Chapter 10

Semi-Analytic Techniques for Solving Quasi-Normal Modes

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Additional information is available at the end of the chapter

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Abstract

In this chapter, we discuss an approach to obtaining black hole quasi-normal modes known as the asymptotic iteration method, which was initially developed in mathematics as a new way to solve for eigenvalues in differential equations. Furthermore, we demonstrate that the asymptotic iteration method allows one to also solve for the radial quasi-normal modes on a variety of black hole spacetimes for a variety of perturbing fields. A specific example for Dirac fields in a general dimensional Schwarzschild black hole spacetime is given, as well as for spin-3/2 field quasi-normal modes.

Keywords: extra-dimensions, quasi-normal modes, quantum fields in curved space, supergravity, blackholes

1. Introduction

Quasi-normal modes (QNMs) are one of the most important theoretical results in modern cosmology, especially for studying the perturbations from various fields on black hole spacetimes. In this theory, the behaviour of a particle around a black hole is dominated by the radial equation, and the evolution of QNMs behaves like damped harmonic oscillators with specific frequencies. The frequencies are constructed by complex modes, where the real part is the actual frequency and the imaginary part represents the damping rate due to the gravitational emission. In lay terms, the QNMs are the characteristic sounds of the black hole.

With the recent ground breaking progress into the detection of gravitational wave data, where it is believed that the last part of the gravitational wave emission, called the ring down phase, is dominated by the QNMs, it is exciting that perturbation theory in curved spacetimes can now be possibly tested in a real experimental system, and that a large number of scientists from all over the world are involved in the data analysis. Related issues of the QNMs within the range of research into cosmology include studying the stability of black holes and probing the dimensionality of spacetime. It is, however, the observable gravitational wave data from the collisions of binary black hole systems, which indicates how the background spacetime will finally become a Kerr black hole spacetime through gravitational wave emission. Perturbation theory on a Kerr black hole spacetime still includes some difficulties in the higher dimensional cases, which will be a challenge for the theoretical community for some time to come.

Methods that are used to obtain QNMs can be both semi-analytic and numerical methods and were introduced by Cho et al. [1], the most famous of these is the WKB approximation methods [2]. Note that, the WKB approximation has been extended to sixth order [3] and is powerful in many cases, but like all methods have several limitations. A new method has been developed in recent years called the asymptotic iteration method (AIM), which is more efficient in some cases. This method was used to solve eigenvalue problems for the second-order homogeneous linear differential equations [4, 5] and also successfully used in calculating QNMs [6]. Reviewing this AIM and providing the tools "in detail" for studying QNMs in the higher dimensional spacetimes are the key focus of this chapter.

As such, this chapter is organised as follows: In the next section, we shall review the recent progress on perturbation theory in curved spacetimes. More precisely, we shall present a comparison of the spin-3/2 field in general dimensional Schwarzschild spacetimes with other spin fields, including the spherical harmonics and the radial equations. In Section 3, we shall review the AIM and present an exercise detailing how the QNMs of Dirac fields are obtained in general dimensional Schwarzschild spacetimes. Furthermore, we can also compare to spin-3/2 QNMs results. We shall conclude with a brief summary.

2. Perturbation theory in a general dimensional Schwarzschild spacetime

2.1. Eigenvalue problem on spheres

For perturbation theory in curved spacetimes, separability can always simplify the equations of motion and plays an important role. For the maximally symmetric spacetime cases, the eigenmodes on spheres allow us to separate the angular part for various spin fields and simplify the equations of motion from the general form into a "radial-time" presentation. For the case of bosonic fields, an earlier study by Rubin and Ordóñez presented a systematic study [7, 8] as well as in a later work by Higuchi [9]. For the case of fermionic fields, Camporesi and Higuchi presented the eigenmodes for spinor fields on arbitrary dimensional spheres [10], and in a recent work by the authors [11], the spinor-vector eigenmodes on arbitrary dimensional spheres were derived using a similar approach to Camporesi and Higuchi's methods. In this section, we review the structure of these eigenmodes, especially for the case of spinor-vector fields, which shall be presented with the characteristics of both spinor and vector fields.

The metric of the N-sphere is given by

$$d\Omega_N^2 = d\theta_N^2 + sin^2\theta_N d\Omega_{N-1}^2, \tag{1}$$

where, in this metric, we restrict the sphere to radius $r = 1$. When we consider the eigenvalue problem in this spacetime, the bosonic field and fermionic field shall be studied with different operators. In the case of bosonic fields, the operator for the eigenvalue equation will be the Laplacian operator $\nabla^\mu \nabla_\mu$, whereas, in the case of fermionic fields, it will be the Dirac operator $\gamma^\mu \nabla_\mu$, where γ^μ is the Dirac gamma matrices. In **Table 1**, we present the structure of the eigenmodes with various spin fields on the sphere and also the conditions on the specific mode, such as the transverse, traceless and symmetric conditions.

Looking first at the longitudinal and non-transverse modes for bosonic fields, the longitudinal and non-transverse eigenfunctions for higher spins are the linear combination of the eigenfunctions for the lower spin one. For example, for the vector fields, the longitudinal eigenvector is the covariant derivative of a scalar eigenfunction. Furthermore, for the symmetric tensor fields, there are three types of non-transverse eigenfunctions. The first one is the metric element multiplied by a scalar eigenfunction, the second one is the longitudinal-longitudinal eigenfunction, which is the linear combination for longitudinal eigenvectors, and the last one is

Fields	**Eigenfunction**	**Eigenvalue**
Scalar	$T^{(l)}$	$-l(l+N-1), l = 0, 1, 2, \ldots.$
Spinor	$\psi^{(j)}$	$\pm i\left(j + \frac{N-1}{2}\right), j = 1/2, 3/2, 5/2, \ldots.$
Vector	Longitudinal eigenvector	
	$L_\mu^{(l)} = \nabla_\mu T^{(l)}$	$-l(l+N-1)+1, l = 1, 2, 3, \ldots.$
	Transverse eigenvector	
	$T_\mu^{(l)}, \nabla^\mu T_\mu^{(l)} = 0$	$-l(l+N-1)+(N-1), l = 1, 2, \ldots.$
Spin.-	Non-transverse-traceless eigenmode	
Vector	$\psi_\mu^{(j)} = \nabla_\mu \psi^{(j)} + a_{(\pm)}\gamma_\mu \psi^{(j)}$	$\pm i\sqrt{j^2 + (N-1)j + \frac{1}{4}(N-5)(N-1)}$
	Transverse and traceless eigenmode	
	$\psi_\mu^{(j)} = (\psi_{\theta_N}, \psi_{\theta_i}), \nabla^\mu \psi_\mu = \gamma^\mu \psi_\mu = 0$	$\pm i\left(j + \frac{N-1}{2}\right), j = 1/2, 3/2\ldots, N \geq 3$
Sym.-	$g_{\mu\nu} T^{(l)}$	$-l(l+N-1), l = 0, 1, 2, \ldots.$
Tensor	Longitudinal-longitudinal (traceless) modes	
	$L_{L^{\mu\nu}}^{(l)} = 2\nabla_\mu \nabla_\nu T^{(l)} - \left(\frac{2}{N}\right) g_{\mu\nu} \nabla^\alpha \nabla_\alpha T^{(l)}$	$-l(l+N-1)+2N, l = 2, 3, 4, \ldots.$
	Longitudinal-transverse (traceless) modes	
	$L_{T^{\mu\nu}}^{(l)} = \nabla_\mu T_\nu^{(l)} + \nabla_\nu T_\mu^{(l)}$	$-l(l+N-1)+(N+2), l = 2, 3, 4, \ldots.$
	Transverse-traceless modes	
	$T_{\mu\nu}^{(l)}, \nabla^\mu T_{\mu\nu}^{(l)} = g^{\mu\nu} T_{\mu\nu}^{(l)} = T_{[\mu\nu]}^{(l)} = 0$	$-l(l+N-1)+2, l = 2, 3, 4, \ldots.$

Table 1. Structure of the eigenvalue problem for various fields on N-sphere.

the longitudinal-transverse eigenfunction, which is the linear combination of transverse eigenvectors. Analogous to the non-transverse-traceless modes for fermionic fields, the non-transverse-traceless eigenspinor-vector is the linear combination of the eigenspinor. We note that there are two non-transverse-traceless eigenspinor-vectors, due to the $a_{(+)}$ and $a_{(-)}$ being different factors where

$$a_{(\pm)} = -\frac{i}{2}\left(j + \frac{N-1}{2}\right) \pm i\sqrt{j^2 + (N-1)j + \frac{1}{4}(N-5)(N-1)}. \tag{2}$$

This indicates a special signature for spinor-vector harmonics that do not have the transverse-traceless eigenmodes for S^2.

Next, we look at the transverse-traceless modes. For the bosonic fields, an unique way to construct this type of eigenfunction was suggested by Higuchi [9]. Analogous to the fermionic case, for the transverse-traceless eigenspinor-vector, $\psi_\mu = (\psi_{\theta_N}, \psi_{\theta_i})$, ψ_{θ_N} behaves like a spinor on $N-1$ spheres, and ψ_{θ_i} behaves like a spinor-vector on $N-1$ spheres. If we let ψ_{θ_i} be the linear combination of the non-transverse-traceless eigenspinor-vector on $N-1$ spheres, ψ_{θ_N} has to be non-zero to satisfy the transverse and traceless conditions. If we let ψ_{θ_i} be the linear combination of the transverse-traceless eigenspinor-vector on $N-1$-spheres, ψ_{θ_N} has to be zero, because the ψ_{θ_i} already satisfies the transverse and traceless condition.

On the other hand, we can take a look at the eigenvalue. For the bosonic fields, all of the eigenvalues contain a similar first term, which is the eigenvalue of the scalar fields; however, the starting value of the angular momentum quantum number l has to be considered case by case, as well as the second term for higher spin cases. For the fermionic cases, the eigenvalue of the non-transverse-traceless eigenspinor-vector has a very different value from the spinor eigenvalue, though the eigenvalue of the transverse and traceless eigenspinor-vector is exactly the same as the eigenspinor for the spin-1/2 field.

As a remark on this section, the eigenfunctions and eigenvalues on N-spheres are independent for bosonic and fermionic fields, even though they have a very similar style of structure, which indicates that the spherical harmonics for a bosonic field cannot be constructed by the spherical harmonics of a fermionic field, and vice versa. In this section, we presented a review of the eigenvalue problem for scalar, spinor, vector, spinor-vector, and symmetric tensor fields on spheres, where further details can be found in the papers referred to in this section.

2.2. Effective potentials

In perturbation theory with various fields in the Schwarzschild black hole spacetime, a radial equation (Schrödinger-like equation) will be derived from the equations of motion, which shall be the master equation of this study. In a general way, the studies of perturbation theory in maximally symmetric spacetimes are well established and include the (A)dS and Reissner-Nordström spacetimes, but not for the spin-3/2 fields yet. With our recent progress in the study of spin-3/2 fields, we may now do a comparison of various massless fields in a general dimensional Schwarzschild black hole spacetime in this section.

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The metric of a general dimensional Schwarzschild spacetime is given by

$$dS^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}^2, \tag{3}$$

where $f(r) = 1 - 2M/r^{D-3}$, D is the dimensional factor and M is the mass of the black hole. The master equation can be obtained from the equations of motion by a change of coordinates to give the Schrödinger-like equation

$$\frac{d^2}{dr_*^2}\Psi_s + (\omega^2 - V_s)\Psi_s = 0, \tag{4}$$

where the subscript s represents the "spin" and r^* represents the "tortoise" coordinate, which can be defined as follows $\frac{d}{dr_*} = f(r)\frac{d}{dr}$. The mathematical meaning of this coordinate is that a mapping of the location of the event horizon of the Schwarzschild black hole from r_0 (where $f(r_0) = 0$) is taken to minus infinity.

We shall first look at the four-dimensional cases, where, for the bosonic fields, the radial equation can be represented with the potential [12]

$$V_s = f\left[\frac{l(l+1)}{r^2} + (1 - s^2)\frac{2M}{r^3}\right], \tag{5}$$

where l is an integer, $s = 0$ represents the effective potential for scalar fields, $s = 1$ for the electromagnetic fields, and $s = 2$ represents the "vector-type" perturbation for gravitational fields (which is the Regge-Wheeler equation). The "scalar-type" perturbation for the gravitational field in the four-dimensional case is the Zerilli equation, with an effective potential

$$V_{Z,s=2} = \frac{2f}{r^3}\left[\frac{9M^3 + 3\lambda^2 M r^2 + \lambda^2(1+\lambda)r^3 + 9M^2\lambda r}{(3M + \lambda r)^2}\right], \tag{6}$$

where $\lambda = (l-1)(l+2)/2$.

For the fermionic fields in the four-dimensional Schwarzschild case, the effective potential can be shown as follows [13, 14]:

$$V_s = \pm f\frac{dW_s}{dr} + W_s^2, \tag{7}$$

where

$$W_{s=\frac{1}{2}} = \frac{\sqrt{f}(j+\frac{1}{2})}{r}, j = \frac{1}{2}, \frac{3}{2}, \ldots. \tag{8}$$

$$W_{s=\frac{3}{2}} = \frac{\sqrt{f}(j-\frac{1}{2})(j+\frac{1}{2})(j+\frac{3}{2})}{r\left[(j+\frac{1}{2})^2 - f\right]}, j = \frac{3}{2}, \frac{5}{2}, \ldots. \tag{9}$$

Note that the "±" represents two isospectral potentials, which were known as the supersymmetric partner potentials.

For the higher dimensional cases, there are more types of effective potentials, which can be presented in the four-dimensional case, which indicates that some special cases will not exist in the four-dimensional cases but will exist in the higher dimensional one. In the following sections, we shall discuss the higher dimensional effective potentials as presented in **Table 2**.

Starting with bosonic fields, we have one effective potential for scalar fields in the higher dimensional Schwarzschild spacetime, and it is necessary to satisfy Eq. (5) when $D = 4$ and $s = 0$. For the case of electromagnetic fields, there are two types of effective potentials, which are the

Fields	V_s
Scalar [12]	$V_{s=0} = f\left[\frac{l(l+D-3)}{r^2} + \frac{D-2}{4}\left(\frac{D-4}{r^2}f + \frac{2f'}{r}\right)\right]$ $l = 0, 1, 2, \ldots.$
Dirac [13]	$V_{s=1/2} = \pm f\frac{dW_{s=1/2}}{dr} + W^2_{s=1/2}$ where $W_{s=1/2} = f\left(\frac{j+\frac{D-3}{2}}{r}\right), j = 1/2, 3/2, 5/2, \ldots.$
Electromagnetic [15]	Scalar-type perturbation $V_{S,s=1} = f\left[\frac{l(l+D-3)+\frac{(D-2)(D-4)}{4}}{r^2} - \frac{(3D-8)(D-4)M}{2r^{D-1}}\right].$ $l = 1, 2, 3, \ldots.$ Vector-type perturbation $V_{V,s=1} = f\left[\frac{l(l+D-3)+\frac{(D-2)(D-4)}{4}}{r^2} - \frac{D(D-4)M}{2r^{D-1}}\right].$ $l = 1, 2, 3, \ldots.$
Rarita-Schwinger [11]	Related to the non-TT eigenmodes $V_{NTT,s=3/2} = \pm f\frac{dW_{NTT,s=3/2}}{dr} + W^2_{NTT,s=3/2}$ where $W_{NTT,s=3/2} = \frac{\sqrt{f}(j+\frac{D-3}{2})}{r}\left[\frac{(\frac{2}{D-2})^2(j+\frac{D-3}{2})^2 - 1 - \frac{D-4}{D-2}(\frac{2M}{r^{D-3}})}{(\frac{2}{D-2})^2(j+\frac{D-3}{2})^2 - f}\right], j = 1/2, 3/2, 5/2\ldots.$ Related to the TT eigenmodes $V_{TT,s=3/2} = \pm f\frac{dW_{TT,s=3/2}}{dr} + W^2_{TT,s=3/2}$ where $W_{TT,s=3/2} = f\left(\frac{j+\frac{D-3}{2}}{r}\right), j = 1/2, 3/2, 5/2, \ldots.$

Fields	V_s
Gravitational [16, 17]	Scalar-type perturbation $V_{S,s=2} = \frac{fH}{16r^2\left[m+\frac{1}{2}N(N+1)(1-f)\right]^2}, N = D-2$ $H = N^4(N+1)^2(1-f)^3 + N(N+1)[4(2N^2-3N+4)m + N(N-2)(N-4)(N+1)](1-f)^2 - 12N[(N-4)m + N(N+1)(N-2)]m(1-f) + 16m^3 + 4N(N+2)m^2.$ $m = l(l+N-1) - N,\ l = 2,3,4\ldots.$ Vector and tensor type perturbation $V_{V/T,s=2} = \frac{f}{r^2}\left[l(l+D-3) + \frac{(D-2)(D-4)}{4} - \frac{\mu_{V/T}}{2}\frac{(D-2)^2M}{r^{D-3}}\right].$ $\mu_V = 3, l_V = 2,3,4\ldots$ for vector-type perturbation. $\mu_T = -1, l_T = 1,2,3,\ldots$ for tensor-type perturbation.

Table 2. The effective potential for various fields in the higher dimensional cases ($D \geq 5$).

"scalar-type perturbation potential" and the "vector-type perturbation potential". It is accidental that both of these effective potentials satisfy Eq. (5) when $D = 4$ and $s = 1$, but it has been shown they have different behaviours in the higher dimensional cases. It is believed that these two types of effective potentials are strongly linked to two types of eigenvectors on spheres, which are longitudinal and transverse ones. For the case of gravitational fields, the "scalar-type perturbation potential" becomes the potential for the Zerilli equation, and the "vector type perturbation potential" becomes the potential for the Regge-Wheeler equation when $D = 4$ and $s = 2$. The "tensor-type perturbation potential" will be present when $D \geq 5$ but absent in the four-dimensional case.

For the fermionic fields, the "$\pm$" still represents two isospectral supersymmetric partner potentials in the higher dimensional cases. We have one set of effective potentials for the Dirac field, which is strongly related to the eigenspinor on the sphere and reduces to Eq. (8) when $D = 4$. For the case of the Rarita-Schwinger field, the potentials related to the non-transverse-traceless eigenmodes are the leading equation both for the four-dimensional case and the higher dimensional one, which are strongly linked to the "non-transverse-traceless" eigenspinor-vector on spheres. Another effective potential for the Rarita-Schwinger fields is the one related to the transverse and traceless eigenmodes; however, this type of eigenspinor-vector was absent on the 2-sphere, which indicates that the potentials related to the transverse and traceless eigenmodes exist for the cases when $D \geq 5$. We must note that, in this case, the effective potentials are exactly the same as the Dirac case.

Lastly, note that, most of the effective potentials in **Table 2** are simple barrier like potentials. Nevertheless, some cases in the higher dimensions, or for the lowest energy state with $j = 1/2$, for the potentials related to the non-transverse-traceless eigenmodes exhibit special behaviours but not a simple barrier potential, which strongly suggests a link with the instabilities of the black hole [18] and warrants further study.

To summarise, we have in this section provided a brief review of the effective potentials, which play an important role in the perturbation theory of various spin fields in the general dimensional Schwarzschild black hole spacetimes.

3. QNM frequencies by AIM

3.1. AIM methods

The AIM is a well-established approach in solving the eigenvalue problem for the second-order differential equations, for example, Schrödinger-like equations. As mentioned in the previous section, the radial equations of the perturbation theory with various spin fields in general dimensional Schwarzschild spacetimes were presented as Schrödinger-like equations. The QNMs, which are the signature modes in the black hole perturbation theory, can be obtained naturally by using the AIM. In this subsection, we shall present a brief review of the AIMs, and in the next subsection, we shall present an example calculation, showing the methods used to obtain the quasi normal frequencies. We shall start with the second-order differential equation for the function $\chi(x)$

$$\chi'' = \lambda_0(x)\chi' + s_0(x)\chi, \tag{10}$$

where $\chi' = d\chi/dx$. The symmetric structure of the right-hand side of Eq. (10) leads to the method, where we differentiate on both sides of the equation we find that

$$\begin{aligned} \chi''' &= \lambda_0\chi'' + (\lambda'_0 + s_0)\chi' + s'_0\chi, \\ &= (\lambda'_0 + s_0 + \lambda_0^2)\chi' + (s'_0 + s_0\lambda_0)\chi, \\ &\equiv \lambda_1\chi' + s_1\chi. \end{aligned} \tag{11}$$

Taking the second derivative of Eq. (10) we have

$$\chi^{(4)} = \lambda_2\chi' + s_2\chi, \tag{12}$$

where

$$\lambda_2 = \lambda'_1 + s_1 + \lambda_0\lambda_1 \ ; \ \ s_2 = s'_1 + s_0\lambda_1. \tag{13}$$

Differentiating iteratively to the $(n+1)^{th}$ and the $(n+2)^{th}$ order, we have

$$\chi^{(n+1)} = \lambda_{n-1}\chi' + s_{n-1}\chi \ ; \ \ \chi^{(n+2)} = \lambda_n\chi' + s_n\chi, \tag{14}$$

where

$$\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1} \ ; \ \ s_n = s'_{n-1} + s_0\lambda_{n-1}. \tag{15}$$

In the AIM, we suppose that for sufficiently large n, which represents the iterating number, the coefficients λ_n and s_n will have the relation

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$$\frac{s_n}{\lambda_n} = \frac{s_{n+1}}{\lambda_{n+1}} = \beta(x), \tag{16}$$

where the general solution of Eq. (10) is

$$\chi(x) = exp\Big(-\int^x \beta(x')dx'\Big)[C_2 + C_1\int^x exp(\int^x (\lambda_0(x'') + 2\beta(x''))dx'')dx']. \tag{17}$$

C_1 and C_2 are constants determined by the normalisation, and the QNMs (or the energy eigenstates) can be obtained by the termination condition

$$s_n\lambda_{n+1} - s_{n+1}\lambda_n = 0. \tag{18}$$

This is the basic idea for the AIM, where to appreciate the effectiveness of this methods, we refer the reader to Ciftci et al. [4, 5], which presented some studies for the constant coefficient, harmonic oscillator, and the energy eigenvalue problem for several well-known potentials. For the study of perturbation theory in curved spacetimes by the AIM, Cho et al. [1] present a review of the QNMs for the bosonic fields in the four-dimensional maximally symmetric and the Kerr black hole spacetimes. In the next subsection, as an example, we shall present how to obtain the QNMs for Dirac fields in the higher dimensional Schwarzschild black hole spacetimes by the AIM and compare these with other numerical semi-analytic results.

3.2. Example: How to obtain the QNMs for Dirac fields in general dimensional Schwarzschild black hole spacetimes by the AIM

The QNMs for Dirac particles in the higher dimensional Schwarzschild black hole spacetimes had been done in the earlier work by some of the authors [13] using the third-order WKB approximation but not the AIM. With recent progress in spin-3/2 fields [11], we find that these results greatly overlap with some of the spin-3/2 particles, which are represented by the relations in the radial equations of the transverse and traceless eigenmodes. In this section, as an example, we are going to show how to reproduce the results by the AIM.

In **Table 2**, the effective potential of the radial equation, Eq. (4), for the Dirac particle is as follows:

$$V_{s=1/2} = f\frac{dW_{s=1/2}}{dr} + W^2_{s=1/2}\ ,\ \ W_{s=1/2} = f\left(\frac{j+\frac{D-3}{2}}{r}\right)\ ,\ \ j = \frac{1}{2}, \frac{3}{2}, \ldots. \tag{19}$$

As we are going to reproduce the results in Ref. [13], a similar choose of $f(r)$ will be

$$f(r) = 1 - \left(\frac{r_H}{r}\right)^{D-3}\ ,\ \ r_H^{D-3} = \frac{8\pi M\Gamma((D-1)/2)}{\pi^{(D-1)/2}(D-2)}\ ,\ \ M \equiv 1, \tag{20}$$

where r_H represents the location of the event horizon and M represents the mass of the black hole. By making a coordinate transformation

$$\xi^2(r) = 1 - \frac{r_H}{r}, \tag{21}$$

the radial equation becomes

$$\left[f\frac{d\xi}{dr}\frac{d}{d\xi}\left(f\frac{d\xi}{dr}\frac{d}{d\xi}\right) + \omega^2 - V_{s=1/2}\right]\Psi_{s=1/2} = 0. \tag{22}$$

Simplifying Eq. (22), we have

$$\left[\frac{d^2}{d\xi^2} + \left(\frac{f'}{f} + \frac{\xi''}{\xi'}\right)\frac{d}{d\xi} + \frac{\omega^2 - V_{s=1/2}}{f^2\xi'^2}\right]\Psi_{s=1/2} = 0, \tag{23}$$

where

$$f' = \frac{d}{d\xi}f(\xi)\ ,\ \ \xi' = \frac{d}{dr}\xi(r)|_{r=\frac{r_H}{1-\xi^2}}\ ,\ \ \xi'' = \frac{d}{d\xi}\xi'. \tag{24}$$

Next, by setting the boundary behaviour of $\Psi_{s=1/2} = \alpha(\xi)\chi(\xi)$ and together with Eq. (23), we have

$$\frac{d^2}{d\xi}\chi = -\left(\frac{f'}{f} + \frac{\xi''}{\xi} + \frac{2\alpha'}{\alpha}\right)\frac{d}{d\xi}\chi - \left[\frac{\omega^2 - V_{s=1/2}}{f'\xi'^2} + \frac{\alpha''}{\alpha} + \left(\frac{f'}{f} + \frac{\xi''}{\xi}\right)\frac{\alpha'}{\alpha}\right]\chi. \tag{25}$$

This is the second-order differential equation, which is the same as Eq. (10) with

$$\lambda_0 = -\left(\frac{f'}{f} + \frac{\xi''}{\xi} + \frac{2\alpha'}{\alpha}\right)\ ,\ \ s_0 = -\left[\frac{\omega^2 - V_{s=1/2}}{f'\xi'^2} + \frac{\alpha''}{\alpha} + \left(\frac{f'}{f} + \frac{\xi''}{\xi}\right)\frac{\alpha'}{\alpha}\right]. \tag{26}$$

Note that the last parameter we have to define is the asymptotic behaviour function $\alpha(\xi)$, and we approach this by starting with the asymptotic behaviour of $\Psi_{s=1/2}$, which can be represented as an outgoing plane wave

$$\begin{array}{ll} \Psi_{s=1/2} \sim e^{i\omega r_*} & \text{for} \quad r \to \infty, \\ \Psi_{s=1/2} \sim e^{-i\omega r_*} & \text{for} \quad r \to -\infty, \end{array} \tag{27}$$

with r^* being the tortoise coordinate, which has the relation with r

$$\frac{d}{dr_*} = f(r)\frac{d}{dr}. \tag{28}$$

Solving Eq. (28) in the four-dimensional case, we have

$$r_* = r + r_H \ln\left(\frac{r_H}{r} - 1\right), \tag{29}$$

together with Eqs. (21) and (27), we have

$$\begin{aligned}\Psi_{s=1/2} &\sim e^{\frac{i\omega r_H}{1-\xi^2}}(1-\xi^2)^{-i\omega r_H}\xi^{2i\omega r_H} \quad \text{for} \quad \xi \to 1,\\ \Psi_{s=1/2} &\sim e^{-\frac{i\omega r_H}{1-\xi^2}}(1-\xi^2)^{i\omega r_H}\xi^{-2i\omega r_H} \quad \text{for} \quad \xi \to 0.\end{aligned} \tag{30}$$

Collecting the leading terms, we can define $\alpha(\xi)$ in the four-dimensional case as follows:

$$\alpha(\xi) = e^{\frac{i\omega r_H}{1-\xi^2}}(1-\xi^2)^{-i\omega r_H}\xi^{-2i\omega r_H}. \tag{31}$$

If we now consider the higher dimensional cases, Eq. (29) will become more complicated with

$$r_* = r + \sum_{a=1}^{D-3}\frac{r_H\varepsilon}{D-3}\ln\left(\frac{r}{r_H\varepsilon}-1\right), \tag{32}$$

where $\varepsilon = e^{\frac{ia2\pi}{D-3}}$. Eq. (27) in the higher dimensions can be presented in the general form

$$\begin{aligned}\Psi_{s=1/2} &\sim e^{\frac{i\omega r_H}{1-\xi^2}}\prod_{a=1}^{D-3}\Big(\varepsilon(1-\xi^2)\Big)^{-\frac{i\omega r_H\varepsilon}{D-3}}\Big(1-\varepsilon(1-\xi^2)\Big)^{\frac{i\omega r_H\varepsilon}{D-3}} \quad \text{for} \quad \xi \to 1,\\ \Psi_{s=1/2} &\sim e^{-\frac{i\omega r_H}{1-\xi^2}}\prod_{a=1}^{D-3}\Big(\varepsilon(1-\xi^2)\Big)^{\frac{i\omega r_H\varepsilon}{D-3}}\Big(1-\varepsilon(1-\xi^2)\Big)^{-\frac{i\omega r_H\varepsilon}{D-3}} \quad \text{for} \quad \xi \to 0.\end{aligned} \tag{33}$$

Next, we shall consider how to define the asymptotic behaviour function $\alpha(\xi)$ in the higher dimensional cases, where in our experience, finding the dominant terms in Eq. (33) is helpful to the AIM calculation. When $D = 5$, Eq. (33) becomes

$$\begin{aligned}\Psi_{s=1/2} &\sim e^{\frac{i\omega r_H}{1-\xi^2}}(\xi^2-2)^{-\frac{i\omega r_H}{2}}(\xi^2)^{\frac{i\omega r_H}{2}} \quad \text{for} \quad \xi \to 1,\\ \Psi_{s=1/2} &\sim e^{-\frac{i\omega r_H}{1-\xi^2}}(\xi^2-2)^{\frac{i\omega r_H}{2}}(\xi^2)^{-\frac{i\omega r_H}{2}} \quad \text{for} \quad \xi \to 0.\end{aligned} \tag{34}$$

By considering the dominant term for the boundary behaviour, a suitable choice of the asymptotic behaviour function is as follows:

$$\alpha_{(D=5)}(\xi) = e^{\frac{i\omega r_H}{1-\xi^2}}(\xi^2)^{-\frac{i\omega r_H}{2}}. \tag{35}$$

For $D = 6$, Eq. (33) becomes

$$\begin{aligned}\Psi_{s=1/2} &\sim e^{\frac{i\omega r_H}{1-\xi^2}}\Big(e^{\frac{i2\pi}{3}}(1-\xi^2)\Big)^{-\frac{i\omega r_H e^{\frac{i2\pi}{3}}}{3}}\Big(1-e^{\frac{i2\pi}{3}}(1-\xi^2)\Big)^{\frac{i\omega r_H e^{\frac{i2\pi}{3}}}{3}}\Big(e^{\frac{i4\pi}{3}}(1-\xi^2)\Big)^{-\frac{i\omega r_H e^{\frac{i4\pi}{3}}}{3}}\\ &\quad\Big(1-e^{\frac{i4\pi}{3}}(1-\xi^2)\Big)^{\frac{i\omega r_H e^{\frac{i4\pi}{3}}}{3}}(1-\xi^2)^{-\frac{i\omega r_H}{3}}(1-\xi^2)^{-\frac{i\omega r_H}{3}} \ \text{for} \ \xi \to 1,\\ \Psi_{s=1/2} &\sim e^{-\frac{i\omega r_H}{1-\xi^2}}\Big(e^{\frac{i2\pi}{3}}(1-\xi^2)\Big)^{\frac{i\omega r_H e^{\frac{i2\pi}{3}}}{3}}\Big(1-e^{\frac{i2\pi}{3}}(1-\xi^2)\Big)^{-\frac{i\omega r_H e^{\frac{i2\pi}{3}}}{3}}\Big(e^{\frac{i4\pi}{3}}(1-\xi^2)\Big)^{\frac{i\omega r_H e^{\frac{i4\pi}{3}}}{3}}\\ &\quad\Big(1-e^{\frac{i4\pi}{3}}(1-\xi^2)\Big)^{-\frac{i\omega r_H e^{\frac{i4\pi}{3}}}{3}}(1-\xi^2)^{-\frac{i\omega r_H}{3}}(1-\xi^2)^{-\frac{i\omega r_H}{3}} \ \text{for} \ \xi \to 0.\end{aligned} \tag{36}$$

The dominant term for the boundary behaviour can be chosen as follows:

$$\alpha_{(D=6)}(\xi) = e^{\frac{i\omega r_H}{1-\xi^2}}(\xi^2)^{-\frac{i\omega r_H}{3}}(1-\xi^2)^{-\frac{i\omega r_H}{3}}. \tag{37}$$

For a similar discussion on the higher dimensional cases, we find that $\alpha(\xi)$ can be defined separately for the odd dimensional and even dimensional cases, where

$$\begin{aligned} \alpha(\xi) &= e^{\frac{i\omega r_H}{1-\xi^2}}(\xi^2)^{-\frac{i\omega r_H}{D-3}}, \qquad D \in \text{odd}; \\ \alpha(\xi) &= e^{\frac{i\omega r_H}{1-\xi^2}}(\xi^2)^{-\frac{i\omega r_H}{D-3}}(1-\xi^2)^{-\frac{i\omega r_H}{D-3}}, \quad D \in \text{even}. \end{aligned} \tag{38}$$

Substituting Eqs. (19), (20), (21) and (38) into Eq. (26), we find λ_1 and s_1 by the relation in Eq. (15). Next, by defining a suitable initial value of ω_0, which represents the parameter ω in s_0, with the termination condition Eq. (18), we can solve the parameter ω_1, which represents the parameter ω in s_1. This loop iterates, as with ω_1 we can solve for ω_2 in the next iteration and so on. For a sufficiently large number of iterations, the $\omega_{n\text{-}1}$ and ω_n will become stable, and it will be the quasi normal frequency we are seeking.

The QNM results are obtained with the iteration number = 200 in **Table 3**, for the D = 5,6 dimensional Schwarzschild spacetimes, and in **Table 4**, for D = 7,8. The results using the third-order WKB methods are the same as presented in our previous work [13], where for completeness we also list the sixth-order WKB results.

		5 Dimensions		
l	*n*	Third-order WKB	Sixth-order WKB	AIM
0	0	0.7247–0.3960 i	0.7823–0.3635 i	0.7252–0.3960 i
1	0	1.3158–0.3839 i	1.3301–0.3852 i	1.3163–0.3839 i
1	1	1.1490–1.2192 i	1.1800–1.2021 i	1.1495–1.2192 i
2	0	1.8754–0.3838 i	1.8801–0.3844 i	1.8802–0.3840 i
2	1	1.7541–1.1818 i	1.7672–1.1791 i	1.7674–1.1779 i
2	2	1.5588–2.0318 i	1.5620–2.0517 i	1.5593–2.0318 i
3	0	2.4251–0.3839 i	2.4271–0.3840 i	2.4256–0.3839 i
3	1	2.3322–1.1686 i	2.3382–1.1674 i	2.3327–1.1686 i
3	2	2.1704–1.9891 i	2.1691–1.9980 i	2.1709–1.9891 i
3	3	1.9611–2.8411 i	1.9385–2.9063 i	1.9616–2.8411 i
4	0	2.9716–0.3839 i	2.9726–0.3839 i	2.9721–0.3839 i
4	1	2.8963–1.1624 i	2.8994–1.1618 i	2.8968–1.1624 i
4	2	2.7596–1.9668 i	2.7574–1.9711 i	2.7601–1.9668 i
4	3	2.5773–2.7984 i	2.5569–2.8330 i	2.5778–2.7984 i
4	4	2.3583–3.6512 i	2.3126–3.7665 i	2.3588–3.6512 i
5	0	3.5166–0.3838 i	3.5171–0.3838 i	3.5171–0.3838 i
5	1	3.4533–1.1591 i	3.4550–1.1588 i	3.4538–1.1591 i
5	2	3.3353–1.9536 i	3.3333–1.956 i	3.3358–1.9536 i

		5 Dimensions		
5	3	3.1741–2.7711 i	3.1580–2.7908 i	3.1746–2.7711 i
5	4	2.9785–3.6089 i	2.9383–3.6779 i	2.9790–3.6089 i
5	5	2.7525–4.4625 i	2.6857–4.6302 i	2.7530–4.4625 i
		6 Dimensions		
0	0	1.2806–0.6391 i	1.4364–0.5821 i	1.2811–0.6391 i
1	0	2.1006–0.6276 i	2.1350–0.6423 i	2.1011–0.6276 i
1	1	1.7071–2.0417 i	1.8139–1.9981 i	1.7076–2.0417 i
2	0	2.8671–0.6308 i	2.8797–0.6354 i	2.8676–0.6308 i
2	1	2.5777–1.962 i	2.6235–1.9629 i	2.5782–1.962 i
2	2	2.1056–3.4196 i	2.1297–3.4716 i	2.1061–3.4196 i
3	0	3.6132–0.6321 i	3.6194–0.6329 i	3.6196–0.6325 i
3	1	3.3917–1.9342 i	3.4143–1.9318 i	3.4180–1.9342 i
3	2	3.0002–3.3204 i	3.0081–3.3404 i	3.0007–3.3204 i
3	3	2.4883–4.7849 i	2.4206–4.9568 i	2.4888–4.7849 i
4	0	4.3518–0.6324 i	4.3550–0.6325 i	4.3523–0.6324 i
4	1	4.1724–1.9214 i	4.1845–1.9192 i	4.1729–1.9214 i
4	2	3.8423–3.2694 i	3.8437–3.2771 i	3.8428–3.2694 i
4	3	3.3973–4.6831 i	3.3411–4.7697 i	3.3978–4.6831 i
4	4	2.8578–6.1516 i	2.6999–6.4718 i	2.8583–6.1516 i
5	0	5.0872–0.6325 i	5.0890–0.6324 i	5.0877–0.6325 i
5	1	4.9359–1.9143 i	4.9428–1.9128 i	4.9364–1.9143 i
5	2	4.6513–3.2399 i	4.6503–3.2435 i	4.6518–3.2399 i
5	3	4.2586–4.6195 i	4.2147–4.6677 i	4.2591–4.6195 i
5	4	3.7773–6.0494 i	3.6482–6.2378 i	3.7778–6.0494 i
5	5	3.2168–7.5218 i	2.9732–8.0104 i	3.2173–7.5218 i

Table 3. Low-lying ($n \leq l$, with $l = j - 1/2$) spin-1/2 field QNM frequencies using the WKB methods and the AIM with D = 5,6.

		7 Dimensions		
l	n	Third-order WKB	Sixth-order WKB	AIM
0	0	1.7861–0.8090 i	2.0640–0.7502 i	1.7866–0.8090 i
1	0	2.7344–0.8066 i	2.7827–0.8558 i	2.7349–0.8066 i
1	1	2.0521–2.6832 i	2.2892–2.6106 i	2.0526–2.6832 i
2	0	3.6130–0.8166 i	3.6327–0.8322 i	3.6135–0.8166 i
2	1	3.1092–2.5590 i	3.2053–2.5943 i	3.1097–2.5590 i
2	2	2.2671–4.5340 i	2.3344–4.6355 i	2.2676–4.5340 i
3	0	4.4610–0.8206 i	4.4730–0.8233 i	4.4615–0.8206 i
3	1	4.0779–2.5187 i	4.1298–2.5218 i	4.0784–2.5187 i

		7 Dimensions		
3	2	3.3809–4.3622 i	3.4158–4.4006 i	3.3814–4.3622 i
3	3	2.4624–6.363 i	2.3163–6.6549 i	2.4629–6.363 i
4	0	5.2968–0.8218 i	5.3035–0.8220 i	5.3033–0.8220 i
4	1	4.9878–2.5009 i	5.0176–2.4966 i	4.9883–2.5009 i
4	2	4.4049–4.2771 i	4.4224–4.2810 i	4.4054–4.2772 i
4	3	3.6066–6.1782 i	3.4954–6.3058 i	3.6071–6.1782 i
4	4	2.6393–8.1950 i	2.2624–8.7432 i	2.6398–8.1950 i
5	0	6.1273–0.8221 i	6.1310–0.8219 i	6.1309–0.8221 i
5	1	5.8671–2.4910 i	5.8847–2.4870 i	5.8676–2.4910 i
5	2	5.3673–4.2295 i	5.3756–4.2263 i	5.3678–4.2295 i
5	3	4.6654–6.0660 i	4.5819–6.1252 i	4.6659–6.0660 i
5	4	3.8002–8.0044 i	3.5047–8.3081 i	3.8007–8.0044 i
5	5	2.7964–10.035 i	2.1874–10.910 i	2.7969–10.0348i
		8 Dimensions		
0	0	2.2437–0.9238 i	2.6486–0.8930 i	2.2442–0.9238 i
1	0	3.2663–0.9359 i	3.3105–1.0469 i	3.2668–0.9359 i
1	1	2.2397–3.1796 i	2.6722–3.0659 i	2.2402–3.1796 i
2	0	4.2077–0.9556 i	4.2279–0.9928 i	4.2082–0.9556 i
2	1	3.4491–3.0091 i	3.6051–3.1428 i	3.4496–3.0091 i
2	2	2.1453–5.4463 i	2.2891–5.5740 i	2.1458–5.4463 i
3	0	5.1097–0.9641 i	5.1285–0.9704 i	5.1102–0.9642 i
3	1	4.5378–2.9607 i	4.6284–2.9895 i	4.5383–2.9607 i
3	2	3.4576–5.1805 i	3.5481–5.2651 i	3.4581–5.1805 i
3	3	2.0383–7.6877 i	1.7944–8.0198 i	2.0388–7.6877 i
4	0	5.9947–0.9669 i	6.0062–0.9669 i	5.9952–0.9669 i
4	1	5.5365–2.9415 i	5.5943–2.9365 i	5.5370–2.9415 i
4	2	4.6421–5.0549 i	4.6967–5.0480 i	4.6426–5.0549 i
4	3	3.4017–7.3877 i	3.2284–7.5066 i	3.4022–7.3877 i
4	4	1.9199–9.9347 i	1.2118–10.564 i	1.9204–9.9347 i
5	0	6.8720–0.9677 i	6.8785–0.9669 i	6.8772–0.9659 i
5	1	6.4875–2.9309 i	6.5231–2.9224 i	6.4880–2.9309 i
5	2	5.7272–4.9883 i	5.7608–4.9630 i	5.7277–4.9883 i
5	3	4.6390–7.2092 i	4.5109–7.2241 i	4.6395–7.2092 i
5	4	3.3023–9.6186 i	2.7516–9.9294 i	3.3028–9.6186 i
5	5	1.7759–12.195 i	0.5680–13.282 i	1.7764–12.1949 i

Table 4. Low-lying ($n \le l$, with $l = j - 1/2$) spin-1/2 field QNM frequencies using the WKB methods and the AIM with D = 7,8.

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4. A remark on the spin-3/2 field QNMs

In this section, we are going to present a discussion of our recent work [11] on the spin-3/2 fields in general dimensional Schwarzschild spacetimes. Note that, this section will not go into as great a detail as the previous one but shall just list some improvements made in light of recent considerations. Starting with the radial equation related to the "non-TT eigenmodes", the radial equation can still be represented as the Schrödinger like one, Eq. (4), where

$$\begin{aligned} V_{NTT,s=3/2} &= \pm f\frac{dW_{NTT,s=3/2}}{dr} + W^2_{NTT,s=3/2}, \\ W_{NTT,s=3/2} &= \frac{\sqrt{f}(j+\frac{D-3}{2})}{r}\left[\frac{(\frac{2}{D-2})^2(j+\frac{D-3}{2})^2 - 1 - \frac{D-4}{D-2}(\frac{2M}{r})^{D-3}}{(\frac{2}{D-2})^2(j+\frac{D-3}{2})^2 - f}\right], \end{aligned} \tag{39}$$

and

$$f(r) = 1 - \left(\frac{2M}{r}\right)^{D-3}. \tag{40}$$

Because the general form of the radial equation is the same as the Dirac case, Eqs. (22)–(38) are still sufficient in this case, but the effective potential will be Eq. (39), not Eq. (19). What we need to consider here is that the asymptotic behaviour function is simpler than what we had used previously and successfully generates better results for the QNMs.

In **Table 5**, for some lower modes in the seven-dimensional spacetime, the new AIM results (with the current choice of boundary behaviour function, Eq. (38)) are better than the previous

		7 Dimensions			
l	n	Third-order WKB	Sixth-order WKB	AIM (earlier)	AIM (new)
0	0	0.7725–0.2978 i	0.7530–0.3037 i	0.7008–0.3036 i	0.7535–0.3036 i
1	0	1.1441–0.2893 i	1.1415–0.2831 i	1.1231–0.2976 i	1.142–0.2831 i
1	1	0.9465–0.9065 i	0.9267–0.8783 i	0.9266–0.8782 i	0.9271–0.8782 i
⋮	⋮	⋮	⋮	⋮	⋮
		8 Dimensions			
⋮	⋮	⋮	⋮	⋮	⋮
4	4	1.0674–3.5290 i	0.7606–3.5381 i	1.0674–3.5290 i	1.0679–3.5291 i
5	4	1.5711–3.4654 i	1.3755–3.4609 i	1.5711–3.4653 i	1.5716–3.4654 i
5	5	1.0321–4.3822 i	0.5837–4.5299 i	1.0321–4.3822 i	1.0326–4.3822 i
⋮	⋮	⋮	⋮	⋮	⋮

Table 5. Selected QNM frequencies of low-lying ($n \leq l$, with $l = j - 3/2$) for spin-3/2 field related to the non-TT eigenmodes by using the WKB methods and the AIM.

results for the first few modes. This is why we believe Eq. (38) is a suitable choice of the boundary behaviour function for the AIM methods in the higher dimensional spacetimes.

5. Summary

In this chapter, we presented a brief review of recent progress on perturbation theory with various fields in curved spacetimes, especially for a systematic comparison to the fermionic and bosonic fields. Generally, the first step in this topic is the obtaining of the radial equations as discussed in Section 2. There are then various methods, but no unique approach for considering a specific field on a specific spacetime, where we strongly suggest the reader follow the references provided to develop a step-by-step approach. On the other hand, the process for obtaining the radial equations always relates to the separability of a spacetime, this being the main reason that we still have difficulties for perturbation theory in the higher dimensional Kerr spacetimes. As such we mentioned in the introduction section that for the gravitational wave experiments, the QNMs in the higher dimensional Kerr spacetimes are definitely an interesting next stage of study for this area, and it shall be interesting seeing further progress in this direction.

In Section 3, we presented the AIM, giving an in detail example to study the perturbation theory in the higher dimensional spacetimes and also presented a remark for our recent work on the spin-3/2 fields. Since there are several numerical methods for obtaining the QNMs, we believe that the AIM is a straightforward way to study the QNMs due to its simple mathematical structure.

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