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Upper Bounds for Randić Spread

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Abstract

The Randić spread of a simple undirected graph G, $spr_{\mathbf{R}}(G)$, is equal to the maximal difference between two eigenvalues of the Randić matrix, disregarding the spectral radius [Gomes et al., *MATCH Commun. Math. Comput. Chem.* **72** (2014) 249–266]. Using a rank-one perturbation on the Randić matrix of G it is obtained a new matrix whose matricial spread coincide with $spr_{\mathbf{R}}(G)$. By means of this result, upper bounds for $spr_{\mathbf{R}}(G)$ are obtained.

1 Basic definitions

The concept of Randić spread was introduced in a previous paper [14]. Here we offer some additional results on this spectral characteristics of the Randić matrix. Our notation and terminology agrees with those in [14], where additional details and references can be found. In this paper G is an undirected simple graph with vertex set and edge set $V(G)$ and $E(G)$, respectively. The vertices of G are labeled by $1, 2, \ldots, n$. If $e \in E(G)$ has end vertices i and j, then we say that i and j are adjacent ($i \sim j$) and that $e = ij$. The set $N_i = \{j \in V(G) : ij \in E(G)\}$ is the set of neighbors of $i \in V(G)$ and its cardinality, denoted by d_i, is its *vertex degree*. The minimum degree is denoted by δ and the maximum degree is Δ. The adjacency matrix $\mathbf{A} = \mathbf{A}(G)$ of G is used to represent the adjacency relations. The elements a_{ij} are equal to 1 if vertices i and j are adjacent and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ are called the eigenvalues of the graph and its set the *spectrum* of G (see [11]).

As in [14], for a real square matrix $\mathbf{M}$ associated to the graph G, we denote by $\lambda_i(\mathbf{M})$ its i-th greatest eigenvalue. The spectrum (the multiset of eigenvalues) of $\mathbf{M}$ is denoted by $\sigma(\mathbf{M}) = \sigma(\mathbf{M}(G))$. The *multiplicity* s of an eigenvalue λ in this spectrum is represented by $\lambda^{(s)}$. The matrix $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$, where $\mathbf{D}(G) = diag(d_1, \ldots, d_n)$ is the diagonal matrix of the vertex degrees, is known as Laplacian matrix or the Laplacian of G. Its spectrum is called the Laplacian spectrum of G. Additional properties for the adjacency and Laplacian matrix can be seen for instance in [11]. The identity matrix of order n (or of appropriate order) is represented by $\mathbf{I}_n$ (or simply by $\mathbf{I}$). The vector $\mathbf{e}_n$ denotes que all one-vector with all its components equal to one.

2 Motivation

The Randić matrix is defined as $\mathbf{R} = \mathbf{R}(G) = (r_{ij})$, where $r_{ij} = 1/\sqrt{d_i\, d_j}$ if $ij \in E(G)$, and zero otherwise. The definition of Randić matrix cames from a molecular structure-descriptor introduced by Milan Randić in 1975, (see [19])

$$\chi = \chi(G) = \sum_{ij \in E(G)} \frac{1}{\sqrt{d_i\, d_j}} \tag{1}$$

known as the Randić index. In fact, the summands on the right hand side of (1) may be understood as matrix entries. In spite of its connection with Randić index this matrix seems to have not been much studied in mathematical chemistry however, the graph invariant Randić energy, defined as the sum of the absolute values of the eigenvalues of the Randić matrix has been recently introduced and some of its properties were established, in particular the study of lower and upper bounds for it, see [3, 4, 6, 13]. Recall that, for graphs without singletons, the diagonal matrix $\mathbf{D}^{-1/2}$ exists. The matrix $\mathcal{L} = \mathcal{L}(G) = \mathbf{D}^{-1/2}\,\mathbf{L}(G)\,\mathbf{D}^{-1/2}$ is the *normalized Laplacian matrix*, [10], and for a graph G without singletons, the following equality can be stated:

$$\mathcal{L}(G) = \mathbf{I}_n - \mathbf{R}(G)\ .$$

Some relations between these two matrices concerning its eigenvalues and its eigenvectors can be found for instance in the previous paper [14].

In 1998 Bollobás and Erdös introduce the *general Randić index* (see [2]) as follows

$$R_\alpha(G) = \sum_{ij \in E(G)} (d_i\, d_j)^\alpha$$

for a fixed real number $\alpha \neq 0$. Note that the topological index introduced in [19] is a particular case of the previous one considering $\alpha = -\frac{1}{2}$. In [9] some known results for the graph invariant $R_{-1}(G)$ are highlighted and the authors provided upper and lower bounds for the energy of a simple graph with respect to the normalized Laplacian eigenvalues (defined as the sum of its absolute values), $E_{\mathcal{L}}(G)$. An upper bound of $R_{-1}(G)$ known for trees is also extended by these authors to connected graphs. Also, in [9] for a graph G of order n with no singletons

$$2R_{-1}(G) \leq E_{\mathcal{L}}(G) \leq \sqrt{2nR_{-1}(G)},$$

and this shows the relevant importance of $R_{-1}(G)$ when related with $E_{\mathcal{L}}(G)$. In [22] it was shown that for G a graph without singletons with minimum vertex degree δ and maximum vertex degree Δ, then

$$\frac{n}{2\Delta} \leq R_{-1}(G) \leq \frac{n}{2\delta}$$

and it was proved that equality occurs in both bounds if and only if G is a regular graph. In [14] it was introduced the concept of *Randić spread* and it was deduced upper and lower bounds for this spectral invariant. Some of the bounds presented are in terms of Randić index of the underlying graph. The Randić spread is defined in [14] as:

$$spr_{\mathbf{R}}(G) = \max\left\{|\lambda_i(\mathbf{R}) - \lambda_j(\mathbf{R})| : \lambda_i(\mathbf{R}), \lambda_j(\mathbf{R}) \in \sigma(\mathbf{R}(G)) \setminus \{1\}\right\}.$$

In this paper it is presented in Section 4 an upper bound for Randić spread using a result due to Scott, see [21], and in addition we also obtain new upper bounds for this spectral invariant in function of $R_\alpha(G)$, for $\alpha = -\frac{1}{2}$. Moreover, in Section 5 we present upper bounds for Randić spread and for the spread of a rank one perturbed matrix in function of $R_{-1}(G)$. In Section 6 an upper bound for the Randić spread of the join of two graphs with disjoint vertex sets is stated.

3 Auxiliary results

We start this section recalling the definition for the spread of an $n \times n$ complex matrix $\mathbf{M}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that is defined by

$$s(\mathbf{M}) = \max_{i,j} |\lambda_i - \lambda_j|$$

where the maximum is taken over all pairs of eigenvalues of $\mathbf{M}$.

This parameter appears in literature in many references, see for instance [1, 15–18]. The following upper bound for the spread of a square matrix $\mathbf{M}$ was given in [17]

$$s^2(\mathbf{M}) \leq 2|\mathbf{M}|^2 - \frac{2}{n}|\mathrm{trace}(\mathbf{M})|^2 \tag{2}$$

with $|M|^2 = \mathrm{trace}(M^*M)$, where M^* is the transconjugate of M.

Now we shall need the following result proved in [20].

Lemma 1. [20] *Let G be an undirected simple and connected graph. For $i \in V(G)$, let N_i be the set of first neighbors of the vertex i of G. If $\xi(\mathbf{R}(G))$ is an eigenvalue with greatest modulus among the Randić eigenvalues of G, then*

$$|\xi(\mathbf{R}(G))| \leq 1 - \min_{i<j}\left\{\frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}\right\}.$$

where the minimum is taken over all pairs (i, j), such that $1 \leq i < j \leq n$.

By using Lemma 1 we directly arrive at:

Theorem 2. *Let G be an undirected simple and connected graph whose Randić matrix is $\mathbf{R}(G)$. Then*

$$spr_{\mathbf{R}}(G) = \lambda_2(\mathbf{R}(G)) - \lambda_n(\mathbf{R}(G)) \leq 2 - 2\min_{i<j}\left\{\frac{|N_i \cap N_j|}{\max\{d_i, d_j\}}\right\}$$

where the minimum is taken over all pairs $(i, j), 1 \leq i < j \leq n$, such that the vertices i and j are adjacent.

The next theorem is due to Brauer [7] which shows how to modify one single eigenvalue of an arbitrary square matrix $\mathbf{M}$ beginning from a rank one perturbation without changing the remaining eigenvalues of $\mathbf{M}$.

Theorem 3. [7] *Let $\mathbf{M}$ be an arbitrary $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let x_k be an eigenvector of $\mathbf{M}$ associated with the eigenvalue λ_k, and let q be any n-dimensional vector. Then the matrix $\mathbf{M} + x_k q^t$ has eigenvalues*

$$\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + x_k^t q, \lambda_{k+1}, \ldots, \lambda_n.$$

Let G be an arbitrary graph of order n with m edges and $pq \in E(G)$. The following statements can be found in [14]. The matrix

$$\mathbf{R}_{pq} = \begin{pmatrix} 0 & (d_p d_q)^{-1/2} \\ (d_p d_q)^{-1/2} & 0 \end{pmatrix}$$

is a principal submatrix of order 2 of $\mathbf{P}\,\mathbf{R}(G)\,\mathbf{P}^t$, where $\mathbf{P}$ is an appropriate permutation matrix of order n. Applying Theorem 3 it was proved in [14], that the matrix

$$\mathbf{B}_{pq} = \mathbf{R}(G) + \beta_{pq}\, w\, w^t$$

has spectrum

$$\begin{aligned} \sigma(\mathbf{B}_{pq}) &= \sigma(\mathbf{R}(G)) \setminus \{1\} \cup \left\{1 + \beta_{pq}\, w^t\, w\right\} \\ &= \sigma(\mathbf{R}(G)) \setminus \{1\} \cup \left\{1 - \left((d_p\, d_q)^{-1/2} + 1\right)\right\} = \sigma(\mathbf{R}(G)) \setminus \{1\} \cup \{\lambda_{pq}\}\,, \end{aligned}$$

where $w = D^{\frac{1}{2}}\,\mathbf{e}\,,\ \lambda_{pq} = -1/\sqrt{d_p\, d_q}\,,$ and

$$\beta_{pq} = -\frac{1}{2m}\left[\frac{1}{\sqrt{d_p\, d_q} + 1}\right].$$

Remark 1. [14] *By Theorem 3, for any given value* ς *such that* $\lambda_n(\mathbf{R}(G)) \le \varsigma \le \lambda_2(\mathbf{R}(G))$*, the equality* $spr_{\mathbf{R}}(G) = s(\mathbf{B}_\varsigma)$ *holds, where* $\mathbf{B}_\varsigma = \mathbf{R}(G) + \kappa\, w\, w^t$ *with*

$$\kappa := \kappa(\varsigma) := \frac{1}{2m}\,(\varsigma - 1)\ .$$

For κ specified in Remark 1 the matrix $\mathbf{B}_\varsigma = (b_{ij}) = \mathbf{R}(G) + \kappa\, w\, w^t$ has the following entries, (see [14]):

$$b_{ij} = \begin{cases} \kappa\, d_i & \text{if } i = j \\ \dfrac{1}{\sqrt{d_i\, d_j}} + \kappa\sqrt{d_i\, d_j} & \text{if } ij \in E(G) \\ \kappa\sqrt{d_i\, d_j} & \text{if } ij \notin E(G)\ . \end{cases} \tag{3}$$

4 An upper bound for Randić spread using a result due to D. Scott

The following upper bound for the spread of a matrix is due to D. Scott and can be easily proved by Gershgorin circle theorem (see also the proof of Theorem 3.1 in [1]). D. Scott, in [21], also study the accuracy of this upper bound. Let $M = (m_{ij})$ be a square matrix. Then

$$s\,(M) \le \max_{i\neq j}\left\{|m_{ii} - m_{jj}| + \sum_{k\neq i}|m_{ik}| + \sum_{k\neq j}|m_{jk}|\right\}.$$

By using this upper bound we establish the next theorem.

Theorem 4. *Let G be a connected graph on n vertices and m edges and $\mathbf{B}_\varsigma$ be the matrix defined in Remark 1, with $\lambda_n(\mathbf{R}(G)) \leq \varsigma \leq \lambda_2(\mathbf{R}(G))$. Then*

$$\begin{aligned} s(\mathbf{B}_\varsigma) \;&\leq\; \max_{i<j}\left\{\left(\frac{1-\varsigma}{2m}\right)|d_i-d_j| + \sum_{l\sim i}\left|\frac{1}{\sqrt{d_i\,d_l}} - \left(\frac{1-\varsigma}{2m}\right)\sqrt{d_i\,d_l}\right|\right. \\ &+\; \frac{(1-\varsigma)}{2m}\sum_{l\not\sim i}\sqrt{d_i\,d_l} + \sum_{l\sim j}\left|\frac{1}{\sqrt{d_j\,d_l}} - \left(\frac{1-\varsigma}{2m}\right)\sqrt{d_j\,d_l}\right| \\ &+\; \left.\frac{(1-\varsigma)}{2m}\sum_{l\not\sim i}\sqrt{d_j\,d_l}\right\}. \end{aligned} \tag{4}$$

The next lemma was proved in [14].

Lemma 5. [14] *Let G be a connected graph of order n. Then $\lambda_2(\mathbf{R}(G)) < 0$ if and only if $G \cong K_n$.*

Taking into account this lemma we establish the next corollary.

Corollary 6. *Let G be a connected graph with n vertices and m edges with $G \ncong K_n$, then*

$$\begin{aligned} spr_R(G) \;&\leq\; \max_{i<j}\left\{\frac{|d_i-d_j|}{2m} + \sum_{l\sim i}\left|\frac{1}{\sqrt{d_i\,d_l}} - \frac{\sqrt{d_i\,d_l}}{2m}\right|\right. \\ &+\; \left.\sum_{l\sim j}\left|\frac{1}{\sqrt{d_j\,d_l}} - \frac{\sqrt{d_j\,d_l}}{2m}\right| + \sum_{l\not\sim j}\frac{\sqrt{d_j\,d_l}}{2m} + \sum_{l\not\sim i}\frac{\sqrt{d_i\,d_l}}{2m}\right\}. \end{aligned} \tag{5}$$

Proof. By Lemma 5 if $G \ncong K_n$ then $\lambda_2(\mathbf{R}(G)) \geq 0 \geq \lambda_n(\mathbf{R}(G))$ which implies that $\varsigma = 0$ satisfies the condition in Remark 1. Thus the upper bound in (4) becomes (5). □

Corollary 7. *Let G be a connected graph with n vertices and m edges. Then the maximum taken over all pairs (i,j), $1 \leq i < j \leq n$ in the set*

$$\begin{aligned} &\left\{\frac{|d_i-d_j|(m+\chi(G))}{2m^2} + \sum_{l\sim i}\left|\frac{1}{\sqrt{d_i\,d_l}} - \frac{m+\chi(G)}{2m^2}\sqrt{d_i\,d_l}\right|\right. \\ +\;& \sum_{l\sim j}\left|\frac{1}{\sqrt{d_j\,d_l}} - \frac{m+\chi(G)}{2m^2}\sqrt{d_j\,d_l}\right| \\ +\;& \left.\left(\frac{m+\chi(G)}{2m^2}\right)\left(\sum_{l\not\sim i}\sqrt{d_i\,d_l} + \sum_{l\not\sim j}\sqrt{d_j\,d_l}\right)\right\} \end{aligned}$$

is an upper bound for $spr_R(G)$.

Proof. The proof follows easily by inequality in (4) by taking $\varsigma = -\frac{\chi(G)}{m}$. □

5 Bounds for Randić spread using a rank one perturbed matrix

In what follows, using the suggestion in Remark 1, we deduce some upper bounds for the Randić spread (and for the spread of a rank one perturbed Randić matrix).

Firstly we need to recall the following result. For an arbitrary graph G, using inequality (2), we present another upper bound for the Randić spread. To this purpose, for $1 \leq j \leq n$ let us define

$$\Gamma(j) = \sum_{s \sim j} \frac{1}{d_s} .$$

Note that if G is a regular graph then $\Gamma(j) = 1$, for all $1 \leq j \leq n$. Moreover define

$$f(G) = 2 \sum_{i \in V(G)} \frac{\Gamma(i)}{d_i} .$$

Then,

$$f(G) = 2 \sum_{i \in V(G)} \sum_{s \sim j} \frac{1}{d_i\, d_s} = 4R_{-1}(G) = 2\left|\mathbf{R}(G)\right|^2 .$$

By using inequality in (2) we get the following result.

Theorem 8. *Let G be a connected graph on n vertices and m edges and $\mathbf{B}_\varsigma$ be the matrix defined in Remark 1. Then*

$$s(\mathbf{B}_\varsigma) \leq \sqrt{4R_{-1}(G) + \frac{2(\varsigma - 1)}{n}\left(1 + n + \varsigma(n-1)\right)} . \tag{6}$$

Moreover, if $\kappa = \kappa(\varsigma) = \frac{1}{2m}(\varsigma - 1)$, with $\lambda_n(\mathbf{R}(G)) \leq \varsigma \leq \lambda_2(\mathbf{R}(G))$, we have

$$spr_R(G) \leq \sqrt{4R_{-1}(G) + 8\kappa m\left(1 + m\kappa - \frac{m\kappa}{n}\right)} . \tag{7}$$

For $\varsigma = \lambda_2(\mathbf{R}(G))$ the upper bound in (6) becomes

$$spr_R(G) \leq \sqrt{4R_{-1}(G) + \frac{2(\lambda_2(\mathbf{R}(G)) - 1)}{n}\left(1 + n + \lambda_2(\mathbf{R}(G)(n-1)\right)} .$$

For $\varsigma = \lambda_n(\mathbf{R}(G))$ the upper bound in (6) becomes

$$spr_R(G) \leq \sqrt{4R_{-1}(G) + \frac{2(\lambda_n(\mathbf{R}(G)) - 1)}{n}\left(1 + n + \lambda_n(\mathbf{R}(G))(n-1)\right)} .$$

Proof. We shall use the upper bound (2) on the matrix $\mathbf{B}_\varsigma$. Taking into account the entries b_{ij} defined in (3), and that $\left|\mathbf{B}_\varsigma\right|^2 = trace(\mathbf{B}_\varsigma^* \mathbf{B}_\varsigma)$, where $\mathbf{B}_\varsigma^*$ denotes the transconjugate of $\mathbf{B}_\varsigma$, we obtain

$$\left|\mathbf{B}_\varsigma\right|^2 = \sum_{i \in V(G)} \frac{\Gamma(i)}{d_i} + 4\kappa m(1 + m\kappa) .$$

By a direct computation we have

$$trace\,(\mathbf{B}_\varsigma) = 2m\kappa\ .$$

Applying the upper bound in (2) at $\mathbf{B}_\varsigma$ we have

$$s^2\,(\mathbf{B}_\varsigma) \leq \sum_{i\in V(G)} \frac{2\Gamma\,(i)}{d_i} + 8\kappa m\,(1+m\kappa) - \frac{8m^2\kappa^2}{n}\ . \tag{8}$$

By elementary algebra, we can carry the term on the right hand side of previous inequality to the upper bound in (7).

If $\kappa = \frac{1}{2m}\,(\varsigma - 1)$ from (8) we get

$$\sum_{i\in V(G)} \frac{2\Gamma\,(i)}{d_i} + 8\kappa m(1+m\kappa) - \frac{8m^2\kappa^2}{n} = \sum_{i\in V(G)} \frac{2\Gamma\,(i)}{d_i} + 2\,(\varsigma^2 - 1) - \frac{2}{n}\,(\varsigma - 1)^2\ .$$

Now, by a standard computation we arrive at (6). □

Corollary 9. *Let G be a connected graph with n vertices with $G \ncong K_n$. Then*

$$spr_R\,(G) \leq \sqrt{4R_{-1}(G) - \frac{2(1+n)}{n}}\ . \tag{9}$$

Proof. By Lemma 5 if $G \ncong K_n$ then $\lambda_2(\mathbf{R}(G)) \geq 0 \geq \lambda_n(\mathbf{R}(G))$ which implies that $\varsigma = 0$ satisfies the condition in Remark 1. Thus the upper bound in (7) becomes (9). □

Corollary 10. *Let G be a connected graph with n vertices and m edges, then*

$$spr_R\,(G) \leq \sqrt{4R_{-1}\,(G) - \frac{2\,(\chi(G)+m)\,((1+n)\,m - \chi(G)\,(n-1))}{nm^2}}.$$

Proof. By inequality in (7) by taking $\varsigma = -\frac{\chi(G)}{m}$. □

6 An upper bound for the Randić spread of join of two graphs

Usually, considering two graphs G_1 and G_2, the join $G_1 \vee G_2$ is the graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. In [14] for $i = 1, 2$ and considering G_i a p_i-regular graph on n_i vertices with $p_i \geq 0$, $n_i \geq 1$, for the matrix

$$\mathbf{R}(G_1 \vee G_2) = \begin{pmatrix} \dfrac{1}{p_1+n_2}\mathbf{A}(G_1) & \dfrac{\mathbf{e}_{n_1}\,\mathbf{e}^t_{n_2}}{\sqrt{(p_1+n_2)(p_2+n_1)}} \\ \dfrac{\mathbf{e}_{n_2}\,\mathbf{e}^t_{n_1}}{\sqrt{(p_1+n_2)(p_2+n_1)}} & \dfrac{1}{p_2+n_1}\mathbf{A}(G_2) \end{pmatrix}.$$

it was obtained

$$\chi\left(G_1 \vee G_2\right)=\frac{1}{2}\left(\frac{n_1\, p_1}{p_1+n_2}+\frac{n_2\, p_2}{p_2+n_1}+\frac{2 n_1\, n_2}{\sqrt{\left(p_1+n_2\right)\left(p_2+n_1\right)}}\right).$$

Using the matrix

$$\mathbf{S}=\left(\begin{array}{cc}\dfrac{p_1}{p_1+n_2} & \dfrac{\sqrt{n_1 n_2}}{\sqrt{\left(p_1+n_2\right)\left(p_2+n_1\right)}} \\ \dfrac{\sqrt{n_1\, n_2}}{\sqrt{\left(p_1+n_2\right)\left(p_2+n_1\right)}} & \dfrac{p_2}{p_2+n_1}\end{array}\right)$$

and $\sigma(\mathbf{S})=\{1, \det \mathbf{S}\}$, and applying a lemma of Fiedler (see [8]), the spectrum of $R(G_1 \vee G_2)$ was stated in [14] as:

$$\sigma\left(\frac{1}{p_1+n_2} \mathbf{A}(G_1)\right) \cup \sigma\left(\frac{1}{p_2+n_1} \mathbf{A}(G_2)\right) \cup \sigma(S) \setminus\left\{\frac{p_1}{p_1+n_2}, \frac{p_2}{p_2+n_1}\right\}. \tag{10}$$

Then, the following lemma was proved.

Lemma 11. [14] *Let G_1 and G_2 be graphs of order n_1 and n_2, respectively. Then*

$$\lambda_2(\mathbf{R}(G_1 \vee G_2))=\max \left\{\lambda_2\left(\frac{1}{p_1+n_2}\, \mathbf{A}(G_1)\right), \lambda_2\left(\frac{1}{p_2+n_1}\, \mathbf{A}(G_2)\right)\right\}. \tag{11}$$

Now, we define the following $(G_1 \vee G_2)$-graph invariant

$$\Upsilon\left(G_1 \vee G_2\right)=\min \left\{-\frac{p_1}{p_1+n_2}, -\frac{p_2}{p_2+n_1}, \frac{p_1\, p_2-n_1\, n_2}{\left(p_2+n_1\right)\left(p_1+n_2\right)}\right\}.$$

The next lemma presents a lower bound for $\lambda_n(\mathbf{R}(G_1 \vee G_2))$ in terms of $\Upsilon\left(G_1 \vee G_2\right)$.

Lemma 12. *Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then*

$$\lambda_n(\mathbf{R}(G_1 \vee G_2)) \geq \Upsilon\left(G_1 \vee G_2\right) \tag{12}$$

Proof: By Eq. (10) it is clear that $\lambda_n(\mathbf{R}(G_1 \vee G_2))$ is equal to

$$\min \left\{\frac{1}{p_1+n_2}\, \lambda_{n_1}\left(A\left(G_1\right)\right), \frac{1}{p_2+n_1}\, \lambda_{n_2}\left(A\left(G_2\right)\right), \frac{p_1\, p_2-n_1\, n_2}{(p_2+n_1)(p_1+n_2)}\right\}.$$

Attending that

$$\frac{1}{p_1+n_2}\, \lambda_{n_1}(A(G_1)) \geq -\frac{p_1}{p_1+n_2}$$

and

$$\frac{1}{p_2+n_1}\, \lambda_{n_2}(A(G_2)) \geq -\frac{p_2}{p_2+n_1}$$

if we have

$$\begin{aligned}\lambda_n(\mathbf{R}(G_1 \vee G_2)) &= \min\left\{\frac{1}{p_1+n_2}\,\lambda_{n_1}\left(A\left(G_1\right)\right),\ \frac{1}{p_2+n_1}\,\lambda_{n_2}\left(A(G_2)\right)\right\}\\ &\geq \min\left\{-\frac{p_1}{p_1+n_2}-\frac{p_2}{p_2+n_1}\right\} \geq \Upsilon(G_1 \vee G_2).\end{aligned}$$

Now, if

$$\lambda_n(\mathbf{R}(G_1 \vee G_2)) = \frac{p_1p_2 - n_1n_2}{\left(p_2+n_1\right)\left(p_1+n_2\right)}$$

then evidently

$$\frac{p_1\,p_2 - n_1\,n_2}{(p_2+n_1)(p_1+n_2)} \geq \Upsilon(G_1 \vee G_2)\ .$$

Thus the result follows.

Theorem 13. *For $i = 1,2$, let G_i be a p_i-regular graph on n_i vertices with $p_i \geq 0$, $n_i \geq 1$. If*

$$\begin{aligned}\widetilde{\delta_1} &= \left|\lambda_2\left(\frac{1}{p_1+n_2}\,\mathbf{A}(G_1)\right) - \Upsilon\left(G_1 \vee G_2\right)\right|\\ \widetilde{\delta_2} &= \left|\lambda_2\left(\frac{1}{p_2+n_1}\,\mathbf{A}(G_2)\right) - \Upsilon\left(G_1 \vee G_2\right)\right|.\end{aligned}$$

then

$$spr_{\mathbf{R}}\left(G_1 \vee G_2\right) \leq \max\left\{\widetilde{\delta_1}, \widetilde{\delta_2}\right\}.$$

Proof. The result follows from (11) by noting that by (10), $\det \mathbf{S} = \frac{p_1\,p_2 - n_1\,n_2}{(p_1+n_2)(p_2+n_1)}$ is a Randić eigenvalue of $G_1 \vee G_2$. □

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