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RELATIONS BETWEEN (κ,τ)-REGULAR SETS AND STAR COMPLEMENTS

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Abstract. Let G be a finite graph with an eigenvalue μ of multiplicity m. A set X of m vertices in G is called a star set for μ in G if μ is not an eigenvalue of the star complement $G \setminus X$ which is the subgraph of G induced by vertices not in X. A vertex subset of a graph is (κ,τ)-regular if it induces a κ-regular subgraph and every vertex not in the subset has τ neighbors in it. We investigate the graphs having a (κ,τ)-regular set which induces a star complement for some eigenvalue. A survey of known results is provided and new properties for these graphs are deduced. Several particular graphs where these properties stand out are presented as examples.

Keywords: eigenvalue, star complement, non-main eigenvalue, Hamiltonian graph

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1. Introduction

We will consider only simple graphs, that is, finite undirected graphs without loops or multiple edges. Let G be a such graph with vertex set $V(G) = \{1,\dots,n\}$, edge set $E(G)$ and $(0,1)$-adjacency matrix $A_G = (a_{ij})$. Since the eigenvalues of A_G are invariant under permutations of $V(G)$, they are called the eigenvalues of G. The spectrum of A_G, that is, the multiset of its eigenvalues (with multiplicities), is also

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called the spectrum of G, and it is denoted by $\sigma(G)$ (to denote the multiplicities, we will use the exponential notation—see Section 3). For an eigenvalue $\mu \in \sigma(G)$, we denote by $\mathscr{E}_G(\mu)$ the eigenspace $\{\mathbf{x} \in \mathbb{R}^n \colon A_G\mathbf{x} = \mu\mathbf{x}\}$. Let P be the matrix of the orthogonal projection of $\mathbb{R}^n$ onto $\mathscr{E}_G(\mu)$ with respect to the standard orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$. The vectors $P\mathbf{e}_1, \ldots, P\mathbf{e}_n$ form a *eutactic star* since they are the orthogonal projection of pairwise orthogonal vectors of the same length. The set of vectors $P\mathbf{e}_j$ $(j = 1, \ldots n)$ spans $\mathscr{E}_G(\mu)$ and therefore there exists $X \subseteq V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathscr{E}_G(\mu)$. Such a set X is called a *star set* for μ in G.

We write $\overline{X}$ for the complement of X in $V(G)$. Also, $G[X]$ denotes the subgraph of G induced by the vertices from X, while $G \setminus X = G[\overline{X}]$. If $X(= X(\mu))$ is a star set for the eigenvalue μ then $\overline{X}(= \overline{X}(\mu))$ is its *co-star set*, while $G \setminus X$ is said to be a *star complement* for μ in G. The following results are fundamental to the theory of star complements (see [9, pp. 136–140]).

Theorem 1.1. *Let G be a graph, let $X \subseteq V(G)$ and let μ be an eigenvalue of G with multiplicity k. Then the following statements are equivalent:*

(i) *$\{P\mathbf{e}_j \colon j \in X\}$ is a basis of $\mathscr{E}_G(\mu)$;*
(ii) *$\mathbb{R}^n = \mathscr{E}_G(\mu) \oplus \mathscr{V}$, where $\mathscr{V} = \langle \mathbf{e}_j \colon j \in \overline{X}\rangle$;*
(iii) *$|X| = k$ and μ is not an eigenvalue of $G \setminus X$.*

Theorem 1.2. *Let X be a set of k vertices in the graph G and suppose that G has the adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{1.1}$$

In this situation, $\mathscr{E}_G(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$, $\mathbf{x} \in \mathbb{R}^k$.

This last result is known as the *Reconstruction theorem*. The columns $\mathbf{b}_u$, $u \in X$ of the matrix B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) \colon u \sim v\}$, where $u \in X$ and $H = G \setminus X$.

We write $t = |\overline{X}|(= n - k)$ and define a bilinear form on $\mathbb{R}^t$ by:

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T(\mu I - C)^{-1}\mathbf{y} \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

By equating the entries in (1.1) we see that X is a star set for μ if and only if μ is not an eigenvalue of $H = G \setminus X$ and the following identities hold:

$$\langle \mathbf{b}_u, \mathbf{b}_u\rangle = \mu \qquad (u \in X), \tag{1.2}$$

and, for $u \neq v$

$$(1.3) \qquad \langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1 \quad \text{if} \quad u \sim v, \qquad \langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0 \quad \text{if} \quad u \nsim v \qquad (u, v \in X).$$

Given a graph H, a subset U of $V(H)$ and a vertex $u \notin V(H)$, denote by $H(U)$ the graph obtained from H by joining u to all vertices of U. Let μ be an eigenvalue of $H(U)$ but not of H. We will say that u is a *good vertex* if $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$; moreover, if u and v are good vertices such that $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{-1, 0\}$ then u and v are *good partners* and $\Delta_H(u), \Delta_H(v)$ are *compatible*. In addition, it follows that any vertex set X in which all vertices are good, both individually or in pairs, gives rise to a good extension, say G, in which X can be viewed as a star set for μ with H as the corresponding star complement.

The above considerations describe in brief the star complement technique. In order to find H-maximal graphs for any $\mu \notin \{-1, 0\}$, i.e. those graphs which are not extendible any further, we form an extendability graph whose vertices are good vertices for a fixed μ and H. We add an edge between two good vertices whenever they are good partners. Therefore the search for maximal cliques in the extendability graph is equivalent to the search for H-maximal extensions.

In the case that $\mu = 0$ (or $\mu = -1$), we can have infinite series of extensions due to the presence of *duplicate* (*co-duplicate*) vertices. Recall that two vertices are *duplicate* (*co-duplicate*) if they have the same open (closed) neighbourhood. So, if v and w are two distinct vertices then they are duplicate if $\Delta(v) = \Delta(w)$, or co-duplicate if $\Delta^*(v) = \Delta^*(w)$ (note that $\Delta^*(u) = \Delta(u) \cup \{u\}$). A graph is called *reduced* (*co-reduced*) with respect to fixed star complement if it has no duplicate (co-duplicate) vertices in the corresponding star set (for more details see [9, Chapter 7]). If $\mu \notin \{-1, 0\}$ then $|X| \leqslant \binom{t}{2}$, [10, Theorem 5.3.1] (recall $t = |V(H)|$). On the other hand, if $\mu \in \{-1, 0\}$ this need not be true in connected graphs even in the case that they are reduced or co-reduced. Moreover, if $\mu = 0$ then this bound can be exponential (see, for example, [12] and references therein).

A (κ, τ)-regular set is a subset of the vertices of a graph inducing a κ-regular subgraph such that every vertex not in the subset has τ neighbors in it. The (κ, τ)-regular sets appeared first in [20] under the designation of eigengraphs and in [13], in both cases in the context of strongly regular graphs and designs. By convention, if G is a κ-regular graph then we say that $V(G)$ is a $(\kappa, 0)$-regular set. Also, (κ, τ)-regular sets were considered in [11], [19], related to the study of graphs with domination constraints, and later, in the general context of arbitrary graphs, in [4], [5], [6].

The eigenvalue μ of a graph G which has an associated eigenspace $\mathscr{E}_G(\mu)$ not orthogonal to the all-one vector $\mathbf{j}$ is said to be *main*, otherwise it is called *non-main*. Recall that μ is a main eigenvalue of G if and only if $P\mathbf{j} \neq 0$ ([8, p. 46]). Also, from the description of $\mathscr{E}_G(\mu)$ we see that μ is non-main if and only if $\langle \mathbf{b}_u, \mathbf{j}\rangle = -1$ for all $u \in X$, where X is a star set. Recently, a nice survey paper on main (and non-main) eigenvalues was published by Rowlinson [17]. Rowlinson also has a survey paper on star complements [14].

A (κ,τ)-regular set in a graph G which is also a star set (co-star set) for some eigenvalue μ of G is called a (κ,τ)-*regular star set* (a (κ,τ)-*regular co-star set*). If H is a star complement for some eigenvalue μ of G, such that $V(H)$ is a (κ,τ)-regular set, then we say that H is a (κ,τ)-*regular star complement*.

2. Survey of known results

Our motivation to consider graphs with (κ,τ)-regular star sets (or star complements) comes from several directions. Here we will mention some of them.

Domination sets. We recall that a *dominating set* in a graph G is a subset D of $V(G)$ such that each vertex of $\overline{D}$ is adjacent to a vertex of D. If X is a (κ,τ)-regular set in the graph G for $\tau \neq 0$ then X is a dominating set in G. On the other hand, if X is a star set for $\mu \neq 0$, then $\overline{X}$ is a dominating set in G (this follows from Theorem 1.1 (i)).

Hamiltonian graphs. Recall that a *line graph* of a graph G, denoted by $L(G)$, has the edge set of G as its vertex set, where two vertices in $L(G)$ are adjacent if the corresponding edges in G have a common end-vertex. Having this in mind, the following necessary and sufficient condition for line graphs of Hamiltonian graphs was obtained in [1]:

Theorem 2.1. *A graph G is the line graph of a Hamiltonian graph of order t, where t is an odd integer greater than 2, if and only if either $G = C_t$ or G has C_t as a star complement for the eigenvalue -2.*

The class of line graphs of Hamiltonian graphs can be characterized using (κ,τ)-regular sets, as follows.

Theorem 2.2. *A graph G which is not a cycle is Hamiltonian if and only if its line graph $L(G)$ has a $(2,4)$-regular set $S \subset V(L(G))$ inducing a connected subgraph of $L(G)$.*

P r o o f. Assume that G is Hamiltonian, that is, it contains a cycle C with all vertices of G. Then all edges not in C have each end-vertex in C and the corresponding vertices in $L(G)$ have 4 neighbors in $V(L(C))$. Since $V(L(C))$ induces a cycle in $L(G)$, hence $S = V(L(C))$ is a $(2,4)$-regular set of $L(G)$ inducing a connected subgraph. Conversely, assume that $L(G)$ has a $(2,4)$-regular set S inducing a connected subgraph. Then S corresponds to a cycle C in G and each edge not in C is connected to 4 edges in C. Therefore, each edge not in C has both end-vertices in C, which means that G is Hamiltonian. $\square$

Perfect matching. Let G be a graph without isolated vertices for which X is a minimal dominating set, where X is the star set for the eigenvalue $\mu \notin \{-1, 0\}$. If $G \setminus X$ has no isolated vertices then the set of all edges between X and $\overline{X}$ is a perfect matching for G (see [9, p. 174]). Our motivation in this case comes from the fact that a connected graph with more than one edge has a perfect matching $M \subseteq E(G)$ if and only if $V(L(M)) \subset V(L(G))$ is a (0,2)-regular set in $L(G)$ (see [3]). Notice that the line graph $L(M)$ has no edges.

3. (κ, τ)-REGULAR SETS AND STAR COMPLEMENTS

Given a vector $\mathbf{x} \in \mathbb{R}^n$ and a non-empty set $S \subset \{1, 2, \dots n\}$, let $\mathbf{x}_S$ denote the characteristic vector of S. The following result was proved in [7]:

Theorem 3.1. *Let G be a graph with a (κ, τ)-regular set $S \subset V(G)$, where $\tau > 0$. Then $\mu \in \sigma(G)$ is non-main if and only if*

(a) *$\mu = \kappa - \tau$, or*
(b) *$\mathbf{x}_S$ is orthogonal to $\mathscr{E}_G(\mu)$, that is $P\mathbf{x}_S = 0$ holds.*

Using examples of graphs with integral spectra, we show that either or both of (a) and (b) can hold.

Let $G = S(K_{1,3})$. So G is a subdivision of a star $K_{1,3}$. Recall that a subdivision $S(G)$ of G is a graph obtained from G by inserting in each of its edges a vertex of degree 2. We can label the vertices of G in breadth-first manner, starting from the vertex of degree 3, labelled 1; its neighbours are labelled $2, 3$ and 4, and their neighbours are labelled respectively $5, 6$ and 7. Let $S = V(G) \setminus \{1\}$. Then S is a $(1,3)$-regular set in G. So $\kappa - \tau = -2$. Note first that $-2 \in \sigma(G)$. It is easy to show that $\mathscr{E}_G(-2)$ is generated by the vector $(3, -2, -2, -2, 1, 1, 1)^{\mathrm{T}}$. So -2 is a non-main eigenvalue of G, but $\mathbf{x}_S$ is not orthogonal to $\mathscr{E}_G(-2)$. Therefore, (a) holds, but (b) does not. Next, it is easy to see that $1 \in \sigma(G)$ (so $1 \neq \kappa - \tau$). Moreover, 1 is a non-main eigenvalue of G. Namely, $\mathscr{E}_G(1)$ is spanned by vectors $(0, 1, -1, 0, 1, -1, 0)^{\mathrm{T}}$ and $(0, 1, 0, -1, 1, 0, -1)^{\mathrm{T}}$, and our claim holds. But now we have that $\mathbf{x}_S$ is orthogonal to $\mathscr{E}_G(1)$. Therefore, (a) does not hold, but (b) holds.

Let G be a graph depicted in Figure 1 and let $S = \{1, 4\}$. Then S is a $(1,1)$-regular set. So $\kappa - \tau = 0$. Note first that $0 \in \sigma(G)$. Also, it is easy to see that $\mathscr{E}_G(0)$

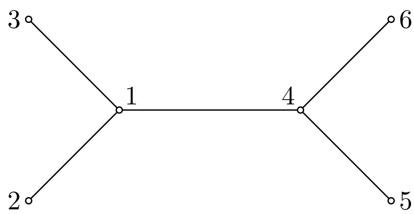

Figure 1. A graph G with a (1,1)-regular set for which (a) and (b) in Theorem 3.1 hold.

is spanned by vectors $(0,1,-1,0,0,0)^{\mathrm{T}}$ and $(0,0,0,0,1,-1)^{\mathrm{T}}$. So 0 is a non-main eigenvalue of G. But now we have that $\mathbf{x}_S$ is orthogonal to $\mathscr{E}_G(1)$. Therefore, both (a) and (b) hold.

We now consider (κ,τ)-regular sets which are star (or co-star) sets, and we prove the following slightly different result. Similar considerations can be found in [18, Proposition 1.5].

Theorem 3.2. *Let G be a graph and $S \subset V(G)$ a star (or co-star) set for an eigenvalue $\mu \in \sigma(G)$. If S or $\overline{S}$ is (κ,τ)-regular in G, with $\tau > 0$, then μ is non-main if and only if $\mu = \kappa - \tau$.*

Proof. Since A has spectral decomposition $A = \sum\limits_{i=1}^{m} \mu_i P_i$, then each P_i commutes with A. Therefore, denoting the characteristic vector of S and $\overline{S}$ by $\mathbf{x} = \mathbf{x}_S$ and $\mathbf{x} = \mathbf{x}_{\overline{S}}$, respectively, we have

$$PA\mathbf{x} = \begin{cases} PA\mathbf{x}_S = \sum\limits_{u\in S} PA\mathbf{e}_u = \sum\limits_{u\in S} \mu P\mathbf{e}_u = \mu P\mathbf{x}_S & \text{if} \quad \mathbf{x} = \mathbf{x}_S, \\ PA\mathbf{x}_{\overline{S}} = \sum\limits_{u\in \overline{S}} PA\mathbf{e}_u = \sum\limits_{u\in \overline{S}} \mu P\mathbf{e}_u = \mu P\mathbf{x}_{\overline{S}} & \text{if} \quad \mathbf{x} = \mathbf{x}_{\overline{S}}. \end{cases}$$

Furthermore, according to the hypothesis, one of the characteristic vectors $\mathbf{x} = \mathbf{x}_S$ or $\mathbf{x} = \mathbf{x}_{\overline{S}}$ satisfies $A\mathbf{x} = (\kappa-\tau)\mathbf{x} + \tau\mathbf{j}$. After multiplying both sides of this equality by P on the left, it follows that

$$\mu P\mathbf{x} = (\kappa-\tau)P\mathbf{x} + \tau P\mathbf{j}, \quad \text{i.e.} \quad (\mu - (\kappa-\tau))P\mathbf{x} = \tau P\mathbf{j}. \tag{3.1}$$

(1) If $\mathbf{x} = \mathbf{x}_S$, since the vectors $P\mathbf{e}_u$ $(u \in S)$ are linearly independent, hence $P\mathbf{x}_S \neq \mathbf{0}$. Therefore, from (3.1) $\mu = \kappa - \tau$, i.e. $P\mathbf{j} = \mathbf{0}$.

(2) Assume that $\mathbf{x} = \mathbf{x}_{\overline{S}}$. If $\mu = \kappa - \tau$, then (3.1) implies $P\mathbf{j} = \mathbf{0}$ and therefore μ is non-main. Conversely, if μ is non-main, then $P\mathbf{j} = \mathbf{0}$ implies $P\mathbf{x}_{\overline{S}} = -P\mathbf{x}_S$. Therefore, $P\mathbf{x}_S \neq \mathbf{0}$ or equivalently $P\mathbf{x}_{\overline{S}} \neq \mathbf{0}$ and, from (3.1), it follows that $\mu = \kappa - \tau$.

In both cases $P\mathbf{x} \neq \mathbf{0}$ and therefore $\mu = \kappa - \tau$ is equivalent to $P\mathbf{j} = \mathbf{0}$. $\square$

Remark 3.3. Note that, under the assumptions of Theorem 3.2, when S or $\overline{S}$ is (κ,τ)-regular, neither $\mathbf{x}_S$ nor $\mathbf{x}_{\overline{S}}$ is orthogonal to $\mathscr{E}_G(\kappa-\tau)$. In fact, according to the proof of Theorem 3.2, $P(\mathbf{x}_S) \neq \mathbf{0}$ and $P(\mathbf{x}_{\overline{S}}) \neq \mathbf{0}$.

Notice that if S ($\overline{S}$) is a (κ,τ)-regular star set (co-star set) with $\tau > 0$, and the eigenvalue $\mu \in \sigma(G)$ is main, then $P\mathbf{x}_S$ ($P\mathbf{x}_{\overline{S}}$) is a scalar multiple of $P\mathbf{j}$. In fact, in such a case,

$$\frac{\mu-(\kappa-\tau)}{\tau}P\mathbf{x}_S = P\mathbf{j} \qquad \Big(\frac{\mu-(\kappa-\tau)}{\tau}P\mathbf{x}_{\overline{S}} = P\mathbf{j}\Big).$$

As an immediate consequence, we have the following corollary.

Corollary 3.4. *If S ($\overline{S}$) is a (κ,τ)-regular star set (co-star set) for the main eigenvalue μ of a graph G, with $\tau > 0$, then $\mu \neq \kappa-\tau$ and the vector*

$$\frac{\mu-(\kappa-\tau)}{\tau}\mathbf{x}_S - \mathbf{j} \qquad \Big(\frac{\mu-(\kappa-\tau)}{\tau}\mathbf{x}_{\overline{S}} - \mathbf{j}\Big) \tag{3.2}$$

is orthogonal to $\mathscr{E}_G(\mu)$.

Remark 3.5. Let G be a graph with a (κ,τ)-regular set S such that $\kappa-\tau \in \sigma(G)$. Then we have:

- ▷ If S is a (κ,τ)-regular set then S may or may not be a star set for $\kappa-\tau$, and may or may not be a co-star set for $\kappa-\tau$. Consider for example the octahedron (Figure 2), with spectrum $\{[-2]^2, [0]^3, [4]^1\}$. The $(2,2)$-regular set $\{1,3,5\}$ is both a star set and a co-star set for 0, while the $(2,4)$-regular set $\{1,2,4,5\}$ is neither a star set nor a co-star set for -2. On the other hand, if $G = K_3$, then any set S with $|S| = 2$ is $(1,2)$-regular, $-1 \in \sigma(G[S])$ and S is a star set.
- ▷ If $\kappa-\tau \notin \sigma(G[S])$, then it is obvious that S can be a co-star set, or it can be shown that the cardinality of $\overline{S}$ is greater than the multiplicity of $\kappa-\tau$. In the former case, S can be both a star set and a co-star set for $\mu = \kappa-\tau$ (see Figure 2).

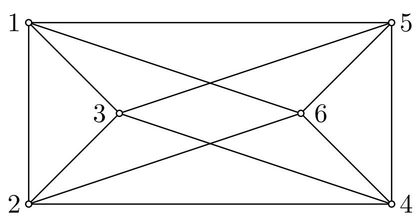

Figure 2. Graph with the $(2,2)$-regular star set (co-star set) $\{1,3,5\}$ for $\mu = 0$.

4. Some particular cases

Let G be a graph having a vertex subset S ($\emptyset \subseteq S \subseteq V(G)$) such that

1° S is a (κ, τ)-regular set in G, with $\tau > 0$;

2° $H = G[S]$ is a star complement for $\mu = \kappa - \tau$.

It is noteworthy that μ (defined in 2°) is a non-main eigenvalue (by Theorem 3.2). Let $H = G[S]$. Then a graph is *S-maximal* if it is H-maximal with respect to τ-good vertices, i.e. good vertices u for which $|U| = \tau$. So, if G is S-maximal then, for any $u \notin V(G)$ with $|U| = \tau$, S is not a co-star set in $G + u$.

In this section we study S-maximal graphs G for which $G[S]$ is κ-regular with $\kappa \in \{0, 1, 2, s-2, s-1\}$, where $s = |S|$.

We first show that S-maximal graphs for $\kappa = k$ and $\kappa = s-k-1$ are complementary graphs.

Theorem 4.1. *Let S be a nonempty set in G. Then S is a (κ, τ)-regular co-star set for the non-main eigenvalue $\mu = \kappa - \tau$ if and only if S is a $(|S| - \kappa - 1, |S| - \tau)$-regular co-star set in $\overline{G}$ for the non-main eigenvalue $-\mu - 1$.*

Proof. First, if S is a (κ, τ)-regular set in G then in its complement $\overline{G}$, S is $(|S| - \kappa - 1, |S| - \tau)$-regular. For any $\mathbf{x} \in \mathscr{E}_G(\kappa - \tau)$ we have $A_{\overline{G}}\mathbf{x} = (J - I - A_G)\mathbf{x} = (-1 - \kappa + \tau)\mathbf{x}$ since $J\mathbf{x} = 0$. By this we have proved that $\mathbf{x} \in \mathscr{E}_G(\kappa - \tau)$ if and only if $\mathbf{x} \in \mathscr{E}_{\overline{G}}(-1 - \kappa + \tau)$. Note that the eutactic stars of both the eigenvalues are the same and therefore all star sets (co-star sets) coincide (see [9, Chapter 7]). Hence $\overline{G}[S]$ is the star complement for the eigenvalue $-1 - \kappa + \tau$ in $\overline{G}$. □

4.1. Case $\kappa \in \{0, s-1\}$. Assume first that $\kappa = 0$. Then $H = sK_1$, $\mu = -\tau$; note that since $\tau > 0$, $-\tau \notin \sigma(H)$. Now, we obtain $\tau = 1$ from $\tau = \sqrt{\tau}$.

Proposition 4.2. *If G is an S-maximal graph with $\kappa = 0$ then $\tau = 1$ and $G = sK_2$ ($s = |S|$).*

Proposition 4.2 can be strengthened: If G is a graph with $\bar{K}_t$ as a star complement H for the non-main eigenvalue μ then $\mu = -1$ and $V(H)$ is $(0, 1)$-regular because $\langle \mathbf{b}_u, \mathbf{j} \rangle = -1$ implies $|\Delta_H(u)| = -\mu$ and $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$ implies $|\Delta_H(u)| = \mu^2$ (cf. [18, Proposition 1.6])

Next we assume $\kappa = s - 1$. By Theorem 4.1 and Proposition 4.2 we also have:

Proposition 4.3. *If G is an S-maximal graph with $\kappa = s - 1$ then $\tau = s - 1$ and $G = \overline{sK_2}$ ($s = |S|$).*

4.2. Case $\kappa \in \{1, s-2\}$. Assume first $\kappa = 1$. In this case $H = hK_2$, where $h = s/2$, $\mu = 1-\tau$ ($\tau \notin \{0,2\}$ since $\mu \notin \sigma(H)$). So, $\tau \in \{0,1,\ldots,2h\} \setminus \{0,2\}$. Following the notation of Theorem 1.2, the submatrix C is block-diagonal with h blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and exactly τ nonzero entries in each column of B. Note that $(\mu I - C)^{-1} = (\mu^2-1)^{-1}(\mu I + C)$. Therefore $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = 1-\tau$ if and only if

$$\mu \sum_{i=1}^{2h} b_i^2 + 2\sum_{i=1}^{h} b_{2i-1}b_{2i} = \mu(\mu^2-1),$$

where $\mathbf{b}_u = (b_1, \ldots, b_{2h})^{\mathrm{T}}$. Since $\sum\limits_{i=1}^{2h} b_i^2 = \tau$ we have

$$2\sum_{i=1}^{h} b_{2i-1}b_{2i} = \tau(1-\tau)(\tau-3). \tag{4.1}$$

The nonnegativity of the left hand side implies $\tau \in \{1, 3\}$.

Subcase $\tau = 1$: Now $\mu = 0$, and from $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -\mathbf{b}_u^T C \mathbf{b}_v$ we immediately obtain:

Proposition 4.4. *If G is an S-maximal graph with $\kappa = 1$ and $\tau = 1$ then $G = hC_4$ ($|S| = 2h$).*

Remark 4.5. Note that $G = hC_4$ has duplicate vertices, but not in the star set consisting of two adjacent vertices.

Subcase $\tau = 3$: Now $\mu = -2$. Then by (4.1) we have $\sum\limits_{i=1}^{h} b_{2i-1}b_{2i} = 0$ and this holds if and only if $(b_{2i-1}, b_{2i}) \neq (1,1)$ for any $i \in \{1, \ldots, h\}$. There are exactly three i's such that (b_{2i-1}, b_{2i}) is equal either to $(1,0)$ or to $(0,1)$. Consequently, there are exactly $8\binom{h}{3}$ good vertices. If u, v are good vertices with $b_u = (a_1, \ldots, a_{2h})^{\mathrm{T}}$ and $b_v = (b_1, \ldots, b_{2h})^{\mathrm{T}}$ then

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \frac{1}{3}\sum_{i=1}^{h}(-2a_{2i-1}b_{2i-1} + a_{2i}b_{2i-1} + a_{2i-1}b_{2i} - 2a_{2i}b_{2i}).$$

This sum can be reduced to the sum of three terms of the form $(-2a_{2i-1}b_{2i-1} + a_{2i}b_{2i-1} + a_{2i-1}b_{2i} - 2a_{2i}b_{2i})$. Each of them is equal to -2, 0 or 1:

▷ -2, if and only if $\begin{cases} (a_{2i-1}, a_{2i}) & = (b_{2i-1}, b_{2i}) = (1,0), \text{ or} \\ (a_{2i-1}, a_{2i}) & = (b_{2i-1}, b_{2i}) = (0,1); \end{cases}$

▷ 0, if and only if $\begin{cases} (a_{2i-1}, a_{2i}) & = (0,0), \text{ or} \\ (b_{2i-1}, b_{2i}) & = (0,0); \end{cases}$

$\triangleright$ 1, if and only if $\begin{cases} (a_{2i-1}, a_{2i}) & = (1,0),\ (b_{2i-1}, b_{2i}) = (0,1), \text{ or} \\ (a_{2i-1}, a_{2i}) & = (0,1),\ (b_{2i-1}, b_{2i}) = (1,0). \end{cases}$

Therefore $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = -1$ if and only if there are exactly two terms equal to -2 and one equal to 1. On the other hand, $\langle \mathbf{b}_u, \mathbf{b}_v \rangle = 0$ if and only if one term is -2 and two are equal to 1 or all three are 0. Now, we can reformulate the obtained results as follows. Two good vertices u and v with the corresponding subsets U and V of hK_2 are good partners in the following three cases:

- $\triangleright$ If U and V have only one vertex in common, then the remaining two are end-vertices of two copies of K_2 and u and v are non-adjacent.
- $\triangleright$ If U and V are disjoint, then there exists no copy of K_2 having one end vertex in U and the other in V and u and v are non-adjacent.
- $\triangleright$ If U and V have exactly two vertices in common, then the remaining ones are different end-vertices of the same copy of K_2, and u and v are adjacent.

Note that H has at least 6 vertices. In the following example we will discuss what happens when $H = 3K_2$.

Example 1. Let $H = 3K_2$ be such that $V(H) = \{1, \ldots, 6\}$ and $E(H) = \{12, 34, 56\}$, and consider good vertices $u_1, \ldots, u_8$, such that $U_1 = \{1,3,5\}$, $U_2 = \{1,4,6\}$, $U_3 = \{2,3,6\}$, $U_4 = \{2,4,5\}$, $U_5 = \{1,3,6\}$, $U_6 = \{1,4,5\}$, $U_7 = \{2,3,5\}$, $U_8 = \{2,4,6\}$. The maximal number of those which are compatible is 4 (for $U_i \cap U_j = \emptyset$, u_i and u_j are not good partners). Up to isomorphism we can add:

- $\triangleright$ u_1, u_2, u_3, u_4;
- $\triangleright$ u_1, u_2, u_3, u_5;
- $\triangleright$ u_1, u_2, u_5, u_6.

This leads to three connected maximal graphs G_1, G_2, G_3, with $(1,3)$-regular star complement $H = 3K_2$ for the eigenvalue $\mu = -2$. (Figures 3–5, where bold edges belong to a star complement.)

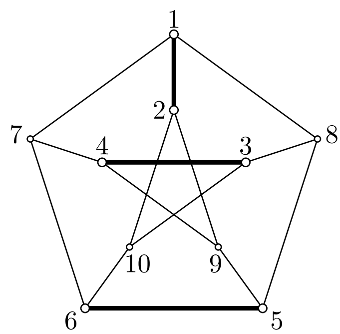

Figure 3. The Petersen graph G_1, where the bold edges 12, 34, and 56 belong to the star complement for $\mu = -2$.

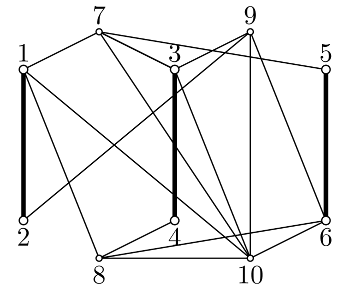

Figure 4. Graph G_2, where the bold edges 12, 34, and 56 belong to the star complement for $\mu = -2$.

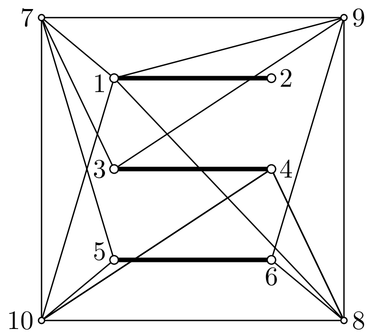

Figure 5. Graph G_3, where the bold edges 12, 34, and 56 belong to the star complement for $\mu = -2$.

Proposition 4.6. *If G is an S-maximal graph with $\kappa = 1$ and $\tau = 3$, then $G = n_1G_1 \cup n_2G_2 \cup n_3G_3 \cup n_4K_2$, where the graphs G_1, G_2, G_3 are depicted in Figures 3–5.*

Moreover, we can conclude that there exists no graph with a $(1,3)$-regular co-star set H if $|V(H)| < 6$. Graphs G_1, G_2, G_3 are the only connected graphs in this class (graphs with a $(1,3)$-regular co-star set).

Now, we switch to the complementary case.

Proposition 4.7. *If G is an S-maximal graph with $\kappa = s-2$, then $\tau \in \{s-1, s-3\}$ and $G = \overline{\frac{1}{2}sC_4}$ for $\tau = s-1$ and $\overline{n_1G_1 \cup n_2G_2 \cup n_3G_3 \cup n_4K_2}$ for $\tau = s-2$, where the graphs G_1, G_2, G_3 are depicted in Figures 3–5.*

4.3. Case $\kappa = 2$. Now H is a disjoint union of cycles. For simplicity, here we will assume that H is connected. So let $H = C_h$, and $\mu = 2 - \tau$, $C = \mathrm{circul}(0, 1, 0, \dots, 0, 1)$, where circul denotes the circulant matrix (in this case of order h). Recall that (see [21, p. 107])

$$\operatorname{circul}(a_1,\ldots,a_h)^{-1}=\frac{1}{h}\overline{F}D^{-1}F,$$

where F is the $h\times h$ matrix with $f_{ij}=\omega^{(i-1)(j-1)}$, $\omega=\mathrm{e}^{2\pi i/h}$ and $D=\operatorname{diag}(\lambda_1,\ldots,\lambda_h)$ is such that $\lambda_i=f(\omega^{(i-1)})$ for $f(\lambda)=\sum\limits_{i=1}^{h}a_i\lambda^{i-1}$. Then

$$\langle\mathbf{b}_u,\mathbf{b}_u\rangle=\mathbf{b}_u^{\mathrm{T}}\operatorname{circul}(\mu,-1,0,\ldots,0,-1)^{-1}\mathbf{b}_u=\mu$$

if and only if

$$\sum_{i=1}^{h}\lambda_i^{-1}|x_i|^2=\mu h, \tag{4.2}$$

where $x_i=\sum\limits_{j=1}^{h}b_j\omega^{(i-1)(j-1)}$. Since $\mathbf{b}_u$ has exactly τ nonzero entries, hence $x_1=\tau$. It is well known (cf. [21, p. 107]) that $\lambda_i=\mu-\omega^{(i-1)}-\omega^{-(i-1)}=\mu-2\cos 2(i-1)\pi/h$, which implies $\mu-2\leqslant\lambda_i\leqslant\mu+2$, that is, $-\tau\leqslant\lambda_i\leqslant 4-\tau$. Suppose $\tau>4$. Hence $1/(4-\tau)\leqslant 1/\lambda_i\leqslant -1/\tau$ and consequently $\sum\limits_{i=2}^{h}\lambda_i^{-1}|x_i|^2\leqslant-(h-1)\tau^2/\tau$, since $|x_i|^2\leqslant\tau^2$. Moreover, $\lambda_1^{-1}|x_1|^2=\tau^2/(\mu-2)$. Summing up all these observations, from (4.2) we obtain

$$\frac{\tau^2}{\mu-2}+\Big(-\frac{(h-1)\tau^2}{\tau}\Big)\geqslant\mu h,$$

an obvious contradiction. Therefore $\tau\in\{1,2,3,4\}$. Since all eigenvalues of C_h are different from $2-\tau$, we conclude the following:

(1) If $h\equiv 0(\text{mod }12)$, then there is no option for τ;
(2) if $h\equiv 6(\text{mod }12)$, then $\tau=2$;
(3) if $h\equiv x(\text{mod }12)$, with $x\in\{1,5,7,11\}$, then $\tau\in\{1,2,3,4\}$;
(4) if $n\equiv x(\text{mod }12)$, with $x\in\{2,10\}$, then $\tau\in\{1,2,3\}$;
(5) if $n\equiv x(\text{mod }12)$, with $x\in\{3,9\}$, then $\tau\in\{1,2,4\}$;
(6) if $n\equiv x(\text{mod }12)$, with $x\in\{4,8\}$, then $\tau\in\{1,3\}$.

Subcase $\tau=1$: Now $\mu=1$. To determine all good vertices we will determine all unicyclic graphs with one pendent edge having 1 as an eigenvalue. These graphs are characterized in the next lemma.

Lemma 4.8. *Let G be the graph obtained from C_h by adding a pendant edge. Then C_h is a star complement for the eigenvalue 1 of G if and only if $h = 6k-1$ for some $k \in \mathbb{N}$.*

P r o o f. Note that $h \pmod 6 \neq 0$ since $1 \in \sigma(C_{6k})$ for any $k \in \mathbb{N}$. Let $\mathbf{x} = (x_0, x_1, \dots x_h)^{\mathrm{T}}$ be the eigenvector for $\mu = 1$ (the pendent vertex is labelled by 0). We may set $x_0 = 1$ since 0 is in the star set. The eigenvalue equations yield

$$\mathbf{x} = (1, \underbrace{1, a, a-1, -1, -a, 1-a}, \dots, \underbrace{1, a, a-1, -1, -a, 1-a}, \dots, -a)^{\mathrm{T}}$$

for some $a \in \mathbb{R}$. The coordinates of $\mathbf{x}$ (starting from x_1) will periodically repeat with period 6. Depending on the remainder of h modulo 6, $x_h = -a$ takes one of the values $1, a, a-1, -1, -a$ and $1-a$. Except for $h = 6k-1$ for some $k \in \mathbb{N}$, this argument leads to a contradiction. $\square$

For $C = A_{C_{6k-1}}$ we have

$$(I-C)^{-1} = \mathrm{circul}(1, \underbrace{\underbrace{0, -1, -1}, \underbrace{0, 1, 1}, \dots, \underbrace{0, -1, -1}}_{2k-1}, 0).$$

Good sets are singletons and hence the corresponding characteristic vectors can be labelled by e_i, $1 \leqslant i \leqslant 6k-1$ and good vertices by u_i. Let $(a_1, \dots, a_{6k-1})^{\mathrm{T}} = (1, 0, -1, -1, 0, 1, 1, \dots, 0, -1, -1, 0)^{\mathrm{T}}$ so that

- ▷ $a_1 = a_{6p} = a_{6p+1} = 1$ for $1 \leqslant p \leqslant k-1$;
- ▷ $a_{3p+2} = 0$ for $0 \leqslant p \leqslant 2k-1$;
- ▷ $a_{6p+3} = a_{6p+4} = -1$ for $0 \leqslant p \leqslant k-1$.

From

$$e_i^{\mathrm{T}}(I-C)^{-1}e_j = \begin{cases} a_{6k-i+j}, & j \leqslant i-1, \\ a_{j-i+1}, & j > i-1 \end{cases}$$

we see that $u_i \sim u_j$ $(i < j)$ if and only if $j - i \equiv 2, 3 \pmod 6$; $u_i \nsim u_j$ if and only if $j - i \equiv 1, 4 \pmod 6$ while i and j are not good partners if and only if $j - i \equiv 0, 5 \pmod 6$. Hence, we can add at most 5 vertices. Moreover, by each rotation of u_i from i to any $6l + i$ the cycle C_{6k-1} remains a $(2,1)$-regular co-star set for $\mu = 1$. Hence:

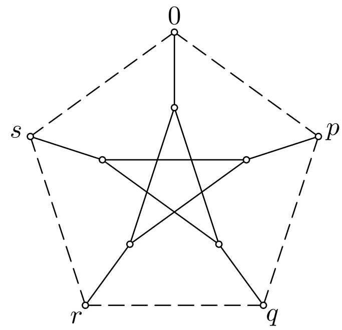

Figure 6. Graph with a $(2,1)$-regular star complement C_{6k-1}.

Theorem 4.9. *Let G be a maximal graph having the cycle C_h as a $(2,1)$-regular co-star set for the non-main eigenvalue μ. Then $\mu = 1$ and $h \equiv 5 \pmod 6$ and G is a graph depicted in Figure 6, where*

$$\{d(0,p), d(0,q), d(0,r), d(0,s)\} = \{1,2,3,4\},$$

and $d(0,v)$ is the reduced modulo 6 clockwise distance between 0 and v. In particular, if $h = 5$ then G is the Petersen graph.

Note that p, q, r, s are not necessarily in cyclic order on C_{6k-1} if $k > 1$.

Subcase $\tau = 2$: Then $\mu = 0$. Let $C = \operatorname{circul}(0,1,0,0,\ldots,0,1)$ of size h with $h \not\equiv 0 \pmod 4$ and let $D = 2C^{-1}$. Here we consider only the case $h \equiv 1 \pmod 4$; the other two cases are quite analogous. Then

$$D = \operatorname{circul}(1,1,-1,-1;1,1,-1,-1;\ldots;1,1,-1,-1;1).$$

For $\mathbf{b} = (b_1,\ldots,b_h)^{\mathrm{T}}$ we have

$$-2\langle \mathbf{b}, \mathbf{b}\rangle = \sum_{i=1}^{h} b_i^2 + 2 \sum_{\substack{1\leqslant i<j\leqslant h \\ j-i\equiv 0 \pmod 4 \\ j-i\equiv 1 \pmod 4}} b_i b_j - 2 \sum_{\substack{1\leqslant i<j\leqslant h \\ j-i\equiv 2 \pmod 4 \\ j-i\equiv 3 \pmod 4}} b_i b_j,$$

which is equal to 0 if and only if

$$b_k = \begin{cases} 0, & k \neq i,j, \\ 1, & k = i,j, \quad \text{where } j-i \equiv 2 \pmod 4 \text{ or } j-i \equiv 3 \pmod 4. \end{cases}$$

First let $\Delta_H(u_{ij}) = \{i,j\}$. Since $h = 4l+1$, we have that

$$4l-1+4l-5+\ldots+3+4l-2+4l-6+\ldots+2 = l(4l+1) = hl.$$

Therefore, there are hl good vertices u_{ij} arising from $\Delta_H(u_{i,j})$, where (i,j) (with $i<j$) corresponds to the position of -1 in the matrix D. Good vertices correspond to the sets

$$\{i, 4k+2+i\},\ 1 \leqslant i \leqslant 4(l-k)-1 \quad \text{and} \quad \{i, 4k+3+i\},\ 1 \leqslant i \leqslant 4(l-k)-2.$$

There are too many different types of vertices and therefore the question of which of them are compatible becomes too messy. Therefore we restrict ourselves to some easier cases. First, it is easy to see that vertices of C_h at distance two are compatible with all other good vertices. If we add all h vertices that arise in this way we get the 4-regular graph G which is the NEPS (non-complete extended p-sum; for more details see [10, p. 43]) of graphs C_h and K_2 with basis $\mathfrak{B} = \{(1,1),(1,0)\}$. For $h=9$ this graph is depicted in Figure 7.

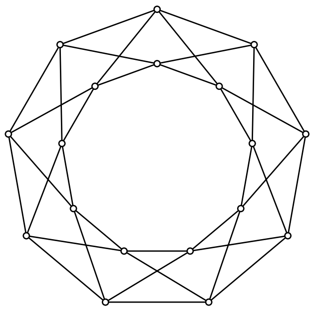

Figure 7. A 4-regular graph with a (2,2)-regular co-star set.

A similar procedure can be applied when $h \equiv 2, 3 \pmod 4$. Thus we have:

Example 2. The NEPS of graphs C_h ($h \not\equiv 0 \pmod 4$) and K_2 with basis $\mathfrak{B} = \{(1,1),(1,0)\}$ is a 4-regular graph having C_h as a $(2,2)$-regular star complement for the eigenvalue 0.

In the following example, we take $h=9$ and in this particular case we determine all maximal graphs having C_h as a $(2,2)$-regular co-star set for the eigenvalue $\mu = 0$.

Example 3. In this situation we divide good sets into two sets

$$S = \{\{1,3\},\{2,4\},\{3,5\},\{4,6\},\{5,7\},\{6,8\},\{7,9\},\{1,8\},\{2,9\}\} \quad \text{and}$$
$$T = \{\{1,4\},\{2,5\},\{3,6\},\{4,7\},\{5,8\},\{6,9\},\{1,7\},\{2,8\},\{3,9\}\}.$$

The vertices associated with subsets of S are compatible with all other subsets, while each vertex associated with a subset in T is not compatible with two vertices. More precisely, vertices arising from each of subsets $T_1 = \{\{1,4\},\{4,7\},\{1,7\}\}$,

$T_2 = \{\{2,5\},\{5,8\},\{2,8\}\}$ and $T_3 = \{\{3,6\},\{6,9\},\{3,9\}\}$ are not compatible. So, we have 3^3 possibilities. However, some give rise to isomorphic graphs.We find that there are three maximal reduced non isomorphic graphs with the desired properties. Their star sets correspond to the following three sets: $S \cup \{\{1,4\},\{2,5\},\{3,6\}\}$, $S \cup \{\{1,4\},\{2,5\},\{6,9\}\}$ and $S \cup \{\{1,4\},\{5,8\},\{3,6\}\}$.

Subcase $\tau = 3$*:* Now $\mu = -1$. Again, by similar calculation, it follows that all good vertices which correspond to the three consecutive vertices of the cycle C_h are compatible with all. If we include all such h vertices we get a 5-regular graph, bearing in mind that $h \not\equiv 0 \pmod 3$. Moreover:

Example 4. NEPS of graphs C_h ($h \not\equiv 0 \pmod 3$) and K_2 with basis $\mathfrak{B} = \{(1,1),(0,1),(1,0)\}$ is a 5-regular graph having C_h as a $(2,3)$-regular co-star set for the eigenvalue -1.

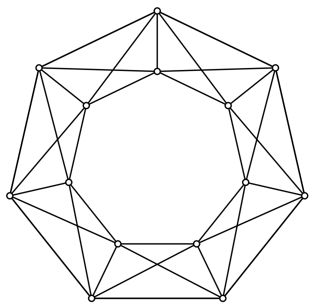

Figure 8. A 5-regular graph with a (2,3)-regular co-star set

Subcase $\tau = 4$: Now $\mu = -2$. Graphs with a $(2,4)$-regular co-star set are determined in [2] as graphs whose star complement for -2 is a cycle. The maximal graph is the line graph of K_h. The construction is possible only for odd cycles. It turns out that the good sets are the end-vertices of two non-adjacent edges of C_h.

We conclude this section with the following two comments.

First, we have also in mind considering the star complement technique in the presence of some regularities (here (κ,τ)-regular sets) in order to see the extent to which maximal extensions are characterized by a simple subgraph (i.e. the star complement) and one eigenvalue, see [15], [16], [18].

Secondly, considering the above examples, it turns out that the Petersen graph (see Theorem 4.9) appears in this context; but also, in the same theorem, we encountered some non-regular graphs as its unusual "generalizations". So far we have tackled some simpler cases for which analytical considerations were possible. It will be interesting to apply this technique to other cases. For instance, we believe that some strongly regular graphs, besides the Petersen graph, can arise. But then the computer aided approach seems to be preferable.

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