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# Differential Equations Arising from the 3-Variable Hermite Polynomials and Computation of Their Zeros

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## Abstract

In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. Finally, we investigate the zeros of the 3-variable Hermite polynomials by using computer.

**Keywords:** differential equations, heat equation, Hermite polynomials, the 3-variable Hermite polynomials, generating functions, complex zeros

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## 1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers see [1–15]. The special polynomials of two variables provided new means of analysis for the solution of a wide class of differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

In [1], the Hermite polynomials are given by the exponential generating function

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}.$$

We can also have the generating function by using Cauchy's integral formula to write the Hermite polynomials as

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$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} = \frac{n!}{2\pi i} \oint_C \frac{e^{2tx-t^2}}{t^{n+1}} dt$$

with the contour encircling the origin. It follows that the Hermite polynomials also satisfy the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

Further, the two variables Hermite Kampé de Fériet polynomials  $H_n(x, y)$  defined by the generating function (see [3])

$$\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = e^{xt+yt^2} \quad (1)$$

are the solution of heat equation

$$\frac{\partial}{\partial y} H_n(x, y) = \frac{\partial^2}{\partial x^2} H_n(x, y), \quad H_n(x, 0) = x^n.$$

We note that

$$H_n(2x, -1) = H_n(x).$$

The 3-variable Hermite polynomials  $H_n(x, y, z)$  are introduced [4].

$$H_n(x, y, z) = n! \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{z^k H_{n-3k}(x, y)}{k!(n-3k)!}.$$

The differential equation and the generating function for  $H_n(x, y, z)$  are given by

$$\left( 3z \frac{\partial^3}{\partial x^3} + 2y \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} - n \right) H_n(x, y, z) = 0$$

and

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \quad (2)$$

respectively.

By (2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x_1 + x_2, y, z) \frac{t^n}{n!} &= e^{(x_1+x_2)t+yt^2+zt^3} \\ &= \sum_{n=0}^{\infty} x_2^n \frac{t^n}{n!} \sum_{n=0}^{\infty} H_n(x_1, y, z) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} H_l(x_1, y, z) x_2^{n-l} \right) \frac{t^n}{n!}. \end{aligned} \quad (3)$$

By comparing the coefficients on both sides of (3), we have the following theorem.

**Theorem 1.** For any positive integer  $n$ , we have

$$H_n(x_1 + x_2, y, z) = \sum_{l=0}^n \binom{n}{l} H_l(x_1, y, z) x_2^{n-l}.$$

Applying Eq. (2), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} H_n(x, y, z_1 + z_2) \frac{t^n}{n!} &= e^{xt+yt^2+(z_1+z_2)t^3} \\ &= \sum_{k=0}^{\infty} z_2^k \frac{t^{3k}}{k!} \sum_{l=0}^{\infty} H_l(x, y, z_1) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{H_{n-3k}(x, y, z_1) z_2^k n!}{k!(n-3k)!} \right) \frac{t^n}{n!}. \end{aligned}$$

On equating the coefficients of the like power of  $t$  in the above, we obtain the following theorem.

**Theorem 2.** For any positive integer  $n$ , we have

$$H_n(x, y, z_1 + z_2) = n! \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{H_{n-3k}(x, y, z_1) z_2^k}{k!(n-3k)!}.$$

Also, the 3-variable Hermite polynomials  $H_n(x, y, z)$  satisfy the following relations

$$\frac{\partial}{\partial y} H_n(x, y, z) = \frac{\partial^2}{\partial x^2} H_n(x, y, z),$$

and

$$\frac{\partial}{\partial z} H_n(x, y, z) = \frac{\partial^3}{\partial x^3} H_n(x, y, z).$$

The following elementary properties of the 3-variable Hermite polynomials  $H_n(x, y, z)$  are readily derived from (2). We, therefore, choose to omit the details involved.

**Theorem 3.** For any positive integer  $n$ , we have

- 1  $H_n(2x, -1, 0) = H_n(x).$
- 2  $H_n(x, y_1 + y_2, z) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{n-2k}(x, y_1, z) y_2^k}{k!(n-2k)!}.$
- 3  $H_n(x, y, z) = \sum_{l=0}^n \binom{n}{l} H_l(x) H_{n-l}(-x, y + 1, z).$

**Theorem 4.** For any positive integer  $n$ , we have

$$1 \quad H_n(x_1 + x_2, y_1 + y_2, z) = \sum_{l=0}^n \binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2, z).$$

$$2 \quad H_n(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{l=0}^n \binom{n}{l} H_l(x_1, y_1, z) H_{n-l}(x_2, y_2, z_2).$$

The 3-variable Hermite polynomials can be determined explicitly. A few of them are

$$H_0(x, y, z) = 1,$$

$$H_1(x, y, z) = x,$$

$$H_2(x, y, z) = x^2 + 2y,$$

$$H_3(x, y, z) = x^3 + 6xy + 6z,$$

$$H_4(x, y, z) = x^4 + 12x^2y + 12y^2 + 24xz,$$

$$H_5(x, y, z) = x^5 + 20x^3y + 60xy^2 + 60x^2z + 120yz,$$

$$H_6(x, y, z) = x^6 + 30x^4y + 180x^2y^2 + 120y^3 + 120x^3z + 720xyz + 360z^2,$$

$$H_7(x, y, z) = x^7 + 42x^5y + 420x^3y^2 + 840xy^3 + 210x^4z + 2520x^2yz + 2520y^2z + 2520xz^2,$$

$$H_8(x, y, z) = x^8 + 56x^6y + 840x^4y^2 + 3360x^2y^3 + 1680y^4 + 336x^5z + 6720x^3yz + 20160xy^2z + 10080x^2z^2 + 20160yz^2.$$

$$H_9(x, y, z) = x^9 + 72x^7y + 1512x^5y^2 + 10080x^3y^3 + 15120xy^4 + 504x^6z + 15120x^4yz + 90720x^2y^2z + 60480y^3z + 30240x^3z^2 + 181440xyz^2 + 60480z^3,$$

$$H_{10}(x, y, z) = x^{10} + 90x^8y + 2520x^6y^2 + 25200x^4y^3 + 75600x^2y^4 + 30240y^5 + 720x^7z + 30240x^5yz + 302400x^3y^2z + 604800xy^3z + 75600x^4z^2 + 907200x^2yz^2 + 907200y^2z^2 + 604800xz^3.$$

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [7, 8, 12, 16–19]). In this paper, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials. We give explicit identities for the 3-variable Hermite polynomials. In addition, we investigate the zeros of the 3-variable Hermite polynomials using numerical methods. Using computer, a realistic study for the zeros of the 3-variable Hermite polynomials is very interesting. Finally, we observe an interesting phenomenon of ‘scattering’ of the zeros of the 3-variable Hermite polynomials.

## 2. Differential equations associated with the 3-variable Hermite polynomials

In this section, we study differential equations arising from the generating functions of the 3-variable Hermite polynomials.

Let

$$F = F(t, x, y, z) = e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}, \quad x, y, z, t \in \mathbb{C}. \quad (4)$$

Then, by (4), we have

$$F^{(1)} = \frac{\partial}{\partial t} F(t, x, y, z) = \frac{\partial}{\partial t} (e^{xt+yt^2+zt^3}) = e^{xt+yt^2+zt^3} (x + 2yt + 3zt^2) = (x + 2yt + 3zt^2)F(t, x, y, z), \quad (5)$$

$$F^{(2)} = \frac{\partial}{\partial t} F^{(1)}(t, x, y, z) = (2y + 6zt)F(t, x, y, z) + (x + 2yt + 3zt^2)F^{(1)}(t, x, y, z) = ((x^2 + 2y) + (6z + 4xy)t + (4y^2 + 6xz)t^2 + (12yz)t^3 + (9z^2)t^4)F(t, x, y, z). \quad (6)$$

Continuing this process, we can guess that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t, x, y, z) = \sum_{i=0}^{2N} a_i(N, x, y, z)t^i F(t, x, y, z), \quad (N = 0, 1, 2, \dots). \quad (7)$$

Differentiating (7) with respect to  $t$ , we have

$$\begin{aligned} F^{(N+1)} &= \frac{\partial F^{(N)}}{\partial t} = \sum_{i=0}^{2N} a_i(N, x, y, z)it^{i-1}F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z)t^i F^{(1)}(t, x, y, z) \\ &= \sum_{i=0}^{2N} a_i(N, x, y, z)it^{i-1}F(t, x, y, z) + \sum_{i=0}^{2N} a_i(N, x, y, z)t^i (x + 2yt + 3zt^2)F(t, x, y, z) \\ &= \sum_{i=0}^{2N} ia_i(N, x, y, z)t^{i-1}F(t, x, y, z) + \sum_{i=0}^{2N} xa_i(N, x, y, z)t^i F(t, x, y, z) \\ &\quad + \sum_{i=0}^{2N} 2ya_i(N, x, y, z)t^{i+1}F(t, x, y, z) + \sum_{i=0}^{2N} 3za_i(N, x, y, z)t^{i+2}F(t, x, y, z) \\ &= \sum_{i=0}^{2N-1} (i+1)a_{i+1}(N, x, y, z)t^i F(t, x, y, z) + \sum_{i=0}^{2N} xa_i(N, x, y, z)t^i F(t, x, y, z) \\ &\quad + \sum_{i=1}^{2N+1} 2ya_{i-1}(N, x, y, z)t^i F(t, x, y, z) + \sum_{i=2}^{2N+2} 3za_{i-2}(N, x, y, z)t^i F(t, x, y, z) \end{aligned}$$

Hence we have

$$\begin{aligned} F^{(N+1)} &= \sum_{i=0}^{2N-1} (i+1)a_{i+1}(N, x, y, z)t^i F(t, x, y, z) \\ &\quad + \sum_{i=0}^{2N} xa_i(N, x, y, z)t^i F(t, x, y, z) \\ &\quad + \sum_{i=1}^{2N+1} 2ya_{i-1}(N, x, y, z)t^i F(t, x, y, z) \\ &\quad + \sum_{i=2}^{2N+2} 3za_{i-2}(N, x, y, z)t^i F(t, x, y, z). \end{aligned} \quad (8)$$

Now replacing  $N$  by  $N + 1$  in (7), we find

$$F^{(N+1)} = \sum_{i=0}^{2N+2} a_i(N+1, x, y, z) t^i F(t, x, y, z). \quad (9)$$

Comparing the coefficients on both sides of (8) and (9), we obtain

$$\begin{aligned} a_0(N+1, x, y, z) &= a_1(N, x, y, z) + xa_0(N, x, y, z), \\ a_1(N+1, x, y, z) &= 2a_2(N, x, y, z) + xa_1(N, x, y, z) + 2ya_0(N, x, y, z), \\ a_{2N}(N+1, x, y, z) &= xa_{2N}(N, x, y, z) + 2ya_{2N-1}(N, x, y, z) + 3za_{2N-2}(N, x, y, z), \\ a_{2N+1}(N+1, x, y, z) &= 2ya_{2N}(N, x, y, z) + 3za_{2N-1}(N, x, y, z), \\ a_{2N+2}(N+1, x, y, z) &= 3za_{2N}(N, x, y, z), \end{aligned} \quad (10)$$

and

$$\begin{aligned} a_i(N+1, x, y, z) &= (i+1)a_{i+1}(N, x, y, z) + xa_i(N, x, y, z) \\ &\quad + 2ya_{i-1}(N, x, y, z) + 3za_{i-2}(N, x, y, z), \quad (2 \leq i \leq 2N-1). \end{aligned} \quad (11)$$

In addition, by (7), we have

$$F(t, x, y, z) = F^{(0)}(t, x, y, z) = a_0(0, x, y, z)F(t, x, y, z), \quad (12)$$

which gives

$$a_0(0, x, y, z) = 1. \quad (13)$$

It is not difficult to show that

$$\begin{aligned} &xF(t, x, y, z) + 2ytF(t, x, y, z) + 3zt^2F(t, x, y, z) \\ &= F^{(1)}(t, x, y, z) \\ &= \sum_{i=0}^2 a_i(1, x, y, z)F(t, x, y, z) \\ &= (a_0(1, x, y, z) + a_1(1, x, y, z)t + a_2(1, x, y, z)t^2)F(t, x, y, z). \end{aligned} \quad (14)$$

Thus, by (14), we also find

$$a_0(1, x, y, z) = x, \quad a_1(1, x, y, z) = 2y, \quad a_2(1, x, y, z) = 3z. \quad (15)$$

From (10), we note that

$$\begin{aligned} a_0(N+1, x, y, z) &= a_1(N, x, y, z) + xa_0(N, x, y, z), \\ a_0(N, x, y, z) &= a_1(N-1, x, y, z) + xa_0(N-1, x, y, z), \dots \\ a_0(N+1, x, y, z) &= \sum_{i=0}^N x^i a_1(N-i, x, y, z) + x^{N+1}, \end{aligned} \quad (16)$$

and

$$\begin{aligned}
 a_{2N+2}(N+1, x, y, z) &= 3za_{2N}(N, x, y, z), \\
 a_{2N}(N, x, y, z) &= 3za_{2N-2}(N-1, x, y, z), \dots \\
 a_{2N+2}(N+1, x, y, z) &= (3z)^{N+1}.
 \end{aligned}
 \tag{17}$$

Note that, here the matrix  $a_i(j, x, y)_{0 \leq i \leq 2N+2, 0 \leq j \leq N+1}$  is given by

$$\begin{pmatrix}
 1 & x & 2y + x^2 & \cdot & \dots & \cdot \\
 0 & 2y & 4xy + 6z & \cdot & \dots & \cdot \\
 0 & 3z & 6xz + 4y^2 & \cdot & \dots & \cdot \\
 0 & 0 & 12yz & \cdot & \dots & \cdot \\
 0 & 0 & (3z)^2 & \cdot & \dots & \cdot \\
 0 & 0 & 0 & \cdot & \dots & \cdot \\
 0 & 0 & 0 & (3z)^3 & \dots & \cdot \\
 \vdots & \vdots & \vdots & \vdots & \ddots & \cdot \\
 0 & 0 & 0 & 0 & \dots & (3z)^{N+1}
 \end{pmatrix}$$

Therefore, we obtain the following theorem.

**Theorem 5.** For  $N = 0, 1, 2, \dots$ , the differential equation

$$F^{(N)} = \left( \frac{\partial}{\partial t} \right)^N F(t, x, y, z) = \left( \sum_{i=0}^N a_i(N, x, y, z) t^i \right) F(t, x, y, z)$$

has a solution

$$F = F(t, x, y, z) = e^{xt+yt^2+zt^3},$$

where

$$\begin{aligned}
 a_0(N+1, x, y, z) &= \sum_{i=0}^N x^i a_1(N-i, x, y, z) + x^{N+1}, \\
 a_1(N+1, x, y, z) &= 2a_2(N, x, y, z) + xa_1(N, x, y, z) + 2ya_0(N, x, y, z), \\
 a_{2N}(N+1, x, y, z) &= xa_{2N}(N, x, y, z) + 2ya_{2N-1}(N, x, y, z) + 3za_{2N-2}(N, x, y, z), \\
 a_{2N+1}(N+1, x, y, z) &= 2ya_{2N}(N, x, y, z) + 3za_{2N-1}(N, x, y, z), \\
 a_{2N+2}(N+1, x, y, z) &= (3z)^{N+1},
 \end{aligned}$$



and

$$a_i(N+1, x, y, z) = (i+1)a_{i+1}(N, x, y, z) + xa_i(N, x, y, z) + 2ya_{i-1}(N, x, y, z) + 3za_{i-2}(N, x, y, z), \quad (2 \leq i \leq 2N-1).$$

From (4), we note that

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t, x, y, z) = \sum_{k=0}^{\infty} H_{k+N}(x, y, z) \frac{t^k}{k!}. \quad (18)$$

By (4) and (18), we get

$$\begin{aligned} e^{-nt} \left(\frac{\partial}{\partial t}\right)^N F(t, x, y, z) &= \left(\sum_{m=0}^{\infty} (-n)^m \frac{t^m}{m!}\right) \left(\sum_{m=0}^{\infty} H_{m+N}(x, y, z) \frac{t^m}{m!}\right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^m \binom{m}{k}\right) (-n)^{m-k} H_{N+k}(x, y, z) \frac{t^m}{m!}. \end{aligned} \quad (19)$$

By the Leibniz rule and the inverse relation, we have

$$\begin{aligned} e^{-nt} \left(\frac{\partial}{\partial t}\right)^N F(t, x, y, z) &= \sum_{k=0}^N \binom{N}{k} n^{N-k} \left(\frac{\partial}{\partial t}\right)^k (e^{-nt} F(t, x, y, z)) \\ &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^N \binom{N}{k}\right) n^{N-k} H_{m+k}(x-n, y, z) \frac{t^m}{m!}. \end{aligned} \quad (20)$$

Hence, by (19) and (20), and comparing the coefficients of  $\frac{t^m}{m!}$  gives the following theorem.

**Theorem 6.** Let  $m, n, N$  be nonnegative integers. Then

$$\sum_{k=0}^m \binom{m}{k} (-n)^{m-k} H_{N+k}(x, y, z) = \sum_{k=0}^N \binom{N}{k} n^{N-k} H_{m+k}(x-n, y, z). \quad (21)$$

If we take  $m = 0$  in (21), then we have the following corollary.

**Corollary 7.** For  $N = 0, 1, 2, \dots$ , we have

$$H_N(x, y, z) = \sum_{k=0}^N \binom{N}{k} n^{N-k} H_k(x-n, y, z).$$

For  $N = 0, 1, 2, \dots$ , the differential equation

$$F^{(N)} = \left(\frac{\partial}{\partial t}\right)^N F(t, x, y, z) = \left(\sum_{i=0}^N a_i(N, x, y, z) t^i\right) F(t, x, y, z)$$

has a solution

$$F = F(t, x, y, z) = e^{xt+yt^2+zt^3}.$$

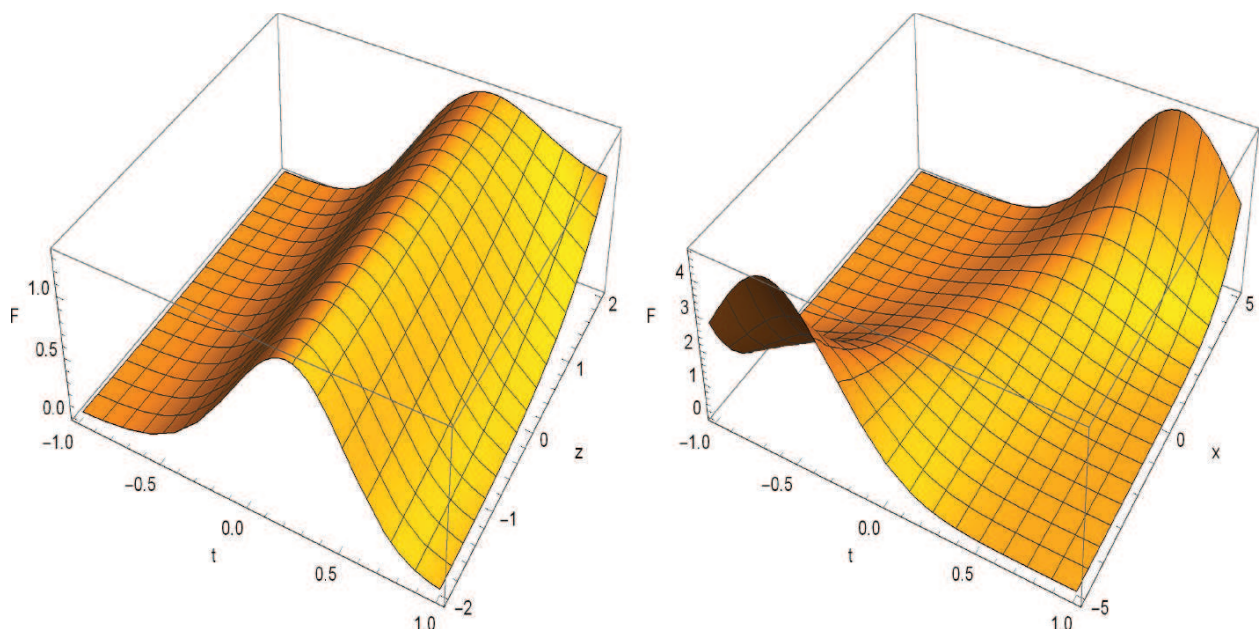
Here is a plot of the surface for this solution. In **Figure 1(left)**, we choose  $-2 \leq z \leq 2$ ,  $-1 \leq t \leq 1$ ,  $x = 2$ , and  $y = -4$ . In **Figure 1(right)**, we choose  $-5 \leq x \leq 5$ ,  $-1 \leq t \leq 1$ ,  $y = -3$ , and  $z = -1$ .

### 3. Distribution of zeros of the 3-variable Hermite polynomials

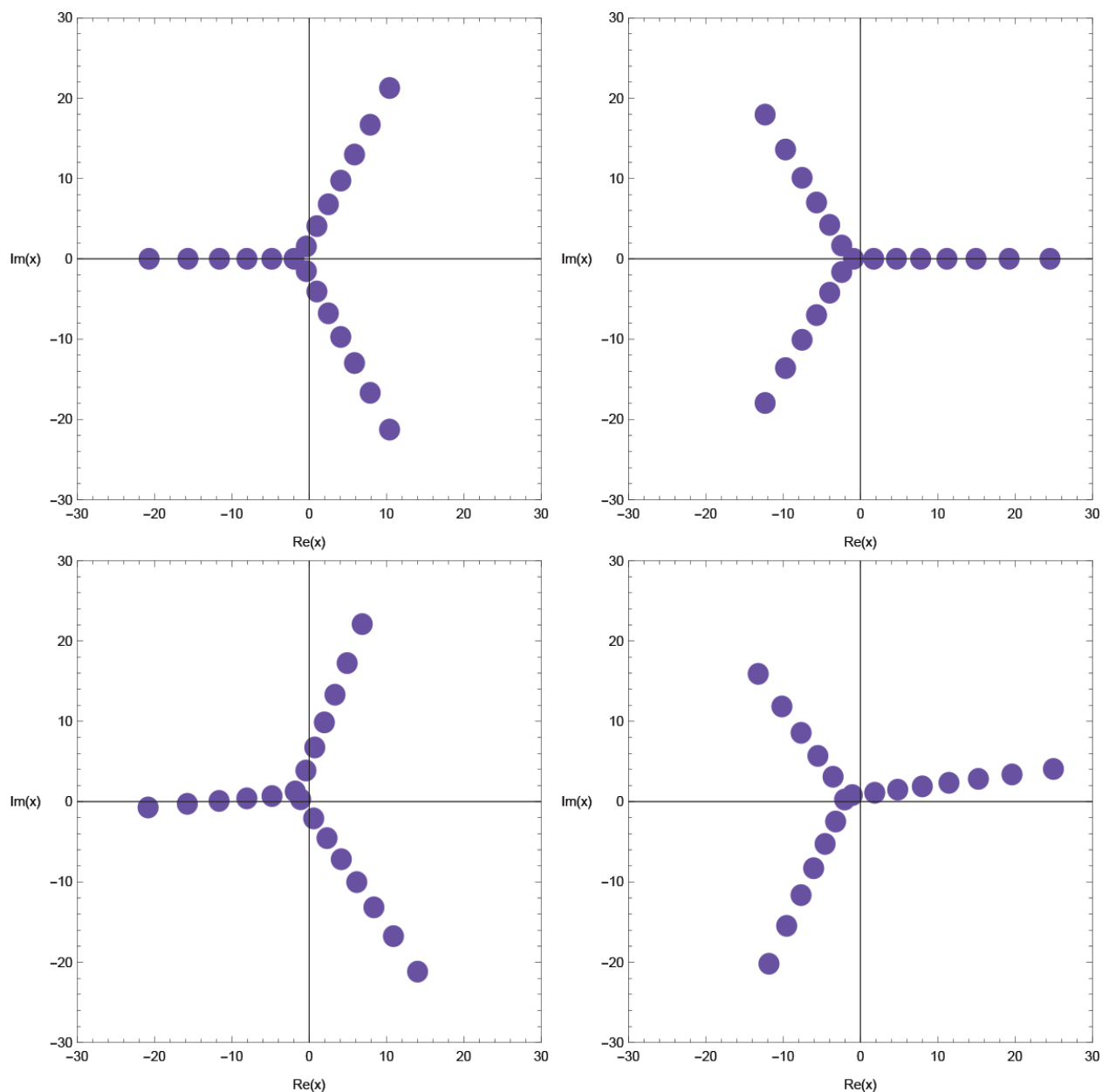
This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$ . By using computer, the 3-variable Hermite polynomials  $H_n(x, y, z)$  can be determined explicitly. We display the shapes of the 3-variable Hermite polynomials  $H_n(x, y, z)$  and investigate the zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$ . We investigate the beautiful zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$  by using a computer. We plot the zeros of the  $H_n(x, y, z)$  for  $n = 20$ ,  $y = 1, -1, 1 + i, -1 - i$ ,  $z = 3, -3, 3 + i, -3 - i$  and  $x \in \mathbb{C}$  (**Figure 2**). In **Figure 2(top-left)**, we choose  $n = 20$ ,  $y = 1$ , and  $z = 3$ . In **Figure 2(top-right)**, we choose  $n = 20$ ,  $y = -1$ , and  $z = -3$ . In **Figure 2(bottom-left)**, we choose  $n = 20$ ,  $y = 1 + i$ , and  $z = 3 + i$ . In **Figure 2(bottom-right)**, we choose  $n = 20$ ,  $y = -1 - i$ , and  $z = -3 - i$ .

In **Figure 3(top-left)**, we choose  $n = 20$ ,  $x = 1$ , and  $y = 1$ . In **Figure 3(top-right)**, we choose  $n = 20$ ,  $x = -1$ , and  $y = -1$ . In **Figure 3(bottom-left)**, we choose  $n = 20$ ,  $x = 1 + i$ , and  $y = 1 + i$ . In **Figure 3(bottom-right)**, we choose  $n = 20$ ,  $x = -1 - i$ , and  $y = -1 - i$ .

Stacks of zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$  for  $1 \leq n \leq 20$  from a 3-D structure are presented (**Figure 3**). In **Figure 4(top-left)**, we choose  $n = 20$ ,  $y = 1$ , and  $z = 3$ . In **Figure 4(top-right)**, we choose  $n = 20$ ,  $y = -1$ , and  $z = -3$ . In **Figure 4(bottom-left)**, we choose  $n = 20$ ,  $y = 1 + i$ , and  $z = 3 + i$ . In **Figure 4(bottom-right)**, we choose  $n = 20$ ,  $y = -1 - i$ , and  $z = -3 - i$ .



**Figure 1.** The surface for the solution  $F(t, x, y, z)$ .



**Figure 2.** Zeros of  $H_n(x, y, z)$ .

Our numerical results for approximate solutions of real zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$  are displayed (**Tables 1–3**).

The plot of real zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$  for  $1 \leq n \leq 20$  structure are presented (**Figure 5**).

In **Figure 5(left)**, we choose  $y = 1$  and  $z = 3$ . In **Figure 5(right)**, we choose  $y = -1$  and  $z = -3$ .

Stacks of zeros of  $H_n(x, -2, 4)$  for  $1 \leq n \leq 40$ , forming a 3D structure are presented (**Figure 6**). In **Figure 6(top-left)**, we plot stacks of zeros of  $H_n(x, -2, 4)$  for  $1 \leq n \leq 20$ . In **Figure 6(top-right)**, we draw  $x$  and  $y$  axes but no  $z$  axis in three dimensions. In **Figure 6(bottom-left)**, we draw  $y$

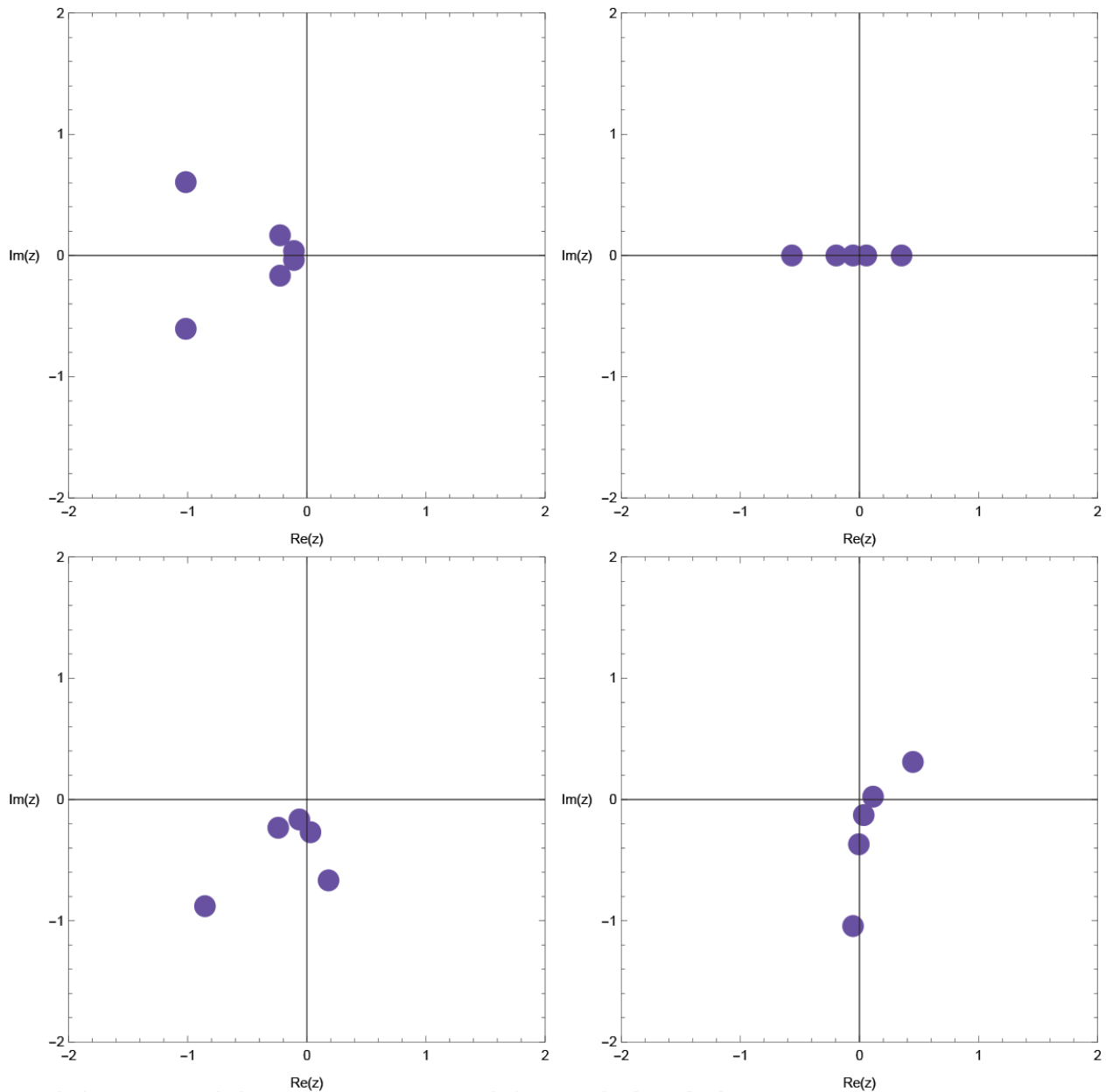


Figure 3. Zeros of  $H_n(x, y, z)$ .

and  $z$  axes but no  $x$  axis in three dimensions. In **Figure 6(bottom-right)**, we draw  $x$  and  $z$  axes but no  $y$  axis in three dimensions.

It is expected that  $H_n(x, y, z)$ ,  $x \in \mathbb{C}$ ,  $y, z \in \mathbb{R}$ , has  $\text{Im}(x) = 0$  reflection symmetry analytic complex functions (see **Figures 2–7**). We observe a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials  $H_n(x, y, z)$  for  $y, z \in \mathbb{R}$ . We also hope to verify a remarkable regular structure of the complex roots of the 3-variable Hermite polynomials  $H_n(x, y, z)$  for  $y, z \in \mathbb{R}$  (**Tables 1 and 2**). Next, we calculated an approximate solution satisfying  $H_n(x, y, z) = 0$ ,  $x \in \mathbb{C}$ . The results are given in **Tables 3 and 4**.

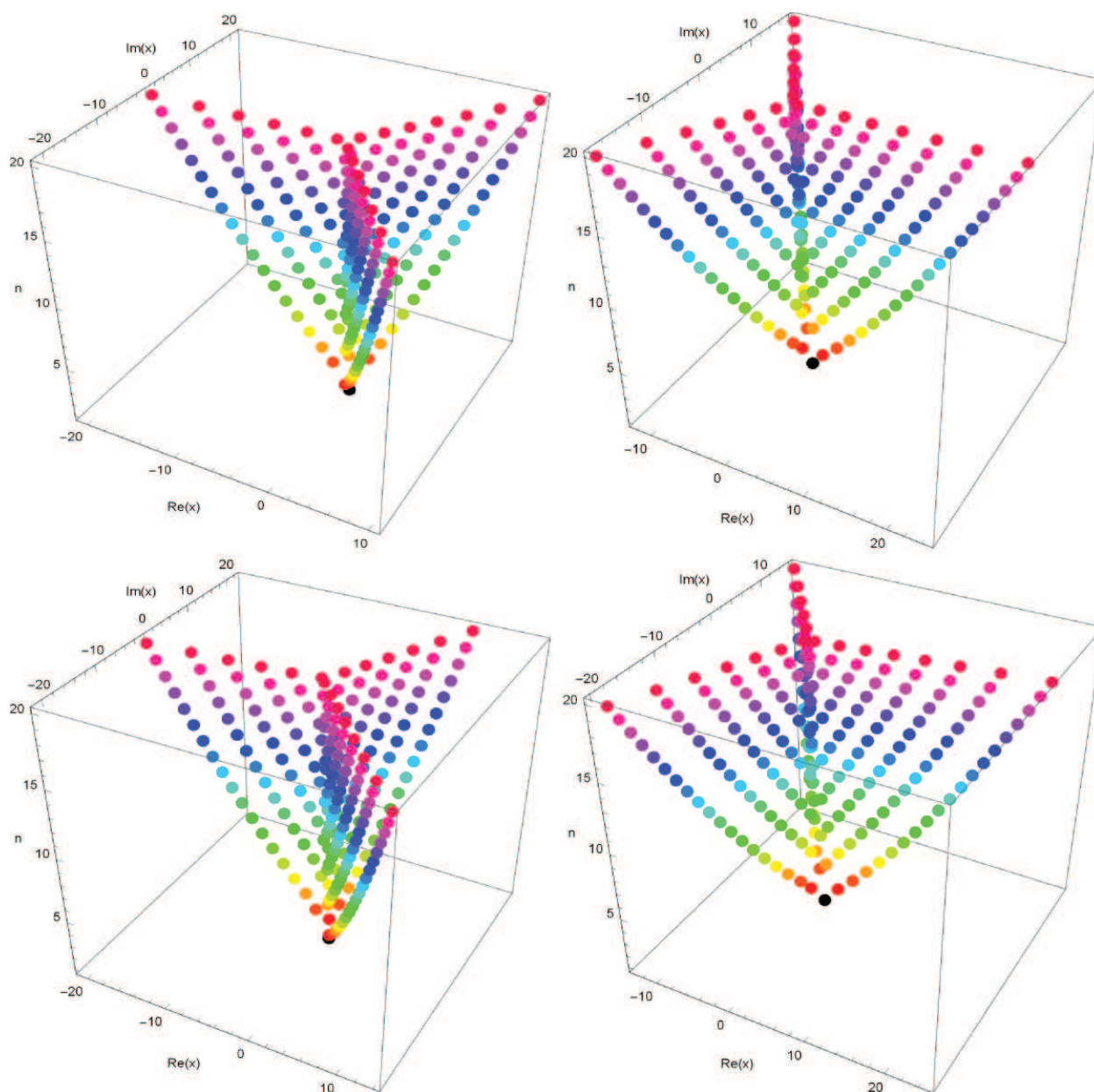


Figure 4. Stacks of zeros of  $H_n(x, y, z)$ ,  $1 \leq n \leq 20$ .

The plot of real zeros of the 3-variable Hermite polynomials  $H_n(x, y, z)$  for  $1 \leq n \leq 20$  structure are presented (Figure 7).

In Figure 7(left), we choose  $x = 1$  and  $y = 2$ . In Figure 7(right), we choose  $x = -1$  and  $y = -2$ .

Finally, we consider the more general problems. How many zeros does  $H_n(x, y, z)$  have? We are not able to decide if  $H_n(x, y, z) = 0$  has  $n$  distinct solutions. We would also like to know the number of complex zeros  $C_{H_n(x,y,z)}$  of  $H_n(x, y, z)$ ,  $\text{Im}(x) \neq 0$ . Since  $n$  is the degree of the polynomial  $H_n(x, y, z)$ , the number of real zeros  $R_{H_n(x,y,z)}$  lying on the real line  $\text{Im}(x) = 0$  is then  $R_{H_n(x,y,z)} = n - C_{H_n(x,y,z)}$ , where  $C_{H_n(x,y,z)}$  denotes complex zeros. See Tables 1 and 2 for

Degree $n$	Real zeros	Complex zeros
1	1	0
2	0	2
3	1	2
4	2	2
5	1	4
6	2	4
7	3	4
8	2	6
9	3	6
10	4	6
11	3	8
12	4	8
13	3	10
14	4	10

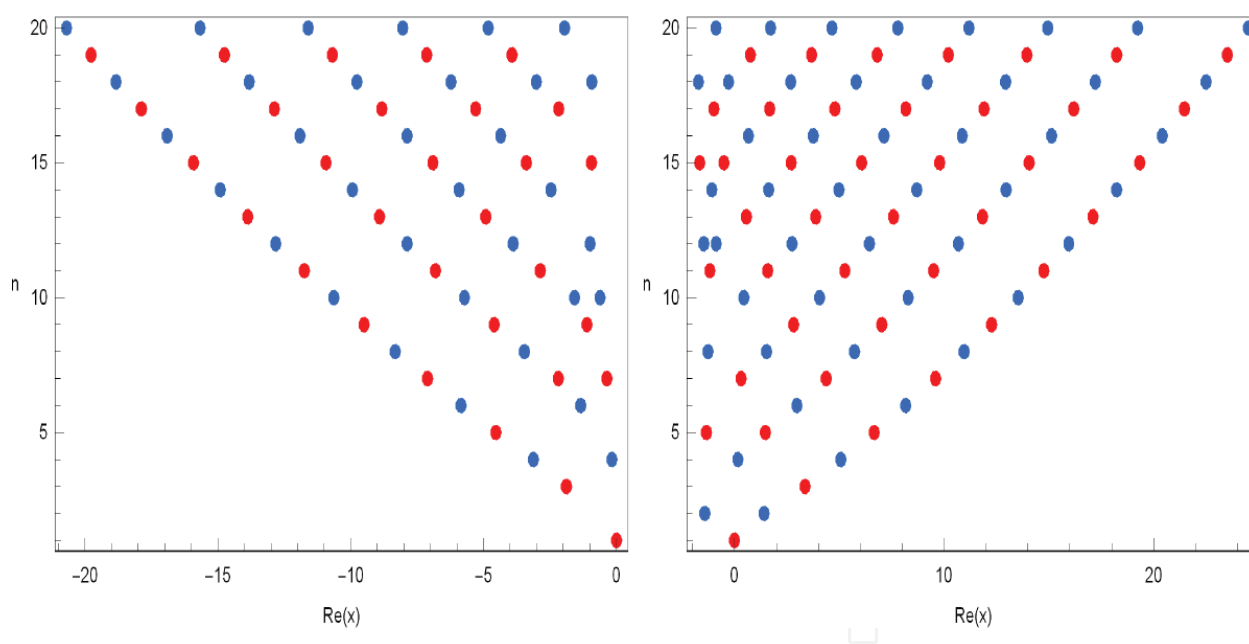
**Table 1.** Numbers of real and complex zeros of  $H_n(x, 1, 3)$ .

Degree $n$	Real zeros	Complex zeros
1	1	0
2	2	0
3	1	2
4	2	2
5	3	2
6	2	4
7	3	4
8	4	4
9	3	6
10	4	6
11	5	6
12	6	6
13	5	8
14	6	8

**Table 2.** Numbers of real and complex zeros of  $H_n(x, -1, -3)$ .

Degree $n$	$x$
1	0
2	—
3	- 1.8845
4	3.1286, -0.17159
5	-4.5385
6	-5.8490, -1.3476
7	-7.1098, -2.1887, -0.36350
8	-8.3241, -3.4645
9	-9.4984, - 4.6021, - 1.1118
10	-10.637, - 5.7212, - 1.5785, -0.61919
11	-11.745, - 6.8105, - 2.8680
12	-12.824, - 7.8743, - 3.8894, - 0.99513

**Table 3.** Approximate solutions of  $H_n(x, 1, 3) = 0, x \in \mathbb{R}$ .



**Figure 5.** Real zeros of  $H_n(x, y, z), 1 \leq n \leq 20$ .

tabulated values of  $R_{H_n(x,y,z)}$  and  $C_{H_n(x,y,z)}$ . The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the 3-variable Hermite polynomials  $H_n(x, y, z)$  which appear in mathematics and physics. The reader may refer to [2, 11, 13, 20] for the details.

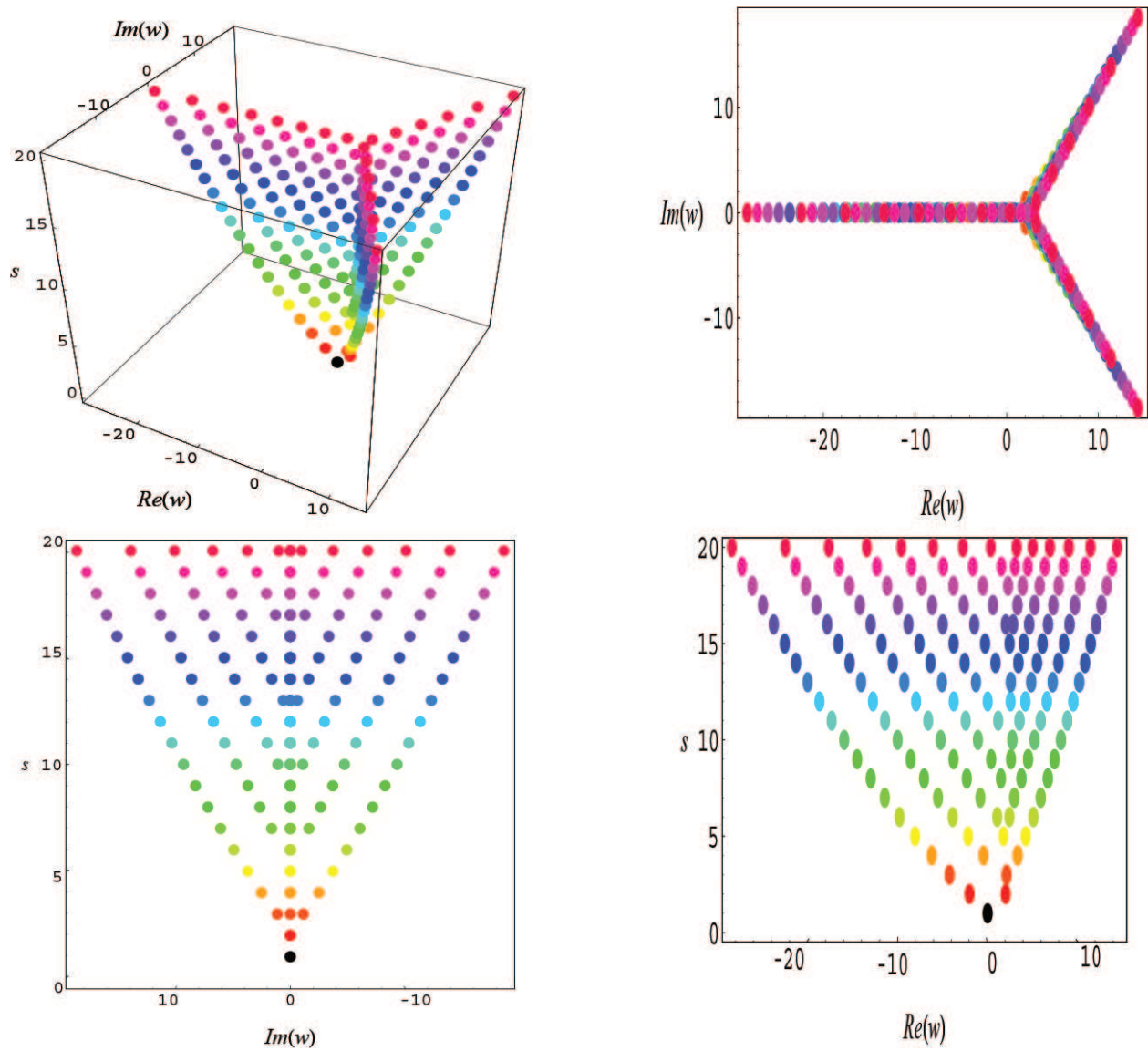


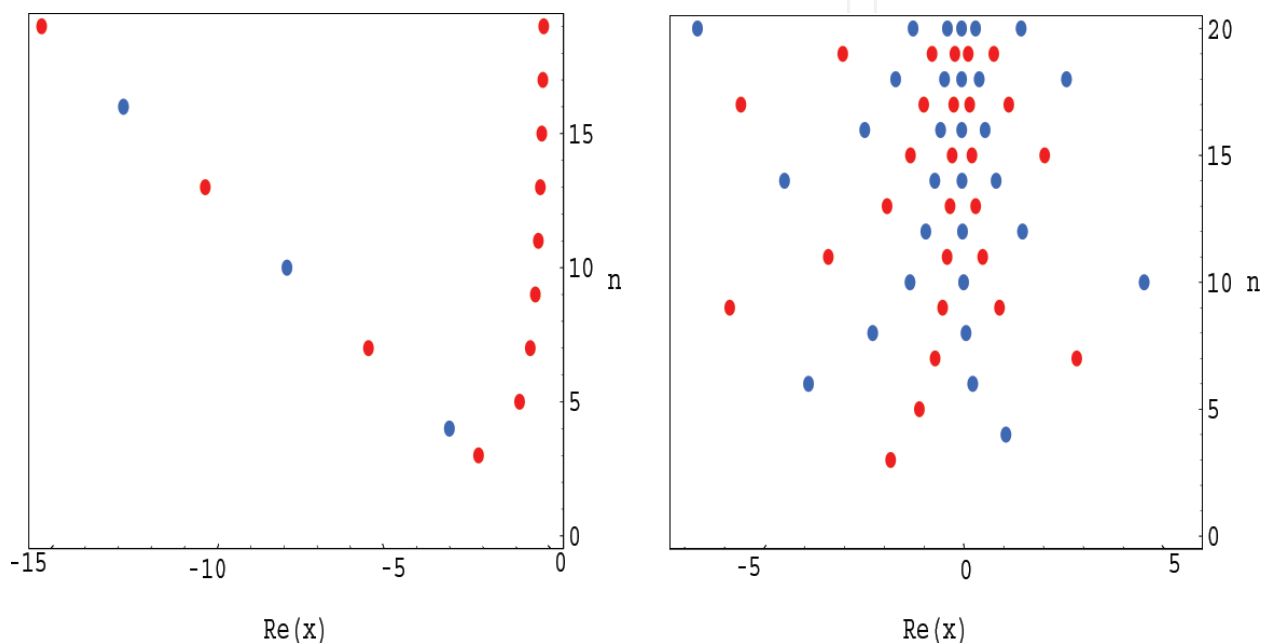
Figure 6. Stacks of zeros of  $H_n(x, -2, 4)$  for  $1 \leq n \leq 20$ .

degree $n$	$x$
1	0
2	-1.4142, 1.4142
3	3.3681
4	0.16229, 5.0723
5	-1.3404, 1.4745, 6.6661
6	2.9754, 8.1678
7	0.31213, 4.3783, 9.5946



degree $n$	$x$
8	-1.2604, 1.5304, 5.7274, 10.959
9	2.8224, 7.0271, 12.270
10	0.44594, 4.0615, 8.2834, 13.535
11	-1.1740, 1.5825, 5.2667, 9.5013, 14.760
12	-1.4659, -0.87728, 2.7469, 6.4398, 10.685, 15.949

**Table 4.** Approximate solutions of  $H_n(x, -1, -3) = 0, x \in \mathbb{R}$ .



**Figure 7.** Real zeros of  $H_n(x, y, z), 1 \leq n \leq 20$ .

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