*

4 Fractional Fueter polynomials

The fractional FCK-extension procedure establishes a homomorphism between the space $\Pi_l^{(d-1)}$ of fractional homogeneous polynomials of degree l in $d-1$ variables and the space $\mathcal{M}_l^{(d)}$ of spherical fractional monogenics of degree l in d variables. From Theorem 3.2 and Corollary 3.3 we get that this homomorphism is injective. Moreover, a basis for the space $\Pi_l^{(d-1)}$ is given by the fractional homogeneous polynomials $(x_2^\alpha)^{\beta_2} \ldots (x_d^\alpha)^{\beta_d}$, with $\beta_2 + \ldots + \beta_d = l$, and its dimension is

$$\dim\left(\Pi_l^{(d-1)}\right) = \frac{(l-d)!}{l!\ (d-2)!},$$

which corresponds to the dimension of $\mathcal{M}_l^{(d)}$ (see last theorem in [11] with $d = d-1$), whence the homomorphism also is surjective. The FCK-extension procedure thus establishes an isomorphism between $\Pi_l^{(d-1)}$ and $\mathcal{M}_l^{(d)}$, allowing us to determine a basis for the space $\mathcal{M}_l^{(d)}$.

Definition 4.1 *Let $\underline{\beta} = (\beta_2, \ldots, \beta_d) \in \mathbb{N}^{d-1}$ with $\beta_2 + \ldots + \beta_d = l$. Then the fractional spherical monogenics*

$$V_{\underline{\beta}} = FCK[(x_2^\alpha)^{\beta_2} \ldots (x_d^\alpha)^{\beta_d}]$$

are called the fractional Fueter polynomials of degree l.

Theorem 4.2 *The set $\{V_{\underline{\beta}} |\ \beta_2 + \ldots + \beta_d = l\}$ is a basis for $\mathcal{M}_l^{(d)}$.*

Proof: The FCK-extension procedure is an isomorphism between both space. In fact the basis

$$\{(x_2^\alpha)^{\beta_2} \ldots (x_d^\alpha)^{\beta_d} |\ \beta_2 + \ldots + \beta_d = l\}$$

of $\Pi_l^{(d-1)}$ is transformed into the basis

$$\{FCK[(x_2^\alpha)^{\beta_2} \ldots (x_d^\alpha)^{\beta_d}] |\ \beta_2 + \ldots + \beta_d = l\}$$

of $\mathcal{M}_l^{(d)}$.

■

Example 4.3 *The space* $\mathcal{M}_2^{(3)}$ *has dimension* 3*. A basis for it is given by the elements*

$$\begin{aligned}
V_{2,0} &= FCK[(x_2^\alpha)^2] = (x_2^\alpha)^2 - 2x_1^\alpha x_2^\alpha - (x_1^\alpha)^2, \\
V_{1,1} &= FCK[x_2^\alpha x_3^\alpha] = x_2^\alpha x_3^\alpha - x_1^\alpha x_3^\alpha + x_1^\alpha x_2^\alpha + (x_1^\alpha)^2, \\
V_{0,2} &= FCK[(x_3^\alpha)^2] = (x_3^\alpha)^2 - 2x_1^\alpha x_3^\alpha - (x_1^\alpha)^2,
\end{aligned}$$

of which it can be checked also directly that they are fractional monogenic, of homogeneity degree 2 *in* $(x_1^\alpha, x_2^\alpha, x_3^\alpha)$ *and linearly independent.*

5 FCK-extension of $(x^\alpha)^s M_l$

In [5] we read that in the Euclidian setting, functions of the form $x^s P_l(x)$ are building blocks of homogeneous polynomials in $\mathbb{R}^d$ and whence, in order to characterize spaces of inner spherical monogenics in $\mathbb{R}^{d+1}$, it suffices to determine the CK-extension of polynomials of the for $x^s P_l(x)$, which was formulated in the following theorem:

Theorem 5.1 *Let* $s \in \mathbb{N}$ *and* $P_l \in M^+(l;d;C)$*. Then the CK-extension of* $x^s P_l(x)$ *has the form* $X_l^s(x_0, x)\ P_l(x)$ *where*

$$X_l^s(x_0,x) = \lambda_l^s r^s \left[C_s^{\frac{d-1}{2}+l}\left(\frac{x_0}{r}\right) + \frac{2l+d-1}{s+2l+d-1}\ C_{s-1}^{\frac{d+1}{2}+l}\left(\frac{x_0}{r}\right)\ \frac{x_0}{r} \right].$$

In this formula, $r^2 = x_0^2 - x^2$ *and the polynomials* $C_n^\lambda(x)$ *are the standard Gegenbauer polynomials [1] given by*

$$C_n^\lambda(x) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^j\ (\lambda)_{n-j}}{j!\ (n-2j)!}\ (2x)^{n-2j}, \tag{9}$$

where the Pochhammer symbol $(a)_n$ *denotes* $a(a+1)\dots(a+n-1)$*. Furthermore, the coefficients* λ_k^s *are*

$$\lambda_l^{2k} = (-1)^k \left(C_{2k}^{\frac{d-1}{2}+l}(0)\right)^{-1}, \qquad \lambda_l^{2k+1} = (-1)^k\ \frac{2k+2l+d}{2l+d-1}\ \left(C_{2k}^{\frac{d+1}{2}+l}(0)\right)^{-1}$$

and explicitly

$$\lambda_l^{2k} = \frac{k!\ \Gamma\left(l+\frac{d-1}{2}\right)}{\Gamma\left(k+l+\frac{d-1}{2}\right)}, \qquad \lambda_l^{2k+1} = \frac{2k+2l+d}{2l+d-1}\ \frac{k!\ \Gamma\left(l+\frac{d+1}{2}\right)}{\Gamma\left(k+l+\frac{d+1}{2}\right)}. \tag{10}$$

We now consider the fractional version of the previous theorem. In our fractional setting we will consider P_l be a fractional homogeneous monogenic function in d variables $x_1^\alpha, \dots, x_d^\alpha$. We will determine the FCK-extension of $(x^\alpha)^s P_l$. The result is a fractional monogenic in $d+1$ variables $x_0^\alpha, x_1^\alpha, \dots, x_d^\alpha$:

$$FCK[f] = \sum_{k=0}^{\infty} \frac{1}{k!}\ x_1^\alpha\ f_k, \qquad f_0 = f, \qquad f_{k+1} = (-1)^{k+1} D^\alpha f_k,$$

where D^α is the fractional Dirac operator in d variables. The operators D^α and x^α satisfy (see [11])

$$D^\alpha (x^\alpha)^s P_l = g_{s,l} (x^\alpha)^{s-1} P_l + (-1)^s (x^\alpha)^s D^\alpha P_l, \tag{11}$$

where $g_{2k,l} = -2k$ and $g_{2k+1,l} = -(2(kK_\alpha + K_{l,\alpha} l) + K_\alpha d)$. Denote by R the fractional vector variable in $d+1$ dimensions, i.e.,

$$R = x_0^\alpha - \sum_{j=1}^{d} e_j x_j^\alpha = x_0^\alpha - x^\alpha,$$

with $R^2 = (x_0^\alpha)^2 - (x^\alpha)^2$. During this section we use the formal notations $\frac{x_0^\alpha}{R}$ and $\frac{x^\alpha}{R}$ as arguments in the Gegenbauer polynomials by which we mean that we first of all expand the Gegenbauer polynomials using $\left(\frac{x_0^\alpha}{R}\right)^k = \frac{(x_0^\alpha)^k}{R^k}$, and then cancel out all appearances of R in the denominators, after which no ambiguity is left.

5.1 Auxiliar results

In this subsection we present some necessary results for the proof, in the next subsection, of the main theorem. We start recalling some multiplications rules

$$\begin{aligned}
{}^{C}_{+}\partial_0^\alpha\ (x_0^\alpha)^s &= sK_\alpha (x_0^\alpha)^{s-1} + (x_0^\alpha)^s\ {}^{C}_{+}\partial_0^\alpha,\\
D^\alpha\ (x^\alpha)^{2s} &= -2sK_\alpha (x^\alpha)^{2s-1} + (x^\alpha)^{2s} D^\alpha,\\
D^\alpha\ (x^\alpha)^{2s+1} &= -(2sK_\alpha + K_\alpha d)(x^\alpha)^{2s-1} - 2(x^\alpha)^{2s}\mathbb{E}^\alpha - (x^\alpha)^{2s+1} D^\alpha.
\end{aligned}$$

Moreover, we present the following auxiliar lemmas:

Lemma 5.2 *Let $k \in \mathbb{N}$, and P_l a fractional spherical monogenic of degree l in the variables $x_1^\alpha, \ldots, x_d^\alpha$. Then*

$$\begin{aligned}
{}^{C}_{+}\partial_0^\alpha\ R^{2k} P_l &= 2K_\alpha k\ x_0^\alpha\ \ R^{2k-2} P_l,\\
D^\alpha R^{2k} P_l &= -2K_\alpha k\ x^\alpha\ \ R^{2k-2} P_l,\\
\mathbb{E}^\alpha R^{2k} P_l &= \left(K_{l,\alpha} l\ R^2 + K_\alpha k\ (x^\alpha)^2\right)\ R^{2k-2} P_l.
\end{aligned}$$

Proof: We start expanding R^{2k} in the following way

$$R^{2k} = \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s} (x^\alpha)^{2s}.$$

Taking into account that x^α and P_l do not depend on x_0^α, we get

$$\begin{aligned}
{}^{C}_{+}\partial_0^\alpha\ R^{2k} P_l &= \sum_{s=0}^{k} \frac{k!}{s!\ (k-s)!}\ (2k-2s)\ (x_0^\alpha)^{2k-2s-1}\ (x^\alpha)^{2s} P_l\\
&= 2K_\alpha k \sum_{s=0}^{k-1} \frac{(k-1)!}{s!\ (k-s-1)!}\ (x_0^\alpha)^{2k-2s-1}\ (x^\alpha)^{2s} P_l\\
&= 2K_\alpha k \sum_{s=0}^{k-1} \begin{pmatrix} k-1 \\ s \end{pmatrix} (x_0^\alpha)^{2(k-1)-2s+1}\ (x^\alpha)^{2s} P_l\\
&= 2K_\alpha k\ x_0^\alpha\ R^{2k-2} P_l.
\end{aligned}$$

The proof of the second statement uses the fact that P_l is a fractional spherical monogenic in the variables $x_1^\alpha, \ldots, x_d^\alpha$ (thus $D^\alpha P_l = 0$).

$$\begin{aligned}
D^\alpha R^{2k} P_l &= \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} D^\alpha \left((x_0^\alpha)^{2k-2s} (x^\alpha)^{2s} P_l\right)\\
&= \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s}\ D^\alpha \left((x^\alpha)^{2s} P_l\right)\\
&= \sum_{s=0}^{k} \frac{k!}{s!\ (k-s)!}\ (x_0^\alpha)^{2k-2s} \left(-2sK_\alpha (x^\alpha)^{2s-1} P_l\right)\\
&= -2sK_\alpha \sum_{s=0}^{k} \frac{k!}{(s-1)!\ (k-s)!}\ (x_0^\alpha)^{2k-2s}\ (x^\alpha)^{2s-1} P_l\\
&= -2sK_\alpha k \sum_{s=1}^{k} \frac{k!}{(s-1)!\ (k-s)!}\ (x_0^\alpha)^{2k-2s}\ (x^\alpha)^{2s-1} P_l\\
&= -2sK_\alpha k \sum_{s=1}^{k} \begin{pmatrix} k-1 \\ s-1 \end{pmatrix} (x_0^\alpha)^{2(k-1)-2(s-1)}\ (x^\alpha)^{2(s-1)+1} P_l\\
&= -2sK_\alpha k \sum_{p=1}^{k-1} \begin{pmatrix} k-1 \\ p \end{pmatrix} (x_0^\alpha)^{2(k-1)-2p}\ (x^\alpha)^{2p+1} P_l\\
&= -2K_\alpha k\ x^\alpha\ R^{2k-2} P_l.
\end{aligned}$$

For the final relation, we use the commutation relation

$$x^\alpha \mathbb{E}^\alpha - \mathbb{E}^\alpha x^\alpha = -K_\alpha x^\alpha \quad \Leftrightarrow \quad \mathbb{E}^\alpha x^\alpha = x^\alpha \mathbb{E}^\alpha + K_\alpha x^\alpha$$

which implies that

$$\mathbb{E}^\alpha (x^\alpha)^{2s} = (x^\alpha)^{2s}\mathbb{E}^\alpha + 2sK_\alpha (x^\alpha)^{2s},$$

to show that

$$\begin{aligned}
\mathbb{E}^\alpha R^{2k} P_l &= \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s}\ \mathbb{E}^\alpha (x^\alpha)^{2s} P_l \\
&= \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s}\ \left((x^\alpha)^{2s}\mathbb{E}^\alpha + 2sK_\alpha\ (x^\alpha)^{2s}\right) P_l \\
&= K_{l,\alpha} l \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s} (x^\alpha)^{2s} P_l + 2K_\alpha \sum_{s=1}^{k} \begin{pmatrix} k \\ s \end{pmatrix} (x_0^\alpha)^{2k-2s} (x^\alpha)^{2s} P_l \\
&= K_{l,\alpha} l\ R^{2k} P_l + 2K_\alpha k \sum_{p=0}^{k-1} \begin{pmatrix} k-1 \\ p \end{pmatrix} (x_0^\alpha)^{2(k-1)-2p} (x^\alpha)^{2p+2} P_l \\
&= \left(K_{l,\alpha} l\ R^2 + 2K_\alpha k (x^\alpha)^2\right) R^{2k-2} P_l.
\end{aligned}$$

■

Lemma 5.3 *For a parameter λ and $k \geq 1$, one has*

$$\left({}^C_+\partial_0^\alpha + D^\alpha\right)\left[C_{2k}^\lambda \left(\frac{x_0^\alpha}{R}\right)\ R^{2k} P_l\right] = 2K_\alpha \lambda \left[C_{2k-1}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right) - C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\right]\ R^{2k-1} P_l.$$

Proof: Taking into account the series expansion (9) for the Gegenbauer polynomials, the relations presented in Lemma 5.2, and after straightforward calculations, we have

$$\begin{aligned}
&\left({}^C_+\partial_0^\alpha + D^\alpha\right)\left[C_{2k}^\lambda \left(\frac{x_0^\alpha}{R}\right)\ R^{2k} P_l\right] \\
&= \sum_{j=0}^{k} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j)!}\ 2^{2k-2j}\ \left({}^C_+\partial_0^\alpha + D^\alpha\right)\left[(x_0^\alpha)^{2k-2j}\ R^{2j} P_l\right] \\
&= \sum_{j=0}^{k} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j)!}\ 2^{2k-2j}\ \big[K_\alpha\ (2k-2j)\ (x_0^\alpha)^{2k-2j-1}\ R^{2j} \\
&\qquad\qquad + (x_0^\alpha)^{2k-2j}\ {}^C_+\partial_0^\alpha\ R^{2j} + (-1)^{2k-2j}\ (x_0^\alpha)^{2k-2j}\ D^\alpha\ R^{2j}\big] P_l \\
&= K_\alpha \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-1)!}\ 2^{2k-2j}\ (x_0^\alpha)^{2k-2j-1}\ R^{2j} P_l \\
&\qquad\qquad + \sum_{j=0}^{k} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j)!}\ 2^{2k-2j}\ (x_0^\alpha)^{2k-2j}\ \left(2j(x_0^\alpha - x^\alpha) R^{2j-2}\right) P_l \\
&= 2K_\alpha\lambda \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda+1)_{2k-j-1}}{j!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1} R^{2j} P_l \\
&\qquad\qquad + 2K_\alpha\lambda \sum_{j=0}^{k} \frac{(-1)^j\ (\lambda+1)_{2k-j-1}}{(j-1)!\ (2k-2j)!}\ (2x_0^\alpha)^{2k-2j}\ R^{2j-1} P_l \\
&= 2K_\alpha\lambda \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda+1)_{2k-j-1}}{j!\ (2k-2j-1)!}\ \left(\frac{2x_0^\alpha}{R}\right)^{2k-2j-1}\ R^{2k-1} P_l \\
&\qquad\qquad - 2K_\alpha\lambda \sum_{p=0}^{k-1} \frac{(-1)^p\ (\lambda+1)_{2k-p-2}}{(p!\ (2k-2p-2)!}\ \left(\frac{2x_0^\alpha}{R}\right)^{2k-2p-2}\ R^{2k-1} P_l \\
&= 2K_\alpha\lambda \left[C_{2k-1}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right) - C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\right]\ R^{2k-1} P_l.
\end{aligned}$$

■

Let us observe that in Lemma 5.3, on the right-hand side, there is no ambiguity about whether the R's should be left or right since the first thing one has to do is to eliminate the powers of R in the denominator which leaves only even powers of R (in the nominator) who commute with x_0^α and x^α. We continue now presenting more auxiliar lemmas:

Lemma 5.4 *For a parameter λ and $k \geq 1$, one has*

$$\left({}_{+}^{C}\partial_0^\alpha + D^\alpha\right)\left[C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k}P_l\right]$$
$$= \left[2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2} - K_\alpha(d+2l)\ C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k-1} + 2K_\alpha\lambda\ C_{2k-3}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k-1}\right]P_l.$$

Proof: Taking into account the series expansion (9) for the Gegenbauer polynomials, the relations presented in Lemma 5.2, and after straightforward calculations, we have

$$\left({}_{+}^{C}\partial_0^\alpha + D^\alpha\right)\left[C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k}P_l\right]$$
$$= \sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\left[K_\alpha\ (2k-2j-1)\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha - (x_0^\alpha)^{2k-2j-1}(x^\alpha\ {}_{+}^{C}\partial_0^\alpha + D^\alpha x^\alpha)\right]R^{2j}P_l$$
$$= K_\alpha\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha\ R^{2j}P_l$$
$$- \sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1}\left[x^\alpha\ {}_{+}^{C}\partial_0^\alpha\ R^{2j} - (2\mathbb{E}^\alpha + K_\alpha d + x^\alpha D^\alpha)\ R^{2j}\right]P_l$$
$$= 2K_\alpha\lambda\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-2}}{j!\ (2k-2j-2)!}\ \left(\frac{2x_0^\alpha}{R}\right)^{2k-2j-2}\ R^{2k-2}\ x^\alpha P_l$$
$$- \sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-2}}{j!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1}\left[2K_\alpha j\ x^\alpha\ x_0^\alpha\ R^{2j-2} - K_\alpha d\ R^{2j} + 2K_\alpha j\ (x^\alpha)^2\ R^{2j-2}\right.$$
$$\left. -2(K_{l,\alpha} l\ R^{2j} + 2K_\alpha j\ (x^\alpha)^2\ R^{2j-2})\right]P_l$$
$$= 2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2}P_l$$
$$- \sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1}\left[2K_\alpha j\ (x_0^\alpha - x^\alpha)\ x^\alpha\ R^{2j-2} - K_\alpha(d+2l)\ R^{2j}\right]P_l$$
$$= 2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2}P_l$$
$$- \sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1}\left[2K_\alpha j\ x^\alpha\ R^{2j-1} - K_\alpha(d+2l)\ R^{2j}\right]P_l$$
$$= 2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2}P_l - K_\alpha(d+2l)\ C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k-1}P_l$$
$$+ 2K_\alpha\lambda\sum_{j=1}^{k-1}\frac{(-1)^{j+1}\ (\lambda)_{2k-j-2}}{(j-1)!\ (2k-2j-1)!}\ (2x_0^\alpha)^{2k-2j-1}\ x^\alpha\ R^{2j-1}P_l$$
$$= 2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2}P_l - K_\alpha(d+2l)\ C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k-1}P_l$$
$$+ 2K_\alpha\lambda\sum_{p=0}^{k-2}\frac{(-1)^p\ (\lambda)_{2k-p-3}}{(p!\ (2k-2p-3)!}\ (2x_0^\alpha)^{2k-2p-3}\ x^\alpha\ R^{2p+1}P_l$$
$$= 2K_\alpha\lambda\ C_{2k-2}^{\lambda+1}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2}P_l - K_\alpha(d+2l)\ C_{2k-1}^{\lambda}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k-1}P_l$$
$$+ 2K_\alpha\lambda\sum_{p=0}^{k-2}\frac{(-1)^p\ (\lambda)_{2k-p-3}}{(p!\ (2k-2p-3)!}\ \left(\frac{2x_0^\alpha}{R}\right)^{2k-2p-3}\ \frac{x^\alpha}{R}\ R^{2k-1}P_l$$

$$= \left[2K_\alpha\lambda\ C^{\lambda+1}_{2k-2}\left(\frac{x_0^\alpha}{R}\right)\ x^\alpha\ R^{2k-2} - K_\alpha(d+2l)\ C^{\lambda}_{2k-1}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k-1} + 2K_\alpha\lambda\ C^{\lambda+1}_{2k-3}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k-1}\right] P_l.$$

■

We remark that in Lemma 5.4 after elimination of the powers of R in the denominator, there is no ambiguity for the first and the last term of the right-hand side. For the second term however, we must clarify how the elimination should be made. For example, let $d = 2$:

$$C^{\lambda+1}_1\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^3 \quad = 2(\lambda+1)\ \frac{x_0^\alpha}{R}\ \frac{x^\alpha}{R}\ R^3 \quad = 2(\lambda+1)\ x_0^\alpha\ x^\alpha R,$$

which is not the same as

$$C^{\lambda+1}_1\left(\frac{x_0^\alpha}{R}\right)\ R^3\ \frac{x^\alpha}{R} \quad = 2(\lambda+1)\ x_0^\alpha\ R\ x^\alpha,$$

or

$$R^3\ C^{\lambda+1}_1\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R} \quad = 2(\lambda+1)\ R\ x_0^\alpha\ x^\alpha.$$

For the second term in the right-hand side we thus put as a convection that (after elimination of R in the denominator), the remaining (odd) powers of R are written on the total right of both x_0^α and x^α.

In a very similar way as we have done in Lemma 5.4, we can prove the following results:

Lemma 5.5 *For a parameter λ and $k \geq 1$, one has*

$$\left({}^C_+\partial_0^\alpha + D^\alpha\right)\left[C^{\lambda}_{2k+1}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k}P_l\right] = 2K_\alpha\lambda\left[C^{\lambda+1}_{2k}\left(\frac{x_0^\alpha}{R}\right) - C^{\lambda+1}_{2k-1}\left(\frac{x_0^\alpha}{R}\right)\left(\frac{x_0^\alpha}{R} - \frac{x^\alpha}{R}\right)\right]\ R^{2k}P_l.$$

Lemma 5.6 *For a parameter λ and $k \geq 1$, one has*

$$\left({}^C_+\partial_0^\alpha + D^\alpha\right)\left[C^{\lambda}_{2k}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k+1}P_l\right]$$
$$= \left[-K_\alpha(d+2l)\ C^{\lambda}_{2k}\left(\frac{x_0^\alpha}{R}\right)\ R^{2k} + 2K_\alpha\lambda\ C^{\lambda+1}_{2k-1}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ R^{2k} + 2K_\alpha\lambda\ C^{\lambda+1}_{2k-2}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\ x^\alpha\ R^{2k-1}\right] P_l.$$

5.2 Main result

We present now the main result of this section:

Theorem 5.7 *For a fractional spherical monogenic P_l of degree l in the fractional variables $x_1^\alpha, \ldots, x_d^\alpha$ and for $s \in \mathbb{N}$, the fractional FCK-extension of $(x^\alpha)^s P_l$ is the fractional monogenic polynomial in $d+1$ variables $x_0^\alpha, x_1^\alpha, \ldots, x_d^\alpha$ given by*

$$FCK\left[(x^\alpha)^{2k}P_l\right] = (-1)^k\ \lambda_l^{2k}\ R^{2k}\left[C^{\frac{d+1}{2}+k}_{2k}\left(\frac{x_0^\alpha}{R}\right) + \frac{2l+d-1}{2l+d-1+2k}\ C^{\frac{d-1}{2}+k}_{2k-1}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\right] P_l, \tag{12}$$

$$FCK\left[(x^\alpha)^{2k+1}P_l\right] = (-1)^k\ \lambda_l^{2k+1}\ R^{2k+1}\left[-C^{\frac{d-1}{2}+k}_{2k+1}\left(\frac{x_0^\alpha}{R}\right) + \frac{2l+d-1}{2l+d-1+2k}\ C^{\frac{d+1}{2}+k}_{2k}\left(\frac{x_0^\alpha}{R}\right)\ \frac{x^\alpha}{R}\right] P_l. \tag{13}$$

In this formula, $r^2 = (x_0^\alpha)^2 - (x^\alpha)^2$, $C_n^\lambda(x)$ are the standard Gegenbauer polynomials given by (9), and the coefficients λ_k^s are given by (10).

Before we present the proof we give the following remark concerning understanding the notation, which is similar to the one presented in [7].

Remark 5.8 *Please note that in the main theorem appear terms like $C_k^\lambda\left(\frac{x_0^\alpha}{R}\right)$. While in fact $\frac{x_0^\alpha}{R}$ is not well-defined, since $Rx_0^\alpha \neq x_0^\alpha R$ and $Rx^\alpha \neq x^\alpha R$ the notation has to be understood in the following way: because $C_k^\lambda(x)$ contains only powers of x of degree at most k, we first multiply it by R^k after which there is no ambiguity left.*

Let us now consider the proof.

Proof: We start with expression (12). The proof has two parts. In the first we show that the restriction of

$$F := (-1)^k\ \lambda_l^{2k} n\ R^{2k} \left[C_{2k}^{\frac{d-1}{2}+k} \left(\frac{x_0^\alpha}{R} \right) + \frac{2l+d-1}{2l+d-1+2k}\ C_{2k-1}^{\frac{d+1}{2}+k} \left(\frac{x_0^\alpha}{R} \right) \frac{x_0^\alpha}{R} \right] P_l$$

to the hyperplane $x_0^\alpha = 0$ is exactly $(x^\alpha)^{2k} P_l$. In fact, $R^{2j}|_{x_0^\alpha=0} = (x^\alpha)^{2j}$ and

$$C_{2k}^\lambda \left(\frac{x_0^\alpha}{R} \right) |_{x_0^\alpha=0} = \sum_{j=0}^{k} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j)!} \left(\frac{2x_0^\alpha}{R} \right)^{2k-2j} |_{x_0^\alpha=0} = \frac{(-1)^k\ (\lambda)_k}{k!},$$

$$C_{2k-1}^\lambda \left(\frac{x_0^\alpha}{R} \right) |_{x_0^\alpha=0} = \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-1)!} \left(\frac{2x_0^\alpha}{R} \right)^{2k-2j-1} |_{x_0^\alpha=0} = 0,$$

which implies that

$$\begin{aligned} F\big|_{x_0^\alpha=0} &= (-1)^k\ \lambda_l^{2k}\ \frac{(-1)^k}{k!} \left(\frac{d-1}{2} + l \right)_k (x^\alpha)^{2k} P_l \\ &= \frac{k!\ \Gamma\left(l+\frac{d-1}{2}\right)}{\Gamma\left(l+\frac{d-1}{2}+k\right)}\ \frac{1}{k!}\ \frac{\Gamma\left(l+\frac{d-1}{2}+k\right)}{\Gamma\left(l+\frac{d-1}{2}\right)} (x^\alpha)^{2k} P_l \\ &= (x^\alpha)^{2k} P_l. \end{aligned}$$

In the second part of the proof we show that F is a fractional monogenic in the $d+1$ variables $x_0^\alpha, x_1^\alpha, \ldots, x_d^\alpha$. By the uniqueness of the FCK-extension, we know that F must be exactly $FCK\left[(x^\alpha)^{2k} P_l\right]$. Since F consists in two terms, we will first consider ${}_+^C\partial_0^\alpha + D^\alpha$ acting on both terms separately via Lemmas 5.3 and 5.4 with $\lambda = l + \frac{d-1}{2}$. We will then continue by combining the obtained results.

$$\begin{aligned} &\left({}_+^C\partial_0^\alpha + D^\alpha\right) F \\ &= (-1)^k\ \lambda_l^{2k} \left[\left({}_+^C\partial_0^\alpha + D^\alpha\right) \left(C_{2k}^\lambda \left(\frac{x_0^\alpha}{R} \right) R^2 k P_l \right) + \frac{\lambda}{\lambda+k} \left({}_+^C\partial_0^\alpha + D^\alpha\right) \left(C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) \frac{x_0^\alpha}{R}\ R^{2k} P_l \right) \right] \\ &= (-1)^k\ \lambda_l^{2k}\ 2K_\alpha \lambda \left[C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) - C_{2k-2}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) \right] R^{2k-1} P_l \\ &\quad + (-1)^k\ \lambda_l^{2k}\ \frac{\lambda}{\lambda+k} \left[2K_\alpha(\lambda+1)\ C_{2k-2}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) x^\alpha\ R^{2k-2} - K_\alpha(2\lambda-1)\ C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) R^{2k-1} \right. \\ &\qquad\qquad \left. + 2K_\alpha(\lambda+1)\ C_{2k-3}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) \frac{x^\alpha}{R}\ R^{2k-1} \right] P_l \\ &= (-1)^k\ \lambda_l^{2k} \left[\left(2K_\alpha\lambda - \frac{K_\alpha\lambda(2\lambda-1)}{\lambda+k} \right) C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) R^{2k-1} - 2K_\alpha\lambda\ C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) R^{2k-1} \right. \\ &\qquad\qquad \left. + \frac{2K_\alpha\lambda(\lambda+1)}{\lambda+k}\ C_{2k-2}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) \frac{x^\alpha}{R}\ R^{2k-1} + \frac{2K_\alpha\lambda(\lambda+1)}{\lambda+k}\ C_{2k-3}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) \frac{x^\alpha}{R}\ R^{2k-1} \right] P_l \\ &= (-1)^k\ \lambda_l^{2k}\ 2K_\alpha\ \lambda \left[\frac{2k+1}{2\lambda+2k}\ C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) R^{2k-1} - C_{2k-1}^{\lambda+1} \left(\frac{x_0^\alpha}{R} \right) R^{2k-1} \right. \\ &\qquad\qquad \left. + \frac{\lambda+1}{\lambda+k}\ C_{2k-2}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) \frac{x^\alpha}{R}\ R^{2k-1} + \frac{\lambda+1}{\lambda+k}\ C_{2k-3}^{\lambda+2} \left(\frac{x_0^\alpha}{R} \right) \frac{x^\alpha}{R}\ R^{2k-1} \right] P_l. \end{aligned}$$

Taking into account the series expansion (9) for the Gegenbauer polynomials, the previous expression becomes equal to:

$$(-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\frac{\lambda(2k+1)}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda+1)_{2k-j-1}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1}R^{2j}\right.$$
$$-2\lambda\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda+1)_{2k-j-2}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-2}R^{2j+1}$$
$$+\frac{2\lambda(\lambda+1)}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda+2)_{2k-j-2}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j}$$
$$\left.+\frac{2\lambda(\lambda+1)}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j\ (\lambda+2)_{2k-j-3}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-3}\ (x_0^\alpha)^{2k-2j-3}\ x^\alpha R^{2j+1}\right] P_l$$
$$=(-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\frac{\lambda(2k+1)}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1}R^{2j}\right.$$
$$-\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}R^{2j+1}$$
$$+\frac{1}{\lambda+k}\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j}$$
$$\left.+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-3}\ x^\alpha R^{2j+1}\right] P_l. \tag{14}$$

From the series expansion of R^{2j} and R^{2j+1}

$$R^{2j}=\sum_{s=0}^{j}\begin{pmatrix} j \\ s \end{pmatrix}\ (x_0^\alpha)^{2j-2s}\ (x^\alpha)^{2s},$$
$$R^{2j+1}=\sum_{s=0}^{j}\begin{pmatrix} j \\ s \end{pmatrix}\ \left((x_0^\alpha)^{2j-2s+1}\ (x^\alpha)^{2s}+(x_0^\alpha)^{2j-2s}\ (x^\alpha)^{2s+1}\right),$$

there are two possible combinations (with respect to the powers of x_0^α and x^α): either an odd power of x_0^α combined with an even power of x^α or vice-versa. We will look at both possibilities separately and show that both must be zero. We first consider the terms of (14) containing a combination of an even power of x_0^α and a odd power of x^α, which we will denote by "even part" (EP).

$$EP=(-1)^k\ \lambda_l^{2k}\ K_\alpha \left[-\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j}\right.$$
$$+\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-2)!}\ \frac{2^{2k-2j-1}}{\lambda+k}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j}$$
$$\left.+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-3}\ x_0^\alpha\ x^\alpha R^{2j}\right] P_l$$
$$=(-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\sum_{j=0}^{k-1}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j}\left(\frac{\lambda+2k-j-1}{\lambda+k}-1\right)\right.$$
$$\left.+\frac{1}{\lambda+k}\sum_{j=0}^{k-2}\frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-3}\ x_0^\alpha\ x^\alpha R^{2j}\right] P_l$$

$$
\begin{aligned}
&= (-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\frac{1}{\lambda+k} \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ (k-j-1)\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j} \right.\\
&\qquad \left. + \frac{1}{\lambda+k} \sum_{j=0}^{k-2} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-2}\ x^\alpha R^{2j} \right] P_l \\
&= 0.
\end{aligned}
$$

For the "odd part" (OP) containing all terms with an odd power of x_0^α and an even power of x^α we have

$$
\begin{aligned}
OP &= (-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\frac{\lambda(2k+1)}{\lambda+k} \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1} R^{2j} \right.\\
&\qquad - \sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-2)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1} R^{2j} \\
&\qquad \left. + \frac{1}{\lambda+k} \sum_{j=0}^{k-2} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-3} R^{2j+1} \right] P_l \\
&= (-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1} R^{2j} \left(\frac{2k-1}{\lambda+k} - \frac{2k-2j-1}{\lambda+2k-j-1} \right) \right.\\
&\qquad \left. + \frac{1}{\lambda+k} \sum_{j=0}^{k-2} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-3} \left(R^{2j+2} - x_0^\alpha R^{2j} \right) \right] P_l \\
&= (-1)^k\ \lambda_l^{2k}\ K_\alpha \left[\sum_{j=0}^{k-1} \frac{(-1)^j\ (\lambda)_{2k-j}}{j!\ (2k-2j-1)!}\ 2^{2k-2j-1}\ (x_0^\alpha)^{2k-2j-1} R^{2j} \left(\frac{2k-1}{\lambda+k} - \frac{2k-2j-1}{\lambda+2k-j-1} \right) \right.\\
&\qquad + \frac{1}{\lambda+k} \sum_{p=1}^{k-1} \frac{(-1)^{p-1}\ (\lambda)_{2k-p}}{(p-1)!\ (2k-2p-1)!}\ 2^{2k-2p}\ (x_0^\alpha)^{2k-2p-1} R^{2p} \\
&\qquad \left. - \frac{1}{\lambda+k} \sum_{j=0}^{k-2} \frac{(-1)^j\ (\lambda)_{2k-j-1}}{j!\ (2k-2j-3)!}\ 2^{2k-2j-2}\ (x_0^\alpha)^{2k-2j-1} R^{2j} \right] P_l.
\end{aligned}
$$

One check for different values of j that the coefficients of R^{2j} will be zero, and hence the total sum will be zero. Regarding (13) we proceed in a similar way and considering Lemmas 5.5 and 5.6.

■

6 Examples

To end this paper, we present some examples of FCK-extension

Example 6.1 *The FCK-extension of $x^\alpha P_l$ is given by*

$$
FCK[x^\alpha P_l] = \sum_{k=0}^{\infty} \frac{(x_0^\alpha)^k}{k!}\ f_k,
$$

where the functions f_k are

$$
f_0 = x^\alpha P_l, \qquad\qquad f_1 = (2K_{l,\alpha} l + K_\alpha d) P_l,
$$

whence explicitly

$$
FCK[x^\alpha P_l] = [x^\alpha + 2K_{l,\alpha} l + K_\alpha d]\, P_l.
$$

Example 6.2 *The FCK-extension of* $(x^\alpha)^2 P_l$ *is given by*

$$FCK[(x^\alpha)^2 P_l] = \sum_{k=0}^{\infty} \frac{(x_0^\alpha)^k}{k!} \ f_k,$$

where the functions f_k *are*

$$f_0 = (x^\alpha)^2 P_l, \qquad f_1 = 2K_\alpha x^\alpha P_l, \qquad f_2 = -2K_\alpha(2K_{l,\alpha} l + K_\alpha d) P_l,$$

whence explicitly

$$FCK[(x^\alpha)^2 P_l] = \left[(x^\alpha)^2 + 2K_\alpha x^\alpha - 2K_\alpha(2K_{l,\alpha} l + K_\alpha d)\right] P_l.$$

Acknowledgement: N. Vieira was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT–Fundação para a Ciência e a Tecnologia"), within project PEst-OE/MAT/UI4106/2014.

References

[1] G. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and Its Applications-Vol.71, Cambridge University Press, Cambridge, 2000.

[2] F. Brackx, H. De Schepper, R. Lávička and V. Souček, *The Cauchy-Kovalevskaya extension theorem in Hermitian Clifford analysis*, J. Math. Anal. Appl., **381**-No.2, (2011), 649-660.

[3] F. Brackx, R. Delanghe and F. Sommen, *Clifford analysis*, Reserach Notes in Mathematics-Vol.76, Pitman Advanced Publishing Program, Boston - London - Melbourne, 1982.

[4] A. Cauchy, *Oeuvres complètes. Série 1-Tome VII*, Gauthier-Villars, Paris, 1882-1974, 17-58.

[5] R. Delanghe, F. Sommen and V. Souček, *Clifford algebras and spinor-valued functions. A function theory for the Dirac operator*, Mathematics and its Applications-Vol.53, Kluwer Academic Publishers, Dordrecht etc., 1992.

[6] B. De Knock, D. Peña-Peña and F. Sommen, *On the CK-extension for a special overdetermined system in complex Clifford analysis*, In: Hypercomplex analysis. Selected papers of the 6th ISAAC conference, Ankara, Turkey, August 1318, Eds: I. Sabadini et al., Birkhäuser, Basel, (2009), 115-124.

[7] H. De Ridder, *Discrete Clifford analysis: function theory and integral transforms*, PhD Thesis, University of Gent, 2013.

[8] H. De Ridder, H. De Schepper and F. Sommen, *The Cauchy-Kovalevskaya extension theorem in discrete Clifford analysis*, Commun. Pure Appl. Anal., **10**-No.4, (2011), 1097-1109.

[9] N. De Schepper and F. Sommen, *Cauchy-Kowalevski extensions and monogenic plane waves using spherical monogenics*, Bull. Braz. Soc., **44**-No.2, (2013), 321-350.

[10] E. Goldfain, *Complexity in quantum field theory and physics beyond the standard model*, Chaos Solitons and Fractals **28**-No.4, (2006), 913-922.

[11] U. Kähler and N. Vieira, *Fractional Clifford analysis*, Hypercomplex Analysis: New perspectives and applications, Trends in Mathematics, Eds: S. Bernstein, U. Kähler, I. Sabadini and F. Sommen, Birkhäuser, Basel, in press.

[12] A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.

[13] S. Kowalevsky, *Zur Theorie der partiellen Differentialgleichungen*, J. Reine Angew. Math., **80**, (1875), 1-32.

[14] M.F. Li, J.R. Ren and T. Zhu, *Fractional vector calculus and fractional special function*, (2010), arXiv:1001.2889v1.

[15] H. Malonek, D. Peña-Peña and F. Sommen, *A Cauchy-Kowalevski theorem for inframonogenic functions*, Math. J. Okayama Univ., **53**, (2011), 167-172.

[16] M.D. Roberts, *Fractional derivative cosmology*, (2009), arXiv:0909.1171v1.

[17] V.E. Tarasov, *Fractional vector calculus and fractional Maxwells equations*, Ann. Physics **323**-No.11, (2008), 2756-2778.