Chapter 3

State-of-the-Art Nonprobabilistic Finite Element Analyses

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Additional information is available at the end of the chapter

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Abstract

The finite element analysis of a mechanical system is conventionally performed in the context of deterministic inputs. However, uncertainties associated with material properties, geometric dimensions, subjective experiences, boundary conditions, and external loads are ubiquitous in engineering applications. The most popular techniques to handle these uncertain parameters are the probabilistic methods, in which uncertainties are modeled as random variables or stochastic processes based on a large amount of statistical information on each uncertain parameter. Nevertheless, subjective results could be obtained if insufficient information unavailable and nonprobabilistic methods can be alternatively employed, which has led to elegant procedures for the nonprobabilistic finite element analysis. In this chapter, each nonprobabilistic finite element analysis method can be decomposed as two individual parts, i.e., the core algorithm and preprocessing procedure. In this context, four types of algorithms and two typical preprocessing procedures as well as their effectiveness were described in detail, based on which novel hybrid algorithms can be conceived for the specific problems and the future work in this research field can be fostered.

Keywords: interval finite element method, fuzzy finite element method, arithmetic approach, perturbation approach, sampling approach, optimization approach, subinterval technique, surrogate model

1. Introduction

The traditional finite element analysis (FEA) was performed in the context of deterministic parameters. However, uncertainties associated with material properties, geometric dimensions, and external loads are always unavoidable in engineering. The ability to include uncertainties is of great value for a design engineer. In the last decade, criticism has arisen regarding the general application of the probabilistic concept. Especially when the statistical information on uncertainties is limited [1], the subjective probabilistic analysis result may be obtained by the probabilistic method [2, 3], which proves to be of little value and does not justify the high computational cost [3–5]. Consequently, nonprobabilistic concepts have been introduced.

In this context, interval and fuzzy approaches are gaining more and more momentum for the uncertainty analysis and optimization of numerical models in their descriptions. In the interval approach, uncertainties are considered to be contained within a predefined range and only the lower and upper bounds are necessary for each uncertain parameter. The fuzzy approach further extends this methodology by the α-level technique, where α stands for the extent that a specific value is member of the range of possible input values. From this viewpoint, a fuzzy analysis requires the consecutive solution for a number of interval analysis based on the α-level technique [6]. For this reason, current researches on nonprobabilistic uncertainty propagation mainly focus on the solution and implementation of the interval analysis. In the past decades, the interval and fuzzy concepts in FEA have been studied extensively and some typical solution schemes for the interval FEA (IFEA) and fuzzy FEA (FFEA) were developed. This chapter is to give an overview of state-of-the-art numerical implementations of IFEA and FFEA in applied mechanics.

FFEA aims to obtain a fuzzy description of an FEA result, starting from fuzzy descriptions of all uncertainties. The α-level technique subdivides the membership function range into a number of discrete α-levels. The α-cuts of the input quantities are defined as $x_{i_\alpha} = \{x_i \in X_i, \mu_{\tilde{x}_i}(x_i) \geq \alpha\}$ where $\mu_{\tilde{x}}(x)$ is the membership function. This means that an α-cut is the interval resulting from intersecting the membership function at $\mu_{\tilde{x}_i}(x_i) = \alpha$. The α-level interval describes the grade of membership to the fuzzy set for each element in the domain and enables the representation of a value that is only to a certain degree member of the set. However, the confidence interval defined in statistics is the range of likely values for a population parameter, such as the population mean. The selection of a confidence level for an interval determines the probability that confidence interval produced will contain the true parameter value. The intersection with the membership function of the input uncertainties on each α-level results in an interval and IFEA is formulated, resulting in an interval for the output on the considered α-level. The fuzzy solution is finally assembled from the resulting intervals on each sublevel. The IFEA is based on the interval description of uncertainties and its goal is to capture the range of specific output quantities of interest that corresponds to a given interval description of input uncertainties. For the sake of simplicity, the static analysis of a mechanical system is adopted in this chapter to explain current IFEA schemes. The FEA equation can be expressed in a general form as follows:

$$\mathbf{K}(\mathbf{p})\mathbf{U}(\mathbf{p}) = \mathbf{F}(\mathbf{p}) \tag{1}$$

where $\mathbf{K}$ and $\mathbf{F}$ stand for the stiffness matrix and load vector, respectively; $\mathbf{U}$ represents the static response vector; and $\mathbf{p}$ is the input parameter vector of the mechanical system. In the IFEA, $\mathbf{p}$ is quantified as an interval input vector $\mathbf{p}^{\mathrm{I}}$ and shown in **Figure 1**.

where p_i^{c} is the nominal value, Δp_i is the interval radius. Then, the IFEA equation is accordingly rewritten as follows:

$$\mathbf{K}(\mathbf{p}^{\mathrm{I}})\mathbf{U}(\mathbf{p}^{\mathrm{I}}) = \mathbf{F}(\mathbf{p}^{\mathrm{I}}) \tag{2}$$

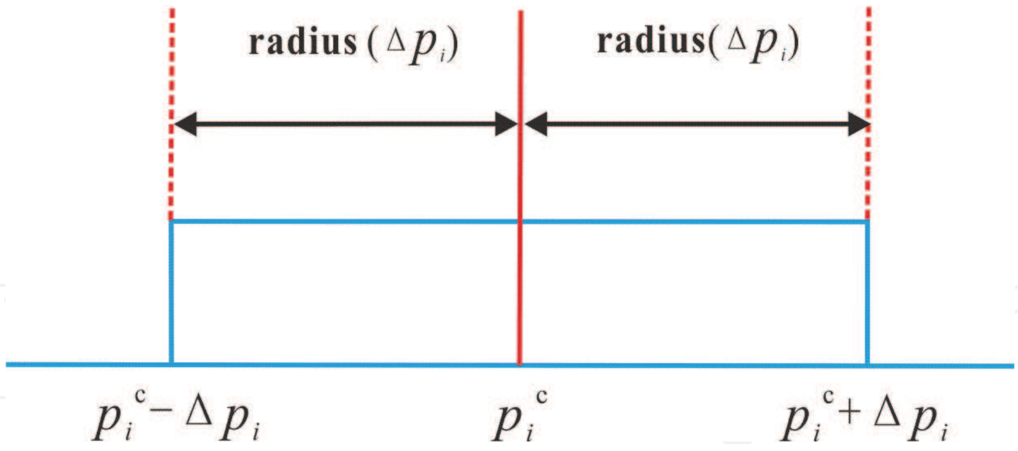

Figure 1. The diagram of interval variable *p*.

where the superscript "I" hereinafter represents an interval input. The exact solution set of this interval equation can be expressed as:

$$\mathbf{U} = \left\{\overline{\mathbf{U}}|\mathbf{K}(\overline{\mathbf{p}})\overline{\mathbf{U}} = \mathbf{F}(\overline{\mathbf{p}}), \forall\overline{\mathbf{p}} \in \mathbf{p}^{\mathrm{I}}\right\} \tag{3}$$

It is noted that interdependencies among entries of the response vector are introduced due to sharing the common input vector and a nonconvex polyhedron is always defined [7], which makes it extremely difficult to obtain the exact solution [5]. However, only individual ranges of some components in the response vector are of interest for real-life problems. Therefore, by neglecting the aforementioned interdependencies, the smallest hypercube approximation denoted as $\mathbf{U}^{\mathrm{I}}$ around the exact solution set is an alternative object for current IFEA. The kth component of $\mathbf{U}^{\mathrm{I}}$ is expressed as follows:

$$U_k^{\mathrm{I}} = \left[U_k^{\mathrm{L}}, U_k^{\mathrm{U}}\right] = \left[\min_{\mathbf{p}\in\mathbf{p}^{\mathrm{I}}} U_k(\mathbf{p}), \max_{\mathbf{p}\in\mathbf{p}^{\mathrm{I}}} U_k(\mathbf{p})\right], \quad k = 1, 2, \ldots, N \tag{4}$$

where superscripts "L" and "U" represent the lower and upper bounds of an interval variable, respectively; N is the total number of response components of interest. Accordingly, the smallest hypercube solution of IFEA equation is expressed as:

$$\mathbf{U}^{\mathrm{I}} = \left[U_1^{\mathrm{I}}, U_2^{\mathrm{I}}, \ldots, U_N^{\mathrm{I}}\right]^{\mathrm{T}} \tag{5}$$

where "T" is a transposition operator.

2. Core algorithms

From published literatures, four types of algorithms for IFEA have been well established. Most of the current schemes are formulated based on these core algorithms.

2.1. Arithmetic approach

The key point of arithmetic approach is to translate the complete deterministic numerical FE procedure to an interval procedure using the arithmetic operations. Each substep of the interval algorithm calculates the range of the intermediate subfunction instead of the deterministic result. Based on this principle, the interval bounds of the output can be obtained. The original solution procedure for IFEA is the interval arithmetic approaches [7–10], in which all basic deterministic algebraic operations are replaced by their interval arithmetic counterparts.

The major advantage of the arithmetic approach is its simplicity. However, the major drawback of this method is its repeated vulnerability to conservatism. It is shown that these methods suffer considerably from the overestimation effect, also referred to as the dependency problem, and for the real-life problems, the resulting overestimation may render the final result totally useless [5]. A simple example is shown as follows. Consider the function

$$f(x) = x^2 - x + 1 \tag{6}$$

applied on the interval $x = [0, 1]$. Applying arithmetic approach, both terms are assumed independently. This leads to the interval solution $f(x) = [0, 2]$. However, the exact range of the function equals $f(x) = \left[\frac{3}{4}, 1\right]$. That is to say, an arithmetic interval operation introduces conservatism in its result if neglecting the correlation that exists between the operands. Besides, the integration of interval arithmetic approaches with software for FEA is also a challenge in real applications.

2.2. Perturbation approach

The perturbation approach has been widely applied in structural response analyses and other applications. Compared to arithmetic approaches, perturbation methods are more popular due to its simplicity and efficiency in IFEA and can be available in the original, improved, and modified versions.

2.2.1. Original version

The first-order Taylor expansions of the interval stiffness matrix and load vector at the nominal (mid-) values of interval parameters were firstly obtained as:

$$\begin{aligned}\mathbf{K}(\mathbf{p}^{\mathrm{I}}) &= \mathbf{K}(\mathbf{p}^{\mathrm{c}}) + \sum_{i=1}^{n} \frac{\partial \mathbf{K}(\mathbf{p}^{\mathrm{c}})}{\partial p_i} \Delta p_i^{\mathrm{I}} = \mathbf{K}^{\mathrm{c}} + \Delta \mathbf{K}^{\mathrm{I}} \\ \mathbf{F}(\mathbf{p}^{\mathrm{I}}) &= \mathbf{F}(\mathbf{p}^{\mathrm{c}}) + \sum_{i=1}^{n} \frac{\partial \mathbf{F}(\mathbf{p}^{\mathrm{c}})}{\partial p_i} \Delta p_i^{\mathrm{I}} = \mathbf{F}^{\mathrm{c}} + \Delta \mathbf{F}^{\mathrm{I}}\end{aligned} \tag{7}$$

where $\mathbf{p}^{\mathrm{c}}$ is the nominal (mid-) value of the interval input vector and $\Delta p_i^{\mathrm{I}} = [-\Delta p_i, \Delta p_i]$ is the interval radius of the ith interval parameter, i.e.,

$$\begin{aligned}\mathbf{p}^{c} &= (\mathbf{p}^{U} + \mathbf{p}^{L})/2 = [p_1^c, p_2^c, \ldots, p_n^c]^T \\ \Delta\mathbf{p} &= (\mathbf{p}^{U} - \mathbf{p}^{L})/2 = [\Delta p_1, \Delta p_2, \ldots, \Delta p_n]^T\end{aligned} \tag{8}$$

And the interval radiuses of the stiffness matrix and load vector in Eq. (7) are expressed as follows, respectively.

$$\begin{aligned}\Delta\mathbf{K}^{I} &= \sum_{i=1}^{n} \frac{\partial \mathbf{K}(\mathbf{p}^c)}{\partial p_i} \Delta p_i^I = [-\Delta\mathbf{K}, \Delta\mathbf{K}] \\ \Delta\mathbf{F}^{I} &= \sum_{i=1}^{n} \frac{\partial \mathbf{F}(\mathbf{p}^c)}{\partial p_i} \Delta p_i^I = [-\Delta\mathbf{F}, \Delta\mathbf{F}]\end{aligned} \tag{9}$$

The FEA model for the perturbed system can be rewritten as follows:

$$(\mathbf{K}^c + \Delta\mathbf{K}^I)(\mathbf{U}^c + \Delta\mathbf{U}^I) = \mathbf{F}^c + \Delta\mathbf{F}^I \tag{10}$$

By expanding Eq. (10) and neglecting the second-order perturbed term, the following equations can be obtained.

$$\begin{aligned}\mathbf{U}^c &= (\mathbf{K}^c)^{-1}\mathbf{F}^c \\ \mathbf{K}^c\Delta\mathbf{U}^I &= \Delta\mathbf{F}^I - \Delta\mathbf{K}^I(\mathbf{K}^c)^{-1}\mathbf{F}^c\end{aligned} \tag{11}$$

Substituting Eq. (10) into Eq. (11) yields the interval radius of the response vector as:

$$\Delta\mathbf{U}^I = (\mathbf{K}^c)^{-1}\sum_{i=1}^{n} \frac{\partial \mathbf{F}(\mathbf{p}^c)}{\partial p_i} \Delta p_i^I - (\mathbf{K}^c)^{-1}\sum_{i=1}^{n} \frac{\partial \mathbf{K}(\mathbf{p}^c)}{\partial p_i} \Delta p_i^I (\mathbf{K}^c)^{-1}\mathbf{F}^c \tag{12}$$

And the radius vector of the response vector is estimated in the original interval perturbation method [11] as follows:

$$\Delta\mathbf{U} = \sum_{i=1}^{n} \left(\left|(\mathbf{K}^c)^{-1}\right| \left|\frac{\partial \mathbf{F}(\mathbf{p}^c)}{\partial p_i}\right| + \left|(\mathbf{K}^c)^{-1}\right| \left|\frac{\partial \mathbf{K}(\mathbf{p}^c)}{\partial p_i}\right| \left|(\mathbf{K}^c)^{-1}\right| \left|\mathbf{F}^c\right| \right) \Delta p_i \tag{13}$$

The smallest hypercube solution can thus be determined as:

$$\mathbf{U}^I = [\mathbf{U}^c - \Delta\mathbf{U}, \mathbf{U}^c + \Delta\mathbf{U}] \tag{14}$$

The major drawback of this method is that a significant overestimation is introduced by the original interval perturbation method, indicating that it applies to the interval analysis of problems with "small" interval parameters.

2.2.2. Improved version

The most typical improved interval perturbation method was proposed in Ref. [12], in which the radius vector of the response vector was calculated as follows:

$$\Delta\mathbf{U} = \sum_{i=1}^{n} \left| (\mathbf{K}^{c})^{-1} \frac{\partial \mathbf{F}(\mathbf{p}^{c})}{\partial p_i} - (\mathbf{K}^{c})^{-1} \frac{\partial \mathbf{K}(\mathbf{p}^{c})}{\partial p_i} (\mathbf{K}^{c})^{-1} \mathbf{F}^{c} \right| \Delta p_i \tag{15}$$

Accordingly, the smallest hypercube solution of IFEA can also be determined by Eq. (14). Although with better accuracy compared to the original one, an interval translation effect, i.e., the translation of the resulting interval w.r.t. the accurate one, is always introduced by the improved interval perturbation method.

2.2.3. Modified versions

Compared with the original version of the perturbation approach where only first-order terms are considered, the main aspect of the following two modified interval perturbation methods [13, 14] is that the interval bounds are calculated by retaining part of higher order terms in Neumann series. Therefore, the modified methods can obtain more accurate response bounds. The key expressions are summarized as follows:

$$\begin{aligned} \mathbf{U}^{c} &= (\mathbf{K}^{c})^{-1} \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{E}_i^{c} \right] \mathbf{F}^{c} \\ \Delta\mathbf{U}^{I} &= \sum_{k=1}^{n} \left\{ (\mathbf{K}^{c})^{-1} \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{E}_i^{c} \right] \frac{\partial \mathbf{F}(\mathbf{p}^{c})}{\partial p_k} \right\} \Delta p_k^{I} + \sum_{i=1}^{n} (\mathbf{K}^{c})^{-1} \Delta\mathbf{E}_i^{I} \mathbf{F}^{c} \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbf{E}_i^{c} &= \left((\mathbf{I} + \Delta p_i \mathbf{K}_i)^{-1} + (\mathbf{I} - \Delta p_i \mathbf{K}_i)^{-1} - 2\mathbf{I} \right) / 2 \\ \Delta\mathbf{E}_i &= \left| (\mathbf{I} + \Delta p_i \mathbf{K}_i)^{-1} - (\mathbf{I} - \Delta p_i \mathbf{K}_i)^{-1} \right| / 2 \end{aligned} \tag{17}$$

and

$$\mathbf{K}_i = \frac{\partial \mathbf{K}(\mathbf{p}^{c})}{\partial p_i} (\mathbf{K}^{c})^{-1} \tag{18}$$

Different estimations of the radius vector of the response vector were, respectively, obtained as follows:

$$\Delta\mathbf{U} = \left| \sum_{k=1}^{n} \left\{ (\mathbf{K}^{c})^{-1} \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{E}_i^{c} \right] \frac{\partial \mathbf{F}(\mathbf{p}^{c})}{\partial p_k} \right\} \Delta p_k + \sum_{i=1}^{n} (\mathbf{K}^{c})^{-1} \Delta\mathbf{E}_i \mathbf{F}^{c} \right| \tag{19}$$

$$\Delta\mathbf{U} = \sum_{k=1}^{n} \left| (\mathbf{K}^{c})^{-1} \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{E}_i^{c} \right] \frac{\partial \mathbf{F}(p^{c})}{\partial p_k} \right| \Delta p_k + \sum_{i=1}^{n} \left| (\mathbf{K}^{c})^{-1} \Delta\mathbf{E}_i \mathbf{F}^{c} \right| \tag{20}$$

It should be pointed out that significant unpredicted estimation is always introduced by Eqs. (19) and (20). A more reasonable estimation of the radius vector of the response vector is simultaneously determined herein as follows:

$$\Delta \mathbf{U} = \sum_{k=1}^{n} \left| (\mathbf{K}^{c})^{-1} \left[\mathbf{I} + \sum_{i=1}^{n} \mathbf{E}_i^{c} \right] \frac{\partial \mathbf{F}(\mathbf{p}^{c})}{\partial p_k} \right| \Delta p_k + \sum_{i=1}^{n} \left| (\mathbf{K}^{c})^{-1} \right| \Delta \mathbf{E}_i \left| \mathbf{F}^{c} \right| \tag{21}$$

And a slight conservatism is alternatively resulted in by Eq. (21). The smallest hypercube solution for the IFEA is finally determined as Eq. (14). It is worth mentioning that the spectral radius of $(\mathbf{K}^{c})^{-1}\Delta\mathbf{K}$ increases with the increase in $\Delta\mathbf{K}^{I}$. $(\mathbf{K}^{c} + \Delta\mathbf{K})^{-1}$ can be expanded with a Neumann series if and only if $\|(\mathbf{K}^{c})^{-1}\Delta\mathbf{K}\|$ is less than 1 based on the criteria of convergence for a Neumann series. Therefore, these methods applies to the interval analysis of nonlinear problems with "small" interval parameters and the accuracy for those with "large" interval inputs can be improved by the subinterval technique in Section 3.1. Furthermore, the integration of all interval perturbation methods with current FEA software for the system simulation remains a great challenge.

2.3. Sampling approach

2.3.1. Vertex method

The vertex method was originally developed in Ref. [15], which can be viewed as a sampling technique with vertices being input samples of the FEA model. This method has been popular for the implementation of IFEA [16–21] due to its main aspect of simple formulation and the black-box property. If the behavior of the target response w.r.t. uncertain parameters can be guaranteed to be monotonic, the vertex method firstly proposed in Ref. [15] yields the exact solution. It should be pointed out that the concept of monotonicity in this section means monotonic along all principal directions where only one parameter is changing at a time. However, it is very hard—if not impossible —to prove the property of monotonicity in a general way, e.g., in the application of structural dynamics [22]. The number of FEA runs necessary for the vertex method is given as:

$$N = 2^{n} \tag{22}$$

where n is the number of interval parameters. It is noted that the computational cost for the vertex method exponentially increases w.r.t. the number of interval parameters, which results in a dimensionality curse.

2.3.2. Transformation method

To promote the accuracy of the vertex method for nonmonotonic problems, transformation methods for the epistemic uncertainty propagation were developed. Its original version was firstly proposed in literature [23]. This method is based on the α-level strategy and on each α-level the interval problem is defined. The interval solution strategy then consists of a dedicated sampling strategy in the space spanned by α-cut of fuzzy parameters. This method is available in a general, a reduced, and an extended form, with the most appropriate form to be selected depending on the type of model to be evaluated [23, 24]. If the behavior of the target response w.r.t. uncertain parameters can be guaranteed to be monotonic, the reduced transformation method yields the exact solution. If it shows nonmonotonic behavior, instead, the extended transformation method can be applied, in which more observation points were added in a well-directed way to the search domain after rating the monotonicity of the response w.r.t. different uncertain parameters on the basis of a classification criterion [24].

The computational cost of the transformation method is governed by the number of FEA runs N to be performed. In the case of the general transformation method, this number is given as:

$$N = \sum_{k=1}^{m+1} k^n \tag{23}$$

where m is the number of discrete α-levels and n is the number of fuzzy parameters. It is noted that the number of FEA runs grows exponentially w.r.t. the number of fuzzy inputs, which makes the general transformation method computational tedious for high-dimensional problems. The main aspect of the transformation method, its characteristic property of reducing fuzzy arithmetic to multiple crisp-number operations entails that this method can be implemented without major problems into an existing software environment for system simulation. Expensive rewriting of the program codes is not required [25]. Some of the most recent applications can be found in Refs. [25–32]. Besides, a program named as FAMOUS (fuzzy arithmetical modeling of uncertain systems) has been developed [25], which provides an interface to commercial software environments. Primarily developed in Matlab environment, FAMOUS actually works as a standalone application on both Windows and Linux platforms.

2.4. Optimization approach

In essence, calculating the solution set expressed in Eq. (3) is equivalent to performing a global optimization, aimed at the minimization and maximization of the components of the deterministic analysis results $\{\boldsymbol{U}\}$. The lower and upper bounds of the output of a classical FEA model are determined by the optimization approach through a search algorithm within the domain spanned by the interval parameters. If the global minimum and maximum of the analysis result are found by the search algorithm, it returns the smallest hypercube solution around the exact one. The optimization is performed independently on each element of the response vector. Furthermore, as the behavior of the target response w.r.t. uncertain parameters is rather unpredictable, the computational cost of the optimization approach in general is strongly problem-dependent. It is noted that the optimization approach is immune to the excessive conservatism for the interval arithmetic approaches because the optimization strategy approaches the smallest hypercube solution from its inside, which means that it does not guarantee conservatism until the actual bounds are captured. Additionally, the smooth behavior of the target response w.r.t. uncertain parameters facilitates the search for the global extrema over the space spanned by uncertain parameters. The directional search-based algorithm [16, 33, 34], linear programming [35], and genetic algorithm [36] were utilized to formulate the procedure of IFEA or FFEA. More applications can be found in [37–39]. It is worth mentioning that the optimization approach and Monte Carlo simulation can be adopted to verify the accuracy of other schemes for IFEA and FFEA.

3. Preprocessing procedures

Except for the aforementioned core algorithms for IFEA/FFEA, two types of preprocessing procedures are always adopted to improve either the accuracy or efficiency.

3.1. Subinterval technique

For the accuracy improvement, the subinterval technique w.r.t. interval inputs is developed [11, 40] and can be integrated with the interval arithmetic and perturbation approaches. The main aspect of the subinterval technique is the ability to relax requirements of "small" or "narrow" interval inputs for nonlinear responses. However, there remain two challenges as follows:

1. *Convergence validation*. Similar to the prior determination of the sample size of MC in the probabilistic analysis, the subinterval number for each interval parameter should be first determined to guarantee the convergence of the analysis result.
2. *Efficiency sacrifice*. An exponential increase of the computational cost is introduced as increasing the subinterval number to guarantee the convergence of the analysis result. For example, the computational cost increases by m^n times where n is the number of parameters and m is the number of subintervals for each interval parameter. Thus, the most dominant advantage in efficiency for the interval arithmetic and perturbation approaches over other interval algorithms is significantly sacrificed.

3.2. Surrogate model

To enhance the efficiency of IFEA and FFEA, many surrogate models of the real numerical model are always adopted when dealing with engineering design problems often involving large-scale FEA models. The main aspect of the surrogate model is to avoid the large amount of computational time. Apart from the conventional surrogate models always used in the optimization procedure of IFEA and FFEA, e.g. response surface models [41, 42], Kriging models [43–45], radial basis function models [46–48] and sparse grid meta-models [49–51], those for the sampling and optimization approaches including the high dimensional model representation (HDMR) and the component mode synthesis (CMS) are gaining momentum in recent years. CMS was originally introduced in Ref. [52], in which a Ritz-type transformation to each individual component of a structure was adopted. The deformation of each component is approximated using a limited number of component modes. For each of these vectors, only a single degree of freedom (DOF) was retained in the reduced component model, yielding a large reduction in DOF for each component and the entire structure. Thus, the computational cost for the FEA is drastically reduced. From this viewpoint, CMS can also be seen as a special surrogate model of the expensive numerical FEA for the improvement in the computational efficiency. The repeated FEAs required in the context of IFEA can benefit from this computational time reduction obtained by CMS.

4. Hybrid algorithms

Numerous schemes for IFEA and FFEA have been developed based on the core algorithms and preprocessing procedure, which can be classified into the following three cases.

4.1. Subinterval-based hybrid algorithms

Divide the large interval parameter $p_i^{\mathrm{I}}(i = 1, 2, \ldots, n)$ into N_i subintervals and its r_ith subinterval can be expressed as follow:

$$(p_i^{\mathrm{I}})_{r_i} = \left[p_i^{\mathrm{L}} + \frac{2(r_i - 1)\Delta p_i}{N_i}, p_i^{\mathrm{L}} + \frac{2r_i\Delta p_i}{N_i}\right], \ \ r_i = 1, 2, \ldots, N_i \tag{24}$$

The number of subintervals for each interval parameter may be different. N_{sub} combinations can be produced by taking a subinterval out of each interval parameter.

$$N_{\mathrm{sub}} = \prod_{i=1}^{n} N_i \tag{25}$$

For each subinterval combination, the IFEA model can be rewritten as:

$$\mathbf{K}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_n})\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_n}) = \mathbf{F}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_n}), \ \ r_i = 1, 2, \ldots, N_i; \ i = 1, 2, \ldots, n \tag{26}$$

where $\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_n}$ stands for a subinterval combination and is composed of the r_1th subinterval of the first interval parameter, the r_2th subinterval of the second one and up to the r_{n}th subinterval of the nth one. In a conclusion, Eq. (26) stands for N_{sub} subinterval IFEA equations. For each subinterval IFEA equation, the response vector can be obtained by using core algorithms in Section 2, e.g., interval arithmetic approaches, perturbation approaches, and vertex method. For two adjacent subinterval vector $\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r \ldots r_n}$ and $\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r+1 \ldots r_n}$, the following formulae hold true, i.e.,

$$\mathbf{K}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r \ldots r_n}) \cap \mathbf{K}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r+1 \ldots r_n}) = \mathbf{K}(p^{\mathrm{I}}_{r_1}, \ldots, p^{\mathrm{U}}_{r_r} = p^{\mathrm{L}}_{r_r}, \ldots, p^{\mathrm{I}}_{n}) \tag{27}$$

$$\mathbf{F}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r \ldots r_n}) \cap \mathbf{F}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r+1 \ldots r_n}) = \mathbf{F}(p^{\mathrm{I}}_{r_1}, \ldots, p^{\mathrm{U}}_{r_r} = p^{\mathrm{L}}_{r_r}, \ldots, p^{\mathrm{I}}_{n}) \tag{28}$$

where $p^{\mathrm{U}}_{r_r}$ and $p^{\mathrm{L}}_{r_r}$ are the upper bound of $p^{\mathrm{I}}_{r_r}$ and lower bound of $p^{\mathrm{L}}_{r_r+1}$, respectively. Thus, the intersection of $\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r \ldots r_n})$ and $\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r+1 \ldots r_n})$ does not equal to an empty set, i.e.,

$$\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r \ldots r_n}) \cap \mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 \ldots r_r+1 \ldots r_n}) \neq \varnothing \tag{29}$$

It is shown from Eq. (29) that the interval response vectors for each subinterval combination are simply connected. Therefore, the interval response vector can be obtained as follows by using the interval union operation.

$$\mathbf{U}(\mathbf{p}^{\mathrm{I}}) = \bigcup_{\substack{r_i = 1, 2, \ldots, N_i \\ i = 1, 2, \ldots, n}} \mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_i \ldots r_n}) = \left[\min_{r_i=1, 2, \ldots, N_i} \left(\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_i \ldots r_n}) \right), \max_{r_i=1, 2, \ldots, N_i} \left(\mathbf{U}(\mathbf{p}^{\mathrm{I}}_{r_1 r_2 \ldots r_i \ldots r_n}) \right) \right] \tag{30}$$

The above subinterval method is shown in **Figure 2** with 50 subintervals when considering one uncertain parameter *x*.

The interval arithmetic approach, subinterval technique and Taylor series expansion were integrated [40]. More applications can be found in [13, 53, 54].

4.2. Surrogate model-based hybrid algorithms

Taylor series expansion was integrated with the interval arithmetic approach in [40] and a method named as Taylor expansion with extrema management was proposed by integrating the higher order Taylor series expansion and the optimization approach [55] to detect possible nonmonotonic influences.

The transformation method was integrated with HDMR in Ref. [25]. And a component mode transformation method was developed [56] by combing the CMS with the transformation method to provide a significant reduction of the computational cost for large mechanical problems with uncertain parameters. Besides, a hybrid method was proposed for the interval frequency response analysis by integrating the optimization and interval arithmetic approach in [57], which was further integrated with CMS in Ref. [22]. An acceptable computational cost and a limited amount of conservatism in the analysis result were achieved by these hybrid algorithms.

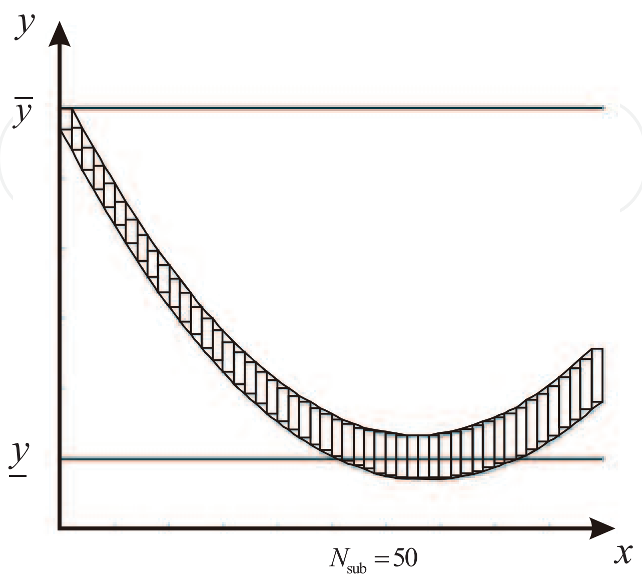

Figure 2. The diagram of subinterval method.

4.3. Hybrid core algorithms

The aforementioned core algorithms can be combined together to achieve a better tradeoff between the accuracy and efficiency, e.g., frameworks [22, 57–60] formulated by the global optimization methods and interval arithmetic ones.

To improve the computational efficiency, any core algorithm in Section 2 can be integrated with reanalysis method [61], which is fundamentally an intrusive FEA. It is noted that the major computational cost of IFEA consists of repeated solutions of the deterministic FEA systems while the main goal of the re-analysis method is to accelerate this conventional FEA solution process. It is shown that the application of the re-analysis method in the context of IFEA can reduce the computational cost by one order of magnitude compared to those based on the conventional FEA strategy [5].

5. Conclusions

This chapter presents the state-of-the-art and recent advances in nonprobabilistic finite element analyses. The main advantages and shortcomings of each nonprobabilistic finite element analysis method are discussed.

The arithmetic approach is the most straightforward strategy for nonprobabilistic finite element analyses. However, this chapter further shows that the interval arithmetic implementation of the finite element procedure is conservative. Therefore, the development of an adequate methodology for solving the uncertain parameter dependency problem is still the main challenge in the domain of arithmetic approach. The perturbation approach has been widely used in structural response analyses and other applications due to its simplicity and efficiency. The accuracy of the original perturbation methods can be improved by retaining part of higher order terms in Neumann series or Taylor series as shown in the improved and modified versions. The sampling approach like vertex method yields the exact solution under the condition that the behavior of the target response w.r.t. uncertain parameters can be guaranteed to be monotonic and has been popular for the implementation of IFEA due to its main aspect of simple formulation and the black-box property. However, when tackling the nonmonotonic problems, the extended transformation methods should be applied by adding more observation points in a well-directed way. The optimization approach is more and more acknowledged as standard procedure in an interval finite element context except for the high computational cost.

Moreover, in this context, two typical preprocessing procedures, e.g., subinterval technique and surrogate model to improve either the accuracy or efficiency are described in detail. Additionally, novel hybrid algorithms, including subinterval-based hybrid algorithms, surrogate model-based hybrid algorithms and hybrid core algorithms can be conceived by combining the aforementioned core algorithms and preprocessing procedures to achieve a better tradeoff between the accuracy and efficiency for the specific problems and the future work in this research field can be fostered.

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