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Soliton-Like Solutions in the Problems of Vibrations of Nonlinear Mechanical Systems: Survey

Yury A. Rossikhin and Marina V. Shitikova

Additional information is available at the end of the chapter

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Abstract

In the given chapter, free vibrations of different nonlinear mechanical systems with one-degree-of-freedom, two-degree-of-freedom, and multiple-degree-of-freedom are reviewed with the emphasis on the vibratory regimes which could go over into the aperiodic motions under certain conditions. Such unfavorable and even dangerous regimes of vibrations resulting in the irreversible process of energy exchange from its one type to another type are discussed in detail. The solutions describing such processes are found analytically in terms of functions, which are in frequent use in the theory of solitons.

Keywords: soliton-like solution, nonlinear mechanical systems, free vibrations, method of multiple time scales, suspension bridge

1. Introduction

It is known [1] that the periodical transfer of energy from one type to another is made possible during vibrational processes occurring in nonlinear mechanical systems. This phenomenon is called *energy exchange* [2, 3].

Investigations on the energy exchange originate from the chapter [4], wherein the authors studied small nonlinear vibrations of a two-degree-of-freedom (2dof) system consisting of a load suspended on a linearly elastic spring and executing pendulum vibrations and vibrations along the spring's axis in the same vertical plane. In spite of the apparent simplicity of that system, it realistically explains some phenomena occurring during vibrations of more complex nonlinear systems and in particular describes all types of energy exchange from pendulum vibratory motions into oscillatory motions along the spring's axis, and vice versa: the periodic

and aperiodic energy interchange, as well as stationary regimes during which the energy exchange is absent.

The energy-exchange mechanism in a similar nonlinear 2dof system has been studied in [5, 6]. The system was made up of two loads, one of which was suspended on a linearly elastic spring and executed vertical vibrations, and the other was suspended on an unstretched rod and executed pendulum vibrations in the same vertical plane. Reviews devoted to nonlinear vibrations of 2dof systems can be found in [2, 3].

However, the energy transfer is observed during free vibrations of different nonlinear mechanical systems: possessing one-degree-of-freedom (1dof), two- (2dof), and more degrees-of-freedom (multiple-dof), and as well as having infinite number of degrees-of-freedom (deformable solids). *The internal resonance* is realized when magnitudes of natural frequencies of two natural modes belonging to the different types of vibrations of the system (*partial subsystems*) are approximately equal to each other or one of them two to three times larger than the other. This phenomenon is particularly evident in modern engineering structures which are very light and flexible due to the application of present-day materials, resulting in finite displacements of individual structural elements as well as of the structure as a whole. Among such constructions are suspension-combined systems: suspension and cable-stayed bridges, suspension roofs in large public and industrial buildings, and so on. Suspension-combined systems and suspension bridges, in particular, are distinguished by high esthetic merits, and many of them are referred to the most remarkable up-to-date engineering structures. For example, "Golden Gate" suspension bridge in San Francisco with the span of 1281 m, cable-stayed bridge in Cologne with the span of 690 m, suspension roofing of Olympic sport complex in Moscow, and many others.

The majority of papers devoted to the dynamic behavior of suspension-combined systems studies free nonlinear vibrations of suspension bridges with a thin-walled stiffening girder [7–11]. Different dynamic loads (wind, seismic excitation, moving loads, etc.) after the completion of acting on a suspended structure setup prolonged free nonlinear vibrations of this structure, in so doing both vertical and flexural-torsional vibrations could be excited. One of the most unfavorable nonlinear effects, which is observed in suspension systems during free vibrations, is just the "energy exchange" from one type of vibratory motions into the other under the conditions of the internal resonance.

The intensity and frequency of energy exchange between strongly coupled modes essentially depend on an absolute level of the initial amplitudes [7, 8, 11, 12] which is governed by the value of the initial mechanical energy of the system.

However, the qualitative character of the energy exchange is dependent on the relative level of initial amplitudes which is independent of the system's initial energy and is defined as the ratio of the initial amplitudes of the two interacting modes [9]. It has been found in [9] that in accordance with a value of that level, three types of an energy-exchange mechanism exist: *two-sided energy exchange* (a periodic energy exchange from one subsystem to another), *one-sided energy exchange* (one subsystem completely or partially transfers the energy to another), and energy exchange does not occur (*stationary vibrations*). Among the three types of the behavior

of the mechanical system, the second one may occur to be the most unfavorable. As for the behavior of a suspension bridge, then the most hazardous type is the irreversible transfer of the energy of vertical vibrations into the energy of its torsional vibrations in the case of a bisymmetrical stiffening girder or into the energy of flexural-torsional vibrations in the case of a mono-symmetrical girder. This is due to the fact that suspension bridges possess a rather higher flexural rigidity than torsional one, that is, they perceive better than those dynamic loads that result in vertical vibrations.

Solutions describing the one-sided energy transfer occurring in mechanical systems we shall call as *soliton-like solutions*, since the functions entering in such solutions are widely met in the theory of solitons [13, 14].

In this chapter, it is shown that solutions of such a type exist both in 1dof systems and in systems possessing two- and more degrees-of-freedom.

2. A one-degree-of-freedom system

The phenomenon of energy transfer, when one type of the energy completely and irreversibly goes into another type of the energy as time passes, can be observed on such a simple object as a mathematical pendulum (**Figure 1**).

In order to demonstrate this, let us consider the expression for the total mechanical energy of the mathematical pendulum which is combined from the kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2(\dot{\varphi})^2 \quad (1)$$

and the potential energy (**Figure 1**)

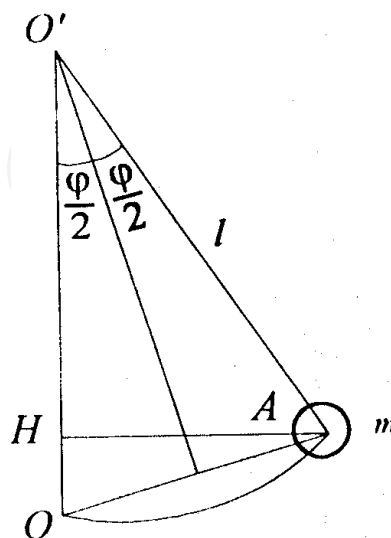


Figure 1. A mathematical pendulum.

$$\Pi = mg(OH) = 2mgl \sin^2\left(\frac{\varphi}{2}\right) \quad (2)$$

and has the form

$$E = T + \Pi = \frac{1}{2} ml^2 (\dot{\varphi})^2 + 2mgl \sin^2\left(\frac{\varphi}{2}\right), \quad (3)$$

where an overdot denotes a time derivative, l is the string length, g is the gravity acceleration, m is the load mass, $v = l\dot{\varphi}$ is its velocity, and φ is the angle of the string's deflection from the vertical.

Rewrite Eq. (3) in the dimensionless form

$$\frac{(\dot{\varphi})^2}{\omega_0^2} + 4 \sin^2\left(\frac{\varphi}{2}\right) = \frac{E}{E_0}, \quad (4)$$

where $E_0 = \frac{1}{2} m\omega_0^2 l^2$ and $\omega_0 = \sqrt{g/l}$.

Consider the case of motion of the mathematical pendulum when its energy E is exactly equal to $4E_0$. Then, the law of conservation of energy Eq. (4) gives the simple relationship [15].

$$\frac{(\dot{\varphi})^2}{\omega_0^2} = 4 \cos^2\left(\frac{\varphi}{2}\right) \quad (5a)$$

or

$$\dot{\varphi} = 2\omega_0 \cos\left(\frac{\varphi}{2}\right). \quad (5b)$$

Dividing the variables in Eq. (5b), integrating separately the right and left parts of the relationship obtained, and considering that $\varphi = 0$ at $t = 0$ yield

$$\ln\left[\tan\left(\frac{\pi - \varphi}{4}\right)\right] = -\omega_0 t \quad (6a)$$

or

$$\varphi = \pi - 4\arctan(e^{-\omega_0 t}). \quad (6b)$$

Differentiating Eq. (6b) over t , we find

$$\dot{\varphi} = \frac{2\omega_0}{\cosh(\omega_0 t)} \quad (7)$$

Reference to Eqs. (6) and (7) shows that if the mathematical pendulum begins its motion from the extreme low position, then at $t \rightarrow \infty$ its velocity $\dot{\varphi} \rightarrow 0$, in so doing does not vanish

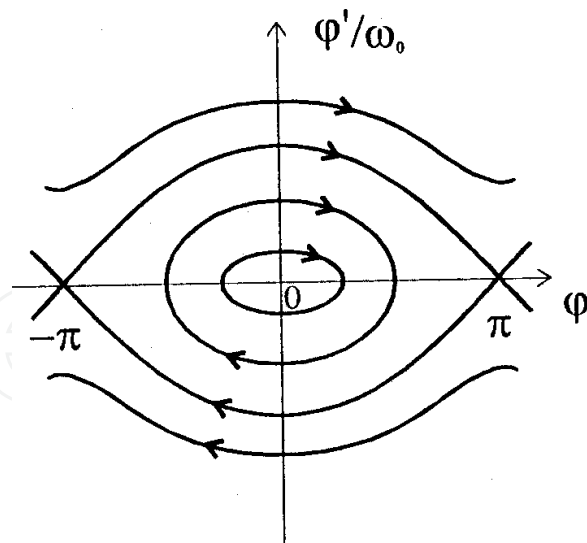


Figure 2. Phase portrait describing vibrations of mathematical pendulum.

anywhere, and the angle $\varphi \rightarrow \pi$, that is, the pendulum, tends to take the upper position of equilibrium which is an unstable one. As this takes place, the kinetic energy completely transforms into the potential energy. This solution is the soliton-like one, since the functions *arctan* and *ch* are frequently met in soliton solutions.

If one represents the phase trajectories of the pendulum motion on the phase plane $\dot{\varphi} / \omega_0 - \varphi$ at different magnitudes of the energy E , then solution (6) will correspond to the phase trajectory which is called as a *separatrix*. This line divides closed trajectories from nonclosed ones (**Figure 2**). Closed and nonclosed trajectories are consistent with the solutions for the periodic transfer of the potential and kinetic energies into each other, in doing so in the first case, the pendulum will vibrate, and in the second one, it will rotate around the point of suspension.

3. A two-degree-of-freedom system

3.1. Governing equations

Now, consider a 2dof system presented in **Figure 3**. The kinetic T and potential Π energies of such a system have the form

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}^2 - m_2 l \dot{y} \dot{\varphi} \sin \varphi + \frac{1}{2} m_2 l^2 \dot{\varphi}^2, \quad (8a)$$

$$\Pi = \frac{1}{2} k(y + y_{cm})^2 - m_1 g(y + y_{cm}) - m_2 g(y + y_{cm} + l \cos \varphi), \quad (8b)$$

where $y_{cm} = (m_1 + m_2)g / k$, k is the elastic spring rigidity, m_1 and m_2 are the masses of the first and second loads, respectively, y is the vertical displacement of the first load, and φ is the angle of the pendulum's deflection.

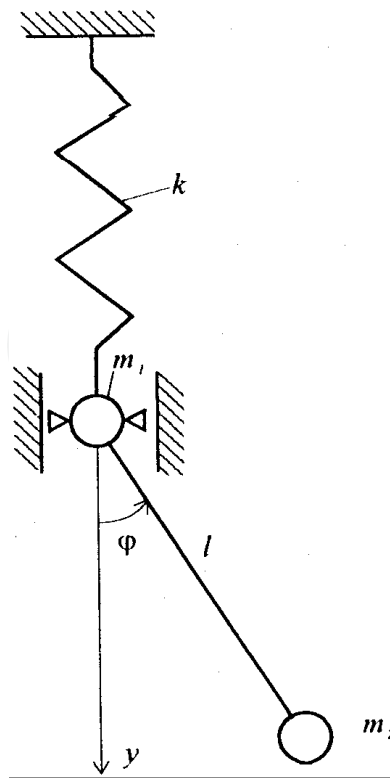


Figure 3. Scheme of a 2dof mechanical system.

Applying Lagrange equations of the second kind [15]

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\lambda}} \right) - \frac{\partial T}{\partial \lambda} = - \frac{\partial \Pi}{\partial \lambda} \quad (\lambda = y, \varphi)$$

and considering Eq. (8), the system's equations of motion in the dimensionless form within an accuracy of the values of the second order of smallness with respect to y and φ can be written as follows:

$$\ddot{y}^* + \omega^{*2} y^* - a \varphi \ddot{\varphi} - a \dot{\varphi}^2 = 0, \quad (9a)$$

$$\ddot{\varphi} + \Omega^{*2} \varphi - b \varphi \ddot{y}^* = 0, \quad (9b)$$

where

$$\omega^{*2} = \omega^2 y_0 g^{-1}, \quad \Omega^{*2} = \Omega y_0 g^{-1}, \quad \omega^2 = k(m_1 + m_2)^{-1}, \quad \Omega^2 = g l^{-1}$$

$$y^* = y y_0^{-1}, \quad t^* = t \sqrt{g y_0^{-1}}, \quad y_0 = m_1 k^{-1} g, \quad a = \frac{m_2}{m_1 + m_2} \cdot \frac{l}{y_0}, \quad b = \frac{y_0}{l}.$$

Suppose that the linear natural frequency ω^* is twice as large than the linear natural frequency Ω^* , that is,

$$\omega^* = 2\Omega^* \tag{10a}$$

or the linear natural frequency ω^* and the linear natural frequency Ω^* are equal to each other, that is,

$$\omega^* = \Omega^* . \tag{10b}$$

It is said that the system is being under the conditions of *the two-to-one internal resonance* or the *one-to-one internal resonance* if the condition Eq. (10a) or (10b) is fulfilled, respectively [2].

For analyzing nonlinear vibrations of the systems subjected to the internal resonance (10), assume that the amplitudes of vibrations are small but finite values and weakly vary with time. Then, perturbation technique could be used to construct the solution of the set of Eq. (9), and, particularly, the method of multiple time scales [16].

3.2. Method of solution

An approximate solution of Eq. (9) can be represented by an expansion in terms of different time scales limiting by the values of the third order of smallness in ε

$$y^*(t) = \varepsilon y_1(T_0, T_1, T_2, \dots) + \varepsilon^2 y_2(T_0, T_1, T_2, \dots) + \varepsilon^3 y_3(T_0, T_1, T_2, \dots) + \dots , \tag{11a}$$

$$\varphi(t) = \varepsilon \varphi_1(T_0, T_1, T_2, \dots) + \varepsilon^2 \varphi_2(T_0, T_1, T_2, \dots) + \varepsilon^3 \varphi_3(T_0, T_1, T_2, \dots) + \dots , \tag{11b}$$

where $T_n = \varepsilon^n t$ ($n = 0, 1, 2, \dots$), and ε is a small parameter.

Substituting Eq. (11) into Eq. (9), considering that

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots, \quad D_n = \frac{\partial}{\partial T_n} \quad (n = 0, 1, 2, \dots)$$

$$\frac{d^2}{dt^2} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots)^2 = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_0 D_2 + D_1^2) + \dots$$

and equating the coefficients of like powers of ε , one obtains, to order ε ,

$$D_0^2 y_1 + \omega^{*2} y_1 = 0, \quad D_0^2 \varphi_1 + \Omega^{*2} \varphi_1 = 0; \tag{12}$$

to order ε^2 ,

$$D_0^2 y_2 + \omega^{*2} y_2 = -2D_0 D_1 y_1 + a \varphi_1 D_0^2 \varphi_1 + a (D_0 \varphi_1)^2, \tag{13}$$

$$D_0^2 \varphi_2 + \Omega^{*2} \varphi_2 = -2D_0 D_1 \varphi_1 + b \varphi_1 D_0^2 y_1;$$

to order ε^3 ,

$$\begin{aligned}
D_0^2 y_3 + \omega^{*2} y_3 = & -2D_0 D_1 y_2 - (D_1^2 + 2D_0 D_2) y_1 + a\varphi_2 D_0^2 \varphi_1 \\
& + 2aD_0 \varphi_1 (D_1 \varphi_1 + D_0 \varphi_2) + a\varphi_1 (D_0^2 \varphi_2 + 2D_0 D_1 \varphi_1),
\end{aligned} \tag{14}$$

$$\begin{aligned}
D_0^2 \varphi_3 + \Omega^{*2} \varphi_3 = & -2D_0 D_1 \varphi_2 - (D_1^2 + 2D_0 D_2) \varphi_1 \\
& + b\varphi_2 D_0^2 y_1 + b\varphi_1 (D_0^2 y_2 + 2D_0 D_1 y_1)
\end{aligned}$$

The solution of Eq. (12) could be sought in the form

$$y_1 = A_1(T_1) \exp(i\omega^* T_0) + \bar{A}_1(T_1) \exp(-i\omega^* T_0), \tag{15}$$

$$\varphi_1 = A_2(T_1) \exp(i\Omega^* T_0) + \bar{A}_2(T_1) \exp(-i\Omega^* T_0),$$

where A_1 and A_2 are unknown complex functions, while \bar{A}_1 and \bar{A}_2 are the complex conjugates of A_1 and A_2 , respectively.

3.2.1. The case of a two-to-one internal resonance

Substituting Eq. (15) into the right-hand sides of Eq. (13) yields

$$\begin{aligned}
D_0^2 y_2 + \omega^{*2} y_2 = & -2i\omega^* D_1 A_1 \exp(i\omega^* T_0) - 2aA_2^2 \Omega^{*2} \exp(2i\Omega^* T_0) + cc, \\
D_0^2 \varphi_2 + \Omega^{*2} \varphi_2 = & -2i\Omega^* D_1 A_2 \exp(i\Omega^* T_0) - bA_1 A_2 \omega^{*2} \exp[i(\omega^* + \Omega^*) T_0] - \\
& - bA_1 \bar{A}_2 \omega^{*2} \exp[i(\omega^* - \Omega^*) T_0] + cc,
\end{aligned} \tag{16}$$

where cc denotes complex conjugate parts of the preceding terms.

The functions $\exp(i\omega^* T_0)$, $\exp(2i\Omega^* T_0) = \exp(i\omega^* T_0)$, $\exp(i\Omega^* T_0)$, $\exp[i(\omega^* - \Omega^*) T_0] = \exp(i\Omega^* T_0)$ entering into the right-hand sides of Eq. (16) produce secular terms in the expression for y_2 and φ_2 , that is, the terms of the type of $T_0 e^{i\omega^* T_0}$ and $T_0 e^{i\Omega^* T_0}$. Since secular terms increase without any limits as time goes on, then there is a need to eliminate them by equating the coefficients standing at the enumerated functions to zero. As a result, we obtain

$$i\omega^* D_1 A_1 + aA_2^2 \Omega^{*2} = 0, \tag{17a}$$

$$2i\Omega^* D_1 A_2 + bA_1 \bar{A}_2 \omega^{*2} = 0. \tag{17b}$$

Multiply Eq. (17a) by \bar{A}_1 and Eq. (17b) by \bar{A}_2 and find the complex conjugate equations. Two mutually conjugated equations first add to each other and then subtract one from another. As a result of such a procedure, we obtain more convenient set of four equations:

$$\begin{aligned}
 i\omega^* (\bar{A}_1 D_1 A_1 - A_1 D_1 \bar{A}_1) + a\Omega^* (\bar{A}_1 A_2^2 + A_1 \bar{A}_2^2) &= 0, \\
 i\omega^* (\bar{A}_1 D_1 A_1 + A_1 D_1 \bar{A}_1) + a\Omega^* (\bar{A}_1 A_2^2 - A_1 \bar{A}_2^2) &= 0, \\
 2i\Omega^* (\bar{A}_2 D_1 A_2 - A_2 D_1 \bar{A}_2) + b\omega^* (A_1 \bar{A}_2^2 + \bar{A}_1 A_2^2) &= 0, \\
 2i\Omega^* (\bar{A}_2 D_1 A_2 + A_2 D_1 \bar{A}_2) + b\omega^* (A_1 \bar{A}_2^2 - \bar{A}_1 A_2^2) &= 0.
 \end{aligned}$$

Representing the functions A_1 and A_2 in a polar form

$$A_1 = a_1(T_1) \exp[i\varphi_1(T_1)], \quad A_2 = a_2(T_1) \exp[i\varphi_2(T_1)], \quad (18)$$

we can rewrite the set of four differential equations as

$$(a_1^2)^\bullet = -\frac{1}{2} a\omega^* a_1 a_2^2 \sin \delta, \quad (19a)$$

$$(a_2^2)^\bullet = 2b\omega^* a_1 a_2^2 \sin \delta, \quad (19b)$$

$$\dot{\varphi}_1 = \frac{1}{4} a\omega^* a_2^2 a_1^{-1} \cos \delta, \quad (19c)$$

$$\dot{\varphi}_2 = b\omega^* a_1 \cos \delta, \quad (19d)$$

where an overdot denotes differentiation with respect to T_1 , and $\delta = 2\varphi_2 - \varphi_1$.

Eliminating the value $\omega^* a_1 a_2^2 \sin \delta$ from Eqs. (19a) and (19b) and integrating the net relationship with respect to T_1 yield

$$a_1^2 + \frac{1}{4} a b^{-1} a_2^2 = E_0, \quad (20)$$

where E_0 is the initial magnitude of the system's energy, which represents the law of conservation of the total mechanical energy of the system under consideration. Expression (20) is the first integral of the set of Eq. (19).

Introducing a new function $\xi(T_1)$ ($0 \leq \xi \leq 1$) such that

$$a_1^2 = E_0 \xi(T_1), \quad a_2^2 = 4ba^{-1} E_0 [1 - \xi(T_1)], \quad (21)$$

and substituting Eq. (21) in Eq. (19a), we have

$$\dot{\xi} = -B\sqrt{\xi}(1 - \xi) \sin \delta, \quad (22)$$

where $B = 2\sqrt{E_0} \omega^* b$.

Doubling both sides of Eq. (19d) and subtracting from the net relationship Eq. (19c) with due account for Eq. (21) and

$$\dot{\delta} = 2\dot{\phi}_2 - \dot{\phi}_1,$$

we obtain

$$\dot{\delta} = B \frac{3\xi - 1}{2\sqrt{\xi}} \cos \delta. \quad (23)$$

Putting

$$\dot{\delta} = \frac{d\delta}{d\xi} \dot{\xi} \quad (24)$$

and substituting Eq. (24) into Eqs. (22) and (23), we are led to the equation

$$\frac{d \cos \delta}{d\xi} + \frac{1 - 3\xi}{2\xi(1 - \xi)} \cos \delta = 0. \quad (25)$$

Separating the variables in Eq. (25) and integrating the equation obtained yield

$$\cos \delta = G_0 \xi^{-1/2} (1 - \xi)^{-1} \quad (26a)$$

or

$$G(\xi, \delta) = \sqrt{\xi} (1 - \xi) \cos \delta = G_0(\xi_0, \delta_0), \quad (26b)$$

where $G_0(\xi_0, \delta_0) = \sqrt{\xi_0} (1 - \xi_0) \cos \delta_0$ is an arbitrary constant determined from the initial conditions, and ξ_0 and δ_0 are the initial magnitudes of the values ξ and δ , respectively. Note that relationship (26b) is the other first integral of the set of Eq. (19).

Finally, let us eliminate the value δ from Eqs. (26a) and (22), resulting in

$$\dot{\xi} = -B \xi^{1/2} (1 - \xi) \sqrt{1 - \frac{G_0^2}{\xi(1 - \xi)^2}}. \quad (27)$$

Separating the variables in Eq. (27) and integrating the net expression, we obtain implicitly the desired function $\xi(T_1)$

$$\int_{\xi_0}^{\xi} \frac{d\xi}{[\xi(1 - \xi)^2 - G_0^2]^{1/2}} = -BT_1, \quad (28)$$

where ξ_0 is the value defining the relative level in the initial amplitudes.

The integral in Eq. (28) can be transformed into an incomplete integral of the first kind, which is tabulated in [17].

At $G_0 = 0$, the integral in Eq. (28) can be calculated, in so doing, it possesses two magnitudes. Really, changing the variable $\sqrt{\xi} = x$ in the integral in Eq. (28) at $G_0 = 0$, we have the first magnitude

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi}(\xi-1)} = 2 \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{x^2-1} = \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{x-1} - \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{x+1} = \ln \left(\frac{\sqrt{\xi}-1}{\sqrt{\xi}+1} \right) \Bigg|_{\xi_0}^{\xi} \quad (29a)$$

and the second magnitude

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi}(1-\xi)} = 2 \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{1-x^2} = \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{1-x} - \int_{\sqrt{\xi_0}}^{\sqrt{\xi}} \frac{dx}{1+x} = \ln \left| \frac{1+\sqrt{\xi}}{1-\sqrt{\xi}} \right| \Bigg|_{\xi_0}^{\xi} . \quad (29b)$$

Considering Eq. (29), the solutions of Eq. (28) may be written in the following form:

the first solution

$$\ln \left| \frac{(\sqrt{\xi}-1)(\sqrt{\xi_0}+1)}{(\sqrt{\xi}+1)(\sqrt{\xi_0}-1)} \right| = -BT_1 \quad (30a)$$

or

$$\sqrt{\xi} = \frac{(1+\sqrt{\xi_0}) - (1-\sqrt{\xi_0})e^{-BT_1}}{(1+\sqrt{\xi_0}) + (1-\sqrt{\xi_0})e^{-BT_1}} , \quad (30b)$$

and the second solution

$$\ln \left| \frac{(1+\sqrt{\xi})(1-\sqrt{\xi_0})}{(1-\sqrt{\xi})(1+\sqrt{\xi_0})} \right| = -BT_1 \quad (31a)$$

or

$$\sqrt{\xi} = \frac{(1+\sqrt{\xi_0})e^{-BT_1} - (1-\sqrt{\xi_0})}{(1+\sqrt{\xi_0})e^{-BT_1} + (1-\sqrt{\xi_0})} \quad (31b)$$

Solutions (30b) and (31b) at $\xi_0 \neq 0$ and 1 describe the motions corresponding to the one-sided energy exchange between pendulum's vibrations and vertical vibration of the load. As this takes place, $\xi \rightarrow 1$ and $\xi \rightarrow 0$ in the first and second solutions, respectively, with the increase in time T_1 . In other words, in the first solution, the energy of vibrations of the pendulum completely transforms into the energy of vertical vibrations of the load, but in the second solution, quite the reverse, the energy of vertical vibrations of the load completely goes into the energy of vibrations of the pendulum.

In the first solution, the process of energy transfer occurs over an infinitely large time interval, which resembles the phenomenon of the transfer of the kinetic energy into potential one, which is described by the soliton-like solution (6b) for the mathematical pendulum.

In the second solution, the process of energy transfer occurs during a finite instant of the time from 0 till T_1^* , where

$$T_1^* = -\frac{1}{B} \ln \left(\frac{1 - \sqrt{\xi_0}}{1 + \sqrt{\xi_0}} \right).$$

According to our classification, both of them are the soliton-like solutions. At $\xi_0 = 0$ from (30b), we obtain the known soliton-like solution in the form of a single kink [13].

$$\sqrt{\xi} = \tanh \left(\frac{1}{2} B T_1 \right). \quad (32)$$

Physically speaking, this solution kink is responsible to the one-sided energy exchange when the energy of the pendulum vibration completely transforms with time into the energy of the vertical vibrations which energy was equal to zero at the initial moment of time, so the pendulum vibrations give way to the vertical vibrations.

In order to understand the physical meaning of the first integral (26b), let us introduce into consideration the phase plane δ - ξ and analyze on this plane the phase fluid flow that interprets the motion of the mechanical system in hand. The velocity vector \mathbf{V} of the phase fluid particles motion has the components $v_\xi = \dot{\xi}$ and $v_\delta = \dot{\delta}$. From Eqs. (22) and (23), it follows that

$$v_\xi = B \frac{\partial G}{\partial \delta}, \quad v_\delta = -B \frac{\partial G}{\partial \xi} \quad (33)$$

Writing the equation of a streamline of the phase fluid $\frac{d\xi}{v_\xi} = \frac{d\delta}{v_\delta}$ and substituting Eq. (33) in it, we obtain that the function $G(\xi, \delta)$ defined by the relationship (26b) is *the stream function of the phase fluid*. In other words, Eq. (26b) at different magnitudes of ξ_0 and δ_0 governs a family of the streamlines of the phase fluid. Since the phase fluid is incompressible ($\text{div } \mathbf{V} = 0$) and its flow is steady and solenoidal ($\text{rot } \mathbf{V} \neq 0$), then streamlines of the phase fluid will coincide with trajectories of the phase fluid particles motion.

Streamlines constructed according to the relationship

$$\sqrt{\xi} (1 - \xi) \cos \delta = \sqrt{\xi_0} (1 - \xi_0) \cos \delta_0 \quad (34)$$

at different magnitudes of ξ_0 and δ_0 are presented in **Figure 4**, where digits near the curves denote the magnitudes of the value $G_0(\xi_0, \delta_0) = \sqrt{\xi_0} (1 - \xi_0) \cos \delta_0$. Reference to **Figure 4** shows that all phase trajectories are closed lines located around the perimeter of the rectangle

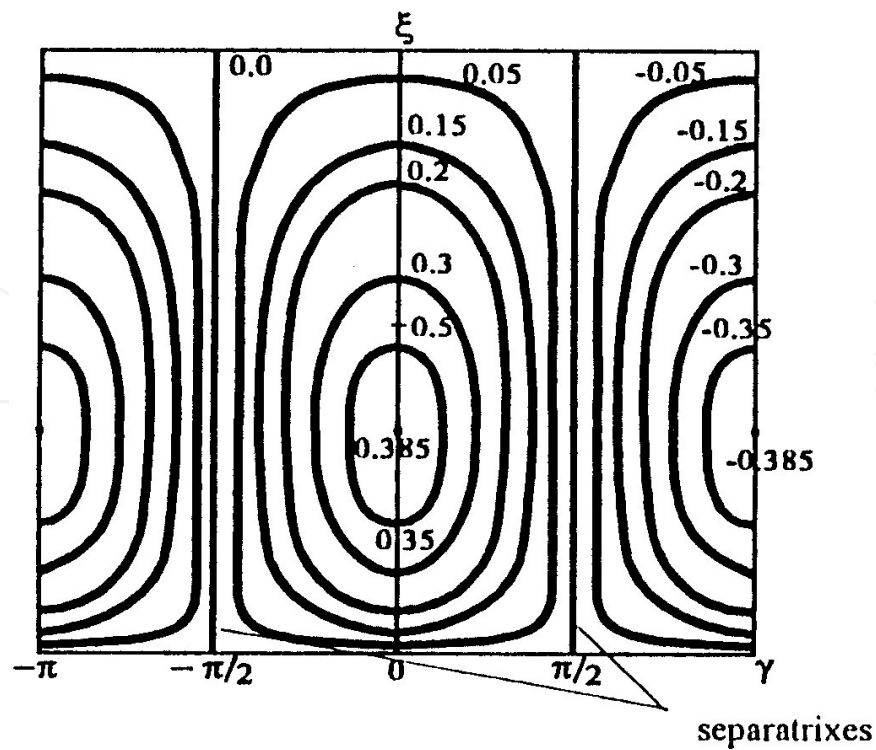


Figure 4. Phase portrait in the case of the two-to-one internal resonance $\omega^* = 2\Omega^*$.

bounded by the lines $\xi = 0, \xi = 1, \delta = \pm\pi/2 + \pi n (n = 0, 1, 2, \dots)$. The flow in each rectangle is isolated. On all four rectangle sides, $G_0 = 0$ and inside it the value G_0 preserves its sign. On the closed streamlines, a two-sided energy exchange takes place between the partial subsystems. Along the lines $\delta = \pm\pi/2 \pm \pi n$ a one-sided energy interchange occurs corresponding to pure amplitude-modulated aperiodic motions, in so doing on the lines with ascending flow of the phase fluid particles (an arrow is directed upwards), the aperiodic regime is described by Eq. (30b), and on the lines with descending flow (an arrow is directed downwards), the aperiodic regime is governed by Eq. (31b). On the line $\xi = 1$, there exists the boundary phase-modulated regime. The transition of fluid elements from the points with the coordinates $\xi = 0, \delta = \pi/2 + \pi n$ to the points $\xi = 0, \delta = -\pi/2 + \pi n$ proceeds instantly. The points with coordinates $\xi = 1/3, \delta = \pi n$ correspond to the stable stationary regimes.

3.2.2. The case of a one-to-one internal resonance

To construct the solution in the case of a one-to-one internal resonance (10b), it will suffice to restrict consideration to the terms of the order of ε^3 and to consider the amplitudes A_1 and A_2 as functions of T_1 and T_2 .

The resonance (10b) is weaker than (10a), since in order to eliminate circular terms arising in the second approximation, it would suffice to consider the functions A_1 and A_2 dependent on T_2 only [18]. Under such an assumption, the set of equations providing the absence of circular terms in the expressions for y_3 and φ_3 has the form

$$\dot{(a_1^2)} = ab\omega^* a_1^2 a_2^2 \sin \delta, \quad (35a)$$

$$\dot{(a_2^2)} = -b^2\omega^* a_1^2 a_2^2 \sin \delta, \quad (35b)$$

$$\dot{\varphi}_1 = -ab\omega^* a_2^2 \left(\frac{1}{3} + \frac{1}{2} \cos \delta \right), \quad (35c)$$

$$\dot{\varphi}_2 = -ab\omega^* \left(\frac{1}{3} a_1^2 - \frac{4}{3} a_2^2 + \frac{1}{2} \frac{b}{a} a_1^2 \cos \delta \right), \quad (35d)$$

where overdots denote differentiation with respect to T_2 , and $\delta = 2(\varphi_2 - \varphi_1)$.

The two first integrals of the system (35) have the following form:

$$a_1^2 + \frac{a}{b} a_2^2 = E_0, \quad (36)$$

$$G(\xi, \delta) = \xi(1 - \xi) \cos \delta - \frac{a}{3b} \xi^2 - \frac{5}{3} (1 - \xi)^2 = G_0(\xi_0, \delta_0), \quad (37)$$

in so doing

$$\dot{\xi} = -\omega^* b^2 E_0 \frac{\partial G}{\partial \delta}, \quad \dot{\delta} = \omega^* b^2 E_0 \frac{\partial G}{\partial \xi},$$

where $G_0(\xi_0, \delta_0) = \xi_0(1 - \xi_0) \cos \delta_0 - \frac{a}{3b} \xi_0^2 - \frac{5}{3} (1 - \xi_0)^2$, the function $\xi(T_2)$ is connected with a_1^2 and a_2^2 by the relationships

$$a_1^2 = E_0 \xi(T_2), \quad a_2^2 = ba^{-1} [1 - \xi(T_2)],$$

and the rest of the values have the same meaning as in the abovementioned case (10a).

Streamlines constructed according to Eq. (37) at different magnitudes of ξ_0 and δ_0 are presented in **Figure 5** when $a/b = 5$ and $\omega^* = b$. Magnitudes of the value $G_0(\xi_0, \delta_0)$ that correspond to the streamlines are indicated by digits near the curves; the flow direction of the phase fluid elements is shown by arrows on the streamlines. Reference to **Figure 5** shows that there exist two types of the streamlines, namely (1) nonclosed which correspond to the periodic change of amplitudes and the aperiodic change of phases and (2) closed ones which correspond to the periodic change of both amplitudes and phases. The alignment of the circulation zones resembles that of Von Karman vortex streets with a symmetric arrangement. The adjacent circulation zones osculate at the *saddle points* with the coordinates $\xi_0 = 0.5$, $\delta_0 = \pi \pm 2\pi n$ ($n = 0, 1, 2, \dots$) and $G_0 = -1.0833$, wherein the *unstable stationary regime* occurs.

On the boundary lines of these zones (separatrixes), the value $G_0 = -13/12$, and the analytical solution corresponding to the soliton-like regime has the form

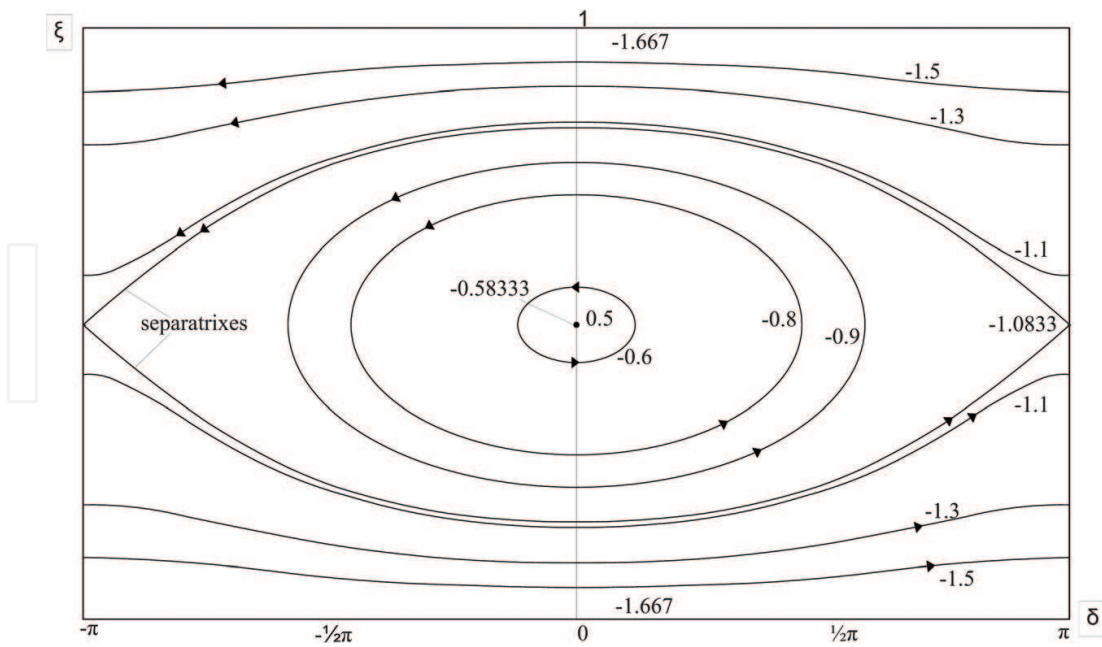


Figure 5. Phase portrait in the case of the one-to-one internal resonance.

$$\ln \left\| \frac{2\sqrt{-0.1154(\xi - 0.5)^2 + 0.3397} + 0.2308}{\xi - 0.5} \right\|_{\xi_0}^{\xi} = \pm 0.083 E_0 T_2, \quad (38)$$

where the sign “+” fits to the initial magnitudes $0.5 < \xi_0 \leq 0.8397$, $-\pi \pm 2\pi n < \delta_0 \leq 2\pi n$, $0.16032 \leq \xi_0 < 0.5$, and $-2\pi \pm 2\pi n < \delta_0 \leq -\pi \pm 2\pi n$, but the sign “-” conforms to the initial magnitudes $0.5 < \xi_0 \leq 0.8397$, $-2\pi \pm 2\pi n < \delta_0 \leq -\pi \pm 2\pi n$ and $0.16032 \leq \xi_0 < 0.5$, $-\pi \pm 2\pi n < \delta_0 \leq \pm 2\pi n$.

The upper branch of the separatrix describes the partial irreversible energy transfer from the vertical vibrations to the pendulum vibrations, but the lower branch, on the contrary, is in compliance with partial irreversible transfer of the energy of the pendulum vibrations to the energy of the vertical vibrations.

The points with coordinates $\xi_0 = 0.5$, $\delta_0 = \pm 2\pi n$, $G_0 = -0.5833$ (points like a center) corresponding to the stable stationary regime are located inside closed streamlines.

4. System with an infinite number of degrees-of-freedom

Similar solutions corresponding to the one-sided energy interchange could be obtained for more complex nonlinear systems that describe dynamic behavior of real structures, as an

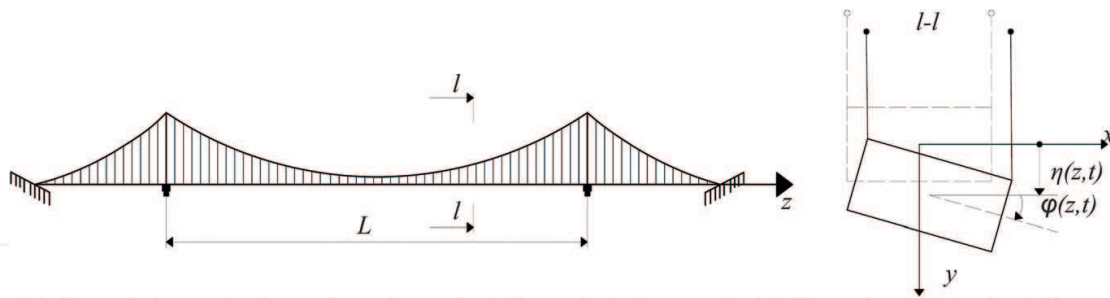


Figure 6. Scheme of a suspension bridge.

example, for systems with an infinite number of degree-of-freedom. Among such systems are suspension bridges, the scheme of one of them is shown in **Figure 6**.

The suspension bridge scheme presents a bisymmetrical thin-walled stiffening girder, which is connected with two suspended cables by virtue of vertical suspensions. The cables are thrown over the pylons and are tensioned by anchor mechanisms. The suspensions are considered as inextensible and uniformly distributed along the stiffening girder. The cables are parabolic, and the contour of the girder's cross section is undeformable. The cross section $l-l$ in **Figure 6** illustrates the displacements of the girder's contour during vibratory motions of the suspension system. Reference to this scheme shows that the girder's contour translates as a rigid body vertically (in the y -axis direction) on the value of $\eta(z, t)$ and rotates with respect to the girder's axis (the z -axis) through the angle of $\varphi(z, t)$. The origin of the frame of references is in the center of gravity of the cross section.

It is known for suspension bridges [8] that some natural modes belonging to different types of vibrations could be coupled with each other, that is, the excitation of one natural mode gives rise to another one. Two modes interact more often than not, although the possibility for the interaction of a greater number of modes is not ruled out.

If only two modes predominate in the vibrational process, namely the vertical n -th mode with linear natural frequency ω_{0n} and the torsional m -th mode with the natural frequency Ω_{0m} such that the modes interaction is observed under the conditions (10a) or (10b), then the functions $\eta(z, t)$ and $\varphi(z, t)$ can be approximately defined as

$$\eta(z, t) \sim v_n(z) x_{1n}(t), \quad \varphi(z, t) \sim \Theta_m(z) x_{2m}(t), \quad (39)$$

where x_{1n} and x_{2m} are the generalized displacements, and $v_n(z)$ and $\Theta_m(z)$ are natural shapes of the two interacting modes of vibrations.

The resolving system of equations in a dimensionless form is written as [7, 8]

$$\begin{aligned} \ddot{x}_{1n} + \omega_{0n}^2 x_{1n} + a_{11}^n x_{1n}^2 + a_{22}^{nm} x_{2m}^2 + (b_{11}^n x_{1n}^2 + b_{22}^{nm} x_{2m}^2) x_{1n} &= 0, \\ \ddot{x}_{2m} + \Omega_{0m}^2 x_{2m} + a_{12}^{nm} x_{1n} x_{2m} + (c_{11}^{nm} x_{1n}^2 + c_{22}^m x_{2m}^2) x_{2m} &= 0, \end{aligned} \quad (40)$$

where the coefficients a_{ij} , b_{ij} , and c_{ij} ($i = 1, 2, j = 2$) are defined in [7]. Subsequently, for the ease of presentation, the indices n and m will be omitted.

An approximate solution of Eq. (40) for small but finite amplitudes could be written as an expansion in terms of different time scales in the following form [16]:

$$\begin{aligned} x_1(t) &= \varepsilon x_{11}(T_0, T_1, T_2, \dots) + \varepsilon^2 x_{12}(T_0, T_1, T_2, \dots) + \dots, \\ x_2(t) &= \varepsilon x_{21}(T_0, T_1, T_2, \dots) + \varepsilon^2 x_{22}(T_0, T_1, T_2, \dots) + \dots \end{aligned} \quad (41)$$

The number of the independent time scales needed depends on the order to which the expansion is carried out. Here, $T_0 = t$ is the first scale characterizing motions with the natural frequencies ω_0 and Ω_0 , and T_n are slow scales characterizing the modulations of the amplitudes and phases.

Substituting Eq. (41) into Eq. (40) and equating the coefficients of like powers of ε , we obtain on each step a set of two linear equations. On the first step, it is convenient to seek the solution in the form:

$$\begin{aligned} x_{11} &= A_1(T_1, T_2) \exp(i\omega_0 T_0) + \bar{A}_1(T_1, T_2) \exp(-i\omega_0 T_0), \\ x_{21} &= A_2(T_1, T_2) \exp(i\omega_0 T_0) + \bar{A}_2(T_1, T_2) \exp(-i\omega_0 T_0), \end{aligned} \quad (42)$$

where A_1 and A_2 are unknown complex functions, and \bar{A}_1 and \bar{A}_2 are the complex conjugates of A_1 and A_2 , respectively.

Substituting Eq. (42) into the set of equations obtained on the first step and using the second step to eliminate secular terms, as well as representing the functions A_1 and A_2 in the polar form $A_1 = a_1 \exp(i\varphi_1)$, $A_2 = a_2 \exp(i\varphi_2)$, we are led to the following system of equations for the case of the two-to-one internal resonance (10a):

$$\begin{aligned} \dot{\xi} &= -b(1 - \xi)\sqrt{\xi} \sin \gamma, \\ \dot{\gamma} &= -\frac{1}{2} b(1 - 3\xi)\xi^{-1/2} \cos \gamma, \end{aligned} \quad (43)$$

where $\xi = \xi(T_1)$ is an unknown function, $\gamma = 2\varphi_2 - \varphi_1$, $b = a_1 a_2 \Omega_0^{-1} \sqrt{E_0}$, $E_0 = a_1^2 + a_2 a_{22} \Omega_0 (a_{12} \omega_0)^{-1} a_2^2$ is the system's initial energy, $a_1 = \sqrt{E_0 \xi}$, $a_2 = \sqrt{E_0 a_{12} \omega_0 (a_{22} \Omega_0)^{-1} (1 - \xi)}$, and an overdot denotes differentiation with respect to T_1 .

Representing $\dot{\gamma} = \xi \dot{\gamma} / d\xi$ and considering Eq. (43) yield

$$\frac{d \cos \gamma}{d \xi} + \frac{1}{2} \frac{1 - 3\xi}{\xi(1 - \xi)} \cos \gamma = 0. \quad (44)$$

The solution to Eq. (44) has the form

$$G_1(\xi, \gamma) = \xi^{1/2}(1 - \xi) \cos \gamma = G_1^0, \quad (45)$$

where G_1^0 is an arbitrary constant determined from the initial conditions. Note that relationship (45) is similar to the first integral (26b) for a two-degree-of-freedom system.

In the case of the one-to-one internal resonance (10b), we seek the solution in the form of Eq. (42) also. Using the procedure for the elimination of secular terms, we obtain the following set of equations:

$$\begin{aligned} \dot{\xi} &= \frac{1}{2} \Gamma_2 E_0 \xi (1 - \xi) \sin \gamma, \\ \frac{1}{2} \dot{\gamma} &= \frac{1}{4} \Gamma_2 E_0 (1 - 2\xi) \cos \gamma - (\lambda_1 - \lambda_3) E_0 \xi - \frac{\Gamma_2}{\Gamma_1} (\lambda_2 - \lambda_4) E_0 (1 - \xi), \end{aligned} \quad (46)$$

where $\xi = \xi(T_2)$ is an unknown function, $\gamma = 2(\varphi_2 - \varphi_1)$, $E_0 = a_1^2 + \Gamma_1 \Gamma_2^{-1} a_2^2$ is the system's initial energy, $a_1 = \sqrt{E_0 \xi}$, $a_2 = \sqrt{\Gamma_2 \Gamma_1^{-1} E_0 (1 - \xi)}$, an overdot denotes differentiation with respect to T_2 , and the coefficients λ_i and Γ_j ($i = 1, \dots, 4; j = 1, 2$) dependent upon the system parameters [8].

Representing $\dot{\gamma} = \dot{\xi} d\gamma / d\xi$ and using Eq. (46) yield

$$\frac{d \cos \gamma}{d\xi} + \frac{1 - 2\xi}{\xi(1 - \xi)} \cos \gamma - \frac{4(\lambda_1 - \lambda_3)}{\Gamma_2(1 - \xi)} - \frac{4(\lambda_2 - \lambda_4)}{\Gamma_1 \xi} = 0. \quad (47)$$

The solution to Eq. (47) has the form

$$G_2(\xi, \gamma) = \xi(1 - \xi) \cos \gamma - 2(\lambda_1 - \lambda_3) \Gamma_2^{-1} \xi^2 + 2(\lambda_2 - \lambda_4) \Gamma_1^{-1} (1 - \xi)^2 = G_2^0, \quad (48)$$

where G_2^0 is an arbitrary constant determined from the initial conditions.

Eliminating the variable γ in Eq. (48) and in the second equation of (46) and integrating over T_2 yield

$$\frac{1}{\sqrt{m_1 m_2}} \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(\xi^2 + p_1 \xi + q_1)(\xi^2 + p_2 \xi + q_2)}} = \frac{E_0}{2\Gamma_1} T_2, \quad (49)$$

where ξ_0 is a value determined by the relative level of the initial amplitudes, and the quantities m_i , p_i , and q_i are the coefficients [11]. The integral in Eq. (49) can be transformed to an incomplete elliptic integral of the first kind [17].

4.1. Soliton-like solutions

As examples, the nonlinear free vibrations of the Golden Gate Bridge in San Francisco are considered. All geometrical data, as well as natural frequency spectra and mode shapes for this one of the most beautiful suspension bridges, are available in [19].

It can be shown that under the relationship among the natural frequencies $\omega_6^s = 2\Omega_1^s = 2.66$ rad/s (a two-to-one internal resonance between the sixth symmetrical mode of vertical vibrations and the first symmetrical mode of torsional vibrations), one can obtain the analytical solution in the form of a single kink (32), where B should be replaced by the coefficient b defined by the system's parameters according to Eq. (43). The physical sense of this solution kink is that it is responsible for the one-sided energy exchange when the energy of the torsional vibrations completely transforms into the energy of the vertical vibrations with time, so that the torsional vibrations initiate the vertical vibrations [20].

Under the relationships among the natural frequencies, $\omega_5^s = \Omega_3^s = 2.61$ rad/s and $\omega_3^s = \Omega_1^s = 1.33$ rad/s (a one-to-one internal resonance), the analytical solutions may be found by solving Eq. (49), respectively, as [20]

$$\begin{aligned} \ln \left| 2\xi^{-1} \sqrt{-0.13\xi^2 - 0.195\xi + 0.019} + 0.27\xi^{-1} - 1.4 \right|_{\xi_0}^{\xi} &= 0.431E_0T_2, \\ \ln \left| 2(1-\xi)^{-1} \sqrt{-0.066(1-\xi)^2 + 0.037(1-\xi) + 0.004} + 0.132(1-\xi)^{-1} + 0.536 \right|_{\xi_0}^{\xi} & \\ &= -0.036E_0T_2, \end{aligned} \quad (50)$$

where $\left|_{\xi_0}^{\xi}\right.$ denotes the evaluation at the upper and lower limits of integration.

In the first case of Eq. (50), the coefficients q_1 and q_2 in the integral (49) become zero, and the analytical solution corresponding to the separatrix $G_2 = -0.354$ describes a one-sided energy transfer from the vertical vibration to the torsional vibration (a low aperiodic regime), which leads in time to the conversion of the flexural-torsional vibrations to the predominantly torsional vibrations. This regime is the most unfavorable and dangerous for suspension bridges.

In the second case of Eq. (50), the analytical solution corresponding to the separatrix $G_2 = 0.487$ describes a one-sided energy transfer from the torsional vibration to the vertical vibration (an upper aperiodic regime), so that the flexural-torsional vibrations evolve into the predominantly vertical vibrations with time.

The solutions obtained may be interpreted on the phase plane $\xi - \gamma$ by virtue of streamlines of the phase fluid which is demonstrated in **Figures 5** and **7** for solutions (32) and (50), respectively. Digits near curves indicate the magnitudes of the values G_1 and G_2 corresponding to the streamlines.

The analysis of the phase portraits in terms of the variables ξ and γ for various oscillatory regimes demonstrates that they contain both closed and nonclosed streamlines which are separated by the curves separatrices. Along the separatrices, one succeeds in finding analytical solutions that are inherently soliton-like solutions in the theory of vibrations and describe the complete one-sided energy transfer from one subsystem to another.

Note that soliton-like solutions could be found also in an analytical form for the case of free damped vibrations of a suspension bridge, when damping features of the system are described

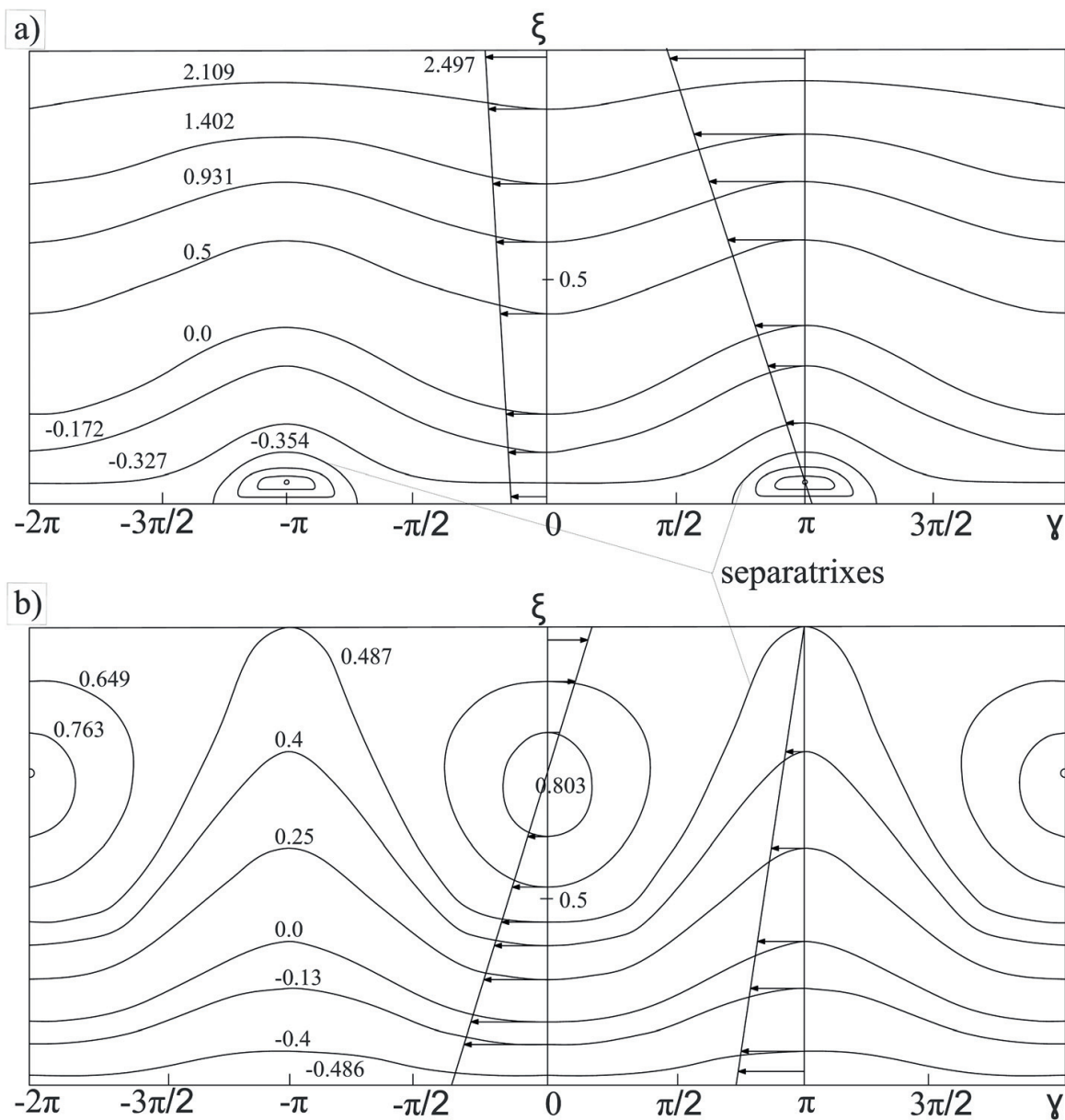


Figure 7. Phase portraits: (a) $\omega_5^s = \Omega_3^s$, and (b) $\omega_3^s = \Omega_1^s$

by ordinary first-order time derivative [21] or defined by a fractional derivative with a fractional parameter (the order of the fractional derivative) changing from zero to one [22].

5. Conclusions

From the review presented, the following conclusions could be deduced. In all considered vibratory systems—1dof, 2dof, and multi-dof—under certain conditions, there exist solutions that describe irreversible processes of energy transfer from its one type to another. Such solutions are called *soliton-like solutions* and could be written in an analytical form.

On the phase plane, these solutions correspond to streamlines which separate closed lines of phase fluid flow from nonclosed ones. These lines are called *separatrixes*.

Since soliton-like solution may describe unfavorable vibratory regimes of real mechanical systems, then they should be investigated systematically by virtue of mathematical models of these systems, in order to avoid, wherever possible, such dangerous vibratory regimes when designing and constructing real structures. A thorough analysis of internal resonances in thin plates and cylindrical shells could be found in [23, 24] and [25, 26], respectively.

Soliton-like solutions in the cases of combinational internal resonances for systems with an infinite number of degrees-of-freedom, when more than two natural modes of vibration are coupled, could be found in sight as well, and such examples for nonlinear plates and cylindrical shells are presented in [27, 28] respectively.

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Author details

Yury A. Rossikhin and Marina V. Shitikova*

*Address all correspondence to: mvs@vgasu.vrn.ru

Voronezh State Technical University, Voronezh, Russian Federation

References

- [1] Mandel'shtam L. Lecture Notes on Theory of Vibrations (in Russian). Moscow: Nauka; 1972
- [2] Nayfeh AH, Balachandran B. Modal interactions in dynamical and structural systems. Applied Mechanics Reviews. 1989;**42**:175-201
- [3] Sado D. Energy transfer in two-degree-of-freedom vibrating systems—A survey. Mechanika Teoretyczna i Stosowana. 1993;**31**:151-173
- [4] Vitt AA, Gorelik GA. Vibrations of an elastic pendulum as an example of vibrations of two parametrically coupled linear systems (in Russian). Journal of Technical Physics. 1933;**2–3**: 294-307
- [5] Sado D. Analysis of vibration of two-degree of freedom system with inertial coupling. Machine Dynamics Problems. 1984;**1**:67-77

- [6] Shitikova MV. Modelling of free nonlinear vibrational processes in suspension bridges by a two-mass system (in Russian). In: *Advanced Methods of Static and Dynamic Analysis of Structures 1*. Voronezh: Voronezh Civil Engineering Institute; 1992. pp. 147-153
- [7] Abdel-Ghaffar AM, Rubin LI. Nonlinear free vibrations of suspension bridges: Theory and application. *ASCE Journal of Engineering Mechanics*. 1983;109:313-345
- [8] Rossikhin YA, Shitikova MV. Nonlinear free spatial vibrations of combined suspension systems. *Applied Mathematics and Mechanics*. 1990;54:825-832
- [9] Rossikhin YA, Shitikova MV. Effect of initial conditions on the behavior of vibrational processes in a combined suspended system. *Mechanics of Solids*. 1991;26:143-154
- [10] Rossikhin YA, Shitikova MV. Effect of viscosity on the vibrational processes in a combined suspension system. *Mechanics of Solids*. 1995;30:157-166
- [11] Rossikhin YA, Shitikova MV. Analysis of nonlinear free vibrations of suspension bridges. *Journal of Sound and Vibration*. 1995;186:369-393
- [12] Goldenblat II. *Dynamic Stability of Structures* (in Russian). Moscow: Stroy Izdat; 1948
- [13] Dodd R, Eilbeck J, Gibbon J, Morris H. *Solitons and Non-linear Wave Equations*. London: Academic Press; 1982. 670 p
- [14] Filippov AT. *Manifold Soliton* (in Russian). "Kvant" Library Series. Moscow: Nauka; 1990. 287 p
- [15] Appell PE. *Theoretical Mechanics* (in Russian). Moscow: Fizmatlit; 1960. 516 p
- [16] Nayfeh AH. *Perturbation Methods*. New York: Wiley; 1973. 450 p
- [17] Abramowitz M, Stegun I, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Tables*. Washington: National Bureau of Standards; 1964. 558 p
- [18] Rossikhin YA, Shitikova MV. Analysis of nonlinear vibrations of a two-degree-of-freedom mechanical systems with damping modeled by a fractional derivative. *Journal of Engineering Mathematics*. 2000;37:343-362
- [19] Abdel-Ghaffar AM, Scanlan RH. Ambient vibration studies of Golden Gate Bridge. I: Suspended structure. *ASCE Journal of Engineering Mechanics*. 1985;111:463-482
- [20] Rossikhin YA, Shitikova MV. Soliton-like solution to the equations of free nonlinear vibrations of suspension bridges. In: *Proceedings of the 1993 International Symposium on Nonlinear Theory and Its Applications*; December 5–10, 1993; Hawaii. pp. 705-708
- [21] Rossikhin YA, Shitikova MV. Soliton-like solutions to the nonlinear damped vibrations of a suspension bridge under an internal resonance. In: *Proceedings of the 2nd European Nonlinear Oscillations Conference*; September 9–13, 1996; Prague. Vol. 2, pp. 203-206
- [22] Rossikhin YA, Shitikova MV. Application of fractional calculus for analysis of nonlinear damped vibrations of suspension bridges. *Journal of Engineering Mechanics*. 1998;124:1029-1036

- [23] Rossikhin YA, Shitikova MV. Analysis of free non-linear vibrations of a viscoelastic plate under the conditions of different internal resonances. *International Journal of Non-Linear Mechanics*. 2006;**41**:313-325
- [24] Rossikhin YA, Shitikova MV, Ngenzi JC. A new approach for studying nonlinear dynamic response of a thin plate with internal resonance in a fractional viscoelastic medium. *Shock and Vibration*. 2015;**2015**:795606
- [25] Rossikhin YA, Shitikova MV. Nonlinear dynamic response of a fractionally damped cylindrical shell with a three-to-one internal resonance. *Applied Mathematics and Computation*. 2015;**257**:498-525
- [26] Rossikhin YA, Shitikova MV. A new approach for studying nonlinear dynamic response of a thin fractionally damped cylindrical shell with internal resonances of the order of ε . In: Altenbach H, Mikhasev GI, editors. *Shell and Membrane Theories in Mechanics and Biology: From Macro- to Nanoscale Structures*. Advanced Structural Materials. Berlin-Heidelberg: Springer; 2015. Chapter 17. pp. 301-321
- [27] Rossikhin YA, Shitikova MV, Ngenzi JC. Fractional calculus application in problems of non-linear vibrations of thin plates with combinational internal resonances. *Procedia Engineering*. 2016;**144**:849-858
- [28] Rossikhin YA, Shitikova MV. Analysis of non-linear vibrations of a fractionally damped cylindrical shell under the conditions of combinational internal resonance. In: Mastorakis N et al., editors. *Computational Problems in Science and Engineering*. Lecture Notes in Electrical Engineering. Vol. 343. Berlin-Heidelberg: Springer; 2015. pp. 59-107

