Chapter 6

Reproducing Kernel Functions

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we obtain some reproducing kernel spaces. We obtain reproducing kernel functions in these spaces. These reproducing kernel functions are very important for solving ordinary and partial differential equations.

Keywords: reproducing kernel functions, reproducing kernel spaces, ordinary and partial differential equations

1. Introduction

Reproducing kernel spaces are special Hilbert spaces. These spaces satisfy the reproducing property. There is an important relation between the order of the problems and the reproducing kernel spaces.

2. Reproducing kernel spaces

In this section, we define some useful reproducing kernel functions [1–23].

Definition 2.1 (reproducing kernel). Let E be a nonempty set. A function $K : E \times E \to \mathbb{C}$ is called a reproducing kernel of the Hilbert space H if and only if

a. $K(\cdot, t) \in H$ for all $t \in E$,

b. $\langle \varphi, K(\cdot, t) \rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called the reproducing property as the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.

Then, we need some notation that we use in the development of this chapter. Next, we define several spaces with inner product over those spaces. Thus, the space defined as

$$W_2^3[0,1] = \left\{ v | v, v', v'' : [0,1] \to \mathbb{R} \text{ are absolutely continuous}, v^{(3)} \in L^2[0,1] \right\} \tag{1}$$

is a Hilbert space. The inner product and the norm in $W_2^3[0,1]$ are defined by

$$\begin{gathered} \langle v, g \rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v^{(3)}(x) g^{(3)}(x) \mathrm{d}x, \quad v, g \in W_2^3[0,1], \\ \|v\|_{W_2^3} = \sqrt{\langle v, v \rangle_{W_2^3}}, \quad v \in W_2^3[0,1], \end{gathered} \tag{2}$$

respectively. Thus, the space $W_2^3[0,1]$ is a reproducing kernel space, that is, for each fixed $y \in [0,1]$ and any $v \in W_2^3[0,1]$, there exists a function R_y such that

$$v(y) = \langle v(x), R_y(x) \rangle_{W_2^3}, \tag{3}$$

and similarly, we define the space

$$T_2^3[0,1] = \left\{ \begin{array}{c} v | v, v', v'' : [0,1] \to \mathbb{R} \text{ are absolutely continuous}, \\ \\ v'' \in L^2[0,1], v(0) = 0, v'(0) = 0 \end{array} \right\} \tag{4}$$

The inner product and the norm in $T_2^3[0,1]$ are defined by

$$\begin{gathered} \langle v, g \rangle_{T_2^3} = \sum_{i=0}^{2} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v'''(t) g'''(t) \mathrm{d}t, \quad v, g \in T_2^3[0,1], \\ \|v\|_{T_2^3} = \sqrt{\langle v, v \rangle_{T_2^3}}, \quad v \in T_2^3[0,1], \end{gathered} \tag{5}$$

respectively. The space $T_2^3[0,1]$ is a reproducing kernel Hilbert space, and its reproducing kernel function r_s is given by [1] as

$$r_s = \begin{cases} \dfrac{1}{4} s^2 t^2 + \dfrac{1}{12} s^2 t^3 - \dfrac{1}{24} s t^4 + \dfrac{1}{120} t^5, & t \le s, \\ \dfrac{1}{4} s^2 t^2 + \dfrac{1}{12} s^3 t^2 - \dfrac{1}{24} t s^4 + \dfrac{1}{120} s^5, & t > s, \end{cases} \tag{6}$$

and the space

$$G_2^1[0,1] = \{ v | v : [0,1] \to \mathbb{R} \text{ is absolutely continuous}, v'(x) \in L^2[0,1] \}, \tag{7}$$

is a Hilbert space, where the inner product and the norm in $G_2^1[0,1]$ are defined by

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$$\langle v, g \rangle_{G_2^1} = v^{(i)}(0) g^{(i)}(0) + \int_0^1 v'(x) g'(x) \mathrm{d}x, \quad v, g \in G_2^1[0,1],$$
$$\|v\|_{G_2^1} = \sqrt{\langle v, v \rangle_{G_2^1}}, \quad v \in G_2^1[0,1], \tag{8}$$

respectively. The space $G_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function Q_y is given by [1] as

$$Q_y = \begin{cases} 1+x, & x \leqslant y \\ 1+y, & x > y. \end{cases} \tag{9}$$

Theorem 1.1. *The space $W_2^3[0,1]$ is a complete reproducing kernel space whose reproducing kernel R_y is given by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y) x^{i-1}, & x \le y, \\ \sum_{i=1}^{6} d_i(y) x^{i-1}, & x > y, \end{cases} \tag{10}$$

where

$$c_1(y) = 1, \quad c_2(y) = y, \quad c_3(y) = \frac{y^2}{4}, \quad c_4(y) = \frac{y^2}{12}, \quad c_5(y) = -\frac{1}{24y}, \quad c_6(y) = \frac{1}{120},$$
$$d_1(y) = 1 + \frac{y^5}{120}, \quad d_2(y) = \frac{-y^4}{24} + y, \quad d_3(y) = \frac{y^2}{4} + \frac{y^3}{12}, \quad d_4(y) = d_5(y) = d_6(y) = 0.$$

Proof. Since

$$\langle v, R_y \rangle_{W_2^3} = \sum_{i=0}^{2} v^{(i)}(0) R_y^{(i)}(0) + \int_0^1 v^{(3)}(x) R_y^{(3)}(x) \mathrm{d}x, \quad \left(v, R_y \in W_2^3[0,1]\right. \tag{11}$$

through iterative integrations by parts for (11), we have

$$\langle v(x), R_y(x) \rangle_{W_2^4} = \sum_{i=0}^{2} v^{(i)}(0) \left[R_y^{(i)}(0) - (-1)^{(2-i)} R_y^{(5-i)}(0) \right]$$
$$+ \sum_{i=0}^{2} (-1)^{(2-i)} v^{(i)}(1) R_y^{(5-i)}(1) + \int_0^1 v(x) R_y^{(6)}(x) \mathrm{d}x. \tag{12}$$

Note, the property of the reproducing kernel as

$$\langle v(x), R_y(x) \rangle_{W_2^3} = v(y). \tag{13}$$

If

$$\begin{aligned} R_y(0) - R_y^{(5)}(0) &= 0, \\ R_y'(0) + R_y^{(4)}(0) &= 0, \\ R_y''(0) - R_y'''(0) &= 0, \\ R_y^{(3)}(1) &= 0, \\ R_y^{(4)}(1) &= 0, \\ R_y^{(5)}(1) &= 0, \end{aligned} \tag{14}$$

Then by (11), we obtain

$$R_y^{(6)}(x) = \delta(x - y), \tag{15}$$

when $x \neq y$,

$$R_y^{(6)}(x) = 0, \tag{16}$$

therefore,

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y) x^{i-1}, & x \le y, \\ \sum_{i=1}^{6} d_i(y) x^{i-1}, & x > y, \end{cases} \tag{17}$$

Since

$$R_y^{(6)}(x) = \delta(x - y), \tag{18}$$

we have

$$\begin{aligned} \partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0, 1, 2, 3, 4, \\ \partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1. \end{aligned} \tag{19}$$

From (14) and (19), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1, 2, \ldots, 6)$ can be obtained. Thus, R_y is given by

$$R_y = \begin{cases} 1 + yx + \frac{1}{4} y^2 x^2 + \frac{1}{12} y^2 x^3 - \frac{1}{24} y x^4 + \frac{1}{120} x^5, & x \le y \\ 1 + yx + \frac{1}{4} y^2 x^2 + \frac{1}{12} y^3 x^2 - \frac{1}{24} x y^4 + \frac{1}{120} y^5, & x > y. \end{cases} \tag{20}$$

Now, we note that the space given in [1] as

$$W(\Omega)=\left\{\begin{array}{c} v(x,t)|\dfrac{\partial^4 v}{\partial x^2\partial t^2}, \text{is completely continuous in }\ \Omega=[0,1]\times[0,1], \\ \\ \dfrac{\partial^6 v}{\partial x^3\partial t^3}\in L^2(\Omega), v(x,0)=0, \dfrac{\partial v(x,0)}{\partial t}=0 \end{array}\right\} \tag{21}$$

is a binary reproducing kernel Hilbert space. The inner product and the norm in $W(\Omega)$ are defined by

$$\begin{aligned} \langle v(x,t), g(x,t)\rangle_W &= \sum_{i=0}^{2}\int_0^1\left[\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}v(0,t)\frac{\partial^3}{\partial t^3}\frac{\partial^i}{\partial x^i}g(0,t)\right]\mathrm{d}t \\ &+\sum_{j=0}^{2}\left\langle\frac{\partial^j}{\partial t^j}v(x,0),\frac{\partial^j}{\partial t^j}g(x,0)\right\rangle_{W_2^3} \\ &+\int_0^1\int_0^1\left[\frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}v(x,t)\frac{\partial^3}{\partial x^3}\frac{\partial^3}{\partial t^3}g(x,t)\right]\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{22}$$

$$\|v\|_w=\sqrt{\langle v,v\rangle_W},\quad v\in W(\Omega),$$

respectively.

Theorem 1.2. *The $W(\Omega)$ is a reproducing kernel space, and its reproducing kernel function is*

$$K_{(y,s)}=R_y r_s \tag{23}$$

such that for any $v\in W(\Omega)$,

$$\begin{gathered} v(y,s)=\left\langle v(x,t),K_{(y,s)}(x,t)\right\rangle_{W'} \\ K_{(y,s)}(x,t)=K_{(x,t)}(y,s). \end{gathered} \tag{24}$$

Similarly, the space

$$\widehat{W}(\Omega)=\left\{v(x,t)|v(x,t)\ \text{ is completely continuous in }\ \Omega=[0,1]\times[0,1],\frac{\partial^2 v}{\partial x\partial t}\in L^2(\Omega)\right\} \tag{25}$$

is a binary reproducing kernel Hilbert space. The inner product and the norm in $\widehat{W}(\Omega)$ are defined by [1] *as*

$$\langle v(x,t), g(x,t)\rangle_{\widehat{W}} = \int_0^1 \left[\frac{\partial}{\partial t} v(0,t) \frac{\partial}{\partial t} g(0,t)\right] \mathrm{d}t + \langle v(x,0), g(x,0)\rangle_{W_2^1}$$

$$+ \int_0^1 \int_0^1 \left[\frac{\partial}{\partial x}\frac{\partial}{\partial t} v(x,t) \frac{\partial}{\partial x}\frac{\partial}{\partial t} g(x,t)\right] \mathrm{d}x\mathrm{d}t, \tag{26}$$

$$\|v\|_{\widehat{W}} = \sqrt{\langle v, v\rangle_{\widehat{W}}}, \quad v \in \widehat{W}(\Omega),$$

respectively. $\widehat{W}(\Omega)$ is a reproducing kernel space, and its reproducing kernel function $G_{(y,s)}$ is

$$G_{(y,s)} = Q_y Q_s. \tag{27}$$

Definition 1.3.

$$W_2^3[0,1] = \left\{ \begin{array}{c} u(x) | u(x), u'(x), u''(x), \text{are absolutely continuous in } [0,1] \\ u^{(3)}(x) \in L^2[0,1], x \in [0,1], u(0) = 0, u(1) = 0. \end{array} \right\}$$

The inner product and the norm in $W_2^3[0,1]$ are defined, respectively, by

$$\langle u(x), g(x)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) g^{(i)}(0) + \int_0^1 u^{(3)}(x) g^{(3)}(x) \mathrm{d}x, u(x), g(x) \in W_2^3[0,1]$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

The space $W_2^3[0,1]$ is a reproducing kernel space, that is, for each fixed $y \in [0,1]$ and any $u(x) \in W_2^3[0,1]$, there exists a function $R_y(x)$ such that

$$u(y) = \langle u(x), R_y(x)\rangle_{W_2^3}.$$

Definition 1.4.

$$W_2^1[0,1] = \left\{ \begin{array}{c} u(x) | u(x), \text{is absolutely continuous in } [0,1] \\ u'(x) \in L^2[0,1], x \in [0,1], \end{array} \right\}$$

The inner product and the norm in $W_2^1[0,1]$ are defined, respectively, by

$$\langle u(x), g(x)\rangle_{W_2^1} = u(0)g(0) + \int_0^1 u'(x)g'(x)\mathrm{d}x, u(x), g(x) \in W_2^1[0,1], \tag{28}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{29}$$

The space $W_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function $T_x(y)$ is given by

$$T_x(y) = \begin{cases} 1+x, & x \le y, \\ 1+y, & x > y. \end{cases} \tag{30}$$

Theorem 1.5. *The space $W_2^3[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $R_y(x)$ can be denoted by*

$$R_y(x) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(y)x^{i-1}, & x \le y, \\ \displaystyle\sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y, \end{cases}$$

where

$$\begin{aligned}
c_1(y) &= 0, \\
c_2(y) &= \frac{5}{516}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y, \\
c_3(y) &= \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 - \frac{5}{312}y^3 - \frac{5}{26}y, \\
c_4(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 + \frac{7}{104}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y, \\
c_5(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 - \frac{1}{104}y, \\
c_6(y) &= \frac{1}{120} + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3 - \frac{1}{156}y, \\
d_1(y) &= \frac{1}{120}y^5, \\
d_2(y) &= -\frac{1}{104}y^4 - \frac{1}{156}y^5 - \frac{5}{26}y^2 - \frac{5}{78}y^3 + \frac{3}{13}y, \\
d_3(y) &= \frac{5}{624}y^4 - \frac{1}{624}y^5 + \frac{21}{104}y^2 + \frac{7}{104}y^3 - \frac{5}{26}y, \\
d_4(y) &= \frac{5}{1872}y^4 - \frac{1}{1872}y^5 - \frac{5}{312}y^2 - \frac{5}{936}y^3 - \frac{5}{78}y, \\
d_5(y) &= -\frac{5}{3744}y^4 + \frac{1}{3744}y^5 + \frac{5}{624}y^2 + \frac{5}{1872}y^3 + \frac{5}{156}y, \\
d_6(y) &= -\frac{1}{156}y + \frac{1}{3744}y^4 - \frac{1}{18720}y^5 - \frac{1}{624}y^2 - \frac{1}{1872}y^3.
\end{aligned}$$

Proof. We have

$$\langle u(x), R_y(x) \rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0) R_y^{(i)}(0) + \int_0^1 u^{(3)}(x) R_y^{(3)}(x) \mathrm{d}x. \tag{31}$$

Through several integrations by parts for (31), we have

$$\begin{aligned} \langle u(x), R_y(x) \rangle_{W_2^6} = & \sum_{i=0}^{2} u^{(i)}(0) \left[R_y^{(i)}(0) - (-1)^{(2-i)} R_y^{(5-i)}(0) \right] \\ & + \sum_{i=0}^{2} (-1)^{(2-i)} u^{(i)}(1) R_y^{(5-i)}(1) \\ & - \int_0^1 u(x) R_y^{(6)}(x) \mathrm{d}x. \end{aligned} \tag{32}$$

Note that property of the reproducing kernel

$$\langle u(x), R_y(x) \rangle_{W_2^3} = u(y),$$

If

$$\begin{cases} R_y''(0) - R_y^{(3)}(0) = 0, \\ R_y'(0) + R_y^{(4)}(0) = 0, \\ R_y^{(3)}(1) = 0, \\ R_y^{(4)}(1) = 0, \end{cases} \tag{33}$$

then by (31), we have the following equation:

$$-R_y^{(6)}(x) = \delta(x-y),$$

$$\text{when } x \neq y,$$

$$R_y^{(6)}(x) = 0,$$

therefore,

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^{6} d_i(y) x^{i-1}, & x > y, \end{cases}$$

Since

$$-R_y^{(6)}(x) = \delta(x-y),$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0, 1, 2, 3, 4, \tag{34}$$

and

$$\partial^5 R_{y^+}(y) - \partial^5 R_{y^-}(y) = -1. \tag{35}$$

Since $R_y(x) \in W_2^3[0,1]$, it follows that

$$R_y(0) = 0, R_y(1) = 0, \tag{36}$$

From (33)–(36), the unknown coefficients $c_i(y)$ and $d_i(y)$ $(i = 1, 2, \ldots, 6)$ can be obtained. Thus $R_y(x)$ is given by

$$R_y(x) = \begin{cases} \frac{5}{516}xy^4 - \frac{1}{156}xy^5 - \frac{5}{26}xy^2 - \frac{5}{78}xy^3 + \frac{3}{13}xy + \frac{5}{624}x^2y^4 - \frac{1}{624}x^2y^5 + \frac{21}{104}x^2y^2 \\ -\frac{5}{312}x^2y^3 - \frac{5}{26}x^2y + \frac{5}{1872}x^3y^4 - \frac{1}{1872}x^3y^5 + \frac{7}{104}x^3y^2 - \frac{5}{936}x^3y^3 - \frac{5}{78}x^3y \\ -\frac{5}{3744}x^4y^4 + \frac{1}{3744}x^4y^5 + \frac{5}{624}x^4y^2 + \frac{5}{1872}x^4y^3 - \frac{1}{104}x^4y - \frac{1}{156}x^5y + \frac{1}{3744}x^5y^4 \\ -\frac{1}{18720}x^5y^5 - \frac{1}{624}x^5y^2 - \frac{1}{1872}x^5y^3, \quad x \le y \\ \frac{5}{516}yx^4 - \frac{1}{156}yx^5 - \frac{5}{26}yx^2 - \frac{5}{78}yx^3 + \frac{3}{13}xy + \frac{5}{624}y^2x^4 - \frac{1}{624}y^2x^5 + \frac{21}{104}x^2y^2 \\ -\frac{5}{312}y^2x^3 - \frac{5}{26}y^2x + \frac{5}{1872}y^3x^4 - \frac{1}{1872}y^3x^5 + \frac{7}{104}y^3x^2 - \frac{5}{936}x^3y^3 - \frac{5}{78}y^3x \\ -\frac{5}{3744}x^4y^4 + \frac{1}{3744}y^4x^5 + \frac{5}{624}y^4x^2 + \frac{5}{1872}y^4x^3 - \frac{1}{104}y^4x - \frac{1}{156}y^5x + \frac{1}{3744}y^5x^4 \\ -\frac{1}{18720}x^5y^5 - \frac{1}{624}y^5x^2 - \frac{1}{1872}y^5x^3, \quad x > y \end{cases}$$

$$W_2^4[0,1] = \left\{ \begin{array}{c} v(x) | v(x), v'(x), v''(x), v'''(x) \\ \text{are absolutely continuous in } [0,1], \\ v^{(4)}(x) \in L^2[0,1], x \in [0,1] \end{array} \right\} \tag{37}$$

The inner product and the norm in $W_2^4[0,1]$ are defined, respectively, by

$$\begin{gathered}\langle v(x), g(x)\rangle_{W_2^4} = \sum_{i=0}^{3} v^{(i)}(0) g^{(i)}(0) + \int_0^1 v^{(4)}(x) g^{(4)}(x) \mathrm{d}x, \quad v(x), g(x) \in W_2^4[0,1], \\ \|v\|_{W_2^4} = \sqrt{\langle v, v\rangle_{W_2^4}}, \quad v \in W_2^4[0,1].\end{gathered} \tag{38}$$

The space $W_2^4[0,1]$ is a reproducing kernel space, that is, for each fixed.

$y \in [0,1]$ and any $v(x) \in W_2^4[0,1]$, there exists a function $R_y(x)$ such that

$$v(y) = \langle v(x), R_y(x)\rangle_{W_2^4} \tag{39}$$

Similarly, we define the space

$$W_2^2[0,T] = \left\{\begin{array}{c} v(t) | v(t), v'(t) \\ \text{are absolutely continuous in } [0,T], \\ v''(t) \in L^2[0,T], t \in [0,T], v(0) = 0 \end{array}\right\} \tag{40}$$

The inner product and the norm in $W_2^2[0,T]$ are defined, respectively, by

$$\begin{gathered}\langle v(t), g(t)\rangle_{W_2^2} = \sum_{i=0}^{1} v^{(i)}(0) g^{(i)}(0) + \int_0^T v''(t) g''(t) \mathrm{d}t, \quad v(t), g(t) \in W_2^2[0,T], \\ \|v\|_{W_1} = \sqrt{\langle v, v\rangle_{W_2^2}}, \quad v \in W_2^2[0,T].\end{gathered} \tag{41}$$

Thus, the space $W_2^2[0,T]$ is also a reproducing kernel space, and its reproducing kernel function $r_s(t)$ can be given by

$$r_s(t) = \begin{cases} st + \dfrac{s}{2} t^2 - \dfrac{1}{6} t^3, & t \leq s, \\ st + \dfrac{t}{2} s^2 - \dfrac{1}{6} s^3, & t > s, \end{cases} \tag{42}$$

and the space

$$W_2^2[0,1] = \left\{\begin{array}{c} v(x) | v(x), v'(x) \\ \text{are absolutely continuous in } [0,1], \\ v''(x) \in L^2[0,1], x \in [0,1] \end{array}\right\} \tag{43}$$

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where the inner product and the norm in $W_2^2[0,1]$ are defined, respectively, by

$$\langle v(t), g(t)\rangle_{W_2^2} = \sum_{i=0}^{1} v^{(i)}(0)g^{(i)}(0) + \int_0^T v''(t)g''(t)\mathrm{d}t, \quad v(t), g(t) \in W_2^2[0,1],$$
$$\|v\|_{W_2} = \sqrt{\langle v, v\rangle_{W_2^2}}, \quad v \in W_2^2[0,1]. \tag{44}$$

The space $W_2^2[0,1]$ is a reproducing kernel space, and its reproducing kernel function $Q_y(x)$ is given by

$$Q_y(x) = \begin{cases} 1 + xy + \frac{y}{2}x^2 - \frac{1}{6}x^3, & x \le y, \\ 1 + xy + \frac{x}{2}y^2 - \frac{1}{6}y^3, & x > y. \end{cases} \tag{45}$$

Similarly, the space $W_2^1[0,T]$ is defined by

$$W_2^1[0,T] = \left\{ \begin{matrix} v(t) | v(t) \text{ is absolutely continuous in } [0,T], \\ \\ v(t) \in L^2[0,T], t \in [0,T] \end{matrix} \right\} \tag{46}$$

The inner product and the norm in $W_2^1[0,T]$ are defined, respectively, by

$$\langle v(t), g(t)\rangle_{W_2^1} = v(0)g(0) + \int_0^T v'(t)g'(t)\mathrm{d}t, \quad v(t), g(t) \in W_2^1[0,T],$$
$$\|v\|_{W_2^1} = \sqrt{\langle v, v\rangle_{W_2^1}}, \quad v \in W_2^1[0,T]. \tag{47}$$

The space $W_2^1[0,T]$ is a reproducing kernel space, and its reproducing kernel function $q_s(t)$ is given by

$$q_s(t) = \begin{cases} 1 + t, & t \le s, \\ 1 + s, & t > s. \end{cases} \tag{48}$$

Further, we define the space $W(\Omega)$ as

$$W(\Omega) = \left\{ \begin{matrix} v(x,t) | \frac{\partial^4 v}{\partial x^3 \partial t}, \text{ is completely continuous,} \\ \\ in\Omega = [0,1] \times [0,T], \\ \\ \frac{\partial^6 v}{\partial x^4 \partial t^2} \in L^2(\Omega), v(x,0) = 0 \end{matrix} \right\} \tag{49}$$

and the inner product and the norm in $W(\Omega)$ are defined, respectively, by

$$
\begin{aligned}
\langle v(x,t), g(x,t)\rangle_W &= \sum_{i=0}^{3}\int_0^T\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}v(0,t)\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial x^i}g(0,t)\right]\mathrm{d}t \\
&+\sum_{j=0}^{1}\left\langle\frac{\partial^j}{\partial t^j}v(x,0),\frac{\partial^j}{\partial t^j}g(x,0)\right\rangle_{W_2^4} \\
&+\int_0^T\int_0^1\left[\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2}v(x,t)\frac{\partial^4}{\partial x^4}\frac{\partial^2}{\partial t^2}g(x,t)\right]\mathrm{d}x\mathrm{d}t, \\
\|v\|_W &= \sqrt{\langle v,v\rangle_W},\quad v\in W(\Omega).
\end{aligned}
\tag{50}
$$

Now, we have the following theorem:

Theorem 1.6. *The space $W_2^4[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $R_y(x)$ can be denoted by*

$$
R_y(x)=\begin{cases}\displaystyle\sum_{i=1}^{8}c_i(y)x^{i-1}, & x\le y,\\ \displaystyle\sum_{i=1}^{8}d_i(y)x^{i-1}, & x>y,\end{cases}
\tag{51}
$$

where

$$
\begin{aligned}
&c_1(y)=1, \quad c_2(y)=y, \quad c_3(y)=\frac{1}{4}y^2,\\
&c_4(y)=\frac{1}{36}y^3, \quad c_5(y)=\frac{1}{144}y^3, \quad c_6(y)=-\frac{1}{240}y^2,\\
&c_7(y)=\frac{1}{720}y, \quad c_8(y)=-\frac{1}{5040},\\
&d_1(y)=1-\frac{1}{5040}y^7, \quad d_2(y)=y+\frac{1}{720}y^6,\\
&d_3(y)=\frac{1}{4}y^2-\frac{1}{240}y^5, \quad d_4(y)=\frac{1}{36}y^3+\frac{1}{144}y^4,\\
&d_5(y)=0, \quad d_6(y)=0, \quad d_7(y)=0, \quad d_8(y)=0.
\end{aligned}
\tag{52}
$$

Proof. Since

$$
\begin{aligned}
\langle v(x),R_y(x)\rangle_{W_2^4}&=\sum_{i=0}^{3}v^{(i)}(0)R_y^{(i)}(0)+\int_0^1 v^{(4)}(x)R_y^{(4)}(x)\mathrm{d}x,\\
&\qquad\left(v(x),R_y(x)\in W_2^4[0,1]\right)
\end{aligned}
\tag{53}
$$

through iterative integrations by parts for (53), we have

$$\begin{aligned}\left\langle v(x), R_y(x)\right\rangle_{W_2^4} = & \sum_{i=0}^{3} v^{(i)}(0)\left[R_y^{(i)}(0) - (-1)^{(3-i)} R_y^{(7-i)}(0)\right] \\ & + \sum_{i=0}^{3} (-1)^{(3-i)} v^{(i)}(1) R_y^{(7-i)}(1) \\ & + \int_0^1 v(x) R_y^{(8)}(x)\mathrm{d}x.\end{aligned} \tag{54}$$

Note that property of the reproducing kernel

$$\left\langle v(x), R_y(x)\right\rangle_{W_2^4} = v(y). \tag{55}$$

If

$$\begin{aligned} R_y(0) + R_y^{(7)}(0) &= 0, \\ R_y'(0) - R_y^{(6)}(0) &= 0, \\ R_y''(0) + R_y^{(5)}(0) &= 0, \\ R_y'''(0) - R_y^{(4)}(0) &= 0, \\ R_y^{(4)}(1) &= 0, \\ R_y^{(5)}(1) &= 0, \\ R_y^{(6)}(1) &= 0, \\ R_y^{(7)}(1) &= 0, \end{aligned} \tag{56}$$

then by (54), we obtain the following equation:

$$R_y^{(8)}(x) = \delta(x - y), \tag{57}$$

when $x \neq y$,

$$R_y^{(8)}(x) = 0; \tag{58}$$

therefore,

$$R_y(x) = \begin{cases} \sum_{i=1}^{8} c_i(y) x^{i-1}, & x \leq y, \\ \sum_{i=1}^{8} d_i(y) x^{i-1}, & x > y. \end{cases} \tag{59}$$

Since

$$R_y^{(8)}(x) = \delta(x-y), \tag{60}$$

we have

$$\partial^k R_{y^+}(y) = \partial^k R_{y^-}(y), \quad k = 0,1,2,3,4,5,6, \tag{61}$$

$$\partial^7 R_{y^+}(y) - \partial^7 R_{y^-}(y) = 1. \tag{62}$$

From (56)–(62), the unknown coefficients $c_i(y)$ ve $d_i(y)(i = 1, 2, \ldots, 8)$ can be obtained. Thus, $R_y(x)$ is given by

$$R_y(x) = \begin{cases} 1 + yx + \frac{1}{4}y^2x^2 + \frac{1}{36}y^3x^3 + \frac{1}{144}y^3x^4 \\ -\frac{1}{240}y^2x^5 + \frac{1}{720}yx^6 - \frac{1}{5040}x^7, \quad x \le y, \\ 1 + xy + \frac{1}{4}x^2y^2 + \frac{1}{36}x^3y^3 + \frac{1}{144}x^3y^4 \\ -\frac{1}{240}x^2y^5 + \frac{1}{720}xy^6 - \frac{1}{5040}y^7, \quad x > y. \end{cases} \tag{63}$$

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