Chapter 3

Preservation of Synchronization Using a Tracy-Singh Product in the Transformation on Their Linear Matrix

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Additional information is available at the end of the chapter

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Abstract

Preservation is related to local asymptotic stability in nonlinear systems by using dynamical systems tools. It is known that a system, which is stable, asymptotically stable, or unstable at origin, through a transformation can remain stable, asymptotically stable, or unstable. Some systems permit partition of its nonlinear equation in a linear and nonlinear part. Some authors have stated that such systems preserve their local asymptotic stability through the transformations on their linear part. The preservation of synchronization is a typical application of these types of tools and it is considered an interesting topic by scientific community. This chapter is devoted to extend the methodology of the dynamical systems through a partition in the linear part and the nonlinear part, transforming the linear part using the Tracy-Singh product in the Jacobian matrix. This methodology preserves the structure of signs through the real part of eigenvalues of the Jacobian matrix of the dynamical systems in their equilibrium points. The principal part of this methodology is that it permits to extend the fundamental theorems of the dynamical systems, given a linear transformation. The results allow us to infer the hyperbolicity, the stability and the synchronization of transformed systems of higher dimension.

Keywords: preservation, synchronization, Tracy-Singh product, chaotic dynamical system

1. Introduction

In nonlinear autonomous dynamical systems, the study of synchronization is not new. We can see several papers about these themes from different approaches. Some examples show the use of change of variables, that is, through a diffeomorphism of the origin. From this, it is possible to say if a system is stable, asymptotically stable, or unstable. Some results are also obtained by the product in a vector field in the nonlinear dynamical system by a continuously differentiable function at the origin [1]. On one hand, there are studies showing the use of statistical properties to characterize the synchronization [2]. The eigenvalues of a system determine a system dynamics, but they are not derivable from the statistical features of such a system. One way to observe the stability is through a linear part of a dynamical system. But the problem to preserve stability by the transformation of its linear part in a nonlinear autonomous system has just been analyzed recently.

In [3], it is presented a methodology under which stability and synchronization of a dynamical master-slave system configuration are preserved under a modification through matrix multiplication. The conservation of stability is important for chaos control. A generalized synchronization can also be derived for different systems by finding a diffeomorphic transformation such as the slave system written as a function of the master system. One example of preservation for asymptotic stability is the use of transformations on rational functions in the frequency domain [4, 5].

This class of transformation can be interpreted as noise in the system or as a simple disturbance on the value of the physical parameters of the model. The chaotic synchronization problem studied in [6] is mainly related to preservation of the stability of the master-slave system presented in it. Results included therein show that stability is preserved by transforming the linear part of system. The same results can also be used in the chaos suppression problem. In [7], the authors show the viability of preserving the hyperbolicity of a master-slave pair of chaotic systems under different types of nonlinear modifications to its Jacobian matrix.

In [8], the developed methodology is used to study the problem of preservation of synchronization in chaotic dynamical systems, in particular the case of dynamical networks. Given a chaotic system, its transformed version is also a chaotic system. By means of a master-slave scheme obtained a controller for the system using a linear-quadratic regulator, preserving the stability even after the master-slave controller is transformed. This chapter is inspired by the same objective, that is, to preserve the stability in a master-slave system even through a transformation is performed over it. One way to achieve it is by extending some of the results in [8], particularly those of the local stable-unstable manifold theorem and extension of the center manifold theorem based in the preservation of the linear part of the vector field in nonlinear dynamical systems. As we will see, these results depart from the hypothesis of the existence of a constant state feedback as anominal synchronization force. In this work, we elaborate another approach to the problem of preservation of synchronization. We focus particularly on autonomous nonlinear dynamical systems, extending the previous results already mentioned.

This chapter is organized as follows: First, in Section 2, we will give basic concepts of dynamical systems. The fundamental theorem for linear systems, the local stable-unstable manifold theorem, the center manifold theorem, the Hartman-Grobman theorem and the concept of group action are introduced. In Section 3, we present some definitions about matrices and Tracy-Singh product of matrices. Also in this section, the main result is presented as a generalization of Proposition 4 in [6]. In Section 4, we will show that it is possible to preserve synchronization under a class of transformations defined under a certain method. Numerical experiments on the stability preservation for chaotic synchronization are shown in Section 5. Finally, a set of concluding remarks is given in Section 6.

2. Classical concepts of dynamical systems

We introduce theorems and classical definitions on properties of dynamical systems in this section. The fundamental theorem for linear systems, the local stable-unstable manifold theorem and the center manifold theorem are those important propositions mainly needed to develop analyses in this chapter. We will combine them with the Hartman-Grobman theorem in order to achieve a necessary generalization for those particular results of this chapter.

Theorem 2.1. (The local stable-unstable manifold theorem [9]). *Let E be an open subset of* $\mathbb{R}^n$

containing the origin. Let $f \in C^1(E)$ *and* ϕ_t *be the flow of the nonlinear system of the form* $\dot{x} = f(x)$. *Suppose that* $f(0) = 0$ *and that* $Df(0)$ *are the Jacobian matrix, which has k eigenvalues with negative real part and n−k eigenvalues with positive real part.*

1. *(Stable manifold) Then, there exists a k-dimensional differentiable manifold S tangent to the stable subspace* E^S *of the linear system* $\dot{x} = A(x)$ *at* x_0 *such that for all* $t \geq 0$, $\phi_t(S) \subset S$ *and for all* $x_0 \in S$, $\lim_{t\to\infty} \phi_t(x_0) = 0$.
2. *(Unstable manifold) Also there exists an n−k dimensional differentiable manifold W tangent to the unstable subspace* E^W *of* $\dot{x} = A(x)$ *at* x_0 *such that for all* $t \leq 0$, $\phi_t(W) \subset W$ *and for all* $x_0 \in W$, $\lim_{t\to-\infty} \phi_t(x_0) = 0$.

It should be noted that the manifolds S and W mentioned in Theorem 2.1 are unique. We define now the central manifold theorem in the following.

Theorem 2.2. (The center manifold theorem [9]). *Let E be an open subset of* $\mathbb{R}^n$ *containing the origin and* $r \geq 1$. *Let* $f \in C^r(E)$, *that is, f is a continuously differentiable function on E of order r. Now we suppose that* $f(0) = 0$ *and that* $Df(0)$ *have k eigenvalues with negative real part, j eigenvalues with positive real part and* $l = n-k-j$ *eigenvalues with zero real part. Therefore, there exists an l*

-dimensional center manifold $W^C(0)$ *of class* C^r *tangent to the center subspace* E^C *of* $\dot{x} = A(x)$ *at* 0 *which is invariant under the flow* ϕ_t *of* $\dot{x} = f(x)$.

By what it is established in Theorem 2.2, the center manifold $W^C(0)$ is not unique, which is an important difference for the stable character of the systems to be studied.

Theorem 2.3. (The Hartman-Grobman theorem [9]). *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x)$. Now, we assume that $f(0) = 0$, that is, the origin is an equilibrium point of the dynamical system; also the Jacobian matrix evaluated at the origin, $A = Df(0)$. If H is an homeomorphism of an open set W onto an open set V such that for each $x_0 \in W$, it exists an open interval $I_0 \subset \mathbb{R}$ such that for all $x_0 \in W$ and $t \in I_0$*

$$H \circ \phi_t(x_0) = e^{At} H(x_0); \tag{1}$$

that is, H maps trajectories of the nonlinear system $\dot{x} = f(x)$ near the origin onto trajectories of $\dot{x} = Ax$ near the origin and preserves the parametrization.

From the following argument, it is show that for any matrix $A = U^T T_A U$, there exists an homeomorphism $\hat{H} = UH$ such that for an open set W containing the origin onto an open set V also containing the origin such that for each $x_0 \in W$ and there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in W$ and $t \in I_0$

$$\hat{H} \circ \phi_t(x_0) = e^{T_A t} \hat{H}(x_0); \tag{2}$$

This last equality is a consequence of the Hartman-Grobman theorem and of the fact of $Ue^{At} = e^{T_A t} U$, that is, $\hat{H}$ maps trajectories of the nonlinear system $\dot{x} = f(x)$ near the origin onto trajectories of $\dot{x} = T_A x$ near the origin and preserves the parametrization.

On the other hand, some classical definitions are now included. A linear system of the form $\dot{x} = Ax$ where $x \in \mathbb{R}^n$, A is a $n \times n$ matrix and $\dot{x} = \frac{dx}{dt}$. It is shown that the solution of the linear system together with the initial condition $x(0) = x_0$ is given by $x(t) = e^{At} x_0$. The mapping $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called *the flow* of the linear system.

Definition 2.1. *For all eigenvalues of a matrix $A(n \times n)$ have nonzero real part, then the flow e^{At} is called a hyperbolic flow and therefore, $\dot{x} = Ax$ is called a hyperbolic linear system* [9].

Definition 2.2. *A subspace $E \subset \mathbb{R}^n$ is said to be invariant with respect to the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if $e^{At} \subset E$ for all $t \in \mathbb{R}$* [9].

Lemma 2.1. Let $\mathrm{A} \in \mathbb{R}^{\mathrm{n} \times \mathrm{n}}$. If $\mathbb{R}^{\mathrm{n}} = \mathrm{E}^{\mathrm{s}} \oplus \mathrm{E}^{\mathrm{u}} \oplus \mathrm{E}^{\mathrm{c}}$ where $\mathrm{E}^{\mathrm{s}}, \mathrm{E}^{\mathrm{u}}$ and E^{c} are the stable, unstable and center subspaces of the linear system $\dot{\mathrm{x}} = \mathrm{Ax}$. By the above,$\mathrm{E}^{\mathrm{s}}, \mathrm{E}^{\mathrm{u}}$ and E^{c} are invariant with respect to the flow e^{At}, respectively [9].

Definition 2.3. *Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$, that is, f is a continuous differentiable function defined on E. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the initial value problem $\dot{x} = f(x), x(0) = x_0$ defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the mapping $\phi_t : E \to E$ defined by $\phi_t(x_0) = \phi_t(t, x_0)$ is called the flow of the differential equation* [9].

Definition 2.4. *For any $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0)$ be the flow of the differential equation through x_0.(i) The local stable set S corresponding to a neighborhood V of x_0 is defined by $S = S(0) = \{x_0 \in \mathbb{R}^n : \phi_t(x_0) \in V, t \geq 0$ and $\phi_t(x_0) \to 0$ as $t \to \infty\}$. (ii) The local unstable set W of x_0 corresponding to a neighborhood V of x_0 is defined by $W = W(0) = \{x_0 \in \mathbb{R}^n : \phi_t(x_0) \in V, t \leq 0$ and $\phi_t(x_0) \to 0$ as $t \to \infty\}$.*

Then, these stable and unstable local sets are submanifolds of $\mathbb{R}^n$ *in a sufficiently small neighborhood V of* x_0[9].

Definition 2.5. *If G is a group and X is a set, then a (left) group action of G on X is a binary function* $G \times X \rightarrow X$, *denoted by* [9]

$$(g, x) \mapsto g \cdot x \tag{3}$$

which satisfies the following two axioms:

1. $(gh) \cdot x = g \cdot (h \cdot x)$ *for all* $g, h \in G$ *and* $x \in X$;
2. $e \cdot x = x$ *for every* $x \in X$ *(where e denotes the identity element of G).*

The action is faithful (or effective) if for any two different $g, h \in G$, *there exists an* $x \in X$ *such that* $g \cdot x \neq h \cdot x$; *or equivalently, if for any* $g \neq e$ *in G, there exists an* $x \in X$ *such that* $g \cdot x \neq x$.

The action is free or semiregular if for any two different $g, h \in G$ and all $x \in X$, we have $g \cdot x \neq h \cdot x$; or equivalently, if $g \cdot x = x$ for some x implies $g = e$.

For every $x \in X$, we define the stabilizer subgroup of x (also called the isotropy group or little group) as the set of all elements in G that fix x:

$$G_x = \{g \in G : g \cdot x = x\} \tag{4}$$

This is a subgroup of G, though typically not a normal one. The action of G on X is free if and only if all stabilizers are trivial.

3. Tracy-Singh product and other mathematical extensions

In this third section, we show a definition and some properties of the Tracy-Singh product. We also include a simple extension of the local stable-unstable manifold theorem and the center manifold theorem, using the tools presented in Section 2. These extensions are tools that will also be used in Section 4, where we will present the results on preservation of synchronization in nonlinear dynamical systems.

Definition 3.1. *Let* λ *be an eigenvalue of the* $n \times n$ *matrix A of multiplicity* $m \leq n$. *Then for* $k = 1, \ldots, m$, *any nonzero solution w of* [9]

$$(A - \lambda I)^k w = 0 \tag{5}$$

is called a generalized eigenvector of A.

In this case, let $w_j = u_j + v_j$ be a generalized eigenvector of the matrix A corresponding to an eigenvalue $\lambda_j = a_j + ib_j$ (note that if $b_j = 0$ then $v_j = 0$). Then, let $B = \{u_1, v_1, \ldots, u_k, v_k, \ldots, u_m, v_m\}$ be a basis of $\mathbb{R}^n$ (with $n = 2m - k$ as established by Theorems 1.7.1 and 1.7.2, see [9]). Now, we introduce the definition of Tracy-Singh product and some properties.

Definition 3.2. *If taken the matrices* $A = (a_{ij})$ *and* $C = (c_{ij})$ *of order* $m\times n$ *and* $B = (b_{kl})$ *of order* $p\times q$. *Let* $A = (A_{ij})$ *be partitioned with* A_{ij} *of order* $m_i\times n_j$ *as the* (i, j) *th block submatrix and* $B = (B_{kl})$ *of order* $p_k\times q_l$ *as the* (k, l) *th block submatrix* $(\sum m_i = m, \sum n_j = n, \sum p_k = p, \sum q_l = q)$. *Then, the definitions of the matrix products or sums of A and B are given as follows* [10].

Tracy-Singh product

$$A\circ B = (A_{ij}\circ B)_{ij} = \left((A_{ij}\otimes B_{kl})_{kl}\right)_{ij} \tag{6}$$

where $A_{ij}\otimes B_{kl}$ *is of order* $m_ip_k\times n_jq_l$, $A_{ij}\circ B$ *is a Kronecker product of order* $m_ip\times n_jq$, *and* $A\circ B$ *is of order* $mp\times nq$.

Tracy-Singh sum

$$A\boxplus B = A\circ I_p + I_m\circ B \tag{7}$$

where $A = (A_{ij})$ *and* $B = (B_{kl})$ *are square matrices of respective order* $m\times m$ *and* $p\times p$ *with* A_{ij} *of order* $m_i\times m_j$ *and* B_{kl} *of order* $p_k\times p_l$; I_p *and* I_m *are compatibly partitioned identity matrices.*

Theorem 3.1. *Let* A, B, C, D, E, *and* F *be compatibly partitioned matrices, then* [10]

1. $(A\circ B)(C\circ D) = (AC)\circ(BD)$.
2. $A\circ B \neq B\circ A$.
3. $(C\circ B = B\circ C)$ *where* $C = (c_{ij})$ *and* c_{ij} *is a scalar.*
4. $(A\circ B)^{'} = A^{'}\circ B'$.
5. $(A + D)\circ(B + E) = A\circ B + A\circ E + D\circ B + D\circ E$.
6. $(A\circ B)\circ F = A\circ(B\circ F)$

The next proposition presents some extensions to the local stable-unstable manifold theorem and to the center manifold theorem.

Proposition 3.1. Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$ and ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. Suppose that $f(0) = 0$ and that $A = Df(0)$ have k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part, that is, the origin is an hyperbolic fixed point. Then for each matrix $M \in \Lambda_U$, there exists a k

-dimensional differentiable manifold S_M tangent to the stable subspace E_M^S of the linear system $\dot{x} = MAx$ at 0 such that for all $t \geq 0$, $\phi_{M,t}(S_M)\subset S_M$ and for all $x_0 \in S_M$ [8],

$$\lim_{t\to\infty} \phi_{M,t}(x_0) = 0, \tag{8}$$

where $\phi_{M,t}$ is the flow of the nonlinear system $\dot{x} = MAx + g(x)$ and there exists an $n-k$ dimensional differentiable manifold W_M tangent to the unstable subspace E_M^W of $\dot{x} = MAx$ at 0 such that for all $t \leq 0$, $\phi_{M,t}(W_M) \subset W_M$ and for all $x_0 \in W_M$,

$$\lim_{t \to -\infty} \phi_{M,t}(x_0) = 0. \tag{9}$$

An interesting property is that Proposition 4.1 is valid for each $\overline{g} \in C^1(E)$ such that $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ and

$$\frac{\| \overline{g}(x) \|_2}{\| x \|_2} \to 0 \, \text{as} \, \| x \|_2 \to 0. \tag{10}$$

In consequence, the set of matrices Λ_U generates the action of the group Λ_U on the set of the hyperbolic nonlinear systems, formally on the set of the hyperbolic vector fields $f \in C^1(E)$, $\dot{x} = \overline{f}(x) = Ax + \overline{g}(x)$ with $\overline{g} \in C^1(E)$ and

$$A \in \Omega_U \equiv \{P \in \mathbb{R}^{n \times n} : P = U^T T_P U \text{ with } T_P \text{ any upper triangular matrix}\} \tag{11}$$

Satisfying the last condition, where U is a fixed unitary matrix, the action is generated by the action of the group Λ_U on the set Ω_U. By that this first action preserves the dimension and a nonlinear systems of the stable and unstable manifolds, that is, an hyperbolic nonlinear system $\left(\dot{x} = Ax + \overline{g}(x)\right)$ is mapped in a hyperbolic nonlinear systems $\left(\dot{x} = MAx + \overline{g}(x)\right)$ and $dimS = dimS_M$ and $dimW = dimW_M$.

The proof of this Proposition 3.1 can be revised in Ref. [8].

Given a particular nonlinear system, the stable and unstable manifolds S and W are unique; then for each matrix $M \in \Lambda_U$, there exists an unique pair of manifolds (S_M, W_M) in such a way that it is possible to define a pair of functions in the following form

$$\begin{gathered} \Theta : \Lambda_U \times Man_S \to Man_S \\ \Theta(M, S) = \mathrm{S_M} \\ \Phi : \Lambda_U \times Man_W \to Man_W \\ \Phi(\mathrm{M}, \mathrm{W}) = \mathrm{W_M} \end{gathered} \tag{12}$$

Where Man_S is the set of stable manifolds and Man_W is the set of unstable manifold for autonomous nonlinear systems.

Therefore, we can say that if $A = Df(0)$ is an stable matrix A has all the n eigenvalues with negative real part, then the origin of the nonlinear system $\dot{x} = M \circ Ax + \overline{g}(x)$ is asymptotically stable; but if $A = Df(0)$ is an unstable matrix A has $n-k$ (with $n > k$) eigenvalues with positive real part, then the origin of the nonlinear system $\dot{x} = M \circ Ax + \overline{g}(x)$ is unstable.

As an extension of the local stable-unstable manifold theorem in terms of Tracy-Singh product of matrices in Λ_N and the matrix A of the vector field $f(x)$, we present the following proposition.

Proposition 3.2.

1. Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f\in C^1(E)$ and let ϕ_t be the flow of the nonlinear system $\dot{x} = f(x) = Ax + g(x)$. We suppose that $f(0) = 0$ and that $A = Df(0)$ have a k eigenvalues with negative real part and n−k eigenvalues with positive real part; thus, the origin is a hyperbolic fixed point. Now, take a fixed continuously differentiable function

$$F : C^1(E) \rightarrow C^1(\overline{E}) \tag{13}$$

such that $F(g) = \hat{g}$ where $\hat{g} : \overline{E}\subset\mathbb{R}^{mn} \rightarrow \mathbb{R}^{mn}$ is a fixed continuously differentiable function with domain all $C^1(E)$; moreover, $\hat{g}\in C^1(\overline{E})$ with $\overline{E}$ an open subset of $\mathbb{R}^n$ containing the origin such that

$$\frac{\| \hat{g}(x) \|_2}{\| x \|_2} \rightarrow 0 \text{ as } \| x \|_2 \rightarrow 0. \tag{14}$$

Then, for each matrix $M\in\Lambda_U$ of $m\times m$, there exists a mk− dimensional differentiable manifold $S_{M\circ A}$ tangent to the stable subspace $E^S_{M\circ A}$ of the linear system $\dot{x} = (M\circ A)x$ at 0 such that for all $t\geq 0$, $\phi_{M\circ A,t}(S_{M\circ A})\subset S_{M\circ A}$ and for all $x_0\in S_{M\circ A}$,

$$\lim_{t\rightarrow\infty} \phi_{M\circ A,t}(x_0) = 0, \tag{15}$$

where $\phi_{M\circ A,t}$ be the flow of the nonlinear system $\dot{x} = (M\circ A)x + \hat{g}(x)$ and there exists an $m(n$−$k)$ dimensional differentiable manifold $\mathrm{W}_{\mathrm{M\circ A}}$ tangent to the unstable subspace $E^W_{M\circ A}$ of $\dot{x} = (M\circ A)x$ at 0 such that for all $t\leq 0$, $\phi_{M\circ A,t}(W_{M\circ A})\subset W_{M\circ A}$ and for all $x_0\in W_{M\circ A}$,

$$\lim_{t\rightarrow-\infty} \phi_{M\circ A,t}(x_0) = 0. \tag{16}$$

2. Also, there exists a function of the group Λ_N and the set of all the autonomous hyperbolic nonlinear systems of dimension n (hyperbolic vector fields of dimension n) denoted by Γ_n, to the set Γ_{mn} of all the autonomous hyperbolic nonlinear systems of dimension mn (hyperbolic vector fields of dimension mn); this function (which is similar to an action of the group Λ_N on the set Γ_n) is defined as follows

$$\begin{gathered} \vartheta : \Lambda_N \times \Gamma_n \rightarrow \Gamma_{mn} \\ \vartheta\Big(M, Ax + g(x)\Big) = (M\circ A)x + \hat{g}(x) \end{gathered} \tag{17}$$

and the new nonlinear system is

$$\begin{aligned}\dot{x} &= \vartheta\Big(M, Ax + g(x)\Big)\\ \dot{x} &= (M\circ A)x + \hat{g}(x))\end{aligned} \tag{18}$$

which satisfies the following two axioms:

1. $(gh)\cdot z = g\bullet(h\cdot z)$ for all $g, h\in\Lambda_N$ and $z\in\Gamma_n$;
2. For every $z\in\Gamma_n$, there exists an unique $\hat{z}\in\Gamma_{mn}$ such that $e\cdot z = \hat{z}$ and $h\bullet\hat{z} = \mathrm{h}\cdot\mathrm{z}$ (e denotes the identity element of Λ_N, that is, is the identity matrix I_m of $m\times m$).

Where z is associated with $Ax + g(x)$ (denoted by $z\stackrel{\circ}{=}Ax + g(x)$); $h\cdot z$ means $(M_h\circ A)x + \hat{g}(x)$ (denoted by $h\cdot z\stackrel{\circ}{=}(M_h\circ A)x + \hat{g}(x)$); gh is associated with the usual product of matrices M_g, M_h, that is, $gh\stackrel{\circ}{=}M_gM_h$ and $e\cdot z$ means $(I_m\circ A)x + \hat{g}(x)$, that is, $\Big(e\cdot z\stackrel{\circ}{=}(I_m\circ A)x + \hat{g}(x)\Big)$ and $g\bullet(h\cdot z)$ means $(M_g\circ I_n)(M_h\circ A)x + \hat{g}(x)$ (denoted by $g\bullet(h\cdot z)\stackrel{\circ}{=}(M_g\circ I_n)(M_h\circ A)x + \hat{g}(x)$).

Proof.

1. Consider a matrix A with eigenvalues λ_i for $i = 1, 2, \ldots, n$ and the matrix M with eigenvalues μ_j for $j = 1, 2, \ldots, m$. Then, the eigenvalues of the matrix $M\circ A$ are the mn numbers $\lambda_i\mu_j$ and taking account that $\mu_j > 0$ for each $j = 1, 2, \ldots, m$. Therefore, the matrix $M\circ A$ has mk eigenvalues with negative real part and $m(n-k)$ eigenvalues with positive real part. For this, the result is a consequence of the stable-unstable manifold theorem.

2. The function $\vartheta : \Lambda_N\times\Gamma_n \to \Gamma_{mn}$ is well defined, since $F : C^1(E) \to C^1(\overline{E})$ is a fixed function; then given $g(x)$, the vector field $\hat{g}(x)$ is unique; for a fixed matrix $M_h\in\Lambda_N$, then $M_h\circ : R^{n\times n} \to \mathbb{R}^{mn\times mn}$ is a fixed function and their matrix $M_h\circ A$ is unique.

Axiom (i): Since Λ_N is a multiplicative group if $M_g, M_h\in\Lambda_N$, then $M_gM_h\in\Lambda_N$.

Then, by Theorem 3.1, we have that for all $g, h\in\Lambda_N$ and $z\in\Gamma_n$

$$(gh)\cdot z\stackrel{\circ}{=}(M_gM_h\circ A)x + \hat{g}(x) = (M_g\circ I_n)(M_h\circ A)x + \hat{g}(x)\stackrel{\circ}{=}g\bullet(h\cdot z) \tag{19}$$

Axiom (ii): For every $z\in\Gamma_n$, there exists an unique $\hat{z}\in\Gamma_{mn}$ such that $e\cdot z\stackrel{\circ}{=}(I_m\circ A)x + \hat{g}(x) = \hat{z}$, then by the Theorem 2.1

$$h\bullet\hat{z}\stackrel{\circ}{=}(M_h\circ I_n)(I_m\circ A)x + \hat{g}(x) = (M_h\circ A)x + \hat{g}(x)\stackrel{\circ}{=}h\cdot z \tag{20}$$

From what it has been said above, we can note that if $A = Df(0)$ is as stable matrix A, it has all the n eigenvalues with negative real part, then the origin of the nonlinear system $\dot{x} = (M\circ A)x + \hat{g}(x)$ is asymptotically stable; if $A = Df(0)$ is an unstable matrix A, it has $n-k(n > k)$ eigenvalues with positive real part, then the origin of the nonlinear system $\dot{x} = (M\circ A)x + \hat{g}(x)$ is unstable.

Now the following Proposition 3.2 is an extension of the center manifold theorem, similar to Proposition 3.1.

Proposition 3.3. Let be $f \in C^r(E)$ where E is an open subset of $\mathbb{R}^n$

containing the origin and $r \geq 1$. Suppose that $f(0) = 0$ and that $Df(0)$ have k eigenvalues with negative real part, j eigenvalues with positive real part and $l = n - k - j$ eigenvalues with zero real part. Then,

1. For each matrix $M \in \Lambda_U$, there exists a $m-$ dimensional differentiable center manifold $W_M^C(0)$ of class C^r tangent to the center subspace E_M^C of the linear system $\dot{x} = MAx + g(x)$ at 0 which is invariant under the flow $\phi_{M,t}$ of the nonlinear system $\dot{x} = MAx + g(x)$.
2. If taken a fixed continuously differentiable function

$$\hat{F} : C^r(E) \to C^r(\overline{E}) \tag{21}$$

such that $F(g) = \hat{g}$ where $\hat{g} : \overline{E} \subset \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ is a fixed continuously differentiable function with domain all $C^r(E)$; moreover, $\hat{g} \in C^r(\overline{E})$ with $\overline{E}$ an open subset of $\mathbb{R}^n$ containing the origin such that

$$\frac{\|\hat{g}(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0. \tag{22}$$

Then for each matrix $M \in \Lambda_N$ of $m \times m$, there exists a $ml-$ dimensional differentiable center manifold $W_{M \circ A}^C(0)$ tangent to the center subspace $E_{M \circ A}^S$ of the linear system $\dot{x} = (M \circ A)x$ at 0 which is invariant under the flow $\phi_{M \circ A,t}$ of the nonlinear system $\dot{x} = (M \circ A)x + \hat{g}(x)$.

Proof.

The proof is similar to proof of Proposition 3.1 and we make use of the center manifold theorem.

Also, there exists a similar function $\hat{\vartheta}$ to ϑ, which satisfies the axiom (i) and axiom (ii) of Proposition 3.2. However, in this case, there does not exist similar functions to Θ and Φ. due to that in general, a center manifold is not unique.

Notice that in this case, if the matrix A has $l = n - k - j \neq 0$ eigenvlues with zero real part, then the origin of the nonlinear system $\dot{x} = MAx + \hat{g}(x)$ and $\dot{x} = (M \circ A)x + \hat{g}(x)$ are not asymptotically stable.

Propositions 3.1 and 3.2 generalize Proposition 3 in Ref. [6] and give new tools for preservation of basic properties of dynamical systems and some of these properties are the stability and instability.

4. Synchronization in nonlinear dynamical system

In this section, we present that it is possible to preserve synchronization even though the dimension of the systems changes by the action of a class of transformation on the linear part to a chaotic nonlinear system. If we consider the following two n-dimensional chaotic systems,

$$\begin{aligned} \dot{x} &= Ax + g(x) \\ \dot{y} &= Ay + f(y) + u(t) \end{aligned} \tag{23}$$

Where $A \in \mathbb{R}^{n \times n}$ is a constant matrix. On the other hand,$u \in R^n$ is the control input and $f, g : R^n \to R^n$ are continuous nonlinear functions. Synchronization considered in this section is through the master and the slave system is synchronized by designing an appropriate nonlinear state-feedback control $u(t)$ attached to slave system such that $\lim_{t \to \infty} x(t) - y(t) \to 0$, where $\| \cdot \|$ is the Euclidean norm of a vector [8]. If we consider the error state vector $e = y - x \in R^n, f(y) - f(x) = L(x, y)$ and an error dynamics equation is $\dot{e} = Ae + L(x, y) + u(t)$. Taking the active control approach [5], to eliminate the nonlinear part of the error dynamics and choosing $u(t) = Bv(t) - L(x, y)$, where B is a constant gain vector which is selected such that (A, B) be controllable, we obtain:

$$\dot{e} = Ae + Bv(t) \tag{24}$$

We can see that the original synchronization problem is equivalent to stabilize the zero-input solution of the slave system through a suitable choice of the state-feedback control [8]. If the pair (A, B) is controllable, then one such suitable choice for state feedback is a linear-quadratic regulator [5], which minimizes the quadratic cost function in the next expression,

$$J\Big(u(t)\Big) = \int_0^\infty (e(t)^\intercal Q e(t) + v(t) R v(t)) dt \tag{25}$$

Where Q and R are positive semi-definite and positive definite weighting matrices, respectively. The state-feedback law is given by $v = -Ke$ with $K = R^{-1} B^\intercal S$ and S the solution to the *Riccati* equation

$$A^\intercal S + SA - SBR^{-1} B^\intercal + Q = 0 \tag{26}$$

This state-feedback law makes the error equation to be $\dot{e} = (A - BK)e$, with $(A - BK)$ a Hurwitz matrix.[1] The linear-quadratic regulator is a technique to obtain feedback gains [5]. It is an interesting property of (LQR) which is robustness. On the other hand, if we consider $T \in R^{m \times m}$ be a matrix with strictly positive eigenvalues, supposing that the following two nm-dimensional systems are chaotic:

$$\begin{aligned} \dot{x} &= (T \circ A)x + \hat{g}(x) \\ \dot{y} &= (T \circ A)y + \hat{f}(y) + \hat{u}(t) \end{aligned} \tag{27}$$

for some $\hat{f}, \hat{g} : R^{nm} \to R^{nm}$ continuous nonlinear functions and $\hat{u} \in R^{nm}$ is the control input. Then, for the Proposition 4.1 and the former procedure, we have that $\hat{u}(t) = \hat{\theta}(t) - \hat{L}(x, y)$ stabilizes the zero solution of the error dynamics system, where $\hat{\theta}(t) = -(BK \circ T)e$, that is, the resultant system

[1] A Hurwitz matrix is a matrix for which all its eigenvalues are such that their real part is strictly less than zero.

$$\dot{e} = (T \circ A)e + \hat{\theta}(t)\dot{e} = (T \circ A - T \circ BK)e \tag{28}$$

is asymptotically stable. Then, by using Lemma 2.1 and $K = -R^{-1}B^{\intercal}S$, we obtain that:

$$\begin{aligned} \dot{e} &= \Big(T \circ (A + BK)\Big)e \\ \dot{e} &= \Big(T \circ (A - BR^{-1}B^{\intercal}S)\Big)e \end{aligned} \tag{29}$$

Now, the original control $u(t) = BKe - L(x, y)$ is preserved in its linear part by the Tracy-Singh product $T \circ (\cdot)$ and the new control is given by $\hat{u}(t) = -(T \circ BK)e - \hat{L}(x, y)$. Therefore, we can interpreted the last procedure as one in which the controller $u(t)$ that achieves the synchronization in the two systems is preserved by the transformation $T \circ (\cdot)$ so that $\hat{u}(t)$ achieves the synchronization in the two resultant systems after the transformation. For that, a similar procedure is possible if we consider the transformation $(\cdot) \circ T$.

In general, under the transformation $(A, g) \to (MA, \overline{g})$ or $(A, g) \to (M \circ A, \overline{g})$ and under the hypothesis of existence of a constant state feedback $U = -Kx$, which achieves synchronization of the original chaotic systems and also that the transformed system is chaotic, then synchronization can be preserved [8]. The major contribution does not refer a better synchronization methodology; it deals that synchronization is preserved when a chaotic system changes from a lower dimension to a higher dimension.

5. Synchronization of the classical Lü system

In this section, we present the synchronization of a chaotic system. First, we propose a master and slave system. Then, from these systems, we will apply a linear transformation that allows us to preserve the synchronization. We will use the well-known Lü and Chen [11] model to show the possibility to preserve synchronization, described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= cx_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{30}$$

which has a chaotic attractor when the parameters are $a = 35, b = 3$ and $c = 14.5$. In order to observe synchronization behavior, we have a modified Lü attractor arranged as a master-slave configuration. The master and the slave systems are almost identical and the only difference is that the slave system includes an extra term which is used for the purpose of synchronization with the master system. The master system is defined by the following equations,

$$\begin{aligned} \dot{x}_1 &= 35(x_2 - x_1) \\ \dot{x}_2 &= 28x_2 - x_1x_3 \\ \dot{x}_3 &= x_1x_2 - 3x_3 \end{aligned} \tag{31}$$

and the slave system is a copy of the master system with a control function $u(t)$ to be determined in order to synchronize the two systems.

$$\begin{aligned} \dot{y}_1 &= 35(y_2 - y_1) + u_1(t) \\ \dot{y}_2 &= 28y_2 - y_1y_3 + u_2(t) \\ \dot{y}_3 &= y_1y_2 - 3y_3 + u_3(t) \end{aligned} \tag{32}$$

Now, we consider the errors $e_1 = x_1 - y_1, e_2 = x_2 - y_2$ and $e_3 = x_3 - y_3$,; then, the error dynamics can be written as:

$$\begin{aligned} \dot{e}_1 &= 35(e_2 - e_1) + u_1(t) \\ \dot{e}_2 &= 28e_2 - y_1y_3 + x_1x_3 + u_2(t) \\ \dot{e}_3 &= y_1y_2 - x_1x_2 - 3e_3 + u_3(t) \end{aligned} \tag{33}$$

If we introduce the matrices

$$A = \begin{pmatrix} -35 & 35 & 0 \\ 0 & 14.5 & 0 \\ 0 & 0 & -3 \end{pmatrix}, L(x,y) = \begin{pmatrix} 0 \\ -y_1y_3 + x_1x_3 \\ y_1y_2 - x_1x_2 \end{pmatrix}, u = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}. \tag{34}$$

and selecting the matrix B such that (A, B) is controllable: $B = I$, the LQR controller is obtained by using weighting matrices $Q = I$ and $R = B^{\intercal}B = I$. Then, state-feedback matrix is given by

$$K = \begin{pmatrix} 0.0143 & 0.0101 & 0 \\ 0.0101 & 29.0587 & 0 \\ 0 & 0 & 0.1623 \end{pmatrix} \tag{35}$$

From the formerly said, we now present simulations made for the synchronized system of Lü and for the system also synchronized, but after the transformation of its linear part. All simulations here presented were made in *Matlab* software. In **Figure 1**, we show the trajectories of the master system of Lü. Each line represents one trajectory of the system along the time, taking an initial condition of $(1, 1, 1)$.

For the case of **Figure 3**, we show the trajectories of the slave system of Lü. As it was in the first case, each line represents one trajectory of the system along the time, taking a initial condition as $(3, 3, 3)$. **Figures 2** and **4** are phase space mappings of each system while maintaining the same initial condition.

On the other hand, in **Figure 5**, we can see the error magnitude between master and slave systems. Phase space of synchronization of the master and slave systems in **Figure 6** is presented. Now, we shall present a system showing modifications performed on the Lü attractor. The modified Lü master and slave systems linear and nonlinear parts may be defined as follows:

$$\begin{aligned} \dot{x} &= (T \circ A)x + [0 \quad -x_1x_3 \quad x_1x_2\, 0 \quad -x_4x_6 \quad x_4x_5\ ]^{\intercal} \\ \dot{y} &= (T \circ A)y + [0 \quad -y_1y_3 \quad y_1y_2\, 0 \quad -y_4y_6 \quad y_4y_5\ ]^{\intercal} + u \end{aligned} \tag{36}$$

Considering the error vector $e = y - x$, then the error dynamics can be written as:

$$\dot{e} = (T \circ A)e + L(x,y) + u \tag{37}$$

with $u = -L(x, y) + v$ and $v = -(T \circ BK)e$ and

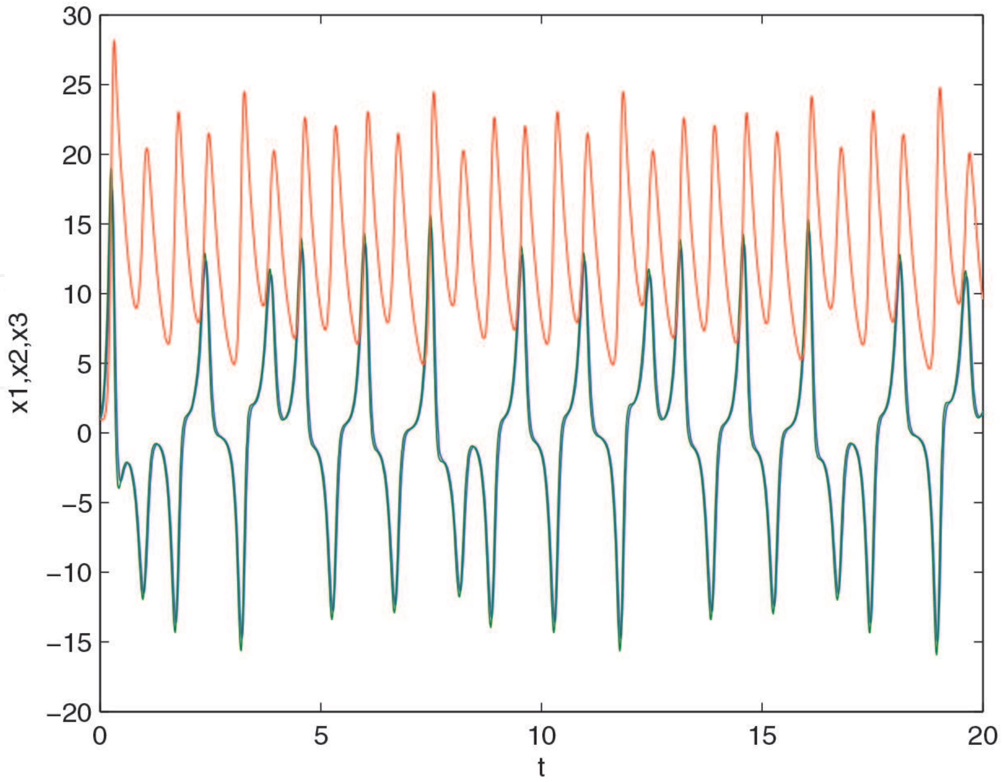

Figure 1. Master system of Lü.

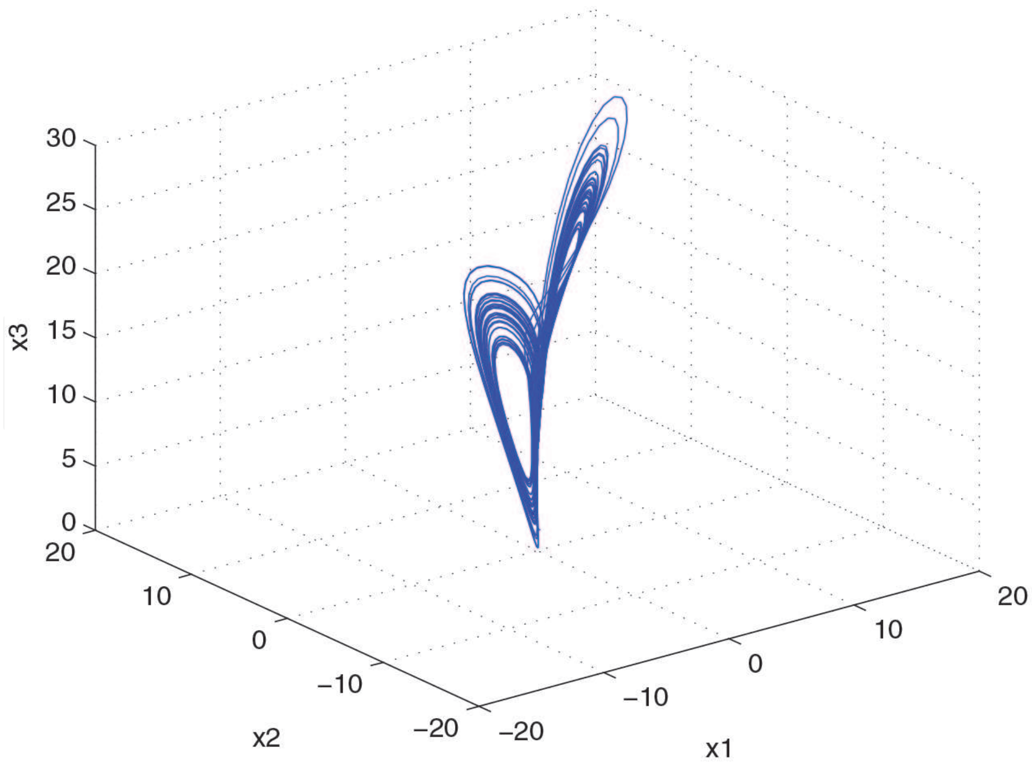

Figure 2. Master system of Lü.

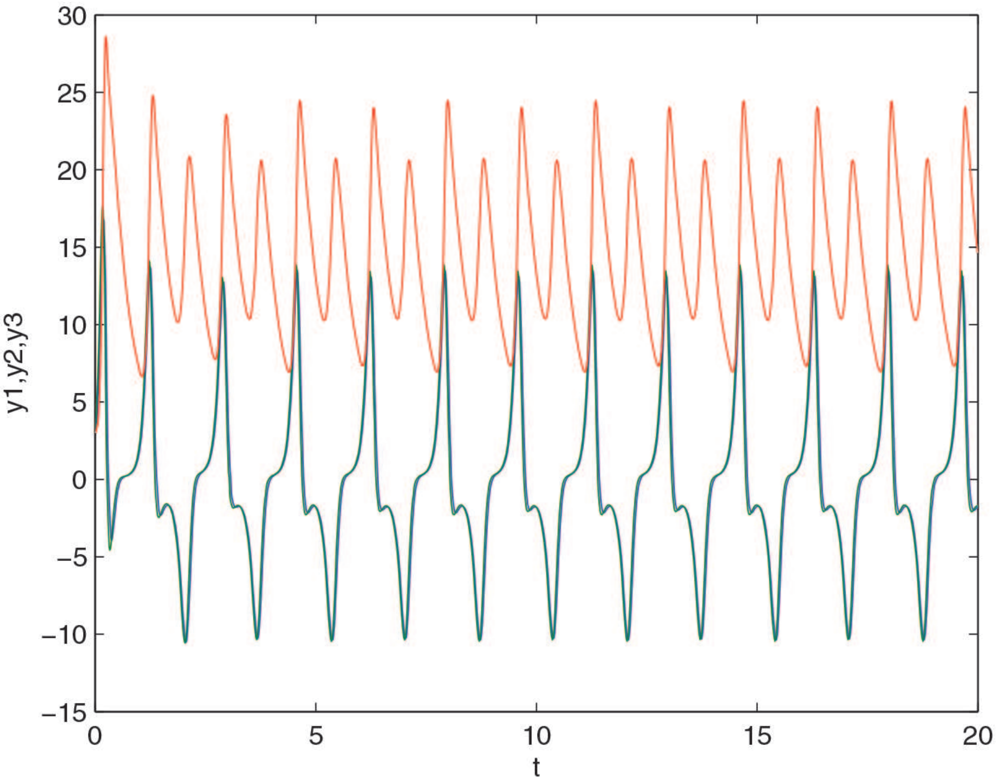

Figure 3. Slave system of Lü.

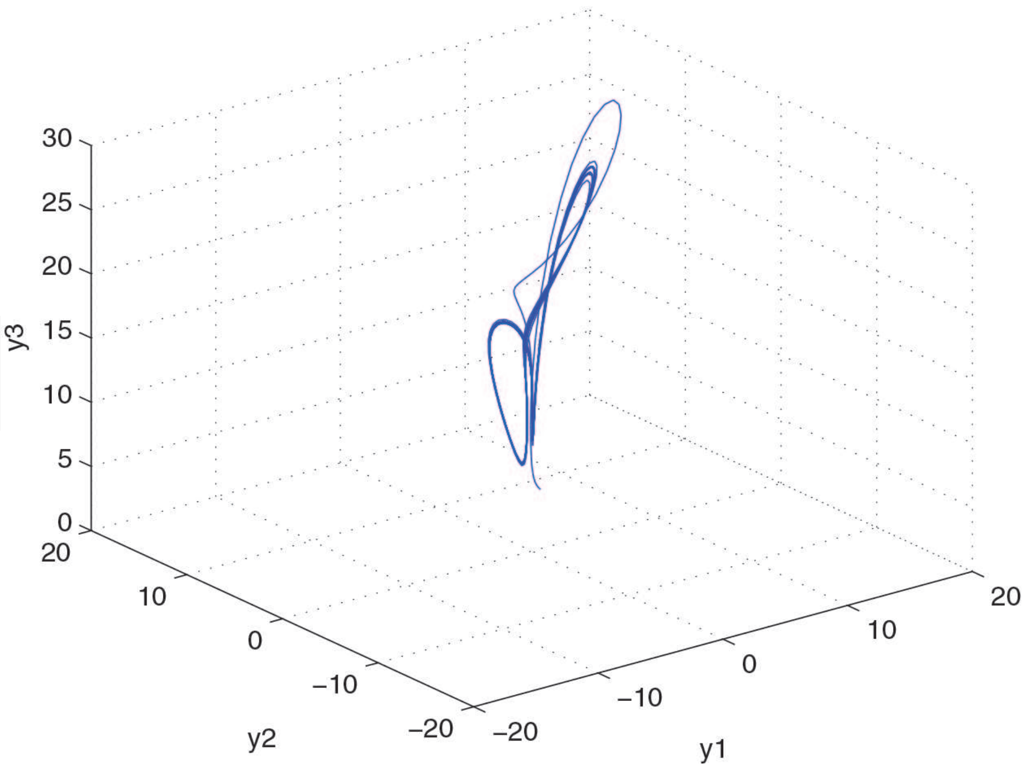

Figure 4. Slave system of Lü.

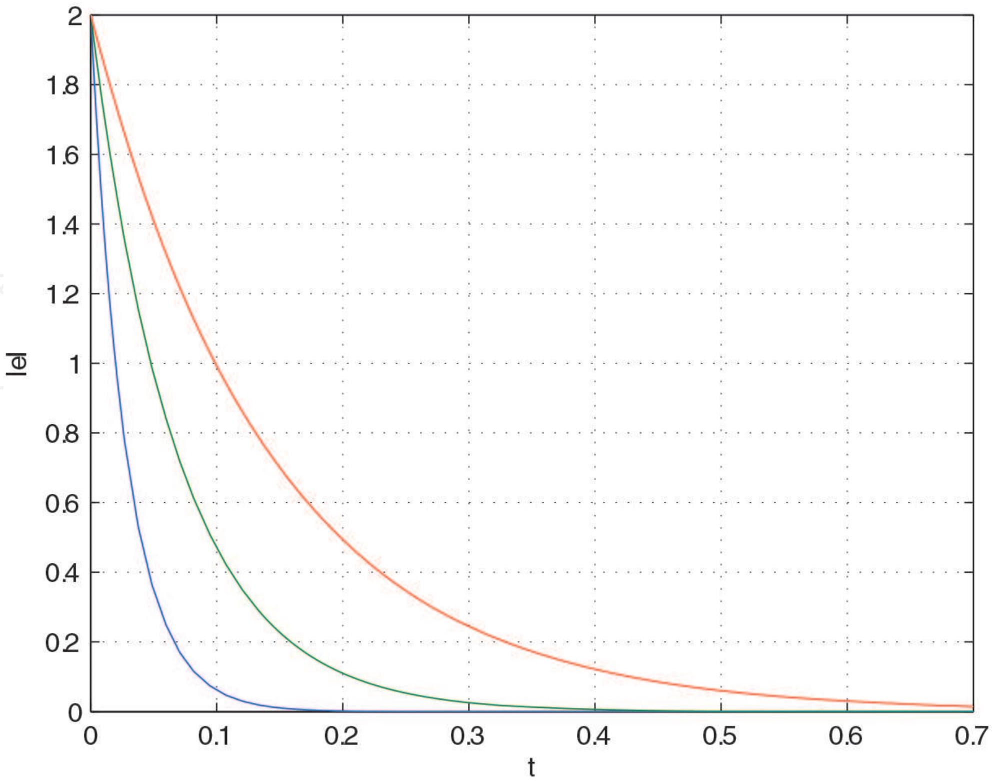

Figure 5. Magnitude of the error between the master and the slave systems.

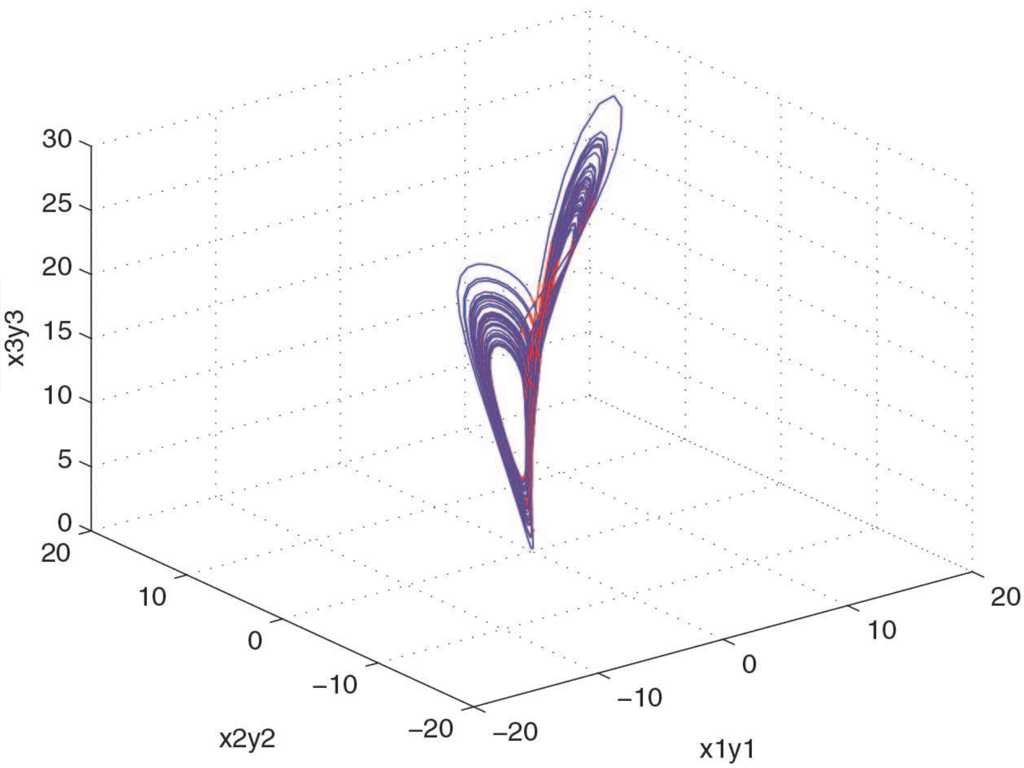

Figure 6. Synchronization of master and slave system of Lü.

$$A = \begin{pmatrix} -35 & 35 & 0 \\ 0 & 14.5 & 0 \\ 0 & 0 & -3 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = [111111]^{\intercal},$$
$$L(x,y) = [0 \quad -y_1y_3 + x_1x_3 \quad y_1y_2 - x_1x_2 \ 0 \quad -y_4y_6 + x_4x_6 \quad y_4y_5 - x_4x_5]^{\intercal} \tag{38}$$

Now, the LQR controller is obtained by using weighting matrices, $B = IQ = I$ and $R = B^{\intercal}B = I$. So the vector $L(x,y)$ takes these values because T is an upper triangular matrix and the value one on the diagonal is repeated.

$$T \circ A = \begin{pmatrix} -35 & 35 & -35 & 35 & 0 & 0 \\ 0 & 14.5 & 0 & 14.5 & 0 & 0 \\ 0 & 0 & -35 & 35 & 0 & 0 \\ 0 & 0 & 0 & 14.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{39}$$

$$K = \begin{pmatrix} 0.0143 & 0.0101 & 0 & -0.0071 & 0.0050 & 0 \\ 0.0101 & 23.3051 & 0 & -0.0151 & 11.5941 & 0 \\ 0 & 0 & 0.1614 & 0 & 0 & -0.0757 \\ -0.0071 & -0.0151 & 0 & 0.0214 & 0.0050 & 0 \\ 0.0050 & 11.5941 & 0 & 0.0050 & 34.8411 & 0 \\ 0 & 0 & -0.0757 & 0 & 0 & 0.2324 \end{pmatrix} \tag{40}$$

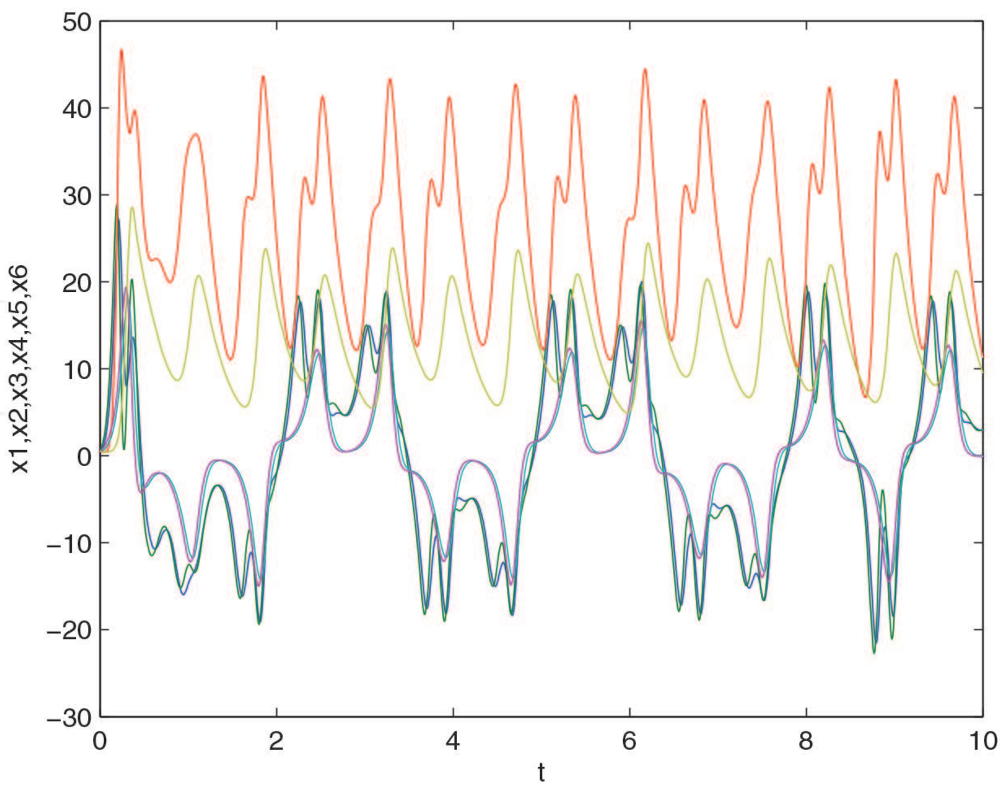

Figure 7. Transformation of the master system of Lü.

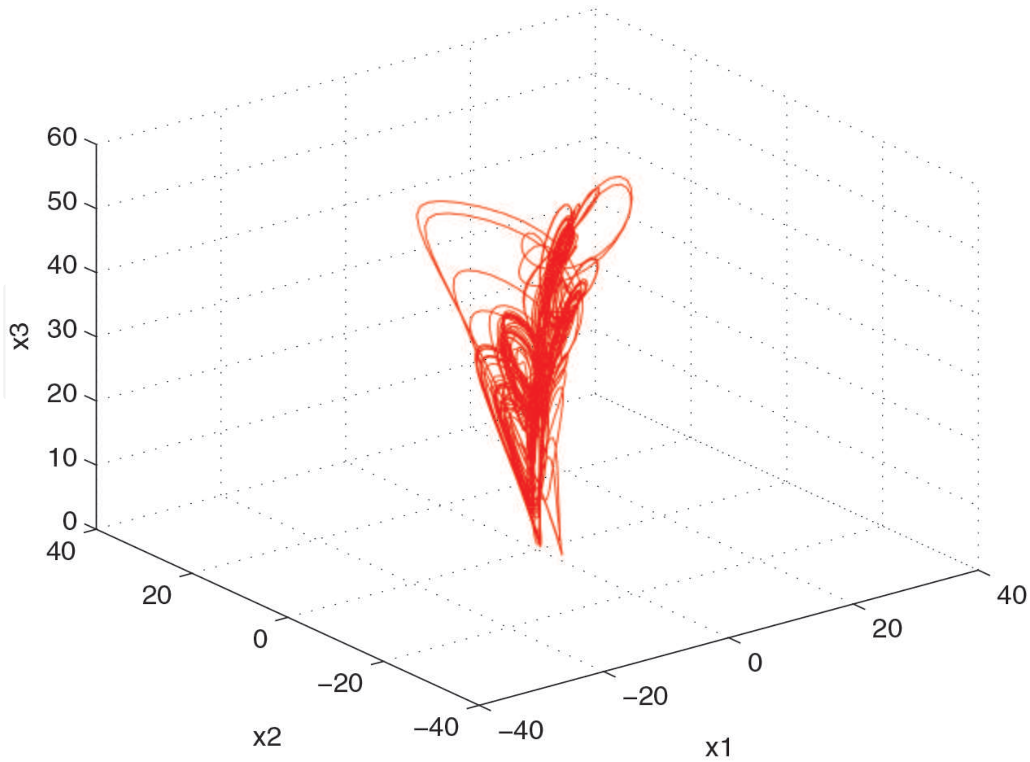

Figure 8. Phase space of the transformation of the master system of Lü.

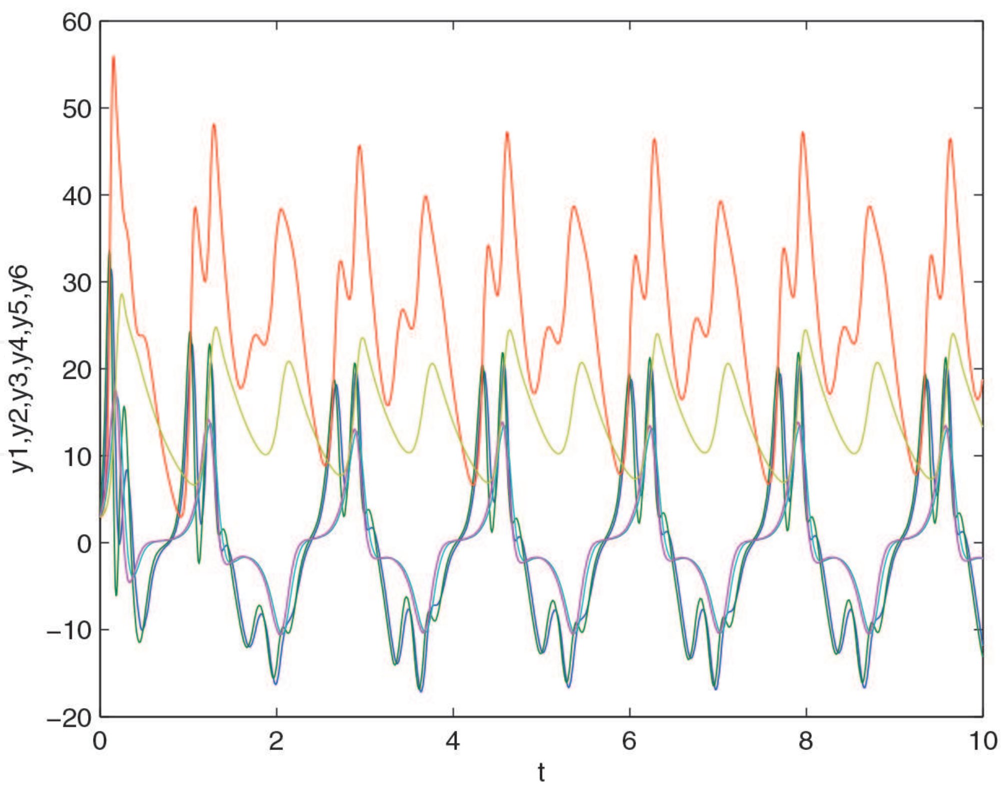

Figure 9. Transformation of the slave system of Lü.

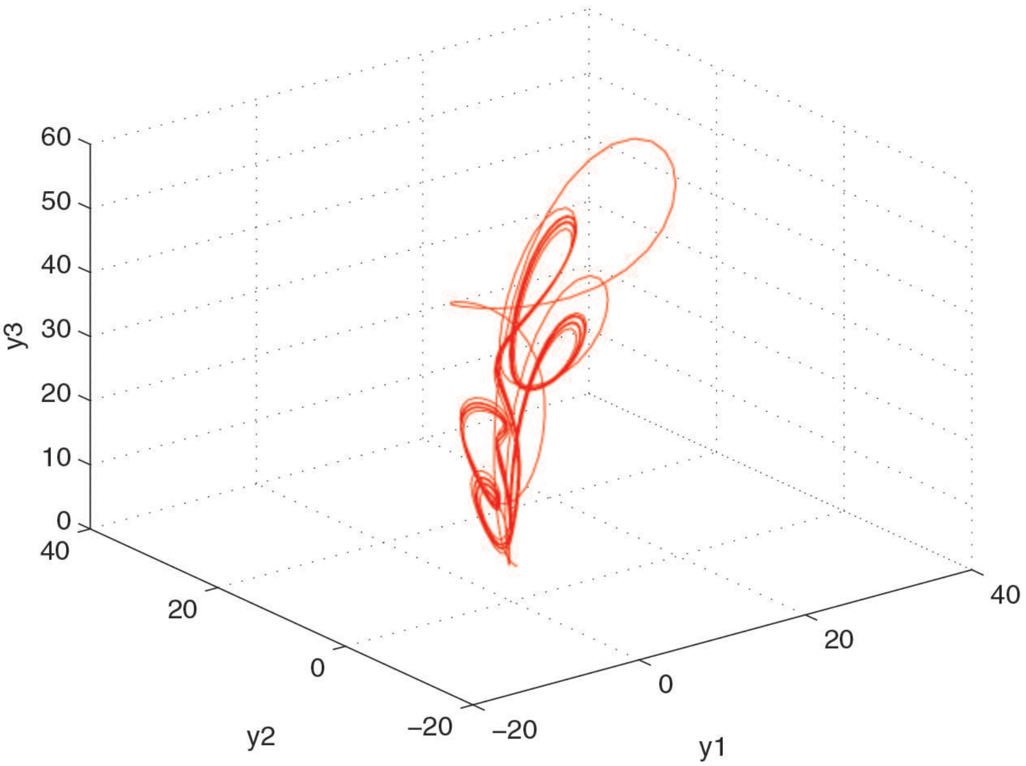

Figure 10. Phase space of the transformation of the slave system of Lü.

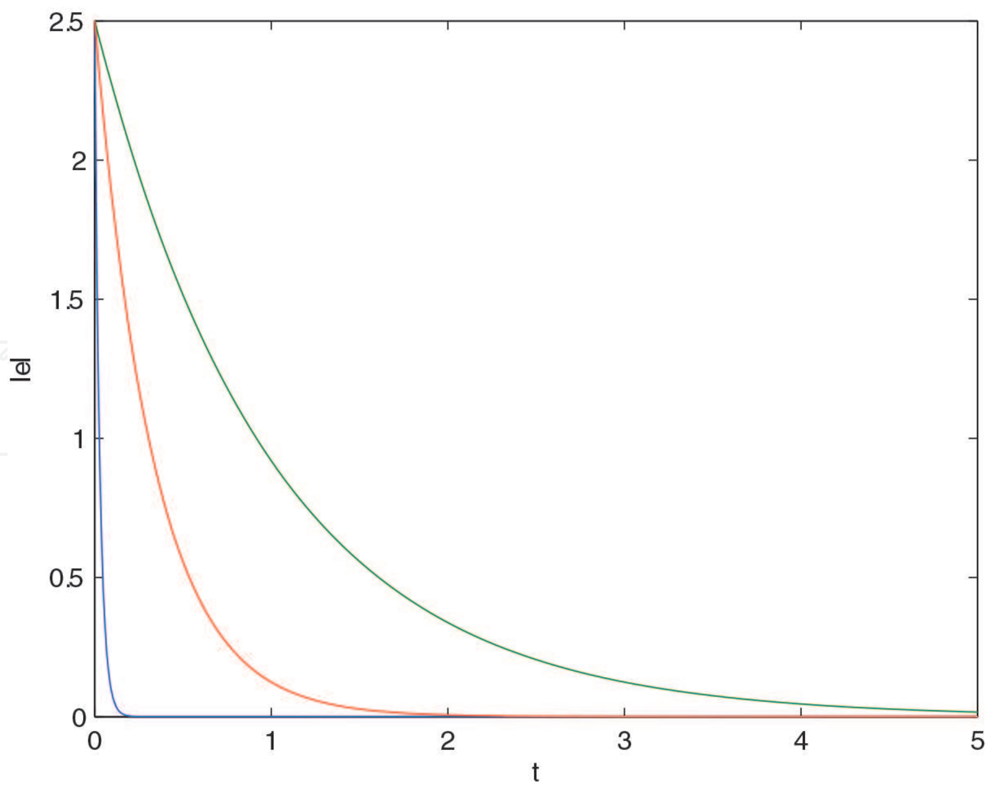

Figure 11. Magnitude of the error between the transformation of master and slave systems.

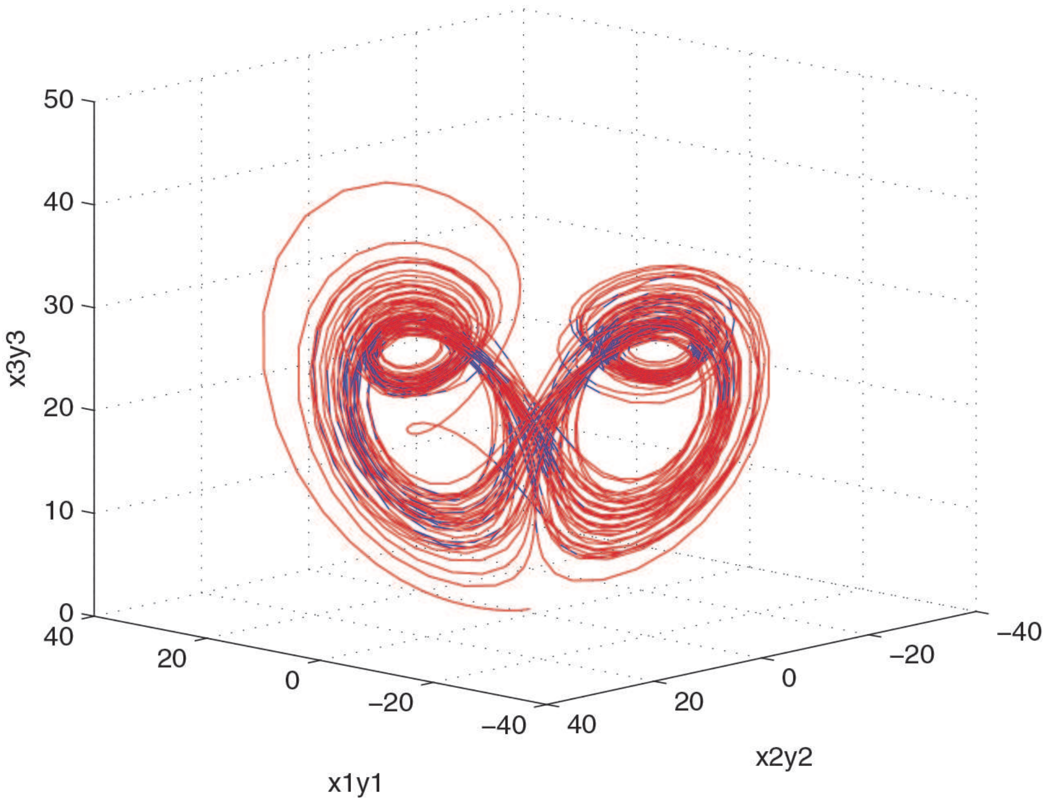

Figure 12. Synchronization of the transformation of the master and slave systems of Lü.

After the transformation in its linear part of Lü attractor, we also have several simulations allowing us to analyze the dynamics of the transformed system. In **Figure 7**, we present the trajectories of the transformation of the master system of Lü. Each line represents one trajectory of the system along with the time taking an initial condition of $(0.5, 0.5, 0.5, 0.5, 0.5, 0.5)$. For the case of **Figure 9**, we show the trajectories of the transformation of the slave system of Lü. Each line represents one trajectory of the system also, along the time, taking an initial condition of $(3, 3, 3, 3, 3, 3)$. **Figures 8** and **10** are the phase space mappings of each transformed system while maintaining the same initial condition. By last, in **Figure 11**, we can see the error magnitude of the transformation of synchronized system. A phase space mapping of the transformation of synchronized system is presented in **Figure 12**.

6. Conclusion

We have studied the preservation of stability of a chaotic dynamic system, from an extension of the stable-unstable manifold theorem and an extension of the center manifold theorem based on the preservation of the linear part in nonlinear dynamical systems. However, we can check that given a chaotic system, its transformed version is also chaotic. A scheme consisting of a master-slave system for which a controller gain is obtained using a linear-quadratic regulator has been presented and synchronization is achieved and preserved even

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after the master-slave controller is transformed, obtaining as a consequence that the chaotic system changes to an higher dimension. It is important to note the transformation of the linear part of the chaotic system from Tracy-Singh product in which it was used to modify a Lü system, showing the effectiveness of the proposed method. The results can be extended to other techniques for feedback design, for example, adaptive control, sliding mode regulator and etcetera.

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