

# We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

**4,800**

Open access books available

**122,000**

International authors and editors

**135M**

Downloads

Our authors are among the

**154**

Countries delivered to

**TOP 1%**

most cited scientists

**12.2%**

Contributors from top 500 universities



**WEB OF SCIENCE™**

Selection of our books indexed in the Book Citation Index  
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?  
Contact [book.department@intechopen.com](mailto:book.department@intechopen.com)

Numbers displayed above are based on latest data collected.

For more information visit [www.intechopen.com](http://www.intechopen.com)



# Relationship between Interpolation and Differential Equations: A Class of Collocation Methods

Francesco Aldo Costabile, Maria Italia Gualtieri and  
Anna Napoli

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/66995>

## Abstract

In this chapter, the connection between general linear interpolation and initial, boundary and multipoint value problems is explained. First, a result of a theoretical nature is given, which highlights the relationship between the interpolation problem and the Fredholm integral equation for high-order differential problems. After observing that the given problem is equivalent to a Fredholm integral equation, this relation is used in order to determine a general procedure for the numerical solution of high-order differential problems by means of appropriate collocation methods based on the integration of the Fredholm integral equation. The classical analysis of the class of the obtained methods is carried out. Some particular cases are illustrated. Numerical examples are given in order to illustrate the efficiency of the method.

**Keywords:** boundary value problem, initial value problem, collocation methods, interpolation, Birkhoff, Lagrange, Peano, Fredholm

## 1. Introduction

The relationship between interpolation and differential equations theories has already been considered. In Ref. ([1], p. 72), Davis observed that the Peano kernel in the interpolation problem

$$y(a) = \alpha, \quad y(b) = \beta, \quad a, b, \alpha, \beta \in \mathbb{R}, \quad (1)$$

is the Green's function of the differential problem

$$\begin{aligned}\phi''(x) &= f(x) \\ \phi(a) &= \phi(b) = 0\end{aligned}$$

where  $\phi(x) = y(x) - P_1[y](x)$ , being  $P_1[y](x)$  the unique interpolatory polynomial for Eq. (1).

He observed that “these remarks indicate the close relationship between Peano kernels and Green’s functions, and hence between interpolation theory and the theory of linear differential equations. Unfortunately, we shall not be able to pursue this relationship” [1].

Later, Agarwal ([2], p. 2), Agarwal and Wong ([3], pp. 21, 151, 186) considered some separate boundary value problems and the related Fredholm integral equation, using only polynomial interpolation, without taking into account the related Peano kernel. They used Fredholm integral equation in order to obtain existence and uniqueness results for the solution of the considered boundary value problems.

Linear interpolation has an important role also in the numerical solution of differential problems. For example, finite difference methods (see, for instance, [4–6] and references therein) approximate the solution  $y(x)$  of a boundary value problem by a sequence of overlapping polynomials which interpolate  $y(x)$  in a set of grid points. This is obtained by replacing the differential equation with finite difference equations on a mesh of points that covers the range of integration. The resultant algebraic system of equations is often solved with iterative processes, such as relaxation methods.

Many authors (see [7–10] and references therein) used linear interpolation with spline functions for the numerical solution of boundary value problems.

Here, we take into account a more general nonlinear initial/boundary/multipoint value problems for high-order differential equations

$$\begin{cases} y^{(r)}(x) = f(x, y(x)), & x \in I = [a, b], \quad r \geq 1 \\ L_i[y](x) = w_i, \quad i = 0, \dots, r-1, \quad x \in I \end{cases} \quad (2)$$

where  $y(x) = (y(x), y'(x), \dots, y^{(q)}(x))$ ,  $0 \leq q < r$ ,  $y \in \mathcal{C}^r(I)$ , and  $L_i$  are  $r$  linearly independent functionals on  $\mathcal{C}^r(I)$ . Moreover, we suppose that the function  $f : [a, b] \times \mathbb{R}^{q+1} \rightarrow \mathbb{R}$  is continuous at least in the interior of the domain of interest, and it satisfies a uniform Lipschitz condition in  $y$ , which means that there exists a nonnegative constant  $\Lambda$ , such that, whenever  $(x, y_0, y_1, \dots, y_q)$  and  $(x, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_q)$  are in the domain of  $f$ , the following inequality holds

$$\left| f(x, y_0, y_1, \dots, y_q) - f(x, \bar{y}_0, \bar{y}_1, \dots, \bar{y}_q) \right| \leq \Lambda \sum_{k=0}^q |y_k - \bar{y}_k|. \quad (3)$$

If  $L_i[y] = \Phi(y(a))$ ,  $i = 0, \dots, r-1$ , then (2) is an initial value problem (IVP); if  $L_i[y] = \Phi(y(a), y(b))$ ,  $i = 0, \dots, r-1$ , then (2) is a boundary value problem (BVP); if  $L_i[y] = \Phi(y(x_j))$ ,  $i = 0, \dots, r-1$ ,  $j = 0, \dots, m \geq 2$ , then (2) is a multipoint value problem (MVP).

In this chapter,

- we assume that the conditions for the existence and uniqueness of solution of problem (2) in a certain appropriate domain of  $[a, b] \times \mathbb{R}^{q+1}$  are satisfied and that the solution  $y(x)$  is differentiable with continuity up to what is necessary;
- we get the Fredholm integral equation related to problem (2), by polynomial interpolation and the Peano kernel of the linear interpolation problem  $L_i[y](x) = w_i, i = 0, \dots, r-1$ . In this way, we point out the close relationship between Green's function and Peano kernel;
- then, we construct a class of spectral collocation (pseudospectral) methods which are derived by a linear interpolation process.

The reason for which we prefer collocation methods is their superior accuracy for problems whose solutions are sufficiently smooth functions. Recently, Boyd ([11], p. 8) observed that "When many decimal places of accuracy are needed, the contest between pseudospectral algorithms and finite difference and finite element methods is not an even battle but a rout: pseudospectral methods win hands-down."

## 2. The Fredholm integral equation for problem (2)

We consider the general differential problem (2), and we prove that it is equivalent to a Fredholm integral equation.

**Proposition 1** [1, p. 35] *The linear interpolation problem*

$$L_i[P](x) = w_i, \quad w_i \in \mathbb{R}, \quad i = 0, \dots, r-1, \quad P \in P_{r-1}, \quad x \in I \quad (4)$$

with  $L_i, i = 0, \dots, r-1$ , linearly independent functionals on  $\mathcal{C}^r(I)$ , has the unique solution

$$P_{r-1}(t) = -\frac{1}{G} \begin{vmatrix} 0 & 1 & t & \dots & \dots & t^{r-1} \\ w_0 & & & & & \\ w_1 & & & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ w_{r-1} & & & & & \end{vmatrix} L_i[x^j], \quad G = |L_i[x^j]|_{i,j=0,\dots,r-1}. \quad (5)$$

*Proof.* Since the  $L_i, i = 0, \dots, r-1$  are linearly independent, the result follows from the general linear interpolation theory.

**Proposition 2** *If  $y \in \mathcal{C}^r(I)$  and  $L_i[y](x) = w_i, i = 0, \dots, r-1, x \in I$ , then*

$$y(x) = P_{r-1}[y](x) + \int_a^b K_r^x(x, t) y^{(r)}(t) dt, \quad \forall x \in I \text{ fixed}, \quad (6)$$

with  $L_i[y] = L_i[P_{r-1}], i = 0, \dots, r-1, P_{r-1}[y](x) = P_{r-1}(x)$ , and

$$K_r^x(x, t) = \frac{1}{(r-1)!} \left[ (x-t)_+^{r-1} - P_{r-1} \left[ (x-t)_+^{r-1} \right] (x) \right], \quad (7)$$

where index  $x$  means that  $K_r^x(x, t)$  is considered as a function of  $x$ .

*Proof.* It follows by observing that  $P_{r-1}[(x)_+^j](t) = (t)_+^j, j = 0, \dots, r-1$  and from Peano kernel Theorem [1].

**Theorem 1** *With the above notations and under the mentioned hypothesis, problem (2) is equivalent to the Fredholm integral equation*

$$y(x) = P_{r-1}[y](x) + \int_a^b K_r^x(x, t) f(t, y(t)) dt. \quad (8)$$

*Proof.* The result follows from the uniqueness of the Peano kernel and from Propositions 1 and 2.

**Corollary 1** *It results  $L_i[K_r^x] = 0, i = 0, \dots, r-1$ .*

From Theorem 1, general results on the existence and uniqueness of solution of problem (2) by standard techniques [2, 3] can be obtained. In the following, we will not linger over them, but we will outline the close relationship between interpolation and differential equations. Particularly, we will use linear interpolation in order to determine a class of collocation methods for the numerical solution of problem (2).

### 3. A class of Birkhoff-Lagrange collocation methods

Integral Eq. (8) allows to determine a very wide class of numerical methods for Eq. (2), which we call *methods of collocation for integration*.

Let  $\{x_i\}_{i=1}^m$  be  $m$  distinct points in  $[a, b]$  and denoted by  $l_i(t), i = 1, \dots, m$ , the fundamental Lagrange polynomials on the nodes  $x_i$ , that is

$$l_i(t) = \frac{\omega_m(t)}{(t-x_i)\omega'_m(x_i)}, \quad \text{where } \omega_m(t) = \prod_{k=1}^m (t-x_k). \quad (9)$$

**Theorem 2** *If the solution  $y(x)$  of Eq. (8) is in  $\mathcal{C}^{r+m}(I)$ , then*

$$y(x) = P_{r-1}[y](x) + \sum_{i=1}^m p_{r,i,m}(x) f(x_i, y(x_i)) + T_{r,m}(y, x), \quad (10)$$

where

$$p_{r,i,m}(x) = \int_a^b K_r^x(x, t) l_i(t) dt, \quad i = 1, \dots, m, \quad (11)$$

and the remainder term  $T_{r,m}(y, x)$  is given by:

$$T_{r,m}(y, x) = \frac{1}{m!} \int_a^b K_r^x(x, t) \omega_m(t) y^{(r+m)}(\xi_x) dt, \quad (12)$$

being  $\xi_x$  a suitable point of the smallest interval containing  $x$  and all  $x_i, i = 1, \dots, m$ .

*Proof.* From Lagrange interpolation

$$y^{(r)}(x) = \sum_{i=1}^m l_i(x) y^{(r)}(x_i) + \bar{R}_m(y, x) \quad (13)$$

where

$$\bar{R}_m(y, x) = \frac{1}{m!} \omega_m(t) y^{(r+m)}(\xi_x) \quad (14)$$

is the remainder term. From (2),  $f(x, y(x)) = \sum_{i=1}^m l_i(x) y^{(r)}(x_i) + \bar{R}_m(y, x)$ . Then, from Theorem 1, inserting Eq. (13) into (8), we obtain Eq. (10).

Theorem 2 suggests to consider the implicitly defined polynomial

$$y_{r,m}(x) = P_{r-1}[y_{r,m}](x) + \sum_{i=1}^m p_{r,i,m}(x) f(x_i, y_{r,m}(x_i)). \quad (15)$$

For polynomial (15), the following theorem holds.

**Theorem 3 (The main Theorem).** *Polynomial (15), of degree  $r + m - 1$ , satisfies the relations*

$$\begin{aligned} L_i[y_{r,m}](x) &= w_i, & i = 0, \dots, r-1, & x \in I, w_i \in \mathbb{R} \\ y_{r,m}^{(r)}(x_j) &= f(x_j, y_{r,m}(x_j)) & j = 1, \dots, m, \end{aligned} \quad (16)$$

that is,  $y_{r,m}(x)$  is a collocation polynomial for Eq. (2) at nodes  $x_j, j = 1, \dots, m$ .

*Proof.* From (15), Corollary 1 and the linearity of operators  $L_i$ , we get  $L_i[y_{r,m}](x) = w_i, i = 0, \dots, r-1$ . By Theorems 1 and 2, we obtain  $y^{(r)}(x_i) = y_{r,m}^{(r)}(x_i)$ , and from Eq. (11),  $p_{r,i,m}^{(r)}(x) = l_i(x)$ . Hence, relations (16) follow.

**Remark 1 (Hermite-Birkhoff-type interpolation).** *Theorem 3 is equivalent to the general Hermite-Birkhoff interpolation problem [12]: given  $w_i \in \mathbb{R}, i = 0, \dots, r-1$ , and  $\alpha_j \in \mathbb{R}, j = 1, \dots, m$ , determine, if there exists, the polynomial  $Q(x) \in \mathcal{P}_{m+r-1}$  such that*

$$\begin{aligned} L_i[Q] &= w_i, & i = 0, \dots, r-1 \\ Q^{(r)}(x_j) &= \alpha_j, & j = 1, \dots, m, x_j \in I. \end{aligned} \quad (17)$$

**Remark 2** *In the case of IVPs, for each method (15), we can derive the corresponding implicit Runge-Kutta method. For example, for  $r = 2$ , let  $b = x_0 + h$  and  $x_i = x_0 + c_i h$  with  $c_i \in [0, 1]$ . With the change of coordinates  $x = x_0 + th, t \in [0, 1]$ , we can write*

$$p_{r,i,m}(x) = p_{r,i,m}(x_0 + th) = h^2 \int_0^t \int_0^r l_i(s) ds dr, \quad l_i(s) = \prod_{\substack{k=1 \\ k \neq i}}^m \frac{s-c_k}{c_i-c_k}. \quad (18)$$

Putting  $f(x_i, y_{r,m}(x_i)) = y''_{r,m}(x_i) \equiv K_i$ ,  $a_{i,j} = p_{r,j}(x_i) = h^2 \int_0^{c_i} (c_i-s)l_j(s) ds$ , we have

$$K_i = f\left(x_0 + c_i h, y_0 + y'_0 th + \sum_{j=1}^m a_{i,j} K_j\right) \quad (19)$$

and

$$\begin{cases} y_1(t) \equiv y_{r,m}(x_0 + th) = y_0 + y'_0 th + h^2 \sum_{i=1}^m p_{r,i,m}(x_0 + th) K_i \\ y'_1(t) \equiv y'_{r,m}(x_0 + th) = y'_0 h + h^2 \sum_{i=1}^m p'_{r,i,m}(x_0 + th) K_i. \end{cases} \quad (20)$$

Eqs. (19) and (20) are the well-known continuous Runge-Kutta method for second-order differential equations. Particularly, for  $t = 1$ , we have the implicit Runge-Kutta-Nystrom method.

### 3.1. A-priori estimation of error

In what follows, we consider the norm

$$\|f\| = \max_{a \leq t \leq b} \sum_{k=0}^q |f^{(k)}(t)|, \quad \forall f \in \mathcal{C}^{(q)}(I). \quad (21)$$

Moreover, we define

$$Q_m = \sum_{i=1}^m \|p_{r,i,m}\|, \quad F(x) = \int_a^b K_r^x(x, t) dt, \quad H = \max_{a \leq t \leq b} |\bar{R}_m(y, t)|, \quad (22)$$

where  $\bar{R}_m(y, t)$  is defined as in (14).

**Theorem 4** *With the previous notations, if  $\Lambda Q_m < 1$ , then*

$$\|y - y_{r,m}\| \leq \frac{H \|F\|}{1 - \Lambda Q_m}. \quad (23)$$

*Proof.* By deriving Eqs. (10) and (15),  $s$  times,  $s = 0, \dots, q$ , we get

$$y^{(s)}(x) - y_{r,m}^{(s)}(x) = \sum_{i=1}^m p_{r,i,m}^{(s)}(x) \left[ f(x_i, y(x_i)) - f(x_i, y_{r,m}(x_i)) \right] + \frac{\partial^s}{\partial x^s} \int_a^b K_r^x(x, t) \bar{R}_m(y, t) dt. \quad (24)$$

It follows that

$$\begin{aligned}
 |y^{(s)}(x) - y_{r,m}^{(s)}(x)| &\leq \sum_{i=1}^m |p_{r,i,m}^{(s)}(x)| \Lambda \sum_{k=0}^q |y^{(k)}(x_i) - y_{r,m}^{(k)}(x_i)| + H |F^{(s)}(x)| \\
 &\leq \Lambda \|y - y_{r,m}\| \sum_{i=1}^m |p_{r,i,m}^{(s)}(x)| + H |F^{(s)}(x)|.
 \end{aligned}
 \tag{25}$$

From this, we obtain inequality (23).

#### 4. Algorithms and implementation

To calculate the approximate solution of problem (2) by polynomial  $y_{r,m}(x)$  at  $x \in I$ , we need the values  $y_{r,m}^{(s)}(x_k)$ ,  $k = 1, \dots, m$ ,  $s = 0, \dots, q$ . In order to get these values, we propose the following algorithm:

- Put  $y_k^{(s)} = y_{r,m}^{(s)}(x_k)$ ,  $k = 1, \dots, m$ ,  $s = 0, \dots, q$  and consider the following system

$$y_k^{(s)} = P_{r-1}^{(s)}[y_k](x_k) + \sum_{i=1}^m p_{r,i}^{(s)}(x_k) f(x_i, y_i), \tag{26}$$

$k = 1, \dots, m$ ,  $s = 0, \dots, q$ , where  $y_i = (y_i, y'_i, \dots, y_i^{(q)})$ .

System (26) can be written in the form

$$Y - AF(Y) = C \tag{27}$$

where

$$A = \begin{pmatrix} A_0 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_q \end{pmatrix}_{m(q+1) \times m(q+1)} \tag{28}$$

with

$$A_s = \begin{pmatrix} \tilde{a}_{1,1}^{(s)} & \dots & \tilde{a}_{1,m}^{(s)} \\ \vdots & & \vdots \\ \tilde{a}_{m,1}^{(s)} & \dots & \tilde{a}_{m,m}^{(s)} \end{pmatrix}_{m \times m} \quad \tilde{a}_{i,j}^{(s)} = p_{r,j}^{(s)}(x_i), \quad s = 0, \dots, q, \tag{29}$$

$$Y = (\bar{Y}_0, \dots, \bar{Y}_q)^T_{m(q+1) \times 1}, \quad \bar{Y}_s = (y_1^{(s)}, \dots, y_m^{(s)})^T, \tag{30}$$

$$F(Y) = \underbrace{(F_m, \dots, F_m)^T}_q, \quad F_m = (f_1, \dots, f_m)^T, \quad f_i = f(x_i, y_i), \tag{31}$$

$$B_s = (P_{r-1}^{(s)}[y_1](x_1), \dots, P_{r-1}^{(s)}[y_m](x_m)), \quad C = (B_0, \dots, B_q)^T_{m(q+1) \times 1}. \tag{32}$$

From Eq. (27), we get



$$Y = AF(Y) + C, \tag{33}$$

or, putting  $G(Y) = AF(Y) + C$ ,

$$Y = G(Y). \tag{34}$$

For the existence and uniqueness of solution of system (34), we can prove, with standard technique, the following theorem.

**Theorem 5** *If  $T = \Lambda \|A\|_\infty < 1$ , system (34) has a unique solution which can be calculated by an iterative method*

$$(Y_m)_{v+1} = G((Y_m)_{(v)}), \quad v \geq 0 \tag{35}$$

with a fixed  $(Y_m)_0 \in \mathbb{R}^{m(q+1)}$  and  $G(Y_m) = AF(Y_m) + C$ .

Moreover, if  $Y$  is the exact solution,

$$\|(Y_m)_{v+1} - Y\|_\infty \leq \frac{T^v}{1-T} \|(Y_m)_1 - (Y_m)_0\|_\infty. \tag{36}$$

**Remark 3** *If  $f$  is linear, then system (27) is a linear system which can be solved by a more suitable method.*

**Remark 4** *System (27) can be considered as a discrete method for the numerical solution of (2).*

**Remark 5** *Method (15) can generate the polynomial sequence*

$$(y_{r,m}(x))_v = P_{r-1}[y_{r,m}](x) + \sum_{i=1}^m p_{r,i,m}(x) f(x_i, (y_{r,m}(x_i))_{v-1}), \quad (y_{r,m})_0 = P_{r-1}[y](x) \tag{37}$$

which is equivalent to the discretization of Picard method for differential equations.

#### 4.1. Numerical computation of the entries of matrix $A$

To calculate the elements  $\tilde{a}_{i,k}^{(s)}$  of the matrix  $A$  in Eq. (27), we have to compute the integrals

$$p_{r,k}^{(s)}(x) = \frac{d^s}{dx^s} \int_a^b K_r^x(x,t) l_i(t) dt \tag{38}$$

for  $x = x_i$ . Integrating by parts, it remains to solve the problem of the computation of

$$F_{i1}(x_j) = \int_a^{x_j} l_i(t) dt, \quad F_{ik}(x_j) = \int_a^{x_j} F_{i,k-1}(t) dt \quad k = 2, \dots, n \tag{39}$$

$$M_{i1}(x_j) = \int_{x_j}^b l_i(t) dt, \quad M_{ik}(x_j) = \int_{x_j}^b M_{i,k-1}(t) dt \quad k = 2, \dots, n \tag{40}$$

$i, j = 1, \dots, m$ . To this aim, it suffices to compute

$$\int_c^{x_j=t_k} \int_c^{t_{k-1}} \cdots \int_c^{t_1} r_{m,i}(t) dt dt_1 \cdots dt_{k-1} \tag{41}$$

where  $c = a$  or  $c = b$ ,  $r_{0,0}(t) = 1$ ,

$$r_{m,i}(t) = (t-x_1) \cdots (t-x_{i-1})(t-x_{i+1}) \cdots (t-x_m) \quad i = 1, 2, \dots, m. \tag{42}$$

For the computation of the integral (41), we use the recursive algorithm introduced in Ref. [13]: for each  $i = 1, \dots, m$ , let us consider the new points  $z_j^{(i)} = x_j$  if  $j < i$ , and  $z_j^{(i)} = x_{j+1}$  if  $j \geq i$ . Moreover, let us define  $g_{0,1,c}^{(i)}(x) = x-c$  and for  $s = 1, \dots, m-1$

$$g_{s,j,c}^{(i)}(x) = \int_c^{x=t_j} \int_c^{t_{j-1}} \cdots \int_c^{t_1} (t-z_1^{(i)}) (t-z_2^{(i)}) \cdots (t-z_s^{(i)}) dt dt_1 \cdots dt_{j-1}. \tag{43}$$

We can easily compute  $g_{0,j,c}^{(i)}(x) = \frac{(x-c)^j}{j!}$ . For the computation of Eq. (43), the following recurrence formula [13] holds

$$g_{s,j,c}^{(i)}(x) = (x-z_s^{(i)}) g_{s-1,j,c}^{(i)}(x) - j g_{s-1,j+1,c}^{(i)}(x). \tag{44}$$

Thus, if  $W_i = \prod_{k=1, k \neq i}^m (x_i - x_k)$ , then

$$F_{ik}(x_j) = \frac{g_{m-1,k,a}^{(i)}(x_j)}{W_i}, \quad M_{ik}(x_j) = (-1)^k \frac{g_{m-1,k,b}^{(i)}(x_j)}{W_i}. \tag{45}$$

**Remark 6** An alternative approach for the exact computation of integrals (39) and (40) is to use a quadrature formula with a suitable degree of precision.

## 4.2. Outline of the method

Summarizing the proposed method consists of the following steps:

1. determine the interpolation polynomial  $P_{r-1}[y](x)$  satisfying the boundary conditions and compute the Peano remainder;
2. approximate  $y^{(r)}(x)$  by Lagrange interpolation on a set of fixed nodal point;
3. compute the elements of matrix A (28) and solve system (26);
4. obtain polynomial (15).

## 5. Some particular cases

Now we consider some special cases of problem (2), and for each case, we determine  $P_{r-1}[y](x)$  and  $K_r^x(x, t)$ . We also show how the proposed class of methods includes the methods presented in previous works [12–24].

### 5.1. Initial value problems

In the case of initial value problems, in Refs. [13, 17, 25], problem

$$y^{(r)}(x) = f(x, y(x)) \quad (46)$$

has been considered, while in Ref. [23], the authors introduced the more general equation

$$y^{(r)}(x) = f(x, y(x), y'(x), \dots, y^{(i)}(x)), \quad q \leq r-1. \quad (47)$$

In both cases

$$P_{r-1}[y](x) = \sum_{i=0}^{r-1} \frac{(x-a)^i}{i!} y^{(i)}(a) \quad (48)$$

and

$$K_r^x(x, t) = \frac{1}{(r-1)!} (x-t)_+^{r-1}. \quad (49)$$

If  $\{x_i\}_{i=1}^m$  are the zeros of Chebyshev polynomials of first and second kind, the explicit expression for polynomials  $p_{r,i,m}(x)$  can be obtained [13, 17, 25] for some values of  $r$ .

Particularly, for  $r = 1$  and  $r = 2$ , in the case of zeros of Chebyshev polynomials of first kind, we get

$$p_{1,i,m}(x) = \frac{1}{m} \sum_{k=2}^{m-1} \left\{ \left[ \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} + 2 \frac{(-1)^{k-1}}{k^2-1} \right] \cos \left( \frac{2i-1}{2m} k\pi \right) \right. \\ \left. + \frac{1}{m} \left[ x+1 + \cos \left( \frac{2i-1}{2m} \pi \right) (x^2-1) \right] \right\} \quad (50)$$

where  $T_{k-1}(x)$  and  $T_{k+1}(x)$  are the Chebyshev polynomials of the first kind and degree  $k-1$  and  $k+1$ , respectively, and

$$p_{2,i,m}(x) = \frac{1}{m} \left\{ \frac{(x+1)^2}{2} + \frac{x^3-3x-2}{3} \left( \frac{\cos \frac{\pi(2i-1)}{2m} + x \cos \pi(2i-1)}{m} \right) \right. \\ \left. + \frac{1}{2} \sum_{k=3}^{m-1} \cos \frac{k\pi(2i-1)}{2m} \left[ \frac{T_{k+2}(x)}{(k+1)(k+2)} - 2 \frac{T_k(x)}{k^2-1} \right. \right. \\ \left. \left. + \frac{T_{k-2}(x)}{(k-1)(k-2)} - \frac{12k(-1)^k}{k(k^2-1)(k^2-4)} - \frac{4(-1)^k}{k^2-1} (x+1) \right] \right\}. \quad (51)$$

In the case of zeros of Chebyshev polynomials of second kind

$$p_{1,i,m}(x) = \frac{2}{m+1} \sin \frac{\pi i}{m+1} \sum_{k=0}^{m-1} \sin \frac{(k+1)\pi i}{m+1} \frac{1}{k+1} [T_{k+1}(x) + (-1)^k] \quad (52)$$

and

$$p_{2,i,m}(x) = \frac{1}{m+1} \sin \frac{\pi i}{m+1} \left\{ \sin \frac{\pi i}{m+1} (x+1)^2 + \sum_{k=2}^m \frac{1}{k} \sin \frac{k\pi i}{m+1} \left[ \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} - 2 \left( x + \frac{k^2}{k^2-1} \right) (-1)^k \right] \right\} \quad (53)$$

In Refs. [13, 25], the authors presented the corresponding implicit Runge-Kutta methods too.

In Ref. [26], Coleman and Booth used also a polynomial interpolant of degree  $n$  for  $y''$ , but they started from an identity different to Eq. (8) and derived a collocation method for which the nodes  $\{x_i\}_{i=1}^m$  are the zeros of Chebyshev polynomials of second kind.

## 5.2. Boundary value problems

### 5.2.1. Case $r = 2n$

For  $n = 1$ , for the exact solution  $y(x)$  of the second-order BVP

$$y''(x) = f(x, y(x), y'(x)), \quad y(-1) = y_0, \quad y(1) = y_1 \quad (54)$$

$x \in [-1, 1]$ , it is known that

$$y(x) = \frac{y_1 + y_0}{2} + x \frac{y_1 - y_0}{2} + \int_{-1}^1 K_2^x(x, t) f(x, y(x), y'(x)) dt \quad (55)$$

where

$$K_2^x(x, t) = \begin{cases} \frac{(t+1)(x-1)}{2} & t \leq x \\ \frac{(x+1)(t-1)}{2} & x < t. \end{cases} \quad (56)$$

By applying method (15), we get [16]

$$y_{2,m}(x) = \frac{y_1 + y_0}{2} + x \frac{y_1 - y_0}{2} + \sum_{i=1}^m p_{r,i,m}(x) f(x_i, y(x_i), y'(x_i)) \quad (57)$$

with  $p_{r,i,m}(x) = \int_{-1}^1 K_2^x(x, t) l_i(t) dt$ .

If  $x_i = \cos \frac{\pi i}{m+1}$ ,  $i = 1, \dots, m$ , we obtain explicitly the expression of  $p_{r,i,m}(x)$  [18]

$$p_{r,i,m}(x) = \frac{1}{m+1} \sin \frac{\pi i}{m+1} \left[ \sum_{k=2}^m \frac{G_k(x)}{k} \sin \frac{k\pi i}{m+1} + (x^2-1) \sin \frac{\pi i}{m+1} \right] \tag{58}$$

where

$$G_k(x) = \frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1} + \begin{cases} \frac{2x}{k^2-1} & \text{even } k \\ k \frac{2}{k^2-1} & \text{odd } k. \end{cases} \tag{59}$$

The same method has been presented in Ref. [24], where also stability has been studied.

Now assume  $[a, b] = [0, 1]$  and  $r > 2$ . Several types of boundary conditions can be considered.

*-Hermite boundary conditions [22]:*

$$y^{(h)}(0) = \alpha_h, \quad y^{(h)}(1) = \beta_h, \quad h = 0, \dots, n-1 \tag{60}$$

with  $\alpha_h, \beta_h, h = 0, \dots, n-1$  real constants.

In this case,  $P_{r-1}$  is the Hermite polynomial of degree  $2n-1$

$$P_{2n-1}[y](x) = \sum_{i=0}^{n-1} (y^{(i)}(0)H_{i1}(x) + y^{(i)}(1)H_{i2}(x)) \tag{61}$$

with

$$H_{i1}(x) = \frac{x^i(1-x)^{n-i-1}}{i!} \sum_{s=0}^{n-i-1} \binom{n+s-1}{n-1} x^s$$

$$H_{i2}(x) = \frac{x^n(1-x)^{i+n-i-1}}{i!} \sum_{s=0}^{n-i-1} \binom{n+s-1}{n-1} (1-x)^s. \tag{62}$$

The kernel is

$$K_{2n}^x(x, t) = \begin{cases} \sum_{i=0}^{n-1} \frac{(-t)^{2n-i-1}}{(2n-i-1)!} H_{i1}(x) & 0 \leq t \leq x \\ -\sum_{i=0}^{n-1} \frac{(1-t)^{2n-i-1}}{(2n-i-1)!} H_{i2}(x) & x \leq t \leq 1. \end{cases} \tag{63}$$

*-Lidstone boundary conditions [19]:*

$$y^{(2h)}(0) = \alpha_h, \quad y^{(2h)}(1) = \beta_h, \quad h = 0, \dots, n-1 \tag{64}$$

where  $\alpha_h, \beta_h, h = 0, \dots, n$  are real constants.

In this case,  $P_{r-1}$  is the Lidstone interpolating polynomial [3] of degree  $2n-1$

$$P_{2n-1}[y](x) = \sum_{k=0}^{n-1} \left[ y^{(2k)}(0)\Lambda_k(1-x) + y^{(2k)}(1)\Lambda_k(x) \right] \quad (65)$$

where  $\Lambda_k(x)$  are the Lidstone polynomials of degree  $2k + 1$  [3], and the function  $K_{2n}^x(x, t)$  is

$$K_{2n}^x(x, t) = \begin{cases} \sum_{k=0}^{n-1} \frac{t^{2n-2k-1}}{(2n-2k-1)!} \Lambda_k(1-x) & t \leq x \\ \sum_{k=0}^{n-1} \frac{(1-t)^{2n-2k-1}}{(2n-2k-1)!} \Lambda_k(x) & x \leq t. \end{cases} \quad (66)$$

### 5.2.2. Case $r = 2n + 1$

If we consider the following boundary conditions

$$y(0) = \alpha_0, \quad y^{(2h-1)}(0) = \alpha_h, \quad y^{(2h-1)}(1) = \beta_h, \quad h = 1, \dots, n \quad (67)$$

with  $\alpha_0, \alpha_h, \beta_h, h = 1, \dots, n$  real constants, then  $P_{r-1}$  is the complementary Lidstone interpolating polynomial [27] of degree  $2n$  [3, 24, 27, 28].

$$P_{2n}[y](x) = y(0) + \sum_{k=1}^n \left[ y^{(2k-1)}(0) \left( v_k(1) - v_k(1-x) \right) + y^{(2k-1)}(1) \left( v_k(x) - v_k(0) \right) \right], \quad (68)$$

where  $v_k(x)$  are the complementary Lidstone polynomials of degree  $k$  [27]. The kernel is

$$K_{2n-1}^x(x, t) = \begin{cases} \frac{t^{2n}}{(2n)!} + \sum_{k=1}^n \frac{t^{2n-2k+1}}{(2n-2k+1)!} \left( v_k(1-x) - v_k(1) \right) & t \leq x \\ - \sum_{k=1}^n \frac{(1-t)^{2n-2k+1}}{(2n-2k+1)!} \left( v_k(x) - v_k(0) \right) & x \leq t. \end{cases} \quad (69)$$

In Ref. [19], the proposed method applied to problem (2) with conditions (64) and (67), respectively, has been examined in detail.

### 5.2.3. Other special boundary conditions

If  $r = n-1$  and  $[a, b] = [0, 1]$ , we can consider *Bernoulli* boundary conditions [21]

$$y(0) = \beta_0, \quad y(1) = \beta_1, \quad y^{(k)}(1) - y^{(k)}(0) = \beta_{k+1}, \quad k = 1, \dots, n-2 \quad (70)$$

with  $\beta_k, k = 0, \dots, n-1$  real constants. The method has been examined in [14].

## 5.3. Multipoint boundary value problems

Let us now consider [15] the following conditions in  $I = [-1, 1]$

$$y^{(k)}(-1) = \alpha_k, \quad k = 0, \dots, s-1, \quad y^{(s)}(x_i) = \omega_i \quad i = 1, \dots, r-s. \tag{71}$$

In this case

$$P_{r-1}[y](x) = \sum_{i=0}^{s-1} \frac{(x+1)^i}{i!} \alpha_i + \frac{1}{(s-1)!} \sum_{k=1}^{r-s} \omega_k p_{r,k}(x), \tag{72}$$

with

$$p_{r,k}(x) = \int_{-1}^x (x-t)^{s-1} l_k(t) dt \tag{73}$$

and  $l_k(t)$  are the fundamental Lagrange polynomials on the points  $x_j, j = 1, \dots, r-s$ .  $P_{r-1}(x)$  is the unique polynomial of degree  $\leq r-1$  which satisfies the Birkhoff interpolation problem

$$P_{r-1}^{(k)}(-1) = \alpha_k, \quad k = 0, \dots, s-1, \quad P_{r-1}^{(s)}(x_i) = \omega_i, \quad i = 1, \dots, r-s, \quad s \leq r-1 \tag{74}$$

with  $-1 < x_1 < \dots < x_{r-s} \leq 1$ . Hence, the solution of problem (2), with multipoint conditions (71), is

$$y(x) = P_{r-1}[y](x) + \int_{-1}^1 K_r^x(x, t) y^{(r)}(t) dt, \tag{75}$$

with  $P_{r-1}[y](x)$  given in Eq. (72) and

$$K_r^x(x, t) = \frac{1}{(r-1)!} \left[ (x-t)_+^{r-1} - \binom{r-1}{s} \sum_{i=1}^{r-s} p_{r,i,m}(x) (x_i-t)_+^{r-s-1} \right]. \tag{76}$$

Observe that Eq. (74) is a special type of Birkhoff interpolation problem with incidence matrix  $E = (e_{ij})$  defined by  $e_{1j} = e_{is} = 1, j = 0, \dots, s-1, i = 2, \dots, r-s + 1, e_{ij} = 0$  otherwise and  $r \geq 1$ .

In Ref. [23],  $P_{r-1}[y](x)$  is presented in a little different form:

$$P_{r-1}[y](x) = \sum_{i=0}^{s-1} \frac{(x+1)^i}{i!} \alpha_i + \sum_{k=1}^{r-s} \omega_k E_s(x, l_k(x)), \tag{77}$$

where  $E_s(x, l_k(x)) = \underbrace{\int_{-1}^x \dots \int_{-1}^x}_{s} l_k(t) dt \dots dt$ .

Let us now consider the following conditions [12, 20]

$$y(-1) = \omega_0, \quad y(1) = \omega_{r-1} \quad y''(x_i) = \omega_i \quad i = 1, \dots, r-2. \tag{78}$$

The solution to the Birkhoff interpolation problem

$$P_{r-1}(-1) = \omega_0, \quad P_{r-1}(1) = \omega_{r-1}, \quad P_{r-1}''(x_i) = \omega_i, \quad i = 1, \dots, r-2 \quad (79)$$

with  $-1 < x_1 < \dots < x_{r-2} < 1$  is [12]

$$P_{r-1}[y](x) = \frac{\omega_{r-1} + \omega_0}{2} + \frac{\omega_{r-1} - \omega_0}{2}x + \sum_{i=1}^{r-2} q_{r,i}(x)\omega_i \quad (80)$$

with

$$q_{r,i}(x) = \int_{-1}^1 K_r^x(x, t)l_i(t)dt \quad (81)$$

and

$$K_r^x(x, t) = \begin{cases} \frac{(t+1)(x-1)}{2} & t \leq x \\ \frac{(x+1)(t-1)}{2} & x < t. \end{cases} \quad (82)$$

Hence, the solution of problem (2) is

$$y(x) = P_{r-1}[y](x) + \int_{-1}^1 K_r^x(x, t)y^{(r)}(t)dt, \quad (83)$$

with  $P_{r-1}[y](x)$  given in Eq. (80) and

$$K_r^x(x, t) = \frac{1}{(r-1)!} \left[ (x-t)_+^{r-1} - \frac{(1-t)^{r-1}(1+x)}{2} - (r-1)(r-2) \sum_{i=1}^{r-2} p_{r,i,m}(x)(x_i-t)_+^{r-3} \right]. \quad (84)$$

## 6. Numerical examples

In this section, we present some numerical results obtained by applying method (15), which we call *CGN method*, to find numerical approximations  $y_{r,m}(x)$  to the solution of some test problems. In order to solve the nonlinear system (19), we use the so-called modified Newton method [29] (the same Jacobian matrix is used for more than one iteration) and we use algorithm (44) for the computation of the entries of the matrix, when polynomials  $p_{r,i,m}(x)$  are not explicitly known. Since the true solutions of the analyzed problems are known, we consider the error function  $e_m(x) = |y(x) - y_{r,m}(x)|$ .

The maximum values of  $e_m(x)$  over the interval  $[a, b]$  have also been calculated by using Matlab, particularly the built-in solvers

- **ode15s**, a variable-step, variable-order multistep solver based on the numerical differentiation formulas of orders 1–5;



- **ode45**, a single-step solver, based on an explicit Runge-Kutta (4, 5) formula, the Dormand-Prince pair

for initial value problems, and the finite difference codes;

- **bvp4c** (with an optional mesh of 200 points) that implements the three-stage Lobatto IIIa formula;
- **bvp5c** that implements the four-stage Lobatto IIIa formula.

for boundary value problems.

All solvers have been used with optional parameters **RelTol=AbsTol=1e-17**.

Moreover, the powerful tool **Chebfun** [30] has been used.

**Example 1** Consider the following linear ninth-order BVP [28]

$$\begin{cases} y^{(9)}(x) = -9e^x + y(x) & x \in [0, 1] \\ y^{(j)}(0) = 1-j & j = 0, \dots, 4 \\ y^{(j)}(1) = -j e & j = 0, \dots, 3 \end{cases} \quad (85)$$

with exact solution  $y(x) = (1-x)e^x$ .

The unique polynomial  $P_8(x) = P_8[y](x)$  of degree 8 satisfying the boundary conditions  $P_8^{(j)}(0) = 1-j$  for  $j = 0, \dots, 4$ , and  $P_8^{(j)}(1) = -j e$   $j = 0, \dots, 3$  is

$$P_8(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \left(\frac{31}{2} - \frac{253}{6}e\right)x^5 + \left(\frac{1321}{12} - \frac{81}{2}e\right)x^6 + \left(\frac{71}{2} - \frac{193}{2}e\right)x^7 + \left(\frac{685}{24} - \frac{21}{2}e\right)x^8. \quad (86)$$

From Eq. (7), we get

$$K_9^x(x, t) = \frac{1}{8!} \cdot \begin{cases} 70t^4(x^4 - 4x^5 + 6x^6 - 4x^7 + x^8) + 56t^5(-x^3 + 10x^5 - 20x^6 + 15x^7 - 4x^8) + \\ 28t^6(x^2 - 20x^5 + 45x^6 - 36x^7 + 10x^8) + 8t^7(-x + 35x^5 - 84x^6 + 70x^7 - 20x^8) + \\ t^8(1 - 56x^5 + 140x^6 - 120x^7 + 35x^8) & 0 \leq t \leq x \\ -x^8 + 8tx^7 - 28t^2x^6 + 56t^3x^5 + 70t^4(-4x^5 + 6x^6 - 4x^7 + x^8) + \\ 56t^5(10x^5 - 20x^6 + 15x^7 - 4x^8) + 28t^6(-20x^5 + 45x^6 - 36x^7 + 10x^8) + \\ 8t^7(35x^5 - 84x^6 + 70x^7 - 20x^8) + \\ t^8(-56x^5 + 140x^6 - 120x^7 + 35x^8) & x \leq t \leq 1. \end{cases} \quad (87)$$

Now we calculate the values of the integrals (39) by using Eq. (45), and we solve system (26). Thus, we obtain the approximate solution (15) to problem (85).

**Table 1** shows the numerical results. The absolute errors are compared with those obtained in Ref. [28], where a modified decomposition method is applied for the solution of problem (85). The second and third columns of **Table 1** show the error, respectively, in the method in Ref. [28] and in the CGN method, using in both cases polynomials of degree 12. The last column contains the error in the approximation

by a polynomial of degree 14 using CGN method. As collocation points, equidistant nodes in  $[0, 1]$  are chosen. Analogous results are obtained by using Chebyshev nodes of first and second kind, and Legendre-Gauss-Lobatto points.

The maximum absolute error  $\max\{e_m(x)\}$  on  $[0, 1]$  has also been calculated by using Matlab (Table 2).

$x$	Method in [28]	CGN $m = 4$	CGN $m = 6$
0.1	$2.0e-10$	$1.45e-14$	0.00
0.2	$2.0e-10$	$3.93e-13$	$1.11e-16$
0.3	$2.0e-10$	$2.16e-12$	$9.99e-15$
0.4	$2.0e-10$	$5.70e-12$	$2.00e-15$
0.5	$2.0e-10$	$9.27e-12$	$2.55e-15$
0.6	$6.0e-10$	$1.00e-11$	$2.66e-15$
0.7	$1.0e-9$	$7.04e-12$	$2.44e-15$
0.8	$2.0e-9$	$2.70e-12$	$2.83e-15$
0.9	$3.4e-9$	$2.98e-13$	$4.91e-15$

Table 1. Absolute error  $e_m(x)$  in MDM and CGN methods for problem (85).

Chebfun	bvp4c	bvp5c
1.46	$1.55e-12$	$4.44e-16$

Table 2. Maximum absolute error in problem (85) using Matlab built-in functions.

$x$	Cheb I	Cheb II	EqPts
	$m = 4$	$m = 6$	$m = 9$
0.1	$1.11e-16$	0.00	0.00
0.2	$9.54e-15$	0.00	0.00
0.3	$5.47e-13$	$3.33e-16$	0.00
0.4	$9.45e-12$	$1.11e-16$	$4.44e-16$
0.5	$8.50e-11$	$4.22e-15$	$1.11e-16$
0.6	$5.05e-10$	$3.47e-14$	$2.11e-15$
0.7	$2.25e-9$	$2.08e-13$	$1.55e-15$
0.8	$8.08e-9$	$9.68e-13$	$1.44e-14$
0.9	$2.74e-8$	$3.72e-12$	$9.18e-15$
1.0	$6.64e-8$	$1.22e-11$	$1.37e-14$

Table 3. Problem (88)—example 2.

**Example 2** Consider the fifth-order initial value problem [13]

$$\begin{cases} y^{(5)} + (32x^5 + 120x)y = 160x^3e^{-x^2} & x \in [0, 1] \\ y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2 \\ y'''(0) = 0, \quad y^{(4)}(0) = 12 \end{cases} \quad (88)$$

with solution  $y(x) = e^{-x^2}$ .

**Table 3** shows the absolute error in some points of the interval  $[0, 1]$  for CGN method in the case, respectively, of Chebyshev nodes of first kind (Cheb I), of second kind (Cheb II) and in the case of equidistant nodes (EqPts).

The maximum absolute errors calculated by using Matlab are displayed in **Table 4**.

Chebfun	ode15s	ode45
2.11e-11	1.35e-13	1.33e-15

**Table 4.** Maximum absolute error in problem (88) using Matlab built-in functions.

**Example 3** Consider now the following nonlinear problem [31]

$$\begin{cases} y^{(4)}(x) = \sin x + \sin^2 x - (y''(x))^2 & x \in [0, 1] \\ y(0) = 0 \quad y'(0) = 1 \\ y(1) = \sin(1) \quad y'(1) = \cos(1) \end{cases} \quad (89)$$

with exact solution  $y(x) = \sin(x)$ .

This kind of problems models several nonlinear phenomena such as traveling waves in suspension bridges [32] or the bending of an elastic beam [33].

Suspension bridges are generally susceptible to visible oscillations, due to the forces acting on the bridge (including the force due to the cables which are considered as a spring with a one-sided restoring, the gravitation force and the external force due to the wind or other external sources).  $f$  represents the forcing term, while  $y$  represents the vertical displacement when the bridge is bending.

In the case of elastic beam,  $f$  represents the force exerted on the beam by the supports.  $x$  measures the position along the beam ( $x = 0$  is the left-hand endpoint of the beam),  $y$  and  $y'$  indicate, respectively, the height and the slope of the beam at  $x$ .  $y''$  measures the curvature of the graph of  $y$ , and, in physical terms, it measures the bending moment of the beam at  $x$ , that is, the torque that the load places on the beam at  $x$ .

The considered boundary conditions state that the beam has both endpoints simply supported. Moreover, the derivative of the deflection function is not zero at those points, and it indicates that the beam at the wall is not horizontal.

**Table 5** shows the comparison between the NMD method presented in Ref. [31] and the CGN method with  $m = 5$  and  $m = 9$ , respectively. The approximating polynomial of NMD method has degree 11, while the polynomial considered in CGN method for  $m = 5$  has degree 8.

The maximum absolute errors calculated by using Matlab are displayed in **Table 6**.

$x$	NMD [31]	CGN $m = 5$	CGN $m = 9$
0.1	$7.78e-8$	$4.45e-10$	$1.53e-15$
0.2	$2.72e-7$	$5.54e-10$	$3.02e-15$
0.3	$5.24e-7$	$8.95e-11$	$7.77e-16$
0.4	$7.77e-7$	$2.03e-10$	$6.66e-16$
0.5	$9.71e-7$	$3.32e-11$	$5.55e-17$
0.6	$1.05e-6$	$1.53e-10$	0
0.7	$9.63e-7$	$9.48e-11$	0
0.8	$6.84e-7$	$5.18e-10$	$1.11e-16$
0.9	$2.71e-7$	$4.15e-10$	0

**Table 5.** Error of NMD and CGN methods—problem (89).

Chebfun	bvp4c	bvp5c
$1.67e-16$	$1.22e-8$	$8.88e-16$

**Table 6.** Maximum absolute error in problem (89) using Matlab build-in functions.

## Author details

Francesco Aldo Costabile, Maria Italia Gualtieri and Anna Napoli\*

\*Address all correspondence to: [anna.napoli@unical.it](mailto:anna.napoli@unical.it)

Department of Mathematics and Informatics, University of Calabria, Rende (Cs), Italy

## References

- [1] Davis P. *Interpolation and approximation*. Dover, New York. 1975.
- [2] Agarwal R. *Boundary value problems for higher order differential equations*. World Scientific Publishing Co., Inc., Teaneck, NJ. 1986.
- [3] Agarwal R, Wong P. *Lidstone polynomials and boundary value problems*. *Computers and Mathematics with Applications*. 1989; **17**(10): 1397–1421.
- [4] Hairer E, Nørsett S, Wanner G. *Solving ordinary differential equations I. Nonstiff problems*. Berlin: Springer-Verlag. 1987.

- [5] Henrici P. *Discrete variable methods in ordinary differential equations*. Wiley, New York. 1962.
- [6] Strikwerda J. *Finite difference schemes and partial differential equations*. SIAM., Philadelphia, PA. 2004.
- [7] Caglar H, Caglar N, Elfaituri K. *B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems*. Applied Mathematics and Computation. 2006; **175**(1): 72–79.
- [8] Chang J, Yang Q, Zhao L. *Comparison of b-spline method and finite difference method to solve bvp of linear odes*. Journal of Computers. 2011; **6**(10): 2149–2155.
- [9] Costabile F, Gualtieri MI, Serafini G. *Cubic Lidstone-Spline for numerical solution of BVPs*. submitted.
- [10] Khan A. *Parametric cubic spline solution of two point boundary value problems*. Applied Mathematics and Computation. 2004; **154**(1): 175–182.
- [11] Boyd J. *Chebyshev and Fourier spectral methods*. 2nd edition, Dover, Mineola, NY. 2000.
- [12] Costabile F, Longo E. *A Birkhoff interpolation problem and application*. Calcolo. 2010; **47**(1): 49–63.
- [13] Costabile F, Napoli A. *A class of collocation methods for numerical integration of initial value problems*. Computers and Mathematics with Applications. 2011; **62**(8): 3221–3235.
- [14] Costabile F, Napoli A. *Numerical solution of high order Bernoulli boundary value problems*. Journal of Applied Mathematics. 2014, Article ID 276585. doi: 10.1155/2014/276585.
- [15] Costabile F, Napoli A. *A method for high-order multipoint boundary value problems with Birkhoff-type conditions*. International Journal of Computer Mathematics. 2015; **92**(1): 192–200.
- [16] Costabile F, Longo E. *A new collocation method for a BVP*. Applied and Industrial Mathematics in Italy III, 289–297. 2009; **3**: 289–297. (Ser. Adv. Math. Appl. Sci., 82, World Sci. Publ., Hackensack, NJ. 2010)
- [17] Costabile F, Napoli A. *A method for global approximation of the solution of second order IVPs*. Rendiconti del Circolo Matematico, Ser. II. 2004; **24**: 239–260.
- [18] Costabile F, Napoli A. *A method for polynomial approximation of the solution of general second order BVPs*. Far East Journal of Applied Mathematics. 2006; **25**(3): 289–305.
- [19] Costabile F, Napoli A. *Collocation for high-order differential equations with Lidstone boundary conditions*. Journal of Applied Mathematics. 2012, Article ID 120792. doi: 10.1155/2012/120792.
- [20] Costabile F, Napoli A. *A multipoint Birkhoff type boundary value problem*. Journal of Numerical Mathematics. 2015; **23**(1): 1–11.
- [21] Costabile F, Serpe A, Bruzio A. *No classic boundary conditions*. In: Proceedings of World Congress on Engineering 2007; July 2–4, 2007, London 918–921.

- [22] Costabile F, Napoli A. *Collocation for high order differential equations with two-points Hermite boundary conditions*. Applied Numerical Mathematics. 2015; **87**: 157–167.
- [23] Dehghan M, Aryanmehr S, Eslahchi M. *A technique for the numerical solution of initial-value problems based on a class of Birkhoff-type interpolation method*. Journal of Computational and Applied Mathematics. 2013; **244**: 125–139.
- [24] Wang L, Samson M, Zhao X. *A well-conditioned collocation method using a pseudospectral integration matrix*. SIAM Journal on Scientific Computing. 2014; **36**(3): 907–929.
- [25] Costabile F, Napoli A. *A method for global approximation of the initial value problem*. Numerical Algorithms. 2001; **27**(2): 119–130.
- [26] Coleman J, Booth A. *The Chebyshev methods of Panovsky and Richardson as Runge-Kutta-Nyström methods*. Journal of Computational and Applied Mathematics. 1995; **61**(3): 245–261.
- [27] Costabile F, DellAccio F, Luceri R. *Explicit polynomial expansions of regular real functions by means of even order Bernoulli polynomials and boundary values*. Journal of Computational and Applied Mathematics. 2005; **176**(1): 77–90.
- [28] Wazwaz AM. *Approximate solutions to boundary value problems of higher order by the modified decomposition method*. Computers and Mathematics with Applications. 2000; **40**(6): 679–691.
- [29] Quarteroni A, Sacco R, Saleri F. *Numerical mathematics*. Second edition. Texts in Applied Mathematics, 37. Springer-Verlag: Berlin. 2007.
- [30] Driscoll TA, Hale N, Trefethen NL. *Chebfun guide*. Pafnuty Publications: Oxford. 2014.
- [31] Noor MA, Mohyud-Din ST. *An efficient method for fourth-order boundary value problems*. Computers and Mathematics with Applications. 2007; **54**(7): 1101–1111.
- [32] Lazer A, McKenna P. *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*. Siam Review. 1990; **32**(4): 537–578.
- [33] Yang B. *Estimates of positive solutions to a boundary value problem for the beam equation*. Communications in Mathematical Analysis. 2007; **2**(1): 13–21.

IntechOpen

