Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 706-717, October 2013

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COPIES OF A ROOTED WEIGHTED GRAPH ATTACHED TO AN ARBITRARY WEIGHTED GRAPH AND APPLICATIONS*

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Abstract. The spectrum of the Laplacian, signless Laplacian and adjacency matrices of the family of the weighted graphs $\mathcal{R}\{\mathcal{H}\}$, obtained from a connected weighted graph $\mathcal{R}$ on r vertices and r copies of a modified Bethe tree $\mathcal{H}$ by identifying the root of the i-th copy of $\mathcal{H}$ with the i-th vertex of $\mathcal{R}$, is determined.

Key words. Weighted graph, Generalized Bethe tree, Laplacian matrix, Signless Laplacian matrix, Adjacency matrix, Randić matrix.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a simple undirected graph with vertex set $V(\mathcal{G}) = \{1, \ldots, n\}$ and edge set $E(\mathcal{G})$. We assume that each edge $e \in E(\mathcal{G})$ has a positive weight $w(e)$. The adjacency matrix $A(\mathcal{G}) = (a_{i,j})$ of $\mathcal{G}$ is the $n \times n$ matrix in which $a_{i,j} = w(e)$ if there is an edge e joining i and j and $a_{i,j} = 0$ otherwise. Let $D(\mathcal{G})$ be the diagonal matrix in which the diagonal entry $d_{i,i} = \sum_e w(e)$ where the sum is over all the edges e incident to the vertex i. The Laplacian matrix and the signless Laplacian matrix of $\mathcal{G}$ are $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$, respectively. The matrices $L(\mathcal{G})$, $Q(\mathcal{G})$ and $A(\mathcal{G})$ are real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(\mathcal{G})$ and $Q(\mathcal{G})$ are nonnegative real numbers. Since the rows of $L(\mathcal{G})$ sum to 0, $(0, \mathbf{e})$ is an eigenpair for $L(\mathcal{G})$, where $\mathbf{e}$ is the all ones vector. Fiedler [9] proved that $\mathcal{G}$ is a connected graph if and only

*Received by the editors on November 1, 2012. Accepted for publication on September 22, 2013. Handling Editor: Xingzhi Zhan.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 706-717, October 2013

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if the second smallest eigenvalue of $L(\mathcal{G})$ is positive. This eigenvalue is called the algebraic connectivity of $\mathcal{G}$. The signless Laplacian matrix has recently attracted the attention of several researchers and some papers on this matrix are [2, 4, 5, 6, 7]. In this paper, $M(\mathcal{G})$ is one of the matrices $L(\mathcal{G})$, $Q(\mathcal{G})$ or $A(\mathcal{G})$. If $w(e) = 1$ for all $e \in E(\mathcal{G})$ then $\mathcal{G}$ is an unweighted graph.

Let $\mathcal{R}$ be a connected weighted graph on r vertices. Let $v_1, v_2, \ldots, v_r$ be the vertices of $\mathcal{R}$. As usual, $v_i \sim v_j$ means that v_i and v_j are adjacent. Let $\varepsilon_{i,j} = \varepsilon_{j,i}$ be the weight of the edge $v_i v_j$ if $v_i \sim v_j$, and let $\varepsilon_{i,j} = \varepsilon_{j,i} = 0$ otherwise. Moreover, for $i = 1, 2, \ldots, r$, let $\varepsilon_i = \sum_{v_j \sim v_i} \varepsilon_{i,j}$. Let $\mathcal{R}\{\mathcal{H}\}$ be the graph obtained from $\mathcal{R}$ and r copies of a rooted weighted graph $\mathcal{H}$ by identifying the root of $i-$copy of $\mathcal{H}$ with v_i.

Example 1.1. If $\mathcal{R}$ is the graph depicted in Figure 1.1 and $\mathcal{H}$ is the graph

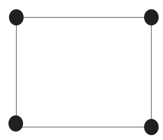

Fig. 1.1. *The cycle C_4.*

depicted in Figure 1.2 then $\mathcal{R}\{\mathcal{H}\}$ is the graph depicted in Figure 1.3.

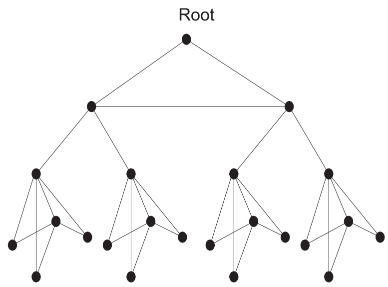

Fig. 1.2. *A modified Bethe tree, $\mathcal{H}$, with four levels.*

We recall that for a rooted graph the level of a vertex is one more than its distance from the root vertex. Let $\mathcal{B}$ be a weighted generalized Bethe tree of $k > 1$ levels, that is, $\mathcal{B}$ is a rooted tree in which vertices at the same level have the same degree and edges connecting vertices at consecutive levels have the same weight. Consider a nonempty subset $\Delta \subseteq \{1, 2, \ldots, k-1\}$ and a family of graphs $F = \{\mathcal{G}_j : j \in \Delta\}$. For $j \in \Delta$, we assume that the edges of $\mathcal{G}_j$ have weight u_j. Let $\mathcal{B}(F)$ be the graph obtained from $\mathcal{B}$ and the graphs in F identifying each set of children of $\mathcal{B}$ at level $k - j + 1$ with the vertices of $\mathcal{G}_j$.

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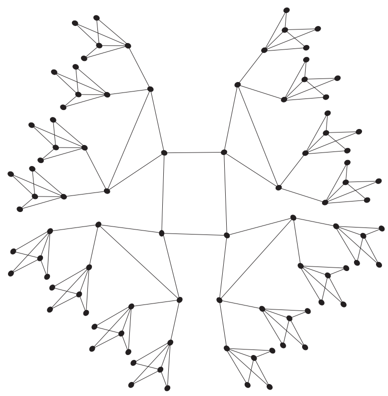

FIG. 1.3. *The graph* $\mathcal{R}\{\mathcal{H}\}$.

EXAMPLE 1.2. Let $\mathcal{B}(F)$ be the graph depicted in Figure 1.2. In this graph, $\mathcal{B}$ is a generalized Bethe tree of $k = 4$ levels, $\Delta = \{1, 3\}$, F=$\{\mathcal{G}_1, \mathcal{G}_3\}$, where $\mathcal{G}_1$ is a star of 4 vertices and $\mathcal{G}_3$ is a path of 2 vertices.

In this paper, we derive a general result on the spectrum of $M(\mathcal{R}\{\mathcal{H}\})$. Using this result, we characterize the eigenvalues of the Laplacian matrix, including their multiplicities, of the graph $\mathcal{R}\{\mathcal{B}(F)\}$; and also of the signless Laplacian and adjacency matrices whenever the subgraphs in F are regular. They are the eigenvalues of symmetric tridiagonal matrices of order j, $1 \le j \le k$. In particular, the Randić eigenvalues are characterized.

Denote by $\sigma(C)$ the multiset of eigenvalues of a square matrix C.

2. A result on the spectrum of $M(\mathcal{R}\{\mathcal{H}\})$**.** Let E be the matrix of order $n \times n$ with 1 in the (n, n) −entry and zeros elsewhere. For $i = 1, 2, \ldots, r$, let $d(v_i)$ be the degree of v_i as a vertex of $\mathcal{R}$ and let n be the order of $\mathcal{H}$. Then $\mathcal{R}\{\mathcal{H}\}$ has rn vertices. We label the vertices of $\mathcal{R}\{\mathcal{H}\}$ as follows: for $i = 1, 2, \ldots, r$, using the labels $(i-1)n+1, (i-1)n+2, \ldots, in$, we label the vertices of the $i-th$ copy of $\mathcal{H}$ from the bottom to the vertex v_i and, at each level, from the left to the right. With this labeling, $M(\mathcal{R}\{\mathcal{H}\})$ is equal to

$$\begin{bmatrix} M(\mathcal{H}) + a\varepsilon_1 E & s\varepsilon_{1,2}E & \cdots & \cdots & s\varepsilon_{1,r}E \\ s\varepsilon_{1,2}E & \ddots & \ddots & & s\varepsilon_{2,r}E \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & M(\mathcal{H}) + a\varepsilon_{i,r-1}E & s\varepsilon_{r-1,r}E \\ s\varepsilon_{1,r}E & s\varepsilon_{2,r}E & \cdots & s\varepsilon_{r-1,r}E & M(\mathcal{H}) + a\varepsilon_r E \end{bmatrix}, \tag{2.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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where

$$s = \begin{cases} -1 & \text{if } M \text{ is the Laplacian matrix,} \\ 1 & \text{if } M \text{ is the signless Laplacian or adjacency matrix} \end{cases} \tag{2.2}$$

and

$$a = \begin{cases} 0 & \text{if } M \text{ is the adjacency matrix,} \\ 1 & \text{if } M \text{ is the Laplacian or signless Laplacian matrix.} \end{cases} \tag{2.3}$$

In this paper, the identity matrix of appropriate order is denoted by I and I_m denotes the identity matrix of order m. Furthermore, $|A|$ denotes the determinant of the matrix A and A^T is the transpose of A.

We recall that the Kronecker product [12] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined as the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}B)$. Then, in particular, $I_n \otimes I_m = I_{nm}$. Some basic properties of the Kronecker product are $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ for matrices of appropriate sizes. Moreover, if A and B are invertible matrices then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Theorem 2.1. *Let $\rho_1(\mathcal{R}), \rho_2(\mathcal{R}), \ldots, \rho_r(\mathcal{R})$ be the eigenvalues of $M(\mathcal{R})$. Then*

$$\sigma(M(\mathcal{R}\{\mathcal{H}\})) = \cup_{s=1}^{r} \sigma(M(\mathcal{H}) + \rho_s(\mathcal{R}) E). \tag{2.4}$$

Proof. From (2.1), it follows

$$M(\mathcal{R}\{\mathcal{H}\}) = I_r \otimes M(\mathcal{H}) + M(\mathcal{R}) \otimes E.$$

Let

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_{r-1} & \mathbf{v}_r \end{bmatrix}$$

be an orthogonal matrix whose columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to the eigenvalues $\rho_1(\mathcal{R}), \rho_2(\mathcal{R}), \ldots, \rho_r(\mathcal{R})$, respectively. Then

$$\begin{aligned}(V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) \left(V^T \otimes I_n\right) &= (V \otimes I_n)(I_r \otimes M(\mathcal{H}) + M(\mathcal{R}) \otimes E)\left(V^T \otimes I_n\right) \\ &= I_r \otimes M(\mathcal{H}) + \left(VM(\mathcal{R})V^T\right) \otimes E.\end{aligned}$$

We have

$$\left(VM(\mathcal{R})V^T\right) \otimes E = \begin{bmatrix} \rho_1(\mathcal{R}) & & & & \\ & \rho_2(\mathcal{R}) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \rho_r(\mathcal{R}) \end{bmatrix} \otimes E$$

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$$= \begin{bmatrix} \rho_1(\mathcal{R})E & & & & \\ & \rho_2(\mathcal{R})E & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \rho_r(\mathcal{R})E \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &(V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) \left(V^T \otimes I_n\right) \\ &= \begin{bmatrix} M(\mathcal{H}) + \rho_1(\mathcal{R})E & & & & \\ & M(\mathcal{H}) + \rho_2(\mathcal{R})E & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M(\mathcal{H}) + \rho_r(\mathcal{R})E \end{bmatrix}. \end{aligned}$$

Since $M(\mathcal{R}\{\mathcal{H}\})$ and $(V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) \left(V^T \otimes I_n\right)$ are similar matrices, we conclude (2.4). □

3. Application to a modified generalized Bethe tree with subgraphs at some levels. Let $\mathcal{B}$ be a weighted generalized Bethe tree of k levels ($k > 1$). For $1 \leq j \leq k$, n_j and d_j are the number and the degree of the vertices of $\mathcal{B}$ at the level $k - j + 1$, respectively. Thus, d_k is the degree of the root vertex (assumed greater than 1), $n_k = 1$, $d_1 = 1$ and n_1 is the number of pendant vertices. For $1 \leq j \leq k - 1$, let w_j be the weight of the edges connecting the vertices of $\mathcal{B}$ at the level $k - j + 1$ with the vertices at the level $k - j$. We consider

$$\delta_j = \begin{cases} w_j & \text{if } j = 1, \\ (d_j - 1) w_{j-1} + w_j & \text{if } 2 \leq j \leq k - 1, \\ d_k w_{k-1} & \text{if } j = k. \end{cases}$$

We observe that δ_j is the sum of the weights of the edges of $\mathcal{B}$ incident with a vertex at the level $k-j+1$ and if $w_1 = w_2 = \cdots = w_{k-1} = 1$ then $\delta_j = d_j$ for $j = 1, 2, \ldots, k$. Let $m_j = \frac{n_j}{n_{j+1}}$ for $j = 1, 2, \ldots, k - 1$. Then

$$\begin{aligned} m_j &= d_{j+1} - 1 \ \ (1 \leq j \leq k - 2), \\ d_k &= n_{k-1} = m_{k-1}. \end{aligned}$$

Note that m_j is the cardinality of each set of children at the level $k - j + 1$.

Let $M_j = M(\mathcal{G}_j)$. From now on, we assume that $\mu_1(M_j), \ldots, \mu_{m_j}(M_j)$ are the eigenvalues of M_j and $\mathbf{e}_{m_j}$ is an eigenvector for $\mu_{m_j}(M_j)$, that is,

$$M_j \mathbf{e}_{m_j} = \mu_{m_j}(M_j)\, \mathbf{e}_{m_j}. \tag{3.1}$$

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We observe that (3.1) holds when $M(\mathcal{G}_j) = L(\mathcal{G}_j)$ and when $M(\mathcal{G}_j) = Q(\mathcal{G}_j)$ or $M(\mathcal{G}_j) = A(\mathcal{G}_j)$ if $\mathcal{G}_j$ is a regular graph.

Assuming (3.1), in [11], we characterize the eigenvalues of the matrix $M(\mathcal{B}(F)) =$

$$\begin{bmatrix} I_{n_2} \otimes S_1 & sI_{n_2} \otimes w_1 \mathbf{e}_{m_1} & & & \\ sI_{n_2} \otimes w_1 \mathbf{e}_{m_1}^T & \ddots & \ddots & & \\ & \ddots & I_{n_{k-1}} \otimes S_{k-2} & sI_{n_{k-1}} \otimes w_{k-2} \mathbf{e}_{m_{k-2}} & \\ & & sI_{n_{k-1}} \otimes w_{k-2} \mathbf{e}_{m_{k-2}} & S_{k-1} & sw_{k-1} \mathbf{e}_{m_{k-1}} \\ & & & sw_{k-1} \mathbf{e}_{m_{k-1}} & a\delta_k \end{bmatrix}$$

in which, for $j = 1, 2, \ldots, k-1$,

$$S_j = \begin{cases} a\delta_j I_{m_j} + M(\mathcal{G}_j) & \text{if } j \in \Delta, \\ a\delta_j I_{m_j} & \text{if } j \notin \Delta, \end{cases} \tag{3.2}$$

with s and a as in (2.2) and (2.3).

The results in [11] generalize several previous contributions (see [3, 8, 10]).

Definition 3.1. For $j = 1, \ldots, k$, let X_j be the $j \times j$ leading principal submatrix of the $k \times k$ symmetric tridiagonal matrix $X_k =$

$$\begin{bmatrix} a\delta_1 + \mu_{m_1}(M_1) & w_1\sqrt{m_1} & & & \\ w_1\sqrt{m_1} & a\delta_2 + \mu_{m_2}(M_2) & \ddots & & \\ & \ddots & \ddots & w_{k-2}\sqrt{m_{k-2}} & \\ & & w_{k-2}\sqrt{m_{k-2}} & a\delta_{k-1} + \mu_{m_{k-1}}(M_{k-1}) & w_{k-1}\sqrt{m_{k-1}} \\ & & & w_{k-1}\sqrt{m_{k-1}} & a\delta_k \end{bmatrix}.$$

Definition 3.2. For $j = 1, 2, \ldots, k-1$ and $i = 1, \ldots, m_j - 1$, let $X_{j,i} =$

$$\begin{bmatrix} a\delta_1 + \mu_{m_1}(M_1) & w_1\sqrt{m_1} & & & \\ w_1\sqrt{m_1} & \ddots & \ddots & & \\ & \ddots & \ddots & w_{j-2}\sqrt{m_{j-2}} & \\ & & w_{j-2}\sqrt{m_{j-2}} & a\delta_{j-1} + \mu_{m_{j-1}}(M_{j-1}) & w_{j-1}\sqrt{m_{j-1}} \\ & & & w_{j-1}\sqrt{m_{j-1}} & a\delta_j + \mu_i(M_j) \end{bmatrix}.$$

Finally, let

$$\Omega = \{j : 1 \le j \le k-1, \ n_j > n_{j+1}\}.$$

We are ready to state the main result published in [11].

Theorem 3.3. [11]

$$\sigma(M(\mathcal{B}(\mathrm{F}))) = \sigma(X_k) \cup \left(\cup_{j \in \Omega - \Delta} \sigma(X_j)^{n_j - n_{j+1}}\right) \cup \left(\cup_{j \in \Delta} \cup_{i=1}^{m_j - 1} \sigma(X_{j,i})^{n_{j+1}}\right),$$

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where $\sigma(X_j)^{n_j-n_{j+1}}$ and $\sigma(X_{j,i})^{n_{j+1}}$ mean that each eigenvalue in $\sigma(X_j)$ and in $\sigma(X_{j,i})$ must be considered with multiplicity n_j-n_{j+1} and n_{j+1}, respectively. Furthermore, the multiplicities of equal eigenvalues obtained in different matrices (if any), must be added.

An equivalent version of Theorem 3.3 is:

Theorem 3.4.

$$|\lambda I - M(\mathcal{B}(\mathrm{F}))| = D_k(\lambda) \prod_{j\in\Omega-\Delta} (D_j(\lambda))^{n_j-n_{j+1}} \prod_{j\in\Delta} \prod_{i=1}^{m_j-1} (D_{j,i}(\lambda))^{n_{j+1}},$$

where, for $j=1,2,\dots,k$ and $i=1,2,\dots,m_j-1$, $D_j(\lambda)$ and $D_{j,i}(\lambda)$ are the characteristic polynomials of the matrices X_j and $X_{j,i}$, respectively.

Let $\widetilde{A}$ be the submatrix obtained from a square matrix A by deleting its last row and its last column.

Moreover, from the proofs of Lemma 2.2, Theorem 2.5 and Lemma 2.7 in [11], we obtain:

Lemma 3.5.

$$\left|\lambda I - \widetilde{M(\mathcal{B}(\mathrm{F}))}\right| = D_{k-1}(\lambda) \prod_{j\in\Omega-\Delta} (D_j(\lambda))^{n_j-n_{j+1}} \prod_{j\in\Delta} \prod_{i=1}^{m_j-1} (D_{j,i}(\lambda))^{n_{j+1}}.$$

Definition 3.6. For $s=1,2,\dots,r$, let $Y(s) =$

$$\begin{bmatrix} a\delta_1+\mu_{m_1}(M_1) & w_1\sqrt{m_1} & & & \\ w_1\sqrt{m_1} & a\delta_2+\mu_{m_2}(M_2) & \ddots & & \\ & \ddots & \ddots & w_{k-2}\sqrt{m_{k-2}} & \\ & & w_{k-2}\sqrt{m_{k-2}} & a\delta_{k-1}+\mu_{m_{k-1}}(M_{k-1}) & w_{k-1}\sqrt{m_{k-1}} \\ & & & w_{k-1}\sqrt{m_{k-1}} & a\delta_k+\rho_s(\mathcal{R}) \end{bmatrix}.$$

We are ready to apply Theorem 2.1 to $\mathcal{H}=\mathcal{B}(F)$.

Theorem 3.7. *If $\mathcal{G}=\mathcal{R}\{\mathcal{B}(\mathrm{F})\}$, then*

$$\sigma(M(\mathcal{G})) = \left(\cup_{j\in\Omega-\Delta}\sigma(X_j)^{r(n_j-n_{j+1})}\right) \cup \left(\cup_{j\in\Delta}\cup_{i=1}^{m_j-1}\sigma(X_{j,i})^{rn_{j+1}}\right) \cup \left(\cup_{s=1}^{r}\sigma(Y(s))\right),$$

where $\sigma(X_j)^{r(n_j-n_{j+1})}$ and $\sigma(X_{j,i})^{rn_{j+1}}$ mean that each eigenvalue in $\sigma(X_j)$ and in $\sigma(X_{j,i})$ must be considered with multiplicity $r(n_j-n_{j+1})$ and rn_{j+1}, respectively. Furthermore, the multiplicities of equal eigenvalues obtained in different matrices (if any), must be added.

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Proof. For $\mathcal{H} = \mathcal{B}(F)$, from Theorem 2.1, we have

$$\sigma\left(M\left(\mathcal{R}\left\{\mathcal{B}\left(F\right)\right\}\right)\right) = \cup_{s=1}^{r} \sigma\left(M\left(\mathcal{B}\left(F\right)\right) + \rho_s\left(\mathcal{R}\right) E\right). \tag{3.3}$$

Let $1 \leq s \leq r$. By linearity on the last column, we have

$$\left|\lambda I - M\left(\mathcal{B}\left(F\right)\right) - \rho_s\left(\mathcal{R}\right) E\right| = \left|\lambda I - M\left(\mathcal{B}\left(F\right)\right)\right| - \rho_s\left(\mathcal{R}\right)\left|\lambda I - \widetilde{M\left(\mathcal{B}\left(F\right)\right)}\right|.$$

Using Theorem 3.4 and Lemma 3.5, $\left|\lambda I - M\left(\mathcal{B}\left(F\right)\right) - \rho_s\left(\mathcal{R}\right) E\right|$ has the form

$$\left(D_k\left(\lambda\right) - \rho_s\left(\mathcal{R}\right) D_{k-1}\left(\lambda\right)\right) \prod_{j \in \Omega - \Delta} \left(D_j\left(\lambda\right)\right)^{n_j - n_{j+1}} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} \left(D_{j,i}\left(\lambda\right)\right)^{n_{j+1}}.$$

Now, by linearity on the last column, we have

$$\left|\lambda I - Y\left(s\right)\right| = \left|\lambda I - X_k\right| - \rho_s\left(\mathcal{R}\right)\left|\lambda I - X_{k-1}\right| = D_k\left(\lambda\right) - \rho_s\left(\mathcal{R}\right) D_{k-1}\left(\lambda\right).$$

Therefore,

$$\left|\lambda I - M\left(\mathcal{B}\left(F\right)\right) - \rho_s\left(\mathcal{R}\right) E\right| = Y\left(s\right) \prod_{j \in \Omega - \Delta} \left(D_j\left(\lambda\right)\right)^{n_j - n_{j+1}} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} \left(D_{j,i}\left(\lambda\right)\right)^{n_{j+1}}.$$

Applying (3.3), $\left|\lambda I - M\left(\mathcal{R}\left\{\mathcal{B}\left(F\right)\right\}\right)\right|$ becomes

$$\prod_{s=1}^{r} Y\left(s\right) \prod_{j \in \Omega - \Delta} \left(D_j\left(\lambda\right)\right)^{r\left(n_j - n_{j+1}\right)} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} \left(D_{j,i}\left(\lambda\right)\right)^{r n_{j+1}}. \quad \square$$

4. On the Laplacian, signless Laplacian and adjacency eigenvalues of $\mathcal{R}\left\{\mathcal{B}\left(\boldsymbol{F}\right)\right\}$. For each j, let

$$\mu_1\left(\mathcal{G}_j\right) \geq \mu_2\left(\mathcal{G}_j\right) \geq \cdots \geq \mu_{m_j - 1}\left(\mathcal{G}_j\right) \geq \mu_{m_j}\left(\mathcal{G}_j\right) = 0,$$

and let

$$\mu_1\left(\mathcal{R}\right) \geq \mu_2\left(\mathcal{R}\right) \geq \cdots \geq \mu_r\left(\mathcal{R}\right) = 0$$

be the Laplacian eigenvalues of $\mathcal{G}_j$ and $\mathcal{R}$, respectively.

Applying Theorem 3.7 to the determination of the Laplacian spectrum of $\mathcal{G}$, we obtain:

THEOREM 4.1. *The Laplacian spectrum of $\mathcal{G} = \mathcal{R}\left\{\mathcal{B}\left(\mathrm{F}\right)\right\}$ is*

$$\begin{aligned} \sigma\left(L\left(\mathcal{G}\right)\right) = \left(\cup_{j \in \Omega - \Delta} \sigma\left(U_j\right)^{r\left(n_j - n_{j+1}\right)}\right) &\cup \left(\cup_{j \in \Delta} \cup_{i=1}^{m_j - 1} \sigma\left(U_{j,i}\right)^{r n_{j+1}}\right) \\ &\cup \left(\cup_{s=1}^{r} \sigma\left(W\left(s\right)\right)\right) \end{aligned} \tag{4.1}$$

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where, for $j = 1, \ldots, k-1$, U_j is the $j \times j$ leading principal submatrix of the $k \times k$ matrix $U_k = X_k$ except for the diagonal entries which in U_k are

$$\delta_1, \delta_2, \ldots, \delta_{k-1}, \delta_k;$$

and, for $i = 1, 2, \ldots, m_j - 1$, $U_{j,i} = X_{j,i}$ except for the diagonal entries which in $U_{j,i}$ are

$$\delta_1, \delta_2, \ldots, \delta_{j-1}, \delta_j + \mu_i\left(\mathcal{G}_j\right);$$

and, for $s = 1, 2, \ldots, r$, $W(s) = Y(s)$ except for the diagonal entries which in $W(s)$ are

$$\delta_1, \delta_2, \ldots, \delta_{k-1}, \delta_k + \mu_s\left(\mathcal{R}\right).$$

The multiplicities of the eigenvalues of $L(\mathcal{G})$ are considered as in Theorem 3.7.

Proof. It must be noted that, since $M(\mathcal{G}) = L(\mathcal{G})$, we have $a = 1$,

$$S_j = \begin{cases} \delta_j I_{m_j} + L\left(\mathcal{G}_j\right) & \text{if } j \in \Delta, \\ \delta_j I_{m_j} & \text{if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = \delta_k I_r + L\left(\mathcal{R}\right).$$

Therefore, $L\left(\mathcal{G}_j\right)\mathbf{e}_{m_j} = \mathbf{0} = 0\mathbf{e}_{m_j}$ and the Laplacian spectrum of $\mathcal{G}$ is given by Theorem 3.7, replacing the matrices X_j, $X_{j,i}$, and $Y(s)$ by the matrices U_j, $U_{j,i}$ and $W(s)$, respectively. ∎

As above, let $\mathcal{G} = \mathcal{R}\left\{\mathcal{B}\left(F\right)\right\}$. We apply now Theorem 3.7 to find the eigenvalues of $Q(\mathcal{G})$ and $A(\mathcal{G})$ whenever each $\mathcal{G}_j$ is a regular graph of degree r_j. For convenience, the signless Laplacian eigenvalues and adjacency eigenvalues are denoted in increasing order. Let

$$q_1(\mathcal{G}) \leq q_2(\mathcal{G}) \leq q_3(\mathcal{G}) \leq \cdots \leq q_{m-1}(\mathcal{G}) \leq q_m(\mathcal{G})$$

and

$$\lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_{m-1}(\mathcal{G}) \leq \lambda_m(\mathcal{G})$$

be the eigenvalues of the signless Laplacian matrix and adjacency matrix of any graph $\mathcal{G}$, respectively. If $\mathcal{G}$ is a regular graph of degree t and order m in which the edges have a weight equal to u then $Q(\mathcal{G})\mathbf{e}_m = 2tu\mathbf{e}_m$ and $A(\mathcal{G})\mathbf{e}_m = tu\mathbf{e}_m$. In this case, we may write $\lambda_m(\mathcal{G}) = tu$ and $q_m(\mathcal{G}) = 2tu$.

THEOREM 4.2. *If for each $j \in \Delta$ the graph $\mathcal{G}_j$ is a regular graph of degree r_j and $r_j = 0$ whenever $j \notin \Delta$, then the signless Laplacian spectrum of $\mathcal{G} = \mathcal{R}\left\{\mathcal{B}\left(\mathrm{F}\right)\right\}$ is*

$$\sigma\left(Q\left(\mathcal{G}\right)\right) = \left(\cup_{j \in \Omega - \Delta}\sigma\left(V_j\right)^{r\left(n_j - n_{j+1}\right)}\right) \cup \left(\cup_{j \in \Delta} \cup_{i=1}^{m_j - 1} \sigma\left(V_{j,i}\right)^{rn_{j+1}}\right) \cup \left(\cup_{s=1}^{r}\sigma\left(U\left(s\right)\right)\right)$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 26, pp. 706-717, October 2013

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where, for $j = 1, 2, 3, \ldots, k-1$, V_j *is the* $j \times j$ *leading principal submatrix of* $V_k = X_k$ *except for the diagonal entries which in* V_k *are*

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \ldots, \delta_{k-1} + 2r_{k-1}u_{k-1}, \delta_k;$$

and, for $i = 1, 2, \ldots, m_j - 1$, $V_{j,i} = X_{j,i}$ *except for the diagonal entries which in* $V_{j,i}$ *are*

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \ldots, \delta_{j-1} + 2r_{j-1}u_{j-1}, \delta_j + q_i\left(\mathcal{G}_j\right);$$

and, for $s = 1, 2, \ldots, r$, $U(s) = Y(s)$ *except for the diagonal entries which in* $U(s)$ *are*

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \ldots, \delta_{k-1} + 2r_{k-1}u_{k-1}, \delta_k + q_s(\mathcal{R}).$$

The multiplicities of the eigenvalues of $Q(\mathcal{G})$ *must be considered as in Theorem 3.7.*

Proof. For $M(\mathcal{G}) = Q(\mathcal{G})$, we have $a = 1$,

$$S_j = \begin{cases} \delta_j I_{m_j} + Q\left(\mathcal{G}_j\right), \\ Q\left(\mathcal{G}_j\right) = 0 \text{ if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = \delta_k I_r + Q(\mathcal{R}),$$

$Q\left(\mathcal{G}_j\right)\mathbf{e}_{m_j} = 2r_ju_j\mathbf{e}_{m_j}$ if $j \in \Delta$ and $r_j = 0$ for $j \notin \Delta$. From Theorem 3.7, we obtain that the set of eigenvalues of $Q(\mathcal{G})$ is given replacing the matrices X_j, $X_{j,i}$, and Y_s by the matrices V_j, $V_{j,i}$ and $U(s)$, respectively. □

THEOREM 4.3. *If for each* $j \in \Delta$ *the graph* $\mathcal{G}_j$ *is a regular graph of degree* r_j *and* $r_j = 0$ *whenever* $j \notin \Delta$, *then the adjacency spectrum of* $\mathcal{G} = \mathcal{R}\{\mathcal{B}(\mathrm{F})\}$ *is*

$$\sigma(A(\mathcal{G})) = \left(\cup_{j\in\Omega-\Delta}\sigma\left(T_j\right)^{r(n_j-n_{j+1})}\right) \cup \left(\cup_{j\in\Delta}\cup_{i=1}^{m_j-1}\sigma\left(T_{j,i}\right)^{rn_{j+1}}\right) \cup \left(\cup_{s=1}^{r}\sigma(R(s))\right)$$

where, for $j = 1, 2, 3, \ldots, k-1$, T_j *is the* $j \times j$ *leading principal submatrix of* $T_k = X_k$ *except for the diagonal entries which in* T_k *are*

$$r_1u_1, r_2u_2, \ldots, r_{k-1}u_{k-1}, 0;$$

and, for $i = 1, 2, \ldots, m_j - 1$, $T_{j,i} = X_{j,i}$ *except for the diagonal entries which in* $T_{j,i}$ *are*

$$r_1u_1, r_2u_2, \ldots, r_{j-1}u_{j-1}, \lambda_i\left(\mathcal{G}_j\right);$$

and, for $s = 1, 2, \ldots, r$, $R(s) = Y(s)$ *except for the diagonal entries which in* $R(s)$ *are*

$$r_1u_1, r_2u_2, \ldots, r_{k-1}u_{k-1}, \lambda_s(\mathcal{R}).$$

The multiplicities of the eigenvalues of $Q(\mathcal{G})$ *must be considered as in Theorem 3.7.*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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Proof. For $M(\mathcal{G}) = A(\mathcal{G})$, we have $a = 0$,

$$S_j = \begin{cases} A(\mathcal{G}_j), \\ A(\mathcal{G}_j) = 0 \text{ if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = A(\mathcal{R}),$$

and $A(\mathcal{G}_j)\mathbf{e}_{m_j} = r_j \mathbf{e}_{m_j}$ if $j \in \Delta$ and $r_j = 0$ for $j \notin \Delta$. Then, from Theorem 3.7, we conclude that the set of eigenvalues of $A(\mathcal{G})$ is obtained replacing the matrices X_j, $X_{j,i}$, and $Y(s)$ by the matrices T_j, $T_{j,i}$ and $R(s)$, respectively. □

5. On the Randić eigenvalues of $\mathcal{R}\{\mathcal{B}(\boldsymbol{F})\}$. Let $\mathcal{H}$ be a simple connected graph with n vertices $v_1, v_2, \ldots, v_n$. Denote by $d(v_1), d(v_2), \ldots, d(v_n)$ the degrees of $v_1, v_2, \ldots, v_n$, respectively. The Randić matrix of $\mathcal{H}$ is the square matrix of order n whose (i,j)−entry is equal to

$$\frac{1}{\sqrt{d(v_i)\,d(v_j)}}$$

if v_i and v_j of $\mathcal{H}$ are connected and 0 otherwise [1]. The Randić eigenvalues of $\mathcal{H}$ are the eigenvalues of the Randić matrix of $\mathcal{H}$. The purpose of this section is to determine the Randić eigenvalues of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$ when each $\mathcal{G}_j$ is regular of degree r_j and $\mathcal{R}$ is a connected regular graph of degree p on r vertices $v_1, v_2, \ldots, v_r$.

Keep in mind that $r_j = 0$ if $j \notin \Delta$. As usual $v_i \sim v_j$ means that v_i and v_j are adjacent. Observe that, for $i = 1, 2, \ldots, r$, the degree of v_k as a vertex of $\mathcal{R}\{\mathcal{B}(F)\}$ is $d_k + p$. The Randić matrix of $\mathcal{R}\{\mathcal{B}(F)\}$ is the adjacency matrix of the weighted graph in which the edges joining the vertices at the level $j+1$ with the vertices at the level j of $\mathcal{B}(F)$ have weights

$$w_{k-j} = \frac{1}{\sqrt{(d_{k-j+1} + r_{k-j+1})(d_{k-j} + r_{k-j})}} \quad (2 \le j \le k-1), \tag{5.1}$$
$$w_{k-1} = \frac{1}{\sqrt{(d_k + p)(d_{k-1} + r_{k-1})}},$$

the edges of graph $\mathcal{G}_j$ have weights

$$u_j = \begin{cases} \frac{1}{d_j + r_j} & \text{if } j \in \Delta, \\ 0 & \text{if } j \notin \Delta, \end{cases} \tag{5.2}$$

and the weights of the edge $v_i v_l$ of $\mathcal{R}$ are

$$\varepsilon_{i,l} = \varepsilon_{l,i} = \begin{cases} \frac{1}{p + d_k} & \text{if } v_i \sim v_l, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3}$$

THEOREM 5.1. *If for each $j \in \Delta$ the graph $\mathcal{G}_j$ is a regular graph of degree r_j and the graph $\mathcal{R}$ is a regular graph of degree p then the Randić spectrum of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(\mathrm{F})\}$*

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is

$$\left(\cup_{j\in\Omega-\Delta}\sigma\left(T_j\right)^{r(n_j-n_{j+1})}\right)\cup\left(\cup_{j\in\Delta}\cup_{i=1}^{m_j-1}\sigma\left(T_{j,i}\right)^{rn_{j+1}}\right)\cup\left(\cup_{s=1}^{r}\sigma\left(R\left(s\right)\right)\right)$$

in which the matrices $T_j, T_{j,i}$ *and* $R(s)$ *are those of Theorem 4.3 with the weights indicated in* $(5.1), (5.2)$ *and* (5.3). *The eigenvalues multiplicities must be considered as in Theorem 3.7.*

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