Chapter 1

Singular Lagrangians and Its Corresponding Hamiltonian Structures

Alvaro Restuccia and Adrián Sotomayor

Additional information is available at the end of the chapter

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Abstract

We present a general procedure to obtain the Lagrangian and associated Hamiltonian structure for integrable systems of the Helmholtz type. We present the analysis for coupled Korteweg-de Vries systems that are extensions of the Korteweg-de Vries equation. Starting with the system of partial differential equations it is possible to follow the Helmholtz approach to construct one or more Lagrangians whose stationary points coincide with the original system. All the Lagrangians are singular. Following the Dirac approach, we obtain all the constraints of the formulation and construct the Poisson bracket on the physical phase space via the Dirac bracket. We show compatibility of some of these Poisson structures. We obtain the Gardner ε-deformation of these systems and construct a master Lagrangian which describe the coupled systems in the weak ε-limit and its modified version in the strong ε-limit.

Keywords: integrable systems, conservation laws, partial differential equations, rings and algebras

1. Introduction

The Lagrangian mechanics has a wide range of applications from classical mechanics to quantum field theory. There are two main reasons to introduce a Lagrangian in order to describe a physical model. Its stationary points, defined in terms of functional derivatives, provide the classical equations of motion or classical field equations governing the evolution of the physical system while the action functional constructed from the Lagrangian provides the path integral approach to quantum mechanics and quantum field theories. In this chapter, we analyze several aspects of singular Lagrangians, which are relevant in various areas of physics. They are essential in the description of the fundamental forces in nature and in the analysis of integrable systems. In this chapter, we consider recent applications of singular Lagrangians in the area of completely integrable systems.

The analysis of integrable systems, in particular the Korteweg-de Vries equation and extensions of it [1–16], have provided a lot of interesting results from both mathematical and physical points of view.

Besides the physical applications of coupled KdV systems at low energies [17–19], one of the Poisson structures of the KdV equation is related to the Virasoro algebra with central terms. The latest is a fundamental symmetry of string theory, a proposal for a consistent quantum gravity theory.

In this chapter, we discuss a general approach based on the Helmholtz procedure to obtain a Lagrangian formulation and the Hamiltonian structure, starting from the system of time evolution partial differential equations describing the coupled KdV systems. Once the Lagrangian, whose stationary points corresponds to the integrable equations, has been obtained we follow the Dirac approach to constrained systems [20] to obtain the complete set of constraints and the Hamiltonian structure of the system. We discuss the existence of more than one Poisson structures associated with the integrable systems. Some of them are compatible Poisson structures and define a pencil of Poisson structures. We also discuss duality relations among the integrable systems we consider. The extensions of the KdV equation include a parametric coupled KdV system [21, 22], which we discuss in Section 3. In Section 8, we present a coupled KdV system arising from the breaking of a $N = 1$ supersymmetric model [15]. In Section 11, we discuss an extension of the KdV equation where the fields are valued on the octonion algebra and the product in the equation is the product on the octonion algebra [23]. This system has a supersymmetric extension which may be directly related to a model of the $D = 11$ supermembrane theory, a relevant sector of $M-$theory. The latest is a proposal of unification of all fundamental forces at very high energy.

2. The Dirac procedure for constrained systems

The Dirac approach for constrained systems [20] is a fundamental tool in the analysis of classical and quantum aspects of a physical theory. From a classical point of view, it provides a precise formulation of the initial valued problem for a time evolution system of partial differential equations. The initial data for the initial valued problem, given in terms of a constrained submanifold of a phase space, defines the physical phase space provided with the corresponding Poisson structure which gives rise to the canonical quantization of the system. In field theory, the starting point is a Lagrangian formulation. Its stationary points determine the classical field equations, generically a time evolution system of partial differential equations. From the Lagrangian density $\mathcal{L}$, one defines the conjugate momenta $p_i, i = 1, \ldots, N,$ associated with the original independent fields $q_i, i = 1, \ldots, N,$ defining the Lagrangian:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{1}$$

$\mathcal{L}$ is assumed to be a function of $\dot{q}_i$ and a finite number of spatial derivatives.

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If the Hessian matrix $\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}$ is singular we cannot express, from the above equation defining the conjugate momenta, all the $\dot{q}_i$ velocities in terms of the conjugate momenta.

The system presents then constraints on the phase space defined by the conjugate pairs $(q_i, p_i), i = 1, \ldots, N$. The phase space is provided with a Poisson structure given by

$$\{q_i,\ p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0. \tag{2}$$

In general, it is a difficult task to disentangle all the constraints on the phase space associated with a given Lagrangian. The Dirac approach provides a systematic way to obtain all the constraints on phase space. Moreover, it determines the Lagrange multipliers associated with the constraints (eventually after a gauge fixing procedure) in a way that if the constraints are satisfied initially then the Hamilton equations ensure that they are satisfied at any time. In this sense, it provides a precise formulation of the initial value problem, the initial data is given by the set of (q_i, p_i) conjugate pairs satisfying the constraints on phase space. The Hamilton equations then provide the time evolution of the system. This constrained initial data, with its associated Poisson structure (also obtained from the Dirac construction) provides the fundamental structure to define the canonical quantization of the original Lagrangian.

From the equation defining momenta one obtains, in the case of singular Lagrangian, a set of constraints $\phi_M(q, p) = 0$, where the argument is a shorthand notation for p, q and their derivatives with respect to the spatial coordinates x_a, $a = 1, \ldots, k$.

Also, by performing a Legendre transformation one gets a Hamiltonian $H_0 = \int_{-\infty}^{+\infty} dx \mathcal{H}_0$, where the Hamilton density is given by

$$\mathcal{H}_0 = \sum_i p_i \dot{q}_i - \mathcal{L}, \tag{3}$$

where $\mathcal{L}$ is the Lagrangian density. Then, we obtain a new Hamiltonian $H = \int_{-\infty}^{+\infty} dx \mathcal{H}$ with a density $\mathcal{H} = \mathcal{H}_0 + \lambda_M \phi_M$. The conservation of the constraints, which have to be satisfied at any time, yields

$$\dot{\phi}_M = \{\phi_M, H\} = 0. \tag{4}$$

$\{\phi_M, H\} = 0$ may (i) be identically satisfied on the constrained surface $\phi_m = 0$,

(ii) determine Lagrange multipliers, or

(iii) give new constraints.

In Case (i) or (ii), the procedure ends; in Case (iii), the iteration follows exactly in the same way. At some step, the procedure ends, assuming that there is a finite of physical degrees of freedom describing the dynamics of the original Lagrangian. In the procedure, a set of Lagrange multipliers may be determined and others may not. The constraints associated with

the ones that have been determined are called second class constraints, the other constraints for which the Lagrange multipliers are not determined are related to first class constraints. The first class constraints are the generators of a gauge symmetry on the time evolution system of partial differential equations. A difficult situation may occur in field theory when there is a combination of first and second class constraints. In order to separate them, one may have to invert some matrix involving fields of the formulation which may render dangerous non-localities in the final formulation.

All physical theories of the known fundamental forces in nature are formulated in terms of Lagrangians with gauge symmetries. All of them have first class constraints in their canonical formulation. In addition, they may also have second class constraints. In the analysis of field theories which are completely integrable systems like the ones we will discuss in this chapter only second class constraint appear. In this case, there are short cut procedures to simplify the Dirac procedure. However, the richness of the Dirac approach is that from its formulation one can extrapolate gauge systems which under a gauge fixing procedure reduce to the given system with second class constraints only. This is one of the main motivations of this chapter, to establish the Lagrangian and Hamiltonian structure for coupled KdV systems, which may allow the construction of gauge systems which are completely integrable.

In the case in which the constrained system has second class constraints, Dirac introduced the Poisson structure on the constrained submanifold in phase space. It determines the "physical" phase space with its Poisson bracket structure given by the Dirac bracket. They are defined in terms of the original Poisson bracket {, } on the full phase space by:

$$\{F, G\}_{DB} = -\{F, \phi_M\}\{\phi_M, \phi_N\}^{-1}\{\phi_N, G\} \tag{5}$$

where $\{\phi_M, \phi_N\}^{-1}$ is the inverse of the matrix $\{\phi_M, \phi_N\}$ which, in the case where $\phi_M = 0$ are second class constraints, is always of full rank.

The difficulty in field theory occurs when the matrix $\{\phi_M, \phi_N\}$ depends on the fields describing the theory and its inverse may lead to nonlocalities in the formulation. In our applications, those difficulties will not be present.

The Dirac bracket of a second class constraint with any other observable is zero. Consequently, the time conservation of the second class constraints is assured by the construction. For the same reason, there is no ambiguity on which Hamiltonian is used in determining the time evolution of observables.

3. A parametric coupled KdV system

A very interesting and well-known integrable system is the Korteweg-de Vries (KdV) equation. It arises from a variational principle of a singular Lagrangian. In what follows, we consider an extension of it. A coupled KdV system formulated in terms of two real differentiable functions $u(x, t)$ and $v(x, t)$ given by the following partial differential equations [21]:

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$$u_t + uu_x + u_{xxx} + \lambda v v_x = 0 \tag{6}$$

$$v_t + u_x v + v_x u + v_{xxx} = 0 \tag{7}$$

where λ is a real parameter.

When discussing conserved quantities, we will assume that u and v belong to the real Schwartz space defined by

$$C^{\infty}_{\downarrow} = \left\{ w \in C^{\infty}(\mathbb{R}) / \lim_{x \to \pm\infty} x^p \frac{\partial^q}{\partial x^q} w = 0; p, q \geq 0 \right\} \tag{8}$$

When $\lambda = +1$ the system is equivalent to two decoupled KdV equations. When $\lambda = -1$ the system is equivalent to a KdV equation valued on the complex algebra. By a redefinition of v given by $v \to \frac{v}{\sqrt{|\lambda|}}$ the system for $\lambda > 0$ reduces to the $\lambda = +1$ case and the system for $\lambda < 0$ reduces to the $\lambda = -1$ case. The case $\lambda = 0$ is an independent integrable system.

The system (6) and (7) for $\lambda = -1$ describes a two-layer liquid model studied in references [17–19]. It is a very interesting evolution system. It is known to have solutions developing singularities on a finite time [24]. Also, a class of solitonic solutions was reported in [25] through the Hirota approach [26] and in [27] via a Bäcklund transformation in the sense of Wahlquist and Estabrook (WE) [28].

The system (6) and (7) for $\lambda = 0$ correspond to the ninth Hirota-Satsuma [6] coupled KdV system given in Ref. [29] (for the particular value of $k = 0$) (see also [30]) and is also included in the interesting study that relates integrable hierarchies with polynomial Lie algebras [31].

4. The Lagrangian associated with the parametric coupled KdV system

In this section, we obtain the Lagrangian and associated Hamiltonian structure of the coupled KdV system. We present the main results in Ref. [22].

The Lagrangian construction requires the introduction of the Casimir potentials w and y given by

$$\begin{aligned} u(x, t) &= w_x(x, t) \\ v(x, t) &= y_x(x, t). \end{aligned} \tag{9}$$

The system (6) and (7) rewritten in terms of w and y is given by

$$\begin{aligned} w_{xt} + F[w, y] = 0, \quad & F[w, y] = w_x w_{xx} + w_{xxxx} + \lambda y_x y_{xx} \\ y_{xt} + G[w, y] = 0, \quad & G[w, y] = w_{xx} y_x + y_{xx} w_x + y_{xxxx}. \end{aligned} \tag{10}$$

We notice that the matrix constructed from the Frechet derivatives of F and G, with respect to w and y, is self-adjoint. We then conclude from the Helmholtz procedure that

$$\mathcal{L}_1 = -\frac{1}{2}w_x w_t - \frac{1}{2}\lambda y_x y_t + \int_0^1 (wF[\mu w, \mu y] + y\lambda G[\mu w, \mu y])d\mu, \tag{11}$$

where $\lambda \neq 0$, and

$$\mathcal{L}_2 = -\frac{1}{2}w_x y_t - \frac{1}{2}w_t y_x + \int_0^1 (yF[\mu w, \mu y] + wG[\mu w, \mu y])d\mu, \tag{12}$$

for every real value of λ, are two Lagrangian densities which give rise, from a variational principle to Eqs. (6) and (7).

The Lagrangians associated with $\mathcal{L}_i$, $i = 1, 2$ are given by $L_i(w, y) = \int_0^T dt \int_{-\infty}^{+\infty} dx \mathcal{L}_i$, $i = 1, 2$.

Independent variations of L_i, for each i, with respect to w and y give rise to the field equations

$$\begin{aligned} \delta_w L_i &= 0 \\ \delta_y L_i &= 0 \end{aligned} \tag{13}$$

which coincide, for each i, with Eqs. (6) and (7). In the above equations δ_w and δ_y denote the Gateaux functional variation defined by

$$\begin{aligned} \delta_w L(w, y) &= \lim_{\epsilon \to 0} \frac{L(w + \epsilon \delta w, y) - L(w, y)}{\epsilon} \\ \delta_y L(w, y) &= \lim_{\epsilon \to 0} \frac{L(w, y + \epsilon \delta y) - L(w, y)}{\epsilon}. \end{aligned} \tag{14}$$

The explicit expressions for $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\mathcal{L}_1 = -\frac{1}{2}w_x w_t - \frac{1}{6}{w_x}^3 + \frac{1}{2}{w_{xx}}^2 - \frac{\lambda}{2}w_x {y_x}^2 - \frac{\lambda}{2}y_x y_t + \frac{\lambda}{2}{y_{xx}}^2, \tag{15}$$

$$\mathcal{L}_2 = -\frac{1}{2}w_x y_t - \frac{1}{2}w_t y_x - \frac{1}{2}w_x^2 y_x - y_x w_{xxx} - \frac{\lambda}{6}y_x^3. \tag{16}$$

The Lagrangians L_i, $i = 1, 2$, are singular Lagrangians, we thus expect a constrained Hamiltonian formulation associated with them. The same happens for the corresponding KdV Lagrangian that can be obtained from $\mathcal{L}_1$ by imposing $\lambda = 0$.

We consider first the Lagrangian L_1. The conjugate momenta associated with w and y, which we denote by p and q, respectively, are given by

$$p = \frac{\partial \mathcal{L}_1}{\partial w_t} = -\frac{1}{2}w_x, \quad q = \frac{\partial \mathcal{L}_1}{\partial y_t} = -\frac{\lambda}{2}y_x. \tag{17}$$

We define

$$\phi_1 \equiv p + \frac{1}{2}w_x, \quad \phi_2 = q + \frac{\lambda}{2}y_x. \tag{18}$$

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Hence, $\phi_1 = \phi_2 = 0$ are constraints on the phase space. We then follow the Dirac procedure to determine the whole set of constraints. It turns out that these are the only constraints on the phase space.

The Hamiltonian density may be obtained directly from $\mathcal{L}_1$ by performing a Legendre transformation,

$$\mathcal{H}_1 = pw_t + qy_t - \mathcal{L}_1. \tag{19}$$

The Hamiltonian density is then given by

$$\mathcal{H}_1 = \frac{1}{6}w_x^3 - \frac{1}{2}w_{xx}^2 + \frac{\lambda}{2}w_x y_x^2 - \frac{\lambda}{2}y_{xx}^2 \tag{20}$$

and the Hamiltonian by $H_1 = \int_{-\infty}^{+\infty} dx\ \mathcal{H}_1$.

We introduce a Poisson structure on the phase space by defining

$$\begin{aligned} \{w(x), p(\hat{x})\}_{PB} &= \delta(x - \hat{x}) \\ \{y(x), q(\hat{x})\}_{PB} &= \delta(x - \hat{x}) \end{aligned} \tag{21}$$

with all other brackets between these variables being zero.

From them we obtain

$$\{\partial_x^n w(x), \partial_{\hat{x}}^m p(\hat{x})\} = \partial_x^n \partial_{\hat{x}}^m \{w(x), p(\hat{x})\}. \tag{22}$$

It turns out that ϕ_1, ϕ_2 are second class constraints. In fact,

$$\begin{aligned} \{\phi_1(x), \phi_1(\hat{x})\}_{PB} &= \delta_x(x - \hat{x}) \\ \{\phi_1(x), \phi_2(\hat{x})\}_{PB} &= 0 \\ \{\phi_2(x), \phi_2(\hat{x})\}_{PB} &= \lambda\delta_x(x - \hat{x}). \end{aligned} \tag{23}$$

In order to define the Poisson structure on the constrained phase space, we need to use the Dirac brackets.

The Dirac bracket between two functionals F and G on phase space is defined by

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \langle\langle \{F, \phi_i(x')\}_{PB} \mathbb{C}_{ij}(x', x'') \{\phi_j(x''), G\}_{PB} \rangle_{x'} \rangle_{x''} \tag{24}$$

where $<>_{x'}$ denotes integration on x' from $-\infty$ to $+\infty$. The indices $i, j = 1, 2$ and the $\mathbb{C}_{ij}(x', x'')$ are the components of the inverse of the matrix whose components are $\{\phi_i(x'), \phi_j(x'')\}_{PB}$.

This matrix becomes

$$\begin{bmatrix} \partial_{x'}\delta(x' - x'') & 0 \\ 0 & \lambda\partial_{x'}\delta(x' - x'') \end{bmatrix} \tag{25}$$

and its inverse is given by

$$[\mathbb{C}_{ij}(x', x'')] = \begin{bmatrix} \int^{x'} \delta(s-x'')ds & 0 \\ 0 & \frac{1}{\lambda}\int^{x'} \delta(s-x'')ds \end{bmatrix}. \tag{26}$$

It turns out, after some calculations, that the Dirac brackets of the original variables are

$$\begin{aligned} &\{u(x), u(\hat{x})\}_{DB} = -\partial_x \delta(x-\hat{x}), \quad \{v(x), v(\hat{x})\}_{DB} = -\frac{1}{\lambda}\partial_x \delta(x-\hat{x}) \\ &\{u(x), v(\hat{x})\}_{DB} = 0. \end{aligned} \tag{27}$$

We remind that this Poisson structure has been constructed assuming $\lambda \neq 0$.

From them, we obtain the Hamilton equations, which of course are the same as Eqs. (6) and (7):

$$\begin{aligned} u_t &= \{u, H_1\}_{DB} = -uu_x - u_{xxx} - \lambda v v_x \\ v_t &= \{v, H_1\}_{DB} = -u_x v - v_x u - v_{xxx}. \end{aligned} \tag{28}$$

We notice that adding any function of the constraints to H_1 does not change the result, since the Dirac bracket of the constraints with any other local function of the phase space variables is zero.

Using the above bracket relations for u and v, we may obtain directly the Dirac bracket of any two functionals $F(u, v)$ and $G(u, v)$. We notice that the observables F and G may be functionals of w, y, p, and q, not only of u and v. In this sense, the phase space approach for singular Lagrangians provides the most general space of observables.

We now consider the action L_2 and its associated Hamiltonian structure. In this case, we denote the conjugate momenta to w and y by $\hat{p}$ and $\hat{q}$, respectively. We have

$$\hat{p} = -\frac{1}{2}y_x, \quad \hat{q} = -\frac{1}{2}w_x. \tag{29}$$

In this case, the constraints become

$$\widehat{\phi_1} = \hat{p} + \frac{1}{2}y_x = 0, \quad \widehat{\phi_2} = \hat{q} + \frac{1}{2}w_x = 0. \tag{30}$$

The corresponding Poisson brackets are given by

$$\begin{aligned} &\{\widehat{\phi_1}(x), \widehat{\phi_1}(x')\}_{PB} = 0, \quad \{\widehat{\phi_2}(x), \widehat{\phi_2}(x')\}_{PB} = 0, \\ &\{\widehat{\phi_1}(x), \widehat{\phi_2}(x')\}_{PB} = \partial_x \delta(x-x'). \end{aligned} \tag{31}$$

From them, we can construct the Dirac brackets after which some calculations yield the Poisson structure for the original variables

$$\begin{aligned}&\{u(x),u(\hat{x})\}_{DB}=0,\quad \{v(x),v(\hat{x})\}_{DB}=0,\\&\{u(x),v(\hat{x})\}_{DB}=-\partial_x\delta(x-\hat{x}).\end{aligned}\tag{32}$$

The Hamiltonian $H_2=\int_{-\infty}^{+\infty}dx\,\mathcal{H}_2$ is given in terms of the Hamiltonian density

$$\mathcal{H}_2=\frac{1}{2}w_x^2y_x+y_xw_{xxx}+\frac{\lambda}{6}y_x^3.\tag{33}$$

The Hamilton equations follow then in terms of the Dirac brackets, they are

$$u_t=\{u,H_2\}_{DB},\quad v_t=\{v,H_2\}_{DB},\tag{34}$$

which coincide with the field Eqs. (6) and (7) for any value of λ. We have thus constructed two Hamiltonian functionals and associated Poisson bracket structures. These two Hamiltonian structures arise directly from the basic actions L_1 and L_2. In Section 6, we will construct two additional Hamiltonian structures by considering a Miura transformation for the coupled system.

5. A pencil of Poisson structures for the parametric coupled KdV system

We have then constructed two Lagrangian densities $\mathcal{L}_i$, $i=1,2$, we may now introduce a real parameter k and define a parametric Lagrangian density

$$\mathcal{L}_k=k\mathcal{L}_1+(1-k)\mathcal{L}_2.\tag{35}$$

The field equations obtained from the corresponding Lagrangian $L_k=\int_0^T dt\int_{-\infty}^{+\infty}dx\mathcal{L}_k$ are equivalent to Eqs. (6) and (7) in the following cases: If $\lambda<0$ for any k. If $\lambda=0$, for $k\neq 1$. If $\lambda>0$ for $k\neq\frac{1}{1+\sqrt{\lambda}}$ and $k\neq\frac{1}{1-\sqrt{\lambda}}$. From now on, we will exclude these particular values of k.

The parametric Lagrangian L_k is singular for any value of k (excluding the above mentioned particular cases). The corresponding phase space formulation contains constraints, which are determined by the use of the Dirac procedure. We denote p and q the conjugate momenta associated with w and y, respectively. From their definition, we obtain the primary constraints.

$$\phi_1\equiv\frac{k}{2}w_x+\frac{(1-k)}{2}y_x+p=0\tag{36}$$

$$\phi_2\equiv\frac{\lambda k}{2}y_x+\frac{(1-k)}{2}w_x+q=0.\tag{37}$$

We may then define the Hamiltonian density $\mathcal{H}_k$ through the Legendre transformation, we get

$$\mathcal{H}_k = pw_t + qy_t - \mathcal{L}_k = k\mathcal{H}_1 + (1-k)\mathcal{H}_2. \tag{38}$$

We now follow the Dirac algorithm to determine the complete set of constraints. It turns out that these are the only constraints in the formulation.

The Poisson brackets of the constraints obtained from the canonical Poisson brackets of the conjugate pairs are

$$\begin{aligned} \{\phi_1(x), \phi_1(\hat{x})\}_{PB} &= k\partial_x \delta(x-\hat{x}) \\ \{\phi_2(x), \phi_2(\hat{x})\}_{PB} &= \lambda k \partial_x \delta(x-\hat{x}) \\ \{\phi_1(x), \phi_2(\hat{x})\}_{PB} &= (1-k)\partial_x \delta(x-\hat{x}). \end{aligned} \tag{39}$$

Hence, they are second class constraints. We will denote by $\{\}^k_{DB}$ the Dirac bracket corresponding to the parameter k. We then proceed to calculate the Dirac brackets of the original fields u and v.

We obtain

$$\begin{aligned} \{u(x), u(\hat{x})\}^k_{DB} &= \frac{\lambda k}{-\lambda k^2 + (1-k)^2} \partial_x \delta(x-\hat{x}) \\ \{v(x), v(\hat{x})\}^k_{DB} &= \frac{k}{-\lambda k^2 + (1-k)^2} \partial_x \delta(x-\hat{x}) \\ \{u(x), v(\hat{x})\}^k_{DB} &= \frac{1-k}{-\lambda k^2 + (1-k)^2} \Big(-\partial_x \delta(x-\hat{x})\Big). \end{aligned} \tag{40}$$

where the denominator is different from zero for the values of k we are considering. The above brackets define the Poisson structure of the corresponding Hamiltonian

$$H_k = \int_{-\infty}^{+\infty} dx \mathcal{H}_k. \tag{41}$$

The Hamilton equations

$$\begin{aligned} u_t &= \{u, H_k\}_{DB} \\ v_t &= \{v, H_k\}_{DB} \end{aligned} \tag{42}$$

coincide, as expected, with the coupled Eqs. (6) and (7).

In Section 3, we constructed two Poisson structures for the coupled system (6) and (7). We now show they are compatible. It follows, for any two functionals F and G that

$$\{F, G\}^k_{DB} = \frac{-\lambda k}{-\lambda k^2 + (1-k)^2} \{F, G\}^1_{DB} + \frac{1-k}{-\lambda k^2 + (1-k)^2} \{F, G\}^0_{DB}, \tag{43}$$

where $\{F, G\}^1_{DB}$, corresponding to $k = 1$, and $\{F, G\}^0_{DB}$, corresponding to $k = 0$, are the two Dirac brackets structures obtained in Section 3. In particular, for any $\lambda \neq 0, 1$ and $k = \frac{1}{1-\lambda}$, we get

$$\{F, G\}^{k}_{DB} = \{F, G\}^{1}_{DB} + \{F, G\}^{0}_{DB}, \tag{44}$$

which implies that any linear combination of $\{F, G\}^{1}_{DB}$ and $\{F, G\}^{0}_{DB}$, for any $\lambda \neq 0, 1$, is a Poisson bracket. That is, the two Poisson structures obtained in Ref. [22], corresponding to $k = 1$ and $k = 0$, are compatible.

For the particular value of $\lambda = 0$, and any $k \neq 1$ we obtain

$$\{F, G\}^{k}_{DB} = \frac{k}{2(1-k)^2}\{F, G\}^{\frac{1}{2}}_{DB} + \frac{1-2k}{(1-k)^2}\{F, G\}^{0}_{DB}. \tag{45}$$

For $k = \frac{2}{5}$ the two coefficients on the right-hand member of Eq. (45) are equal. It implies that the Poisson structures for $k = \frac{1}{2}$ and $k = 0$ are compatible.

We have thus constructed a pencil of Poisson structures, except for $\lambda = 1$, for which the coupled system reduces to two decoupled KdV equations.

6. The Miura transformation for the parametric coupled KdV system

It is well known that the KdV equation admits two Hamiltonian structures, one of them is a particular case of our previous construction. It is obtained by considering only the $u(x, t)$ field, imposing $v(x, t) = 0$. In this case, the two previous Hamiltonians structures reduce to only one and there is no pencil of Poisson structures. The second Hamiltonian structure for the KdV equation arises from a Miura transformation, which is also a particular case of the following construction. The corresponding Miura transformation for our coupled system becomes

$$\begin{aligned} u &= \mu_x - \frac{1}{6}\mu^2 - \frac{\lambda}{6}\nu^2 \\ v &= \nu_x - \frac{1}{3}\mu\nu. \end{aligned} \tag{46}$$

and the modified KdV system (MKdVS)

$$\begin{aligned} &\mu_t + \mu_{xxx} - \frac{1}{6}\mu^2\mu_x - \frac{\lambda}{6}\nu^2\mu_x - \frac{\lambda}{3}\mu\nu\nu_x = 0 \\ &\nu_t + \nu_{xxx} - \frac{1}{6}\mu^2\nu_x - \frac{\lambda}{6}\nu^2\nu_x - \frac{1}{3}\mu\nu\mu_x = 0. \end{aligned} \tag{47}$$

It is interesting that from Eq. (47), following the Helmholtz procedure, which is also valid for the MKdVS system, we obtain two singular Lagrangians densities $\mathcal{L}^{M}_{i}$, $i = 1, 2$, expressed in terms of the Casimir potentials σ, ρ where $\mu = \sigma_x$, $\nu = \rho_x$:

$$\mathcal{L}_1^M = -\frac{1}{2}\sigma_t\sigma_x - \frac{\lambda}{2}\rho_t\rho_x - \frac{1}{2}\sigma_x\sigma_{xxx} - \frac{\lambda}{2}\rho_x\rho_{xxx} + \frac{1}{72}{\sigma_x}^4 + \frac{\lambda^2}{72}{\rho_x}^4 + \frac{\lambda}{12}\rho_x^2\sigma_x^2 \tag{48}$$

and

$$\mathcal{L}_2^M = -\frac{1}{2}\sigma_t\rho_x - \frac{1}{2}\sigma_x\rho_t - \sigma_{xxx}\rho_x + \frac{1}{18}{\sigma_x}^3\rho_x + \frac{\lambda}{18}{\rho_x}^3\sigma_x, \tag{49}$$

Eq. (48) being valid only for $\lambda \neq 0$.

Each of them has a Poisson structure that follows from the Dirac approach. The Dirac brackets, for the original fields u, v in the coupled system (6) and (7) are given by

$$\begin{aligned}
\{u(x), u(\hat{x})\}_{DB} &= \partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) + \frac{2}{3}u\partial_x\delta(x-\hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{\lambda}\partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3\lambda}u_x\delta(x-\hat{x}) + \frac{2}{3\lambda}u\partial_x\delta(x-\hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x}),
\end{aligned} \tag{50}$$

which is the Poisson structure associated with $\mathcal{L}_1^M$ and

$$\begin{aligned}
\{u(x), u(\hat{x})\}_{DB} &= \frac{\lambda}{3}v_x\delta(x-\hat{x}) + \frac{2\lambda}{3}v\partial_x\delta(x-\hat{x}) \\
\{v(x), v(\hat{x})\}_{DB} &= \frac{1}{3}v_x\delta(x-\hat{x}) + \frac{2}{3}v\partial_x\delta(x-\hat{x}) \\
\{u(x), v(\hat{x})\}_{DB} &= \partial_{xxx}\delta(x-\hat{x}) + \frac{1}{3}u_x\delta(x-\hat{x}) + \frac{2}{3}u\partial_x\delta(x-\hat{x}),
\end{aligned} \tag{51}$$

the Poisson structure associated with $\mathcal{L}_2^M$.

The corresponding Hamiltonian densities $\mathcal{H}_1^M$ and $\mathcal{H}_2^M$ are given in terms of the fields u and v by

$$\begin{aligned}
\mathcal{H}_1^M &= v^2 - u^2 \\
\mathcal{H}_1^M &= -uv.
\end{aligned} \tag{52}$$

The Hamilton equations obtained from these Hamiltonian structures coincide, of course, with Eqs. (6) and (7).

From these two Poisson structures, we may construct a pencil of Poisson structures as we described in the previous section, see Ref. [22] for the details of the construction. We notice that $\mathcal{L}_1^M$ and $\mathcal{L}_2^M$ in the construction are of the same dimension. It is then not possible to construct a hierarchy of higher order Hamiltonians from them. The same occurs with $\mathcal{L}_1$ and $\mathcal{L}_2$. However, the two pencils are of different dimensions and we may obtain from them a hierarchy of higher order Hamiltonians which extends the hierarchy of the KdV equation.

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7. A duality relation among the Lagrangians of the parametric coupled KdV system

We consider a generalization of the Gardner construction for the KdV equation. The Gardner transformation for the system (6) and (7) is given by

$$u = r + \varepsilon r_x - \frac{1}{6}\varepsilon^2(r^2 + \lambda s^2) \tag{53}$$

$$v = s + \varepsilon s_x - \frac{1}{3}\varepsilon^2 rs, \tag{54}$$

where ε is a real parameter and $r(x, t), s(x, t)$ are the fields which describe the Gardner ε-deformation. The Gardner equations are

$$r_t + r_{xxx} + rr_x + \lambda s s_x - \frac{1}{6}\varepsilon^2[(r^2 + \lambda s^2)r_x + 2\lambda rss_x] = 0 \tag{55}$$

$$s_t + s_{xxx} + rs_x + sr_x - \frac{1}{6}\varepsilon^2[(r^2 + \lambda s^2)s_x + 2rsr_x] = 0. \tag{56}$$

Any solution of Eqs. (55) and (56) define through Eqs. (53) and (54) a solution of the system (6), (7).

$\int_{-\infty}^{+\infty} dx\, r(x, t)$ and $\int_{-\infty}^{+\infty} dx\, s(x, t)$ are conserved quantities of the system (55) and (56). Assuming a formal power series on ε of the solutions of Eqs. (55) and (56) and inverting Eqs. (53) and (54), one obtains an infinite sequence of conserved quantities for the system (6), (7). It is an integrable system in this sense.

If we consider the $\varepsilon \to 0$ limit for the Gardner transformation Eqs. (53), (54) and Gardner Eqs. (55) and (56), we get the original system (6) and (7). On the other side, if we redefine

$$\mu \equiv \varepsilon r \tag{57}$$

$$\nu \equiv \varepsilon s \tag{58}$$

and rewrite Eqs. (53) and (54), we get

$$u = \frac{\mu}{\varepsilon} + \mu_x - \frac{1}{6}\mu^2 - \frac{1}{6}\lambda\nu^2 \tag{59}$$

$$v = \frac{\nu}{\varepsilon} + \nu_x - \frac{1}{3}\mu\nu. \tag{60}$$

Taking the limit $\varepsilon \to \infty$ we obtain

$$\hat{u} = \mu_x - \frac{1}{6}\mu^2 - \frac{1}{6}\lambda\mu^2 \tag{61}$$

$$\hat{v} = \nu_x - \frac{1}{3}\mu\nu \tag{62}$$

which is exactly the Miura transformation. In the same limit, we obtain from Eqs. (55), (56) the Miura equations given by Eq. (47).

We now construct using the Helmholtz approach a master Lagrangian for the Gardner equations. The master Lagrangians, there are two of them, are ε dependent and following the above limits we obtain all the Lagrangian structures we discussed previously. The KdV coupled system and the modified KdV coupled system are then dual constructions corresponding to the weak coupling limit $\varepsilon \to 0$ and to the strong coupling limit $\varepsilon \to \infty$ respectively, of the master construction. A direct relation of these two systems arises from the present construction.

We introduce the Casimir potentials

$$r = w_x, \quad s = y_x \tag{63}$$

and using the Helmholtz approach we obtain the Lagrangian densities

$$\begin{aligned}\mathcal{L}_{G1} = &-\frac{1}{2}w_x w_t - \frac{1}{6}(w_x)^3 + \frac{1}{2}(w_{xx})^2 - \frac{\lambda}{2}w_x(y_x)^2 - \frac{\lambda}{2}y_x y_t + \frac{\lambda}{2}(y_{xx})^2 \\ &-\frac{1}{6}\varepsilon^2\left[-\frac{1}{12}(w_x)^4 - \frac{\lambda}{2}(w_x)^2(y_x)^2\right] + \frac{\varepsilon^2}{72}\lambda^2(y_x)^4,\end{aligned} \tag{64}$$

$$\begin{aligned}\mathcal{L}_{G2} = &-\frac{1}{2}w_x y_t - \frac{1}{2}w_t y_x - \frac{1}{2}(w_x)^2 y_x - y_x w_{xxx} - \frac{\lambda}{6}(y_x)^3 \\ &+\frac{1}{18}\varepsilon^2(w_x)^3 y_x + \frac{1}{18}\varepsilon^2\lambda(y_x)^3 w_x.\end{aligned} \tag{65}$$

If we take the weak coupling limit $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0}\mathcal{L}_{G1} = \mathcal{L}_1 \quad , \quad \lim_{\varepsilon\to 0}\mathcal{L}_{G1} = \mathcal{L}_2 \tag{66}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ were defined in Section 3.

If we redefine

$$\begin{aligned}\sigma = \varepsilon w \quad &, \quad \rho = \varepsilon y \\ \mathcal{L}^M_{G1} = \varepsilon^2\mathcal{L}_{G1} \quad &, \quad \mathcal{L}^M_{G2} = \varepsilon^2\mathcal{L}_{G2}\end{aligned} \tag{67}$$

and take the strong coupling limit $\varepsilon \to \infty$, we get

$$\begin{aligned}\lim_{\varepsilon\to\infty}\mathcal{L}^M_{G1}(\sigma,\rho) &= \mathcal{L}^M_1(\sigma,\rho) \\ \lim_{\varepsilon\to\infty}\mathcal{L}^M_{G2}(\sigma,\rho) &= \mathcal{L}^M_2(\sigma,\rho),\end{aligned} \tag{68}$$

where $\mathcal{L}_1^M$ and $\mathcal{L}_2^M$ were defined in Section 5. Consequently, all the Lagrangian structure and the associated Hamiltonian structure of the coupled system (6), (7) arises from the master Lagrangians. They can also be combined to a unique master Lagrangian depending on a parameter k as was done in Section 4. The field equations of the master Lagrangians are the Gardner equations, the spatial integral of $r(x, t)$ and $s(x, t)$ define an ε-deformed conserved quantity of the Gardner equations which implies an infinite sequence of conserved quantities of the original coupled KdV system (6), (7).

8. Hamiltonian structure for a KdV system valued on a Clifford algebra

In this section, we continue the discussion of the Lagrangian and Hamiltonian structures for the coupled KdV systems. We discuss a coupled system arising from the breaking of the supersymmetry on the $N = 1$ supersymmetric KdV equation. The details of this system may be found in Ref. [15]. The system is formulated in terms of a real valued field $u(x, t)$ and a Clifford algebra valued field $\xi(x, t)$. The field $\xi(x, t)$ is expressed in terms of an odd number of generators e_i, $i = 1, \ldots$ of the Clifford algebra

$$\xi = \sum_{i=1}^{\infty} \varphi_i e_i + \sum_{ijk} \varphi_{ijk} e_i e_j e_k + \cdots \tag{69}$$

where

$$e_i e_j + e_j e_i = -2\delta_{ij}, \tag{70}$$

and φ_i, φ_{ijk}, … are real valued fields. We define by $\overline{\xi}$ the conjugate of ξ,

$$\overline{\xi} = \sum_{i=1}^{\infty} \varphi_i \overline{e}_i + \sum_{ijk} \varphi_{ijk} \overline{e}_k \overline{e}_j \overline{e}_i + \cdots \tag{71}$$

where $\overline{e}_i = -e_i$. We denote by $\mathcal{P}(\xi\overline{\xi})$ the projector of the product $\xi\overline{\xi}$ to the identity element of the algebra

$$\mathcal{P}(\xi\overline{\xi}) = \sum_{i=1}^{\infty} \varphi_i^2 + \sum_{ijk} \varphi_{ijk}^2 + \cdots \tag{72}$$

We proposed in Ref. [15] the following coupled KdV system arising from the breaking of the supersymmetry in the $N = 1$ supersymmetric equation [9]:

$$\begin{aligned} u_t &= -u_{xxx} - uu_x - \frac{1}{4}\left(\mathcal{P}(\xi\overline{\xi})\right)_x \\ \xi_t &= -\xi_{xxx} - \frac{1}{2}(\xi u)_x. \end{aligned} \tag{73}$$

In distinction to the $N = 1$ supersymmetric KdV equation the coupled system (73) has only a finite number of local conserved quantities,

$$\begin{aligned}
&\hat{H}_{\frac{1}{2}} = \int_{-\infty}^{+\infty} \xi dx, \\
&\hat{H}_1 = \int_{-\infty}^{+\infty} u dx, \\
&V \equiv \hat{H}_3 = \int_{-\infty}^{+\infty} \left(u^2 + \mathcal{P}(\xi\overline{\xi}) \right) dx, \\
&M \equiv \hat{H}_5 = \int_{-\infty}^{+\infty} \left(-\frac{1}{3}u^3 - \frac{1}{2}u\mathcal{P}(\xi\overline{\xi}) + (u_x)^2 + \mathcal{P}(\xi_x\overline{\xi}_x) \right) dx.
\end{aligned} \tag{74}$$

It is interesting to remark that the following nonlocal conserved charge of Super KdV [32] is also a nonlocal conserved charge for the system (73), in terms of the Clifford algebra valued field ξ,

$$\int_{-\infty}^{\infty} \xi(x) \int_{-\infty}^{x} \xi(s) ds dx. \tag{75}$$

However, the nonlocal conserved charges of Super KdV in Ref. [33] are not conserved by the system (73). For example,

$$\int_{-\infty}^{\infty} u(x) \int_{-\infty}^{x} \xi(s) ds dx. \tag{76}$$

is not conserved by Eq. (73).

The system (73) has multisolitonic solutions. In Ref. [34], we showed that the soliton solution is Liapunov stable under perturbation of the initial data.

9. The Lagrangian and Hamiltonian structure of the Clifford valued system

We introduce the Casimir potentials w and η defined by

$$u = w_x \text{ and } \xi = \eta_x. \tag{77}$$

We notice, as in the previous sections, that Eq. (73) may be expressed as stationary points of a singular Lagrangian constructed following the Helmholtz approach. We denote

$$\begin{aligned}
P(w, \eta) &= w_{xxxx} + w_x w_{xx} + \frac{1}{4}(\mathcal{P}(\eta_x \overline{\eta}_x))_x \\
Q(w, \eta) &= \eta_{xxxx} + \frac{1}{2}(w_x \eta_x)_x
\end{aligned} \tag{78}$$

The Lagrangian becomes $L = \int_0^T dt \int_{-\infty}^{+\infty} dx \mathcal{L}$ in terms of the Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2} w_x w_t + \frac{1}{2}(\mathcal{P}(\eta_x \overline{\eta}_t) - \int_0^1 wP(\mu w, \mu\eta) d\mu - \int_0^1 \mathcal{P}(Q(\mu w, \mu\eta)\overline{\eta}) d\mu. \tag{79}$$

From the Lagrangian L, we may construct its Hamiltonian structure using the Legendre transformation. We denote (p, σ) the conjugate momenta to (w, η):

$$\begin{aligned} p &:= \frac{\partial \mathcal{L}}{\partial(\partial_t w)} = \frac{1}{2} w_x = \frac{1}{2} u \\ \sigma &:= \frac{\partial \mathcal{L}}{\partial(\partial_t \eta)} = \frac{1}{2} \eta_x = \frac{1}{2} \varphi. \end{aligned} \tag{80}$$

Eq. (80) describes constraints on the phase space.

Performing the Legendre transformation we obtain the Hamiltonian of the system

$$H = \int_{-\infty}^{+\infty} dx \left(p w_t + \mathcal{P}(\sigma \overline{\eta}_t) - \mathcal{L} \right) \tag{81}$$

where $H = \frac{1}{2}\hat{H}_5$ in (74).

Following the Dirac approach, the conservation of the primary constraints (80) determines the Lagrange multipliers associated with the constraints (80). There are no more constraints on the phase space. It turns out that both constraints are second class ones. The Poisson structure of the constrained Hamiltonian is then determined by the Dirac brackets, see Ref. [15] for the details. We identify by an index i the independent components of a field η or σ valued on the Clifford algebra. We may rewrite the constraints as

$$\begin{aligned} v &:= p - \frac{1}{2} w_x \\ v_i &:= \sigma_i - \frac{1}{2} \eta_{ix}. \end{aligned} \tag{82}$$

Introducing $v_I := (v, v_i)$, we then have

$$\{v_I(x), v_J(x')\} = -\delta_{IJ} \partial_x \delta(x - x'). \tag{83}$$

The Poisson structure of the constrained Hamiltonian is then determined by the Dirac brackets [20]. For any two functionals on the phase space F and G, the Dirac bracket is defined as

$$\{F, G\}_{DB} := \{F, G\} - \langle \langle \{F, v_I(x')\} \{v_I(x'), v_J(x'')\}^{-1} \rangle_{x'} \{v_J(x''), G\} \rangle_{x''}, \tag{84}$$

where

$$\langle \{v_I(x'), v_J(x'')\}^{-1} g(x'') \rangle_{x''} = -\delta_{IJ} \int_{-\infty}^{x'} g(\tilde{x}) d\tilde{x}. \tag{85}$$

We then have

$$\begin{aligned}
&\{u(x), u(y)\}_{DB} = \partial_x \delta(x, y),\\
&\{\varphi_i(x), \varphi_j(y)\}_{DB} = \delta_{ij}\partial_x \delta(x, y),\\
&\{u(x), \varphi_i(y)\}_{DB} = 0.
\end{aligned} \tag{86}$$

Consequently,

$$\begin{aligned}
\partial_t u = \{u, H\}_{DB} &= -\frac{1}{2}(u^2)_x - u_{xxx} - \frac{\lambda}{4}(\varphi_i^2)_x\\
\partial_t \varphi_i = \{\varphi_i, H\}_{DB} &= -\varphi_{ixxx} - \frac{\lambda}{2}(u\varphi_i)_x,
\end{aligned} \tag{87}$$

where H is given by the last conserved quantity in Eq. (74) and can be directly expressed in terms of u and ξ.

10. Positiveness of the Hamiltonian for the Clifford valued system

An interesting property of the Hamiltonian H of the Clifford coupled system (73) is its *a priori* positiveness. In fact,

$$\hat{H}_3 + \hat{H}_5 = \|(u, \xi)\|_{H_1}^2 + \int_{-\infty}^{+\infty} \left(-\frac{1}{3}u^3 - \frac{1}{2}u\mathcal{P}(\xi\overline{\xi})\right) dx \tag{88}$$

where the Sobolev norm $\|\|_{H_1}$ is defined by

$$\|(u, \xi)\|_{H_1}^2 := \int_{-\infty}^{+\infty} [u^2 + \mathcal{P}(\xi\overline{\xi}) + {u_x}^2 + \mathcal{P}(\xi_x\overline{\xi}_x)]dx. \tag{89}$$

We also noticed that

$$\hat{H}_3 = \|(u, \xi)\|_{L^2}^2 \tag{90}$$

where $\|\|_{L^2}$ is the L^2 norm.

We then have

$$\hat{H}_3 + \hat{H}_5 \geq \|(u, \xi)\|_{H_1}^2 - \frac{1}{2}\int_{-\infty}^{+\infty} |u| \left(u^2 + \mathcal{P}(\xi\overline{\xi})\right) dx. \tag{91}$$

We now use the bound

$$\sup|u| \leq \frac{\|u\|_{H_1}}{\sqrt{2}} \leq \frac{\|(u, \xi)\|_{H_1}}{\sqrt{2}}, \tag{92}$$

to obtain

$$\hat{H}_3 + \hat{H}_5 \geq \|(u, \xi)\|_{H_1}^2 - \frac{1}{2\sqrt{2}}\|(u, \xi)\|_{H_1}\|(u, \xi)\|_{L^2}. \tag{93}$$

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Consequently,

$$\hat{H}_3 + \hat{H}_5 + \left(\frac{1}{4\sqrt{2}}\right)^2 \hat{H}_3 \geq \left(\|(u,\xi)\|_{H_1} - \frac{1}{4\sqrt{2}}\|(u,\xi)\|_{L^2}\right)^2 \geq 0. \tag{94}$$

Finally,

$$\hat{H}_5 \geq -\left(1 + \left(\frac{1}{4\sqrt{2}}\right)^2\right)\hat{H}_3, \tag{95}$$

Hence, for a normalized state satisfying $\|(u,\xi)\|_{L^2} = 1$, we have

$$\hat{H}_5 \geq -\left(1 + \left(\frac{1}{4\sqrt{2}}\right)^2\right). \tag{96}$$

The Hamiltonian is then manifestly bounded from below in the space of normalized L_2 configurations and it is thus physically admissible.

The property of the Hamiltonian is relevant from the physical point of view. In particular, in showing that the soliton solution of the Clifford coupled system is Liapunov stable. The stability analysis follows ideas introduced in Ref. [35] for the KdV equation. It is based on the use of the conserved quantities of the system. It is interesting that only the first few of them, in the case of the KdV equation, are needed. In the case of the Clifford coupled system these are all the local conserved quantities of the system.

11. The KdV equation valued on the octonion algebra

A famous theorem by Hurwitz establishes that the only real normalized division algebras are the reals $\mathbb{R}$, the complex $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions O. In particular, these division algebras are directly related to the existence of super Yang-Mills in several dimensions: 3, 4, 6, and 10 dimensions [36]. The octonion algebra may be explicitly used in the formulation of superstring theory in 10 dimensions and in the supermembrane theory in 11 dimensions, relevant theories in the search for a unified theory of all the known fundamental forces in nature.

The extension of the KdV equation to a partial differential equation for a field valued on a octonion algebra is then an interesting goal [23].

We showed in the previous sections that an extension of the KdV equation to the field valued on a Clifford algebra give rise to a coupled system with Liapunov stable soliton solution but without an infinite sequence of local conserved quantities.

In the present section, we analyze the KdV extension where the field is valued on the octonion algebra. The system shares several properties of the original real KdV equation. It has soliton solutions and also has an infinite sequence of local conserved quantities derived from a

Bäcklund transformation and a bi-Lagrangian and bi-Hamiltonian structure [23]. We will show in this section the construction of the bi-Lagrangian structure.

The octonion algebra contains as subalgebras all other division algebras, hence our construction may be reduced to any of them.

The KdV equation on the octonion algebra can be seen as a coupled KdV system, as we will see it has some similarities to the construction in the previous sections. However, it is invariant under the exceptional Lie group G_2, the automorphisms of the octonions, and under the Galileo transformations. Those symmetries are not present in the model constructed on a Clifford algebra.

We denote $u = u(x, t)$ a function with domain in $\mathbb{R}\times\mathbb{R}$ valued on the octonionic algebra. If we denote e_i, $i = 1, \ldots, 7$ the imaginary basis of the octonions, u can be expressed as

$$u(x, t) = b(x, t) + \vec{B}(x, t) \tag{97}$$

where $b(x, t)$ is the real part and $\vec{B} = \sum_{i=1}^{7} B_i(x, t)e_i$ its imaginary part.

The KdV equation formulated on the algebra of octonions, or simply the octonion KdV equation, is given by

$$u_t + u_{xxx} + \frac{1}{2}(u^2)_x = 0, \tag{98}$$

when $\vec{B} = \overrightarrow{0}$ it reduces to the scalar KdV equation. In terms of b and $\vec{B}$ the equation can be reexpressed as

$$b_t + b_{xxx} + bb_x - \sum_{i=1}^{7} B_i B_{ix} = 0, \tag{99}$$

$$(B_i)_t + (B_i)_{xxx} + (bB_i)_x = 0. \tag{100}$$

Eq. (98) is invariant under the Galileo transformation given by

$$\begin{aligned} \tilde{x} &= x + ct, \\ \tilde{t} &= t, \\ \tilde{u} &= u + c \end{aligned} \tag{101}$$

where c is a real constant.

Additionally, Eq. (98) is invariant under the automorphisms of the octonions, that is, under the group G_2. If under an automorphism

$$u \to \phi(u) \tag{102}$$

then

$$u_1 u_2 \to \phi(u_1 u_2) = \phi(u_1)\phi(u_2) \tag{103}$$

and consequently

$$[\phi(u)]_t + [\phi(u)]_{xxx} + \frac{1}{2}\left([\phi(u)]^2\right)_x = 0. \tag{104}$$

12. The Gardner formulation for the octonion valued algebra KdV equation

Associated with the real KdV equation, there is a Gardner ε-transformation and a Gardner equation which allows to obtain in a direct way the corresponding infinite sequence of conserved quantities. There exists a generalization of this approach for the KdV valued on the octonion algebra. The generalized Gardner transformation, expressed in terms of a new field $r(x, t)$ valued on the octonion is given by

$$u = r + \varepsilon r_x - \frac{1}{6}\varepsilon^2 r^2. \tag{105}$$

The generalized Gardner equation is then

$$r_t + r_{xxx} + \frac{1}{2}(rr_x + r_x r) - \frac{1}{12}\left((r^2)r_x + r_x(r^2)\right)\varepsilon^2 = 0 \tag{106}$$

where ε is a real parameter.

If $r(x, t)$ is a solution of the generalized Gardner equation (106), then $u(x, t)$ is a solution of the octonion algebra valued KdV equation (98). It has been shown in Ref. [23] that $\int_{-\infty}^{+\infty} \mathbb{R}e[r(x, t)]dx$ is a conserved quantity of Eq. (106). We can then invert Eq. (105), assuming a formal ε-expansion of the solution $r(x, t)$, to obtain an infinite sequence of conserved quantities for the KdV equation valued on the octonion algebra.

13. The master Lagrangian for the KdV equation valued on the octonion algebra

We may now use the Helmholtz procedure to obtain a Lagrangian density for the generalized Gardner equation. The master Lagrangian formulated in terms of the Casimir potential $s(x, t)$,

$$r(x, t) = s_x(x, t), \tag{107}$$

is

$$L_\varepsilon(s) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} \mathcal{L}_\varepsilon(s)dx \tag{108}$$

where the Lagrangian density is given by

$$\mathcal{L}_\varepsilon(s) = \mathbb{R}e\left[-\frac{1}{2}s_x s_t - \frac{1}{6}(s_x)^3 + \frac{1}{2}(s_{xx})^2 + \frac{1}{72}\varepsilon^2(s_x)^4\right]. \tag{109}$$

The Lagrangian density $\mathcal{L}_\varepsilon(s)$ is invariant under the action of the exceptional Lie group G_2.

Independent variations with respect to s yields

$$\begin{aligned}\delta\mathcal{L}_\varepsilon(s) = \mathbb{R}e&\left[-\frac{1}{2}(\delta s)_x s_t - \frac{1}{2}s_x(\delta s)_t - \frac{1}{6}\left((\delta s)_x(s_x)^2 + s_x(\delta s)_x s_x + (s_x)^2(\delta s)_x\right)\right]\\ + \mathbb{R}e&\left[\frac{1}{2}\left((\delta s)_{xx}s_{xx} + (\delta s)_{xx}s_{xx}\right) + \frac{1}{72}\varepsilon^2\left((\delta s)_x(s_x)^3 + s_x(\delta s)_x(s_x)^2 + (s_x)^2(\delta s)_x s_x + (s_x)^3(\delta s)_x\right)\right].\end{aligned} \tag{110}$$

Using properties of the octonion algebra we obtain from the stationary requirement $\delta L_\varepsilon(s) = 0$ the generalized Gardner equation (106).

In the calculation the property to be a division algebra of the octonions is explicitly used.

If we take the limit $\epsilon \to 0$, we obtain a first Lagrangian for the KdV equation valued on the octonion algebra,

$$L(w) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} dx \mathbb{R}e\left[-\frac{1}{2}w_x w_t - \frac{1}{6}(w_x)^3 + \frac{1}{2}(w_{xx})^2\right]. \tag{111}$$

Independent variations with respect to w yields, using $u = w_x$, the octonionic KdV equation (98). If we consider the following redefinition

$$\begin{aligned} s &\to \hat{s} = \varepsilon s\\ \mathcal{L}_\varepsilon(s) &\to \varepsilon^2\mathcal{L}_\varepsilon(\hat{s})\end{aligned} \tag{112}$$

and take the limit $\epsilon \to \infty$ we obtain

$$\lim_{\epsilon\to\infty} \epsilon^2\mathcal{L}_\epsilon(\hat{s}) = \mathcal{L}^M(\hat{s}), \tag{113}$$

where

$$\mathcal{L}^M(\hat{s}) = \mathbb{R}e\left[-\frac{1}{2}\hat{s}_x\hat{s}_t + \frac{1}{2}(\hat{s}_{xx})^2 + \frac{1}{72}(\hat{s}_x)^4\right]. \tag{114}$$

We get in this limit the generalized Miura Lagrangian

$$L^M(\hat{s}) = \int_{t_i}^{t_f} dt \int_{-\infty}^{+\infty} dx \mathcal{L}^M(\hat{s}). \tag{115}$$

The Miura equation is then obtained by taking variations with respect to $\hat{s}$, we get

$$\hat{r}_t + \hat{r}_{xxx} - \frac{1}{18}(\hat{r})_x^3 = 0, \quad \hat{r} \equiv \hat{s}_x, \tag{116}$$

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while the Miura transformation arises after the redefinition process, it is $u = \hat{r}_x - \frac{1}{6}\hat{r}^2$.

Any solution of the Miura equation, through the Miura transformation, yields a solution of the KdV equation valued on the octonion algebra. Since $L_\varepsilon(s)$ is invariant under G_2, the same occurs for $L(w)$ and $L^M(\hat{s})$, and consequently for the equations arising from variations of them.

The Lagrangian formulation of the octonionic KdV equation may be used as the starting step to obtain the Hamiltonian structure of the octonion algebra valued KdV equation.

14. Conclusions

We analyzed the relevance of the Dirac approach for constraint systems applied to singular Lagrangians. Several interesting theories are described by singular Lagrangians, notoriously the gauge theories describing the known fundamental forces in nature. In this chapter, we emphasized its relevance in the formulation of completely integrable field theories. We discussed extensions of the Korteweg-de Vries equation in different contexts. All these extensions, together with the KdV equation, allow a construction of a Lagrangian and a Hamiltonian structure arising from the application of the Helmholtz procedure. That is, starting with a time evolution partial differential system we construct, following the Helmholtz procedure, a Lagrangian associated with it. We present the construction of several Lagrangians and their corresponding Hamiltonian structures associated with the coupled KdV systems. All of them are characterized by second class constraints. The physical phase space is obtained by the determination of the complete set of constraints and the corresponding Dirac brackets. We established the relation between the several constructions by obtaining a pencil of Poisson structures. The application includes systems with an infinite sequence of conserved quantities together with a system with finite number of conserved quantities but presenting soliton solutions with nice stability properties. The final application is an extension of the KdV equation to the case in which the fields are valued on the octonion algebra. We constructed a master formulation from which two dual Lagrangian formulations are obtained , one corresponding to the KdV valued on the octonions and the other one corresponding to the extension of the modified KdV equation to fields valued on the octonions.

One important extrapolation of the analysis we have presented is the construction of gauge theories describing completely integrable systems. In fact, it is natural to extend the analysis by constructing a gauge theory which under a gauge fixing procedure reduces to the completely integrable systems of the KdV type we have discussed.

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Author details

Alvaro Restuccia[1,2]* and Adrián Sotomayor[3]*

*Address all correspondence to: arestu@usb.ve and adrian.sotomayor@uantof.cl

1 Departamento de Física, Universidad de Antofagasta, Antofagasta, Chile

2 Departamento de Física, Universidad Simón Bolívar, Caracas, Venezuela

3 Departamento de Matemáticas, Universidad de Antofagasta, Antofagasta, Chile

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