Universidade de Aveiro Departamento de Matemática
2014

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Cálculo das Variações em Escalas Temporais e Aplicações à Economia

Calculus of Variations on Time Scales and Applications to Economics

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Tese de doutoramento apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Associado com Agregação no Departamento de Matemática da Universidade de Aveiro, e co-orientação da Doutora Agnieszka Barbara Malinowska, Professora Auxiliar com Agregação na Faculdade de Ciências da Computação, Universidade Técnica de Bialystok, Polónia.

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palavras-chave

Cálculo em escalas temporais, cálculo das variações, condições necessárias de otimalidade do tipo de Euler-Lagrange, problemas inversos em escalas temporais, aplicações à economia.

resumo

Consideramos alguns problemas do cálculo das variações em escalas temporais. Primeiramente, demonstramos equações do tipo de Euler-Lagrange e condições de transversalidade para problemas de horizonte infinito generalizados. De seguida, consideramos a composição de uma certa função escalar com os integrais delta e nabla de um campo vetorial. Presta-se atenção a problemas extremais inversos para funcionais variacionais em escalas de tempo arbitrárias. Começamos por demonstrar uma condição necessária para uma equação dinâmica integro-diferencial ser uma equação de Euler-Lagrange. Resultados novos e interessantes para o cálculo discreto e quantum são obtidos como casos particulares. Além disso, usando a equação de Euler-Lagrange e a condição de Legendre fortalecida, obtemos uma forma geral para uma funcional variacional atingir um mínimo local, num determinado ponto do espaço vetorial. No final, duas aplicações interessantes em termos económicos são apresentadas. No primeiro caso, consideramos uma empresa que quer programar as suas políticas de produção e de investimento para alcançar uma determinada taxa de produção e maximizar a sua competitividade no mercado futuro. O outro problema é mais complexo e relaciona a inflação e o desemprego, que inflige uma perda social. A perda social é escrita como uma função da taxa de inflação p e a taxa de desemprego u, com diferentes pesos. Em seguida, usando as relações conhecidas entre p, u, e a taxa de inflação esperada π, reescrevemos a função de perda social como uma função de π. A resposta é obtida através da aplicação do cálculo das variações, a fim de encontrar a curva ótima π que minimiza a perda social total ao longo de um determinado intervalo de tempo.

keywords

Time-scale calculus, calculus of variations, necessary optimality conditions of Euler-Lagrange type, inverse problems on time scales, applications to economics.

abstract

We consider some problems of the calculus of variations on time scales. On the beginning our attention is paid on two inverse extremal problems on arbitrary time scales. Firstly, using the Euler-Lagrange equation and the strengthened Legendre condition, we derive a general form for a variation functional that attains a local minimum at a given point of the vector space. Furthermore, we prove a necessary condition for a dynamic integro-differential equation to be an Euler-Lagrange equation. New and interesting results for the discrete and quantum calculus are obtained as particular cases. Afterwards, we prove Euler-Lagrange type equations and transversality conditions for generalized infinite horizon problems. Next we investigate the composition of a certain scalar function with delta and nabla integrals of a vector valued field. Euler-Lagrange equations in integral form, transversality conditions, and necessary optimality conditions for isoperimetric problems, on an arbitrary time scale, are proved. In the end, two main issues of application of time scales in economic, with interesting results, are presented. In the former case we consider a firm that wants to program its production and investment policies to reach a given production rate and to maximize its future market competitiveness. The model which describes firm activities is studied in two different ways: using classical discretizations; and applying discrete versions of our result on time scales. In the end we compare the cost functional values obtained from those two approaches. The latter problem is more complex and relates to rate of inflation, p, and rate of unemployment, u, which inflict a social loss. Using known relations between p, u, and the expected rate of inflation π, we rewrite the social loss function as a function of π. We present this model in the time scale framework and find an optimal path π that minimizes the total social loss over a given time interval.

2010 Mathematics Subject Classification: 26E70; 34N05; 49K05; 49K21.

Contents

Introduction

This thesis is devoted to the study of the calculus of variations on time scales and its applications to economics. It is a discipline with many opportunities of applications of this branch of mathematics. When I started my Ph.D. Doctoral Programme, I had not decided in which direction I would like to develop my research. However, everything became clear after the first semester of the first year of my studies. The doctoral programme PDMA Aveiro–Minho provided one-semester course called Research Lab where five different areas of mathematics were covered. I enrolled to a course led by my current supervisor Prof. Delfim F. M. Torres and I was introduced to the calculus of variations on time scales. It was the first time I met this subject and I found it interesting and full of possibilities in research. I shown a great interest of this theme and due to that Prof. Torres agreed to be my supervisor. At the same time I was studying economics at University of Białystok, Poland, and the title of my Ph.D. thesis emerged naturally — application of the time-scale calculus in economics. Since my second studies took place in Poland, Prof. Torres suggested to ask Prof. Agnieszka B. Malinowska (Faculty of Computer Science, Białystok University of Technology, Poland) to be my co-supervisor. After her acceptance, I got a great support in Portugal and Poland.

The main idea of this thesis is to convert classical economic problems into time scale models. This procedure gives a lot of possibilities. Instead of considering two different models (continuous and/or discrete) we deal with only one model and we are able to apply it to different, even hybrid, time scales. Moreover, we can create mixed models, i.e., combination of two different discretizations (both forward and backward). Finally, we obtain better results comparing with traditional discretizations. Our contribution on this area appears in Section 7.1 of this thesis. We started working on a variational model that describes the relation between inflation and unemployment (see Section 7.2), which was converted into a more general time-scale problem of the calculus of variations. The next question is: based on the set of real data and having a time scale model, is it possible to find a time scale which describes the reality better than classical methods? Usually the procedure is opposite: a time scale is already given (see Conclusions and Future Work). Those two conceptions were the main motivation of this thesis.

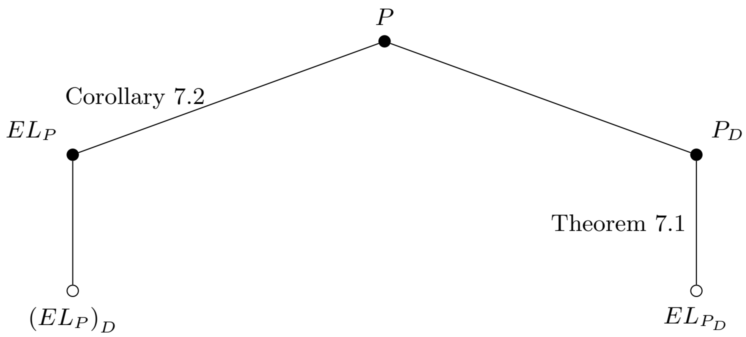
The calculus of variations is one of the classical branches of mathematics with ancient origins in questions of Aristotle and Zenodoros. However, its mathematical principles first emerged in the post-calculus investigations of Newton, the Bernoullis, Euler, and Lagrange. A very strong influence on the development of the calculus of variations had the brachistochrone problem (see, e.g., [90]), stated by Bernoulli in 1696, which most authors consider as the birth of this research area. This problem excited great interest among the mathematicians of XVII century and gave conception to new research which is still continuing. A decisive step was achieved with the work of Euler and Lagrange who found a systematic way of dealing with variational problems by introducing what is now known as the Euler–Lagrange equations (see [53]). The calculus of variations is focused on finding extremum (i.e., maximum or minimum) values of functionals not treatable by the methods of elementary calculus, usually given in the form of an integral that involves an unknown function and its derivatives [33, 49, 61, 90]. The variational integral may represent an action, energy, or cost functional [42, 91]. The variational calculus, in its present form, provides powerful methods for the treatment of differential equations, the theory of invariants, existence theorems in geometric function theory, variational principles in mechanics, that possess also important connections with other fields of mathematics, e.g., analysis, geometry, physics, and many different areas – engineering, biology and economics. The classical theory of the calculus of variations is presented briefly in the beginning of this thesis, in Chapter 2.

The theory of time scales is a relatively new area, that bridges, generalize and extends the traditional discrete dynamical systems (i.e., difference equations) and continuous dynamical systems (i.e., differential equations) [24] and the various dialects of q-calculus [41, 74] into a single unified theory [24, 25, 65]. It was introduced in 1988 by Stefan Hilger in his Ph.D. thesis [55–57] as the "Calculus on Measure Chains" [10, 65]. Today it is better known as the time-scale calculus. In Stefan Hilger's Ph.D. thesis a time scale is defined as a nonempty closed subset of the real numbers. The term "time scales" describes the behavior of a dynamical systems over time, hence, a time scale is a model of time. In the past, engineers and mathematicians considered a time as a variable in two ways. A continuous time was associated to hands of an old-fashioned clock which sweeps continuously, while a discrete time (with steps equal to one) was related to a digital clock. The time-scale theory unify those two approaches to describe systems which may be partly continuous and partly discrete, and in that case having features of both – analog and digital. Currently, this subject is researched in many different fields in which dynamic processes can be described with discrete or continuous models [1]. However, there are infinitely many time scales and due to that it is possible to state more general results.

The calculus of variations on time scales was introduced by Bohner [18] and by Hilscher and Zeidan [2, 58] and has been developing rapidly in the past ten years and is now a fertile area of research – see, e.g., [19, 47, 51, 67, 79]. The time-scale variational calculus has a great potential for applications, e.g., in biology [24] or in economics [4, 6, 9, 45, 75]. In order to deal with nontraditional applications in economics, where the system dynamics are described on a time scale partly continuous and partly discrete, or to accommodate nonuniform sampled systems, one needs to work with variational problems defined on a time scale [6, 8, 35].

This Ph.D. thesis consists of two parts. The former one, named 'Synthesis', gives preliminary definitions (Chapter 1) and properties of classical and time-scale calculi of variations (Chapters 2 and 3). The latter part, called 'Original Work', is divided into four chapters, containing new results published during my Ph.D. project in peer reviewed international journals [35–39]. Moreover, the work has been recognized by the Awarding Committee of the Symposium on Differential Equations and Difference Equations (SDEDE 2014), Homburg/Germany, 5th-8th September 2014, and awarded with the Bernd Aulbach Prize 2014 for students.

In Chapter 4 we describe, in two different ways, inverse problems of the calculus of variations. First we consider a fundamental inverse problem of the calculus of variations subject to the boundary conditions $y(a) = y_a$ and $y(b) = y_b$ on a given time scale $\mathbb{T}$. We describe a general form of a variational functional which attains a local minimum at a given function y_0 under Euler–Lagrange and strengthened Legendre conditions. In order to illustrate our results, we present the form of the Lagrangian L on an isolated time scale (Corollary 4.4). We end by presenting the form of the Lagrangian L in the periodic time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$ (Example 4.6) and in the q-scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ (Example 4.7). In the latter part of Chapter 4 we also consider an inverse problem but stated as an integro-differential dynamic equation. Compared with the direct problem, that establish dynamic equations of Euler–Lagrange type to time-scale variational problems, the inverse problem has not been studied before in the framework of time scales. On the beginning we define self-adjointness of a first order integro-differential equation (Definition 4.8) and its equation of variation (Definition 4.9). To the best of our knowledge, those definitions, in integro-differential form, are new. Using those features (see Lemma 4.11) we prove a necessary condition for a general (non-classical) inverse problem of the calculus of variations on an arbitrary time scale (Theorem 4.12). In order to illustrate our results we present Theorem 4.12 in the particular time scales $\mathbb{T} \in \{\mathbb{R}, h\mathbb{Z}, \overline{q^{\mathbb{Z}}}\}$, $h > 0$, $q > 1$ (Corollaries 4.16, 4.17, and 4.18). The last part of this chapter contains a discussion about equivalences between: (i) the integro-differential equation (4.22) and the second order differential equation (4.35) (Proposition 4.19), and (ii) equations of variations of them ((4.23) andd (4.38), respectively) on an arbitrary time scale $\mathbb{T}$. We found that the first equivalence is easy to prove while the latter one is even irrealizable on an arbitrary time scale (it is only possible in $\mathbb{R}$). It is shown that the absence of a general chain rule on an arbitrary time scale causes this impossibility [20, 24]. Chapter 5 is dedicated to infinite horizon problems of the calculus of variations with nabla derivatives and nabla integrals (see (5.2)). The motivation is that infinite horizon models are often considered in macroeconomics and, moreover, the nabla approach has been recently considered preferably when applied to economic problems [6, 7]. We proved necessary optimality conditions to problem (5.2) obtaining Euler–Lagrange equations and transversality conditions (Theorems 5.6 and 5.7). In Chapter 6 we consider a variational problem which often may be found in economics (see [71] and references therein). We extremize a functional of the calculus of variations that is the composition of a certain scalar function with the delta and nabla integrals of a vector valued field, possibly subject to boundary conditions and/or isoperimetric constraints. Depending on the given boundary conditions, we can distinguish four different problems: with two boundary conditions, with just initial or terminal point, or with none. Euler–Lagrange equations in integral form, transversality conditions, and necessary optimality conditions for isoperimetric problems, on an arbitrary time scale, are proved. At the end, interesting corollaries and examples are presented. The last chapter, Chapter 7, is devoted to two economic models. Both of them are presented with a time-scale formulation. In the former, we consider a general (non-classical) mixed delta-nabla problem (see (7.1)–(7.2)) of the calculus of variations on time scales, as in Chapter 6. However, here we prove general necessary optimality conditions of Euler–Lagrange type in differential form (Theorem 7.1). We consider a firm that wants to program its production and investment policies in order to gain a desirable production level and maximize its market competitiveness. Our idea is to discretize necessary optimality conditions of Euler–Lagrange type (EL_P) and the (continuous) problem P in different ways, combining forward (Δ) and backward (∇) discretization operators into a mixed operator D. One can apply the variational principle to problem P obtaining the respective Euler–Lagrange equation EL_P (Corollary 7.2), and then discretize it using D, obtaining $(EL_P)_D$; or we can begin by discretizing problem P into P_D and then develop the respective variational principle, obtaining EL_{P_D} (Theorem 7.1). This is illustrated in the following diagram.



Note that, in general, $(EL_P)_D$ is different from EL_{P_D}. Four different problems P_D, four Euler–Lagrange equations EL_{P_D} and four Euler–Lagrange equations $(EL_P)_D$ are discussed and investigated in Section 7.1. For our problem a time-scale approach leads to better results: the approach on the right hand side of the diagram gives candidates to minimizers for which the value of the functional is smaller than the values obtained from the approach on the left hand side of the diagram. Appendix A provides all calculations made using the Computer Algebra System Maple, version 10. The latter economic model (Section 7.2) is more complex and relates to inflation and unemployment, which inflicts a social loss. There exists a strict relation between both, which is described by the Phillips curve. Economists use a continuous or a discrete variational problem. In this thesis a time-scale model is presented, which unifies available results in the literature. We apply the theory of the calculus of variations in order to find an optimal path of expected rate of inflation that minimizes the total social loss over a given time interval.

We finish the thesis with a conclusion, pointing also some important directions of future research.

Part I

Synthesis

Chapter 1

Time-scale Calculus

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The sets of real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, and the nonnegative integers $\mathbb{N}_0$, an union of closed intervals $[1,2] \cup [4,5]$ or Cantor set are examples of time scales. While the sets of rational numbers $\mathbb{Q}$, the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, the complex numbers $\mathbb{C}$ or an open interval $(4,5)$ are not time scales. Throughout this thesis for $a, b \in \mathbb{T}$, $a < b$, we define the interval $[a,b]$ in $\mathbb{T}$ by $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T} = \{t \in \mathbb{T} : a \le t \le b\}$.

Definition 1.1 (See Section 1.1 of [24])**.** *Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$. Accordingly, we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) := \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$. The forward graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$, while the backward graininess function $\nu : \mathbb{T} \to [0,\infty)$ is defined by $\nu(t) := t - \rho(t)$.*

Example 1.2. *The two classical time scales are $\mathbb{R}$ and $\mathbb{Z}$, representing the continuous and the purely discrete time, respectively. The other standard examples are periodic numbers $h\mathbb{Z} = \{hk : h > 0, k \in \mathbb{Z}\}$, and q-scale $\overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\} = \{q^k : q > 1, k \in \mathbb{Z}\} \cup \{0\}$ (however, here we also consider a time scale $q^{\mathbb{N}_0} = \{q^k : q > 1, k \in \mathbb{N}_0\}$). We can also define the following time scale: $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$, $a, b > 0$.*

The Table 1.1 and Example 1.3 present different forms of jump operators σ and ρ, and graininiess functions μ and ν, in specified time scales.

Example 1.3 (See Example 1.3 of [93])**.** *Let $a, b > 0$ and consider the time scale*

$$\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a].$$

$\mathbb{T}$	$\mathbb{R}$	$h\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$
$\sigma(t)$	t	$t+h$	qt
$\rho(t)$	t	$t-h$	$\frac{t}{q}$
$\mu(t)$	0	h	$t(q-1)$
$\nu(t)$	0	h	$\frac{t(q-1)}{q}$

Table 1.1: Examples of jump operators and graininess functions on different time scales.

Then

$$\sigma(t) = \begin{cases} t & \text{if } t \in A_1, \\ t+b & \text{if } t \in A_2, \end{cases} \qquad \rho(t) = \begin{cases} t-b & \text{if } t \in B_1, \\ t & \text{if } t \in B_2 \end{cases}$$

and

$$\mu(t) = \begin{cases} 0 & \text{if } t \in A_1, \\ b & \text{if } t \in A_2, \end{cases} \qquad \nu(t) = \begin{cases} b & \text{if } t \in B_1, \\ 0 & \text{if } t \in B_2, \end{cases}$$

where

$$\bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a] = A_1 \cup A_2 = B_1 \cup B_2,$$

with

$$A_1 = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a), \quad B_1 = \bigcup_{k=0}^{\infty} \{k(a+b)\},$$
$$A_2 = \bigcup_{k=0}^{\infty} \{k(a+b)+a\}, \qquad B_2 = \bigcup_{k=0}^{\infty} (k(a+b), k(a+b)+a].$$

In the time-scale theory the following classification of points is used:

- A point $t \in \mathbb{T}$ is called *right-scattered* or *left-scattered* if $\sigma(t) > t$ or $\rho(t) < t$, respectively.
- A point t is *isolated* if $\rho(t) < t < \sigma(t)$.
- If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*.
- We say that t is *dense* if $\rho(t) = t = \sigma(t)$.

Definition 1.4 (See Section 1 of [80])**.** *A time scale $\mathbb{T}$ is said to be an isolated time scale provided given any $t \in \mathbb{T}$, there is a $\delta > 0$ such that $(t-\delta, t+\delta) \cap \mathbb{T} = \{t\}$.*

Remark 1.5. *If the graininess function is bounded from below by a strictly positive number, then the time scale is isolated [22]. Therefore, $h\mathbb{Z}$, $h > 0$, and $q^{\mathbb{N}_0}$, $q > 1$, are examples of isolated time scales. Note that the converse is not true. For example, $\mathbb{T} = \log(\mathbb{N})$ is an isolated time scale but its graininess function is not bounded from below by a strictly positive number.*

Definition 1.6 (See [13])**.** *A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously for all $t \in \mathbb{T}$: $\sigma(\rho(t)) = t$ and $\rho(\sigma(t)) = t$.*

In the following example we present two regular time scales ($\mathbb{R}$ and $\overline{q^{\mathbb{Z}}}$) and an irregular time scale ($\mathbb{P}_{a,b}$).

Example 1.7. *For real numbers $\mathbb{R}$ and q-numbers $\overline{q^{\mathbb{Z}}}$ we have the required equivalence $\sigma(\rho(t)) = \rho(\sigma(t)) = t$ for a time scale to be regular. Considering the time scale $\mathbb{P}_{a,b}$, we get*

$$\sigma(\rho(t)) = \rho(\sigma(t)) = \begin{cases} t-b & \text{if } t \in t \in \bigcup\limits_{k=0}^{\infty} \{k(a+b)\}, \\ t & \text{if } t \in \bigcup\limits_{k=0}^{\infty} (k(a+b), k(a+b)+a), \\ t+b & \text{if } t \in t \in \bigcup\limits_{k=0}^{\infty} \{k(a+b)+a\} \end{cases}$$

and we conclude that $\mathbb{P}_{a,b}$ is irregular.

1.1 The delta derivative and the delta integral

In this section we collect the necessary theorems and properties concerning delta differentiation and delta integration on a time scale. The delta approach is based on the forward jump operator σ. If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function, then we define $f^\sigma : \mathbb{T} \longrightarrow \mathbb{R}$ by $f^\sigma(t) := f(\sigma(t))$ for all $t \in \mathbb{T}$. The delta derivative (or *Hilger derivative*) of function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is defined for points in the set $\mathbb{T}^\kappa$, where

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus \{\sup \mathbb{T}\} & \text{if } \rho(\sup \mathbb{T}) < \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \text{ or } \rho(\sup \mathbb{T}) = \sup \mathbb{T}. \end{cases}$$

Let us define the sets $\mathbb{T}^{\kappa^n}$, $n \geq 2$, inductively: $\mathbb{T}^{\kappa^1} := \mathbb{T}^\kappa$ and $\mathbb{T}^{\kappa^n} := \left(\mathbb{T}^{\kappa^{n-1}}\right)^\kappa$, $n \geq 2$. We define the delta differentiability in the following way.

Definition 1.8 (Section 1.1 of [24])**.** *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U ($U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) of t such that*

$$\left|f^\sigma(t) - f(s) - f^\Delta(t)\,(\sigma(t) - s)\right| \leq \varepsilon\,|\sigma(t) - s| \ \textit{for all } s \in U.$$

A function f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. Then, $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is called the delta derivative of f on $\mathbb{T}^\kappa$.

Theorem 1.9 (Theorem 1.16 of [24])**.** *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The following hold:*

1. *If f is delta differentiable at t, then f is continuous at t.*

2. *If f is continuous at t and t is right-scattered, then f is delta differentiable at t with*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

3. *If t is right-dense, then f is delta differentiable at t if and only if the limit*

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

4. *If f is delta differentiable at t, then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t).$$

The next example is a consequence of Theorem 1.9 and presents different forms of the delta derivative on specific time scales.

Example 1.10. *Let $\mathbb{T}$ be a time scale.*

1. *If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if*

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists, i.e., if and only if f is differentiable (in the ordinary sense) at t and in this case we have $f^\Delta(t) = f'(t)$.

2. *If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $f : h\mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in h\mathbb{Z}$ with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h} =: \Delta_h f(t). \tag{1.1}$$

In the particular case for $h = 1$, we have $f^\Delta(t) = \Delta f(t)$, where Δ is the usual forward difference operator.

3. *If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, then for a delta differentiable function $f : \overline{q^{\mathbb{Z}}} \longrightarrow \mathbb{R}$ we have*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t} =: \Delta_q f(t), \tag{1.2}$$

for all $t \in \overline{q^{\mathbb{Z}}} \setminus \{0\}$, i.e., we get the usual Jackson derivative of quantum calculus [62, 74].

Now we formulate the basic properties of the delta derivative on a time scale.

Theorem 1.11 (Theorem 1.20 of [24]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^{\kappa}$. Then,*

1. *the sum $f + g : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t);$$

2. *for any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t);$$

3. *the product $fg : \mathbb{T} \to \mathbb{R}$ is delta differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t);$$

4. *if $g(t)g^{\sigma}(t) \neq 0$, then f/g is delta differentiable at t with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}.$$

Below we generalize the product rule (the third item of Theorem 1.11) for n functions.

Example 1.12 (Cf. Exercise 1.22 of [24]). *Let $\mathbb{T}$ be a time scale. If x_k is a delta differentiable function at t, $k = 1, \ldots n$, then*

$$\left(\prod_{k=1}^{n} x_k(t)\right)^{\Delta} = \sum_{k=1}^{n} \left(x_k^{\Delta}(t) \prod_{i=1}^{k-1} x_i^{\sigma}(t) \prod_{m=k+1}^{n} x_m(t) \right) \tag{1.3}$$

holds at t, for $i, k, m, n \in \mathbb{N}$, $1 \leq i \leq k-1$, $2 \leq m \leq n$, $1 \leq k \leq n$.

Proof. The proof is made using mathematical induction. First we consider the basic step. For $n = 1$ we get easily that $x_1^{\Delta} = x_1^{\Delta}$. For $n = 2$ we get the formula presented in the third item of Theorem 1.11. Next, in the inductive step, we have to prove that if (1.3) holds for $n = j-1$, then it also holds for $n = j$, $j \in \mathbb{N}$. Hence, our induction hypothesis is the following:

$$\left(\prod_{k=1}^{j-1} x_k(t)\right)^{\Delta} = \sum_{k=1}^{j-1} \left(x_k^{\Delta}(t) \prod_{i=1}^{k-1} x_i^{\sigma}(t) \prod_{m=k+1}^{j-1} x_m(t) \right).$$

It must be shown that (1.3) holds for $n = j$. Observe that

$$\begin{aligned}
\left(\prod_{k=1}^{j} x_k(t)\right)^{\Delta} &= (x_1(t)x_2(t)\cdots x_{j-1}(t)x_j(t))^{\Delta} \\
&= (x_1(t)x_2(t)\cdots x_{j-1}(t))^{\Delta}\, x_j(t) + x_1^{\sigma}(t)x_2^{\sigma}(t)\cdots x_{j-1}^{\sigma}(t)x_j^{\Delta}(t) \\
&= \sum_{k=1}^{j-1}\left(x_k^{\Delta}(t)\prod_{i=1}^{k-1}x_i^{\sigma}(t)\prod_{m=k+1}^{j-1}x_m(t)\right)x_j(t) + x_1^{\sigma}(t)x_2^{\sigma}(t)\cdots x_{j-1}^{\sigma}(t)x_j^{\Delta}(t) \\
&= \sum_{k=1}^{j}\left(x_k^{\Delta}(t)\prod_{i=1}^{k-1}x_i^{\sigma}(t)\prod_{m=k+1}^{j}x_m(t)\right).
\end{aligned}$$

Thereby our statement is proved for $n = j$. By mathematical induction, the statement (1.3) holds for all $n \in \mathbb{N}$. □

Now we introduce the theory of delta integration on time scales. We start by defining the necessary class of functions.

Definition 1.13 (Section 1.4 of [25]). *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 1.14 (Definition 1.71 of [24]). *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$.*

Definition 1.15. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. If $f : \mathbb{T}^{\kappa} \to \mathbb{R}$ is a rd-continuous function and $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of f, then the Cauchy delta integral is defined by*

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

Theorem 1.16 (Theorem 1.74 of [24]). *Every rd-continuous function f has an antiderivative F. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad t \in \mathbb{T},$$

is an antiderivative of f.

Theorem 1.17 (Theorem 1.75 of [24]). *If $f \in C_{rd}$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Let us see a few examples.

Example 1.18. *Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ be rd-continuous.*

1. *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral.

2. *If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum\limits_{t\in[a,b)} \mu(t)f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum\limits_{t\in[b,a)} \mu(t)f(t), & \text{if } a > b. \end{cases}$$

3. *If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum\limits_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum\limits_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h, & \text{if } a > b. \end{cases}$$

4. *If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a < b$, then*

$$\int_a^b f(t)\Delta t = (q-1) \sum_{t\in[a,b)\cap\mathbb{T}} tf(t).$$

Now we present some useful properties of the delta integral.

Theorem 1.19 (Theorem 1.77 of [24]). *If $a, b, c \in \mathbb{T}$, $a < c < b$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then:*

1. $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t,$

2. $\int\limits_a^b \alpha f(t)\Delta t = \alpha \int\limits_a^b f(t)\Delta t$,

3. $\int\limits_a^b f(t)\Delta t = -\int\limits_b^a f(t)\Delta t$,

4. $\int\limits_a^b f(t)\Delta t = \int\limits_a^c f(t)\Delta t + \int\limits_c^b f(t)\Delta t$,

5. $\int\limits_a^a f(t)\Delta t = 0$,

6. *if* $f, g \in C^1_{rd}(\mathbb{T}, \mathbb{R})$, *then* $\int\limits_a^b f(t)g^\Delta(t)\Delta t = f(t)g(t)|_{t=a}^{t=b} - \int\limits_a^b f^\Delta(t)g^\sigma(t)\Delta t$,

7. *if* $f, g \in C^1_{rd}(\mathbb{T}, \mathbb{R})$, *then* $\int\limits_a^b f^\sigma(t)g^\Delta(t)\Delta t = f(t)g(t)|_{t=a}^{t=b} - \int\limits_a^b f^\Delta(t)g(t)\Delta t$,

8. *if* $f(t) \geqslant 0$ *for all* $a \leqslant t < b$, *then* $\int\limits_a^b f(t)\Delta t \geqslant 0$.

1.2 The nabla derivative and the nabla integral

The nabla calculus is similar to the delta one of Section 1.1. The difference is that the backward jump operator ρ takes the role of the forward jump operator σ. For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ we define $f^\rho : \mathbb{T} \longrightarrow \mathbb{R}$ by $f^\rho(t) := f(\rho(t))$. If $\mathbb{T}$ has a right-scattered minimum m, then we define $\mathbb{T}_\kappa := \mathbb{T} - \{m\}$; otherwise, we set $\mathbb{T}_\kappa := \mathbb{T}$. In summary,

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T} \setminus \{\inf \mathbb{T}\} & \text{if } -\infty < \inf \mathbb{T} < \sigma(\inf \mathbb{T}), \\ \mathbb{T} & \text{otherwise.} \end{cases}$$

Let us define the sets $\mathbb{T}_\kappa$, $n \geq 2$, inductively: $\mathbb{T}_{\kappa^1} := \mathbb{T}_\kappa$ and $\mathbb{T}_{\kappa^n} := (\mathbb{T}_{\kappa^{n-1}})_\kappa$, $n \geq 2$. Finally, we can define $\mathbb{T}^\kappa_\kappa := \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$.

The definition of the nabla derivative of a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ at point $t \in \mathbb{T}_\kappa$ is similar to the delta case (Definition 1.8).

Definition 1.20 (Section 3.1 of [25])**.** *We say that a function* $f : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable at* $t \in \mathbb{T}_\kappa$ *if there is a number* $f^\nabla(t)$ *such that for all* $\varepsilon > 0$ *there exists a neighborhood* U *of* t *(i.e.,* $U = (t - \delta, t + \delta) \cap \mathbb{T}$ *for some* $\delta > 0$*) such that*

$$|f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s| \text{ for all } s \in U.$$

We say that $f^\nabla(t)$ *is the nabla derivative of* f *at* t*. Moreover,* f *is said to be nabla differentiable on* $\mathbb{T}$ *provided* $f^\nabla(t)$ *exists for all* $t \in \mathbb{T}_\kappa$*.*

Below we present the basic properties of the nabla derivative.

Theorem 1.21 (Theorem 8.39 of [24]). *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$. Then we have the following:*

1. *If f is nabla differentiable at t, then f is continuous at t.*
2. *If f is continuous at t and t is left-scattered, then f is nabla differentiable at t with*
$$f^\nabla(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$
3. *If t is left-dense, then f is nabla differentiable at t if and only if the limit*
$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$
exists as a finite number. In this case
$$f^\nabla(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$
4. *If f is a nabla differentiable at t, then*
$$f^\rho(t) = f(t) - \nu(t) f^\nabla(t).$$

Example 1.22. *If $\mathbb{T} = \mathbb{R}$, then*
$$f^\nabla(t) = f^{'}(t),$$
and if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then
$$f^\nabla(t) = \frac{f(t) - f(t-h)}{h} =: \nabla_h f(t).$$
In the case $h = 1$, ∇_h is the standard backward difference operator $\nabla f(t) = f(t) - f(t-1)$.

Now we present some useful properties of the nabla derivative.

Theorem 1.23 (Theorem 8.41 of [24]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then,*

1. *the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t with*
$$(f + g)^\nabla(t) = f^\nabla(t) + g^\nabla(t);$$
2. *for any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and*
$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t);$$

3. *the product* $fg : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable at* t *with*

$$(fg)^{\nabla}(t) = f^{\nabla}(t)g(t) + f^{\rho}(t)g^{\nabla}(t) = f^{\nabla}(t)g^{\rho}(t) + f(t)g^{\nabla}(t);$$

4. f/g *is nabla differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)g(t) - f(t)g^{\nabla}(t)}{g(t)g^{\rho}(t)},$$

if $g(t)g^{\rho}(t) \neq 0$.

Now we formulate the theory of nabla integration on time scales. Similarly as in the delta case, first we define an associated class of functions.

Definition 1.24 (Section 3.1 of [25]). *Let* $\mathbb{T}$ *be a time scale and* $f : \mathbb{T} \to \mathbb{R}$. *We say that* f *is ld-continuous if it is continuous at left-dense points* $t \in \mathbb{T}$ *and its right-sided limits exist (finite) at all right-dense points.*

Remark 1.25. *If* $\mathbb{T} = \mathbb{R}$, *then* f *is ld-continuous if and only if* f *is continuous. If* $\mathbb{T} = \mathbb{Z}$, *then any function is ld-continuous.*

The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by

$$C_{ld} = C_{ld}(\mathbb{T}) = C_{ld}(\mathbb{T}, \mathbb{R})$$

and the set of all nabla differentiable functions with ld-continuous derivative by

$$C^1_{ld} = C^1_{ld}(\mathbb{T}) = C^1_{ld}(\mathbb{T}, \mathbb{R}).$$

Now we present the definition of nabla integral on time scales.

Definition 1.26 (Definition 8.42 of [24]). *A function* $F : \mathbb{T} \to \mathbb{R}$ *is called a nabla antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ *provided* $F^{\nabla}(t) = f(t)$ *for all* $t \in \mathbb{T}_{\kappa}$. *In this case we define the nabla integral of* f *from* a *to* b ($a, b \in \mathbb{T}$) *by*

$$\int_a^b f(t)\nabla t := F(b) - F(a), \text{ for all } t \in \mathbb{T}.$$

Theorem 1.27 (Theorem 8.45 of [24] or Theorem 11 of [70]). *Every ld-continuous function* f *has a nabla antiderivative* F. *In particular, if* $a \in \mathbb{T}$, *then* F *defined by*

$$F(t) = \int_a^t f(\tau)\nabla\tau, \quad t \in \mathbb{T},$$

is a nabla antiderivative of f.

Theorem 1.28 (Theorem 8.46 of [24])**.** *If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is ld-continuous and $t \in \mathbb{T}_\kappa$, then*

$$\int_{\rho(t)}^{t} f(\tau)\nabla\tau = \nu(t)f(t).$$

Properties of the nabla integral are analogous to properties of the delta integral.

Theorem 1.29 (See Theorem 8.47 of [24])**.** *If $a, b, c \in \mathbb{T}$, $a < c < b$, $\alpha \in \mathbb{R}$, and $f, g : \mathbb{T} \longrightarrow \mathbb{R}$, $f, g \in C_{ld}(\mathbb{T}, \mathbb{R})$, then:*

1. $\int_a^b (f(t) + g(t))\nabla t = \int_a^b f(t)\nabla t + \int_a^b g(t)\nabla t$;

2. $\int_a^b \alpha f(t)\nabla t = \alpha \int_a^b f(t)\nabla t$;

3. $\int_a^b f(t)\nabla t = -\int_b^a f(t)\nabla t$;

4. $\int_a^b f(t)\nabla t = \int_a^c f(t)\nabla t + \int_c^b f(t)\nabla t$;

5. *if* $f, g \in C^1_{ld}(\mathbb{T}, \mathbb{R})$, *then* $\int_a^b f^\rho(t) g^\nabla(t)\nabla t = f(t)g(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g(t)\nabla t$;

6. *if* $f, g \in C^1_{ld}(\mathbb{T}, \mathbb{R})$, *then* $\int_a^b f(t) g^\nabla(t)\nabla t = f(t)g(t)|_{t=a}^{t=b} - \int_a^b f^\nabla(t) g(\rho(t))\nabla t$;

7. $\int_a^a f(t)\nabla t = 0$.

Theorem 1.30 (See Theorem 8.48 of [24])**.** *Assume $a, b \in \mathbb{T}$ and $f : \mathbb{T} \longrightarrow \mathbb{R}$ is ld-continuous.*

1. *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t)\nabla t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the Riemann integral.

2. *If $\mathbb{T}$ consists of only isolated points, then*

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{t\in(a,b]} \nu(t)f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum\limits_{t\in(b,a]} \nu(t)f(t), & \text{if } a > b. \end{cases}$$

3. If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then

$$\int_a^b f(t)\nabla t = \begin{cases} \sum\limits_{k=\frac{a+h}{h}}^{\frac{b}{h}} f(kh)h, & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum\limits_{k=\frac{b+h}{h}}^{\frac{a}{h}} f(kh)h, & \text{if } a > b. \end{cases}$$

1.3 Relation between delta and nabla operators

It is possible to relate the approach of Section 1.1 with that of Section 1.2.

Theorem 1.31 (See Theorems 2.5 and 2.6 of [7]). *If $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$ and if f^Δ is continuous on $\mathbb{T}^\kappa$, then f is nabla differentiable on $\mathbb{T}_\kappa$ with*

$$f^\nabla(t) = \left(f^\Delta\right)^\rho(t) \;\; \text{for all } t \in \mathbb{T}_\kappa. \tag{1.4}$$

If $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_\kappa$ and if f^∇ is continuous on $\mathbb{T}_\kappa$, then f is delta differentiable on $\mathbb{T}^\kappa$ with

$$f^\Delta(t) = \left(f^\nabla\right)^\sigma(t) \;\; \text{for all } t \in \mathbb{T}^\kappa. \tag{1.5}$$

Theorem 1.32 (Proposition 7 of [54]). *If function $f : \mathbb{T} \to \mathbb{R}$ is continuous, then for all $a, b \in \mathbb{T}$ with $a < b$ we have*

$$\int_a^b f(t)\Delta t = \int_a^b f^\rho(t)\nabla t, \tag{1.6}$$

$$\int_a^b f(t)\nabla t = \int_a^b f^\sigma(t)\Delta t. \tag{1.7}$$

For a more general theory relating delta and nabla approaches, we refer the reader to the duality theory of Caputo [29].

1.4 Delta dynamic equations

In the beginning of this section we introduce a generalized delta exponential function defined for an arbitrary time scale $\mathbb{T}$. This function is used to solve an initial value problem presented in the further part of this section. The nabla exponential function can be defined analogously [25]. Next we present a second order linear dynamic homogenous equation with constant coefficients and its solution.

Definition 1.33 (Definition 2.25 of [24])**.** *We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive if*

$$1 + \mu(t)p(t) \neq 0$$

for all $t \in \mathbb{T}^\kappa$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 1.34 (Definition 2.30 of [24])**.** *If $p \in \mathcal{R}$, then we define the delta exponential function by*

$$e_p(t,s) := exp\left(\int_s^t \xi_{\mu(\tau)}\left(p(\tau)\right)\Delta\tau\right), \quad s, t \in \mathbb{T},$$

where ξ_μ is the cylinder transformation (see Definition 2.21 of [24]).

Example 1.35 (Section 2.3 of [24])**.** *Let $\mathbb{T}$ be a time scale, $t, t_0 \in \mathbb{T}$, and $\alpha \in \mathcal{R}(\mathbb{T}, \mathbb{R})$ be a constant. If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$ for all $t \in \mathbb{R}$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $\alpha \in \mathbb{C}\setminus\left\{-\frac{1}{h}\right\}$, then*

$$e_\alpha(t, t_0) = (1 + \alpha h)^{\frac{t-t_0}{h}} \text{ for all } t \in h\mathbb{Z}. \tag{1.8}$$

If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then for all $t \in q^{\mathbb{N}_0}$

$$e_\alpha(t, t_0) = \prod_{s \in [t_0, t)} [1 + (q-1)\alpha s], \text{ if } t > t_0. \tag{1.9}$$

If $\mathbb{T} = \left\{\sum\limits_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N}\right\}$, then $e_\alpha(t, t_0) = \frac{(n+\alpha)^{(n-n_0)}}{n^{(n-n_0)}}$ if $t = \sum\limits_{k=1}^{n} \frac{1}{k}$.

The delta exponential function has the following properties.

Theorem 1.36 (Theorem 2.36 of [24])**.** *Let $p, q \in \mathcal{R}$ and $r, s, t \in \mathbb{T}$. We define $\ominus p(t) := \frac{-p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}^\kappa$. The following holds:*

1. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$;
4. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
5. $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
6. $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = \frac{-p(t)}{e_p^\sigma(\cdot,s)}$.

Next we study the first order nonhomogeneous linear equation $y^{\Delta}(t) = p(t)y + f(t)$ on a time scale $\mathbb{T}$. This equation with given initial condition $y(t_0) = y_0$ forms a initial value problem. Under some assumptions (e.g., regressivity) and using the exponential function, we can find its unique solution.

Theorem 1.37 (Variation of Constants – see Theorem 2.1 of [25])**.** *Let $p \in \mathcal{R}$, $f \in C_{rd}$, $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then, the unique solution of the initial value problem (IVP)*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0, \tag{1.10}$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Remark 1.38 (See Remark 2.75 of [24])**.** *An alternative form of the solution of the initial value problem* (1.10) *is given by*

$$y(t) = e_p(t, t_0)\left[y_0 + \int_{t_0}^{t} e_p(t_0, \sigma(\tau))f(\tau)\Delta\tau\right].$$

In the following example three initial value problems on specific time scales are presented.

Example 1.39 (Cf. Exercise 2.79 of [24])**.**

1. *IVP: $y^{\Delta} = 2y + t$, $y(0) = 0$, with $\mathbb{T} = \mathbb{R}$.* Solution*: For $\mathbb{T} = \mathbb{R}$ we get the first order differential equation $y'(t) = 2y(t) + t$, where functions p and f (of equation* (1.10)*) are: $p(t) = 2$, $f(t) = t$. In order to write the solution to our problem we need the exponential function $e_p(t, \sigma(\tau))$, which in this case is equal to $e^{2(t-\tau)}$. Hence, the solution to the IVP is given by $y(t) = \int_0^t e^{2(t-\tau)}\tau d\tau = \frac{1}{4}e^{2t} - \frac{1}{2}t - \frac{1}{4}$.*

2. *IVP: $y^{\Delta} = 2y + 3^t$, $y(0) = 0$, with $\mathbb{T} = \mathbb{Z}$.* Solution*: For integers $\mathbb{Z}$ our problem is the first order difference equation $\Delta y(t) = 2y(t) + 3^t$ with functions $p(t) = 2$, $f(t) = 3^t$. In this case the exponential function $e_p(t, \sigma(\tau))$ is equal to $3^{t-\tau-1}$. Hence, we get as solution $y(t) = \sum_{k=0}^{t-1} 3^{t-k-1}3^k = t3^{t-1}$.*

3. *IVP: $y^{\Delta}(t) = p(t)y + e_p(t, t_0)$, $y(t_0) = 0$, with an arbitrary time scale $\mathbb{T}$ and a regressive function p.* Solution: *First we modify the function $e_p(t, \sigma(\tau))$ using the properties of the exponential function gathered in Theorem 1.36: $e_p(t, \sigma(\tau)) = \frac{1}{e_p(\sigma(\tau), t)} =$*

$\frac{1}{(1+\mu(\tau)p(\tau))e_p(\tau,t)}$. *Then the solution to our problem is given by*

$$y(t) = \int_{t_0}^{t} e_p(t,\sigma(\tau))e_p(\tau,t_0)\Delta\tau = \int_{t_0}^{t} \frac{1}{(1+\mu(\tau)p(\tau))e_p(\tau,t)} e_p(\tau,t_0)\Delta\tau$$
$$= \int_{t_0}^{t} \frac{1}{(1+\mu(\tau)p(\tau))} e_p(t,\tau)e_p(\tau,t_0)\Delta\tau = e_p(t,t_0)\int_{t_0}^{t} \frac{1}{1+\mu(\tau)p(\tau)}\Delta\tau$$

Now let us consider the following second-order linear dynamic homogeneous equation with constant coefficients:

$$y^{\Delta\Delta} + \alpha y^{\Delta} + \beta y = 0, \quad \alpha,\beta \in \mathbb{R}, \tag{1.11}$$

on a time scale $\mathbb{T}$. We say that the dynamic equation (1.11) is regressive if $1-\alpha\mu(t)+\beta\mu^2(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, i.e., $\beta\mu - \alpha \in \mathcal{R}$.

Definition 1.40 (Definition 3.5 of [24]). *Let y_1 and y_2 be delta differentiable functions. For them we define the Wronskian $W(y_1,y_2)(t)$ by*

$$W(y_1,y_2)(t) := \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1^{\Delta}(t) & y_2^{\Delta}(t) \end{bmatrix}.$$

We say that two solutions y_1 and y_2 of (1.11) form a fundamental set of solutions (or a fundamental system) for (1.11), provided $W(y_1,y_2)(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.

Theorem 1.41 (Theorem 3.7 of [24]). *If functions y_1 and y_2 form a fundamental system of solutions for (1.11), then $y(t) = \gamma_1 y_1(t) + \gamma_2 y_2(t)$, where γ_1, γ_2 are constants, is a general solution to (1.11), i.e., every function of this form is a solution to (1.11) and every solution of (1.11) is of this form.*

In order to solve equation (1.11) we have to build its characteristic equation

$$\lambda^2 + \alpha\lambda + \beta = 0, \tag{1.12}$$

where $\lambda \in \mathbb{C}$, $1+\lambda\mu(t) \neq 0$, $t \in \mathbb{T}^\kappa$. Then $y(t) = e_\lambda(t,t_0)$ is a solution to (1.11) if and only if λ satisfies (1.12). The solutions λ_1, λ_2 of (1.12) are given by

$$\lambda_1 := \frac{-\alpha - \sqrt{\alpha^2-4\beta}}{2} \text{ and } \lambda_2 := \frac{-\alpha + \sqrt{\alpha^2-4\beta}}{2}. \tag{1.13}$$

Theorem 1.42 (Theorem 3.16 of [24]). *If (1.11) is regressive and $\alpha^2 - 4\beta \neq 0$, then a fundamental system of (1.11) is given by*

$$e_{\lambda_1}(\cdot,t_0) \text{ and } e_{\lambda_2}(\cdot,t_0),$$

where $t_0 \in \mathbb{T}^\kappa$ and λ_1 and λ_2 are given by (1.13).

In order to formulate the fundamental system of (1.11) in case $\alpha^2 - 4\beta < 0$, we have to introduce the trigonometric functions $\cos_p$ and $\sin_p$.

Definition 1.43 (See Definition 3.25 of [24])**.** *If $p \in C_{rd}$ and $\mu p^2 \in \mathcal{R}$, then we define the trigonometric functions $\cos_p$ and $\sin_p$ by*

$$\cos_p = \frac{e_{ip} + e_{-ip}}{2} \quad \text{and} \quad \sin_p = \frac{e_{ip} - e_{-ip}}{2i}.$$

Remark 1.44. *Let us consider an arbitrary time scale $\mathbb{T}$, $p \in C_{rd}$ and $t, t_0 \in \mathbb{T}$. Then Euler's formula $e_{ip}(t,t_0) = \cos_p(t,t_0) + i\sin_p(t,t_0)$ holds. This is easy to show using Definition 1.43 of trigonometric functions. However, the identity*

$$[\sin_p(t,t_0)]^2 + [\cos_p(t,t_0)]^2 = 1 \tag{1.14}$$

is not necessarily true on an arbitrary time scale. When we extend the left-hand side of equation (1.14), we get

$$\begin{aligned}[\sin_p(t,t_0)]^2 + [\cos_p(t,t_0)]^2 &= \left(\frac{e_{ip}(t,t_0) - e_{-ip}(t,t_0)}{2i}\right)^2 + \left(\frac{e_{ip}(t,t_0) + e_{-ip}(t,t_0)}{2}\right)^2 \\ &= e_{ip}(t,t_0)e_{-ip}(t,t_0).\end{aligned}$$

The value of $e_{ip}(t,t_0)e_{-ip}(t,t_0)$ depends on the time scale and it is not necessary to be equal to one. In order to show this we consider the two classical time scales $\mathbb{R}$ and $\mathbb{Z}$, for $t \neq t_0$ (otherwise it is trivial, since $e_p(t,t) \equiv 1$), and to simplify calculations (without loss of generality) we set $p(t) \equiv 2$. For $\mathbb{T} = \mathbb{R}$ we get $e^{2i(t-t_0)}e^{-2i(t-t_0)} = e^0 = 1$. However, for $\mathbb{T} = \mathbb{Z}$ we obtain $(1+2i)^{(t-t_0)}(1-2i)^{(t-t_0)} = (1-(2i)^2)^{(t-t_0)} = (-3)^{(t-t_0)} \neq 1$.

Theorem 1.45 (See Theorem 3.32 of [24])**.** *Suppose that $\alpha^2 - 4\beta < 0$. Define $p = \frac{-\alpha}{2}$ and $q = \frac{\sqrt{4\beta-\alpha^2}}{2}$. If p and $\mu\beta - \alpha$ are regressive, then a fundamental system of (1.11) is given by*

$$\cos_{\frac{q}{1+\mu p}}(\cdot,t_0)e_p(\cdot,t_0) \quad \text{and} \quad \sin_{\frac{q}{1+\mu p}}(\cdot,t_0)e_p(\cdot,t_0),$$

where $t_0 \in \mathbb{T}^\kappa$ and the Wronskian of these solutions is $qe_{\mu\beta-\alpha}(\cdot,t_0)$.

Finally, we consider the case $\alpha^2 - 4\beta = 0$.

Theorem 1.46 (Theorem 3.34 of [24])**.** *Suppose $\alpha^2 - 4\beta = 0$. Define $p = \frac{-\alpha}{2}$. If $p \in \mathcal{R}$, then a fundamental system of (1.11) is given by*

$$e_p(t,t_0) \quad \text{and} \quad e_p(t,t_0)\int_{t_0}^{t} \frac{1}{1+p\mu(\tau)}\Delta\tau,$$

where $t_0 \in \mathbb{T}^\kappa$, and the Wronskian of these solutions is equal to $e_{\alpha-\mu\alpha^2/4}(\cdot,t_0)$.

Chapter 2

Classical Calculus of Variations

The history of the calculus of variations begins with a problem posed by Johann Bernoulli (1696) as a challenge to the mathematical community and in particular to his brother Jacob. The problem that Johann posed is as follows. Two points a and b are given in a vertical plane. It is required to determine the shape of a curve along which a bead slides initially at rest under gravity from one end to the other in minimal time. The endpoints of the curve are specified and absence of friction is assumed.

The problem attracted the attention of a number of a mathematicians including Huygens, L'Hôpital, Leibniz, Newton and the Bernoulli brothers, and later Euler and Lagrange; and was called the brachistochrone problem or the problem of the curve of fastest descent. The solution was provided by Johann and Jacob Bernoulli, Newton, Euler, and Leibniz. All of them reached the same conclusion – that the brachistochrone is not the circular arc (as predicted by Galileo), but a cycloid. Afterwards, a student of Bernoulli, the brilliant mathematician Leonhard Euler, considered the general problem of finding a function extremizing (minimizing or maximizing) an integral

$$\mathcal{L}[y] = \int_a^b L(t, y(t), y'(t))dt \tag{2.1}$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b \tag{2.2}$$

with $y \in C^1([a,b];\mathbb{R}^n)$, $a, b \in \mathbb{R}$, $y_a, y_b \in \mathbb{R}^n$ and $L(t, y, v) \in C^2([a,b] \times \mathbb{R}^n \times \mathbb{R}^n)$. We say that $y_0 \in C^1([a,b];\mathbb{R}^n)$ is a global minimizer (respectively, global maximizer) of the variational problem (2.1)–(2.2) if $\mathcal{L}[y] \geq \mathcal{L}[y_0]$ (respectively, $\mathcal{L}[y] \leq \mathcal{L}[y_0]$) for all $y \in C^1([a,b];\mathbb{R}^n)$ satisfying (2.2).

Definition 2.1 (See, e.g., Definition 1.1 of [94])**.** *The functional $\mathcal{L}$ is said to attain a local minimum (respectively, local maximum) at $\hat{y} \in C^1([a,b];\mathbb{R}^n)$ if there exists $\delta > 0$ such that for* *any* $y \in C^1([a,b];\mathbb{R}^n)$ *with* $||y - \hat{y}|| < \delta$, $y(a) = y_a$ *and* $y(b) = y_b$, *the inequality* $\mathcal{L}[y] \geq \mathcal{L}[\hat{y}]$ *(respectively,* $\mathcal{L}[y] \leq \mathcal{L}[\hat{y}]$*) holds, where*

$$||y|| = \max_{t\in[a,b]} ||y(t) - \hat{y}(t)|| + \max_{t\in[a,b]} ||y'(t) - \hat{y}'(t)||.$$

The following necessary optimality condition, first proved by Euler and called Euler–Lagrange equation, holds:

$$L_y(t, y(t), y'(t)) - \frac{d}{dt} L_v(t, y(t), y'(t)) = 0, \tag{2.3}$$

where L_y and L_v are partial derivatives of the Lagrangian L with respect to its second and third arguments, respectively. Solutions $y(t)$ of (2.3) are called extremals.

The calculus of variations has a long history of interaction with other branches of mathematics such as geometry and differential equations, and with physics, particularly mechanics. More recently, the calculus of variations has found applications in other fields such as economics or electrical engineering.

The next example demonstrates an application of the calculus of variations in economics. We present an economic model related to the tradeoff between inflation and unemployment [84]. The inflation rate, p, affects decisions of the society regarding consumption and saving, and therefore aggregated demand for domestic production, which, in turn, affects the rate of unemployment, u. A relationship between the inflation rate and the rate of unemployment is described by the Phillips curve, the most commonly used term in the analysis of inflation and unemployment [84]. Having a Phillips tradeoff between u and p, what is then the best combination of inflation and unemployment over time? To answer this question, we follow here the formulations presented in [31, 87]. The Phillips tradeoff between u and p is defined by

$$p := -\beta u + \pi, \quad \beta > 0, \tag{2.4}$$

where π is the expected rate of inflation that is captured by the equation

$$\pi' = j(p - \pi), \quad 0 < j \leq 1. \tag{2.5}$$

The government loss function, λ, is specified in the following quadratic form:

$$\lambda = u^2 + \alpha p^2, \tag{2.6}$$

where $\alpha > 0$ is the weight attached to government's distaste for inflation relative to the loss from income deviating from its equilibrium level. Combining (2.4) and (2.5), and substituting the result into (2.6), we obtain that

$$\lambda\left(\pi(t), \pi'(t)\right) = \left(\frac{\pi'(t)}{\beta j}\right)^2 + \alpha\left(\frac{\pi'(t)}{j} + \pi(t)\right)^2,$$

where α, β, and j are real positive parameters that describe the relations between all variables that occur in the model [87]. The problem is to find the optimal path π that minimizes the total social loss over the time interval $[0, T]$. The initial and the terminal values of π, π_0 and π_T, respectively, are given with π_0, $\pi_T > 0$. To express the importance of the present relative to the future, all social losses are discounted to their present values via a positive discount rate δ. The problem is the following:

$$\Lambda_C[\pi] = \int_0^T \lambda(t, \pi(t), \pi'(t)) e^{-\delta t} dt \longrightarrow \min \tag{2.7}$$

subject to given boundary conditions

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T, \tag{2.8}$$

where the Lagrangian is given by

$$\lambda(t, \pi, \upsilon) := \left(\frac{\upsilon}{\beta j}\right)^2 + \alpha \left(\frac{\upsilon}{j} + \pi\right)^2. \tag{2.9}$$

The Euler–Lagrange equation for the continuous model (2.7) has the form

$$\frac{d}{dt} L_v(t, \pi(t), \pi'(t)) = L_y(t, \pi(t), \pi'(t)).$$

Now we present the discrete calculus of variations [63]. Assume that $f(t, y, v)$ is a C^2 function of (y, v) for each fixed $t \in [a, b] \cap \mathbb{Z}$, $a, b \in \mathbb{Z}$. Let

$$\mathcal{D} := \{y : [a, b] \cap \mathbb{Z} \to \mathbb{R}^n : y(a) = y_a, y(b) = y_b\}.$$

We call $\mathcal{D}$ the set of admissible functions. The simplest variational problem is to extremize (maximize or minimize) the finite sum

$$\mathcal{L}[y] = \sum_{t=a}^{b-1} L(t, y(t+1), \Delta y(t)) \longrightarrow \text{extr} \tag{2.10}$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{2.11}$$

where $y \in \mathcal{D}$. We say that $y_0 \in \mathcal{D}$ is a global minimizer (respectively, global maximizer) of the variational problem (2.10)–(2.11) if $\mathcal{L}[y] \geq \mathcal{L}[y_0]$ (respectively, $\mathcal{L}[y] \leq \mathcal{L}[y_0]$) for all $y \in \mathcal{D}$.

Definition 2.2. *We say that $\mathcal{L}$ has a local minimum (respectively, local maximum) at y_0 provided there is a $\delta > 0$ such that $\mathcal{L}[y] \geq \mathcal{L}[y_0]$ (respectively, $\mathcal{L}[y] \leq \mathcal{L}[y_0]$) for all $y \in \mathcal{D}$ with $||y(t) - y_0(t)|| < \delta$, $t \in [a, b] \cap \mathbb{Z}$ and $|| \cdot ||$ the Euclidian norm. If, in addition, $\mathcal{L}[y] > \mathcal{L}[y_0]$ for all $y \neq y_0$ in $\mathcal{D}$ with $||y(t) - y_0(t)|| < \delta$, $t \in [a, b] \cap \mathbb{Z}$, then we say that $\mathcal{L}$ has a proper (strict) local minimum at y_0.*

Next we state a necessary optimality condition for the variational problem (2.10)–(2.11).

Theorem 2.3 (Cf. [3,63]). *If $\hat{y}$ is a local extremizer for the variational problem* (2.10)–(2.11)*, then $\hat{y}$ satisfies the discrete Euler–Lagrange equation*

$$\Delta L_v(t, y(t+1), \Delta y(t)) = L_y(t, y(t+1), \Delta y(t)) \tag{2.12}$$

for $t \in [a, b-1] \cap \mathbb{Z}$.

Remark 2.4. *There are two formulations for discrete-time variational problems. The difference basically lies in the presence of either $y(t+1)$ or $y(t)$ in the data of the problem (in the Lagrangian). The formulation with $y(t+1)$ is commonly used in discrete and time-scale theories. However, the presence of $y(t)$ is traditionally used in the classical discrete optimal control setting. The definitions of admissibility and local minimum (or maximum) for a variational problem*

$$L[y] = \sum_{t=a}^{b-1} L\left(t, y(t), \Delta y(t)\right) \longrightarrow extr \tag{2.13}$$

are similar to those for (2.10)*. Problem* (2.10) *can be transformed into problem* (2.13) *by using relation $y(t+1) = \Delta y(t) + y(t)$.*

Now we consider the discrete-time economic model, presented for the continuous case in (2.7)–(2.8), which describes the tradeoff between inflation and unemployment. The problem is to minimize the discrete functional

$$\Lambda_D[\pi] = \sum_{t=0}^{T-1} \lambda(\pi(t), \Delta\pi(t))(1+\delta)^{-t} \longrightarrow \min \tag{2.14}$$

subject to the boundary conditions (2.8), where the Lagrangian $\lambda(t, \pi, v)$ is given by (2.9) as well. The Euler–Lagrange equation for the discrete model has the form

$$\Delta L_v(t, \pi(t), \Delta\pi(t)) = L_y(t, \pi(t), \Delta\pi(t)).$$

Often in economics, dynamic models are set up in either continuous or discrete time [8,84]. Since the calculus on time scales can be used to model dynamic processes whose time domains are more complex than the set of integers or real numbers, the use of time scales in economics is a flexible and capable modeling technique. Some advantages of using time-scale models in economics will be showed in Chapter 7.

Chapter 3

Calculus of Variations on Time Scales

There are two available approaches to the calculus of variations on time scales. The first one, the delta approach, is widely described in literature (see, e.g., [18, 23–25, 44, 45, 58, 70, 77, 88, 93]). The latter one, the nabla approach, was introduced mainly due to its applications in economics (see, e.g., [6–9]). However, it has been shown that these two types of calculus of variations on time scales are dual [29, 52, 72].

Let $\mathbb{T}$ be a time scale with $a = \min \mathbb{T}$ and $b = \max \mathbb{T}$. For abbreviation, we use $[a, b]$ instead of $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$.

3.1 The delta approach to the calculus of variations

In this section we present the basic information about the delta calculus of variations on time scales. First and second order necessary optimality conditions are formulated (Theorem 3.4 and Theorem 3.6, respectively). Furthermore, transversality conditions are given. Let $\mathbb{T}$ be a given time scale with at least three points, and $a, b \in \mathbb{T}$, $a < b$. Consider the following (so-called shifted[1]) variational problem on the time scale $\mathbb{T}$:

$$\mathcal{L}[y] = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t \longrightarrow \min \tag{3.1}$$

in the class of functions $y \in C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad y_a, y_b \in \mathbb{R}^n, \quad n \in \mathbb{N}. \tag{3.2}$$

[1] A shifted problem of the calculus of variations considers a shifted Lagrangian: the Lagrangian L of functional (3.1) depends on y^σ instead of y.

Definition 3.1. *A function $y \in C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ is said to be an admissible path (function) to problem* (3.1)–(3.2) *if it satisfies the given boundary conditions $y(a) = y_a$ and $y(b) = y_b$.*

In what follows the Lagrangian L is understood as a function $L : \mathbb{T} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, $(t, y, v) \to L(t, y, v)$ and by L_y and L_v we denote the partial derivatives of L with respect to y and v, respectively. Similar notation is used for second order partial derivatives. We assume that $L(t, \cdot, \cdot)$ is differentiable in (y, v); $L(t, \cdot, \cdot)$, $L_y(t, \cdot, \cdot)$ and $L_v(t, \cdot, \cdot)$ are continuous at $\left(y^\sigma(t), y^\Delta(t)\right)$ uniformly at t and rd-continuous at t for any admissible path y. Let us consider the following norm in C^1_{rd}:

$$\|y\|_{C^1_{rd}} = \sup_{t \in [a,b]} \|y(t)\| + \sup_{t \in [a,b]^\kappa} \|y^\triangle(t)\|,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

Definition 3.2. *We say that an admissible function $\hat{y} \in C^1_{rd}(\mathbb{T}; \mathbb{R}^n)$ is a local minimizer (respectively, a local maximizer) to problem* (3.1)–(3.2) *if there exists $\delta > 0$ such that $\mathcal{L}[\hat{y}] \leq \mathcal{L}[y]$ (respectively, $\mathcal{L}[\hat{y}] \geq \mathcal{L}[y]$) for all admissible functions $y \in C^1_{rd}(\mathbb{T}; \mathbb{R}^n)$ satisfying the inequality $||y - \hat{y}||_{C^1_{rd}} < \delta$.*

Local minimizers (or maximizers) to problem (3.1)–(3.2) fulfill the delta differential Euler–Lagrange equation.

Theorem 3.3 (Delta differential Euler–Lagrange equation – see Theorem 4.2 of [18])**.** *If $\hat{y} \in C^1_{rd}(\mathbb{T}; \mathbb{R}^n)$ is a local minimizer to* (3.1)–(3.2)*, then the Euler–Lagrange equation (in the delta differential form)*

$$L_v^\Delta(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = L_y(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) \tag{3.3}$$

holds for $t \in [a, b]^\kappa$.

The next theorem provides a delta integral Euler–Lagrange equation.

Theorem 3.4 (Delta integral Euler–Lagrange equation – see Theorem 1 of [58])**.** *If $\hat{y}(t) \in C^1_{rd}(\mathbb{T}; \mathbb{R}^n)$ is a local minimizer of the variational problem* (3.1)–(3.2)*, then there exists a vector $c \in \mathbb{R}^n$ such that the Euler–Lagrange equation (in the delta integral form)*

$$L_v\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right) = \int_a^t L_y(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau))\Delta\tau + c^T \tag{3.4}$$

holds for $t \in [a, b]^\kappa$.

In the proof of Theorem 3.3 and Theorem 3.4 a time scale version of the Dubois–Reymond lemma is used.

Lemma 3.5 (See [18, 45]). *Let $f \in C_{rd}$, $f : [a, b] \longrightarrow \mathbb{R}^n$. Then*

$$\int_a^b f^T(t)\eta^\Delta(t)\Delta t = 0$$

holds for all $\eta \in C^1_{rd}([a, b], \mathbb{R}^n)$ with $\eta(a) = \eta(b) = 0$ if and only if $f(t) = c$ for all $t \in [a, b]^\kappa$, $c \in \mathbb{R}^n$.

The next theorem contains the second order necessary optimality condition for problem (3.1)–(3.2).

Theorem 3.6 (Legendre condition – see Result 1.3 of [18]). *If $\hat{y} \in C^2_{rd}(\mathbb{T}; \mathbb{R}^n)$ is a local minimizer of the variational problem* (3.1)–(3.2)*, then*

$$A(t) + \mu(t)\left\{C(t) + C^T(t) + \mu(t)B(t) + (\mu(\sigma(t)))^\dagger A(\sigma(t))\right\} \geq 0, \tag{3.5}$$

$t \in [a, b]^{\kappa^2}$, where

$$\begin{aligned} A(t) &= L_{vv}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right), \\ B(t) &= L_{yy}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right), \\ C(t) &= L_{yv}\left(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)\right) \end{aligned}$$

and where $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$.

Remark 3.7. *If* (3.5) *holds with the strict inequality ">", then it is called* the strengthened Legendre condition.

The shifted calculus of variations on time scales was introduced in the pioneering work of Bohner [18]. Since then, it has been developed in several directions, e.g., for problems with double integrals [21], with higher-order delta derivatives [46], with non fixed boundary conditions [58], and many other extensions [38, 59, 60]. However, there are just few papers related to the non shifted calculus of variations on a general time scale [26, 32, 43, 44].[2] In those papers, the non shifted basic variational problem is defined as

$$\mathcal{L}[y] = \int_a^b L(t, y(t), y^\Delta(t))\Delta t \longrightarrow \min \tag{3.6}$$

in the class of functions $y \in C^1_{rd}(\mathbb{T}; \mathbb{R}^n)$ subject to the boundary conditions

$$y(a) = y_a, \qquad y(b) = y_b. \tag{3.7}$$

[2]Non shifted in the sense that Lagrangian L depends on $(t, y(t), y^\Delta(t))$ instead of $(t, y^\sigma(t), y^\Delta(t))$ as in problem (3.1)–(3.2).

Theorem 3.8 (Euler–Lagrange equation for (3.6)–(3.7) – see Theorem 2 of [44])**.** *If $\hat{y} \in C^1_{rd}(\mathbb{T};\mathbb{R}^n)$ is a local minimizer to problem* (3.6)*–*(3.7)*, then $\hat{y}$ satisfies the Euler–Lagrange equation (in delta integral form)*

$$L_v(t, y(t), y^{\Delta}(t)) = \int\limits_a^{\sigma(t)} L_y(\tau, y(\tau), y^{\Delta}(\tau))\Delta\tau + c \tag{3.8}$$

for all $t \in [a,b]^{\kappa}$ and some $c \in \mathbb{R}^n$.

3.2 The nabla approach to the calculus of variations

In this section we consider a problem of the calculus of variations which involves a functional with a nabla derivative and a nabla integral. The motivation to study such variational problems is coming from applications, in particular from economics [6, 9]. Let $\mathbb{T}$ be a given time scale, which has sufficiently many points in order for all calculations to make sense, and let $a, b \in \mathbb{T}$, $a < b$. The problem consists of minimizing or maximizing[3]

$$\mathcal{L}[y] = \int\limits_a^b L(t, y^{\rho}(t), y^{\nabla}(t))\nabla t \tag{3.9}$$

in the class of functions $y \in C^1_{ld}(\mathbb{T};\mathbb{R}^n)$ subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad y_a, y_b \in \mathbb{R}^n, n \in \mathbb{N}. \tag{3.10}$$

Definition 3.9. *A function $y \in C^1_{ld}(\mathbb{T},\mathbb{R}^n)$ is said to be an admissible path (function) to problem* (3.9)*–*(3.10) *if it satisfies the given boundary conditions $y(a) = y_a$ and $y(b) = y_b$.*

In what follows the Lagrangian L is understood as a function $L : \mathbb{T}\times\mathbb{R}^{2n} \longrightarrow \mathbb{R}$, $(t, y, v) \to L(t, y, v)$, and by L_y and L_v we denote the partial derivatives of L with respect to y and v, respectively. Similar notation is used for second order partial derivatives. We assume that $L(t,\cdot,\cdot)$ is differentiable in (y, v); $L(t,\cdot,\cdot)$, $L_y(t,\cdot,\cdot)$ and $L_v(t,\cdot,\cdot)$ are continuous at $\left(y^{\sigma}(t), y^{\Delta}(t)\right)$ uniformly at t and ld-continuous at t for any admissible path y. Let us consider the following norm in C^1_{ld}:

$$\|y\|_{C^1_{ld}} = \sup_{t\in[a,b]} \|y(t)\| + \sup_{t\in[a,b]_{\kappa}} \|y^{\nabla}(t)\|,$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^n$.

[3] In this section we consider the so-called shifted calculus of variations, where Lagrangian L depends on $(t, y^{\rho}(t), y^{\nabla}(t))$ in functional (3.9).

Definition 3.10 (See [4]). *We say that an admissible function $y \in C^1_{ld}(\mathbb{T};\mathbb{R}^n)$ is a local minimizer (respectively, a local maximizer) for the variational problem* (3.9)–(3.10) *if there exists $\delta > 0$ such that $\mathcal{L}[\hat{y}] \leq \mathcal{L}[y]$ (respectively, $\mathcal{L}[\hat{y}] \geq \mathcal{L}[y]$) for all $y \in C^1_{ld}(\mathbb{T};\mathbb{R}^n)$ satisfying the inequality $||y - \hat{y}||_{C^1_{ld}} < \delta$.*

In case of first order necessary optimality condition for the nabla variational problem on time scales, the Euler–Lagrange equation takes the following form.

Theorem 3.11 (Nabla Euler–Lagrange equation – see [88]). *If a function $\hat{y} \in C^1_{ld}(\mathbb{T};\mathbb{R}^n)$ provides a local extremum to the variational problem* (3.9)–(3.10)*, then $\hat{y}$ satisfies the Euler–Lagrange equation (in the nabla differential form)*

$$L_v^{\nabla}(t, y^{\rho}(t), y^{\nabla}(t)) = L_y(t, y^{\rho}(t), y^{\nabla}(t)) \tag{3.11}$$

for all $t \in [a, b]_{\kappa}$.

Now we present the fundamental lemma of the nabla calculus of variations on time scales.

Lemma 3.12 (See [75]). *Let $f \in C_{ld}([a, b], \mathbb{R}^n)$. If*

$$\int_a^b f(t)\eta^{\nabla}(t)\nabla t = 0$$

for all $\eta \in C^1_{ld}([a, b], \mathbb{R}^n)$ with $\eta(a) = \eta(b) = 0$, then $f(t) = c$ for all $t \in [a, b]_{\kappa}$, $c \in \mathbb{R}^n$.

For a good survey on the calculus of variations on time scales, covering both delta and nabla approaches, we refer the reader to [88].

Part II

Original Work

Chapter 4

Inverse Problems of the Calculus of Variations on Arbitrary Time Scales

This chapter is devoted to the inverse problem of the calculus of variations on an arbitrary time scale. First we present a classical approach [90]. The typical inverse problem consists to determine a function (Lagrangian) $F(t, y, y')$ such that y is a solution to the given differential equation

$$y'' - f(t, y, y') = 0 \tag{4.1}$$

if and only if y is a solution to the Euler–Lagrange equation

$$\frac{d}{dt} F_{y'} - F_y = 0. \tag{4.2}$$

In this chapter we consider two inverse problems of the calculus of variations on time scales. To our best knowledge, the inverse problem has not been studied before in the framework of time scales, in contrast with the direct problem, that establishes dynamic equations of Euler–Lagrange type to time-scale variational problems. The classical approach relies on using the chain rule, which is not valid in the general context of time scales [20, 24]. It seems that the absence of a general chain rule on an arbitrary time scale is the main reason explaining the lack of a general theory for the inverse time-scale variational calculus. To begin (Section 4.1) we consider an inverse extremal problem associated with the following fundamental problem of the calculus of variations: to minimize

$$\mathcal{L}[y] = \int\limits_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t \tag{4.3}$$

subject to the boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$ on a given time scale $\mathbb{T}$. The Euler–Lagrange equation and the strengthened Legendre condition are used in order to describe a general form of a variational functional (4.3) that attains an extremum at a given function y_0. In the latter Section 4.2, we introduce a completely different approach to the inverse problem of the calculus of variations, using an integral perspective instead of the classical differential point of view [27,34]. We present a sufficient condition of self-adjointness for an integro-differential equation (Lemma 4.11). Using this property, we prove a necessary condition for an integro-differential equation on an arbitrary time scale $\mathbb{T}$ to be an Euler–Lagrange equation (Theorem 4.12), related to a property of self-adjointness (Definition 4.8) of the equation of variation (Definition 4.9) of the given dynamic integro-differential equation.

4.1 A general form of the Lagrangian

The problem under our consideration is to find a general form of the variational functional

$$\mathcal{L}[y] = \int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right) \Delta t \tag{4.4}$$

subject to the boundary conditions $y(a) = y(b) = 0$, possessing a local minimum at zero, under the Euler–Lagrange and the strengthened Legendre conditions. We assume that $L(t, \cdot, \cdot)$ is a C^2-function with respect to (y, v) uniformly in t, and L, L_y, L_v, $L_{vv} \in C_{rd}$ for any admissible path $y(\cdot)$. Observe that under our assumptions, by Taylor's theorem, we may write L, with the big O notation, in the form

$$L(t, y, v) = P(t, y) + Q(t, y)v + \frac{1}{2}R(t, y, 0)v^2 + O(v^3), \tag{4.5}$$

where

$$\begin{aligned} P(t, y) &= L(t, y, 0), \\ Q(t, y) &= L_v(t, y, 0), \\ R(t, y, 0) &= L_{vv}(t, y, 0). \end{aligned} \tag{4.6}$$

Let $R(t, y, v) = R(t, y, 0) + O(v)$. Then, one can write (4.5) as

$$L(t, y, v) = P(t, y) + Q(t, y)v + \frac{1}{2}R(t, y, v)v^2. \tag{4.7}$$

Now the idea is to find general forms of $P(t, y^\sigma(t))$, $Q(t, y^\sigma(t))$ and $R(t, y^\sigma(t), y^\Delta(t))$ using the Euler–Lagrange and the strengthened Legendre conditions. Note that the Euler–Lagrange equation (3.4) at the null extremal, with notation (4.6), is

$$Q(t, 0) = \int_a^t P_y(\tau, 0)\Delta\tau + C, \tag{4.8}$$

$t \in [a,b]^\kappa$, where $P(t, y^\sigma(t))$ is chosen arbitrarily such that $P(t,\cdot) \in C^2$ with respect to the second variable, uniformly in t, P and P_y are rd-continuous in t for all admissible y. From (4.8) we can write a general form of Q:

$$Q(t, y^\sigma(t)) = C + \int_a^t P_y(\tau, 0)\Delta\tau + q(t, y^\sigma(t)) - q(t, 0), \tag{4.9}$$

where $C \in \mathbb{R}$ and q is an arbitrarily function such that $q(t,\cdot) \in C^2$ with respect to the second variable, uniformly in t, q and q_y are rd-continuous in t for all admissible y. With notation (4.6), the strengthened Legendre condition (3.5) at the null extremal has the form

$$R(t,0,0) + \mu(t)\left\{2Q_y(t,0) + \mu(t)P_{yy}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\} > 0, \tag{4.10}$$

$t \in [a,b]^{\kappa^2}$, where $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Hence, we set

$$R(t,0,0) + \mu(t)\left\{2Q_y(t,0) + \mu(t)P_{yy}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\} = p(t) \tag{4.11}$$

with $p \in C_{rd}([a,b])$, $p(t) > 0$ for all $t \in [a,b]^\kappa$, chosen arbitrary. Note that there exists a unique solution to (4.11) with respect to $R(t,0,0)$. If t is a right-dense point, then $\mu(t) = 0$ and $R(t,0,0) = p(t)$. Otherwise, $\mu(t) \neq 0$, and using Theorem 1.9 with $f(t) = R(t,0,0)$ we modify equation (4.11) into a first order delta dynamic equation, which has a unique solution $R(t,0,0)$ in agreement with Theorem 1.37 (see details in the proof of Corollary 4.4). We derive a general form of R from Legendre's condition (4.10), as a sum of the solution $R(t,0,0)$ of equation (4.11) and function w, which is chosen arbitrarily in such a way that $w(t,\cdot,\cdot) \in C^2$ with respect to the second and the third variables, uniformly in t; w, w_y, w_v and w_{vv} are rd-continuous in t for all admissible y. Concluding: a general form of the integrand L for functional (4.4) follows from (4.7), (4.9) and (4.11), and is given by

$$\begin{aligned}
L\left(t, y^\sigma(t), y^\Delta(t)\right) &= P\left(t, y^\sigma(t)\right)\\
&+ \left(C + \int_a^t P_y(\tau,0)\Delta\tau + q(t, y^\sigma(t)) - q(t,0)\right) y^\Delta(t)\\
&+ \Bigg(p(t) - \mu(t)\left\{2Q_y(t,0) + \mu(t)P_{yy}(t,0) + (\mu^\sigma(t))^\dagger R(\sigma(t),0,0)\right\}\\
&+ w(t, y^\sigma(t), y^\Delta(t)) - w(t,0,0)\Bigg)\frac{(y^\Delta(t))^2}{2}.
\end{aligned} \tag{4.12}$$

We have just proved the following result.

Theorem 4.1. *Let $\mathbb{T}$ be an arbitrary time scale. If functional (4.4) with boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition, then its Lagrangian L takes the form (4.12), where $R(t,0,0)$ is a solution of equation*

(4.11), $C \in \mathbb{R}$, $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$. Functions P, p, q and w are arbitrary functions satisfying:

(i) $P(t,\cdot), q(t,\cdot) \in C^2$ with respect to the second variable uniformly in t; P, P_y, q, q_y are rd-continuous in t for all admissible y; $P_{yy}(\cdot,0)$ is rd-continuous in t; $p \in C_{rd}$ with $p(t) > 0$ for all $t \in [a,b]^\kappa$;

(ii) $w(t,\cdot,\cdot) \in C^2$ with respect to the second and the third variable, uniformly in t; w, w_y, w_v, w_{vv} are rd-continuous in t for all admissible y.

Now we consider the general situation when the variational problem consists in minimizing (4.4) subject to arbitrary boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$, for a certain given function $y_0 \in C^2_{rd}([a,b])$.

Theorem 4.2. *Let $\mathbb{T}$ be an arbitrary time scale. If the variational functional* (4.4) *with boundary conditions $y(a) = y_0(a)$, $y(b) = y_0(b)$, attains a local minimum for a certain given function $y_0(\cdot) \in C^2_{rd}([a,b])$ under the strengthened Legendre condition, then its Lagrangian L has the form*

$$\begin{aligned} &L\left(t, y^\sigma(t), y^\Delta(t)\right) = P\left(t, y^\sigma(t) - y_0^\sigma(t)\right) + \left(y^\Delta(t) - y_0^\Delta(t)\right) \\ &\times \left(C + \int_a^t P_y\left(\tau, -y_0^\sigma(\tau)\right)\Delta\tau + q\left(t, y^\sigma(t) - y_0^\sigma(t)\right) - q\left(t, -y_0^\sigma(t)\right)\right) + \frac{1}{2}\Bigg(p(t) \\ &- \mu(t)\left\{2Q_y(t, -y_0^\sigma(t)) + \mu(t)P_{yy}(t, -y_0^\sigma(t)) + \left(\mu^\sigma(t)\right)^\dagger R(\sigma(t), -y_0^\sigma(t), -y_0^\Delta(t))\right\} \\ &+ w(t, y^\sigma(t) - y_0^\sigma(t), y^\Delta(t) - y_0^\Delta(t)) - w\left(t, -y_0^\sigma(t), -y_0^\Delta(t)\right)\Bigg)\left(y^\Delta(t) - y_0^\Delta(t)\right)^2, \end{aligned}$$

where $R(t,0,0)$ is the solution to equation (4.11), *$C \in \mathbb{R}$, and functions P, p, q, w satisfy conditions (i) and (ii) of Theorem 4.1.*

Proof. The result follows as a corollary of Theorem 4.1. In order to reduce the problem to the case of the zero extremal considered above, it suffices to introduce the auxiliar variational functional

$$\begin{aligned} \tilde{\mathcal{L}}[y] := \mathcal{L}[y + y_0] &= \int_a^b L\left(t, y^\sigma(t) + y_0^\sigma(t), y^\Delta(t) + y_0^\Delta(t)\right)\Delta t \\ &=: \int_a^b \tilde{L}\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t \end{aligned}$$

subject to boundary conditions $y(a) = 0$ and $y(b) = 0$. The result follows by application of Theorem 4.1 to the auxiliar Lagrangian $\tilde{L}$. □

For the classical situation $\mathbb{T} = \mathbb{R}$, Theorem 4.2 gives a recent result of [81].

Corollary 4.3 (Theorem 4 of [81]). *If the variational functional*

$$\mathcal{L}[y] = \int_a^b L(t, y(t), y'(t))dt$$

attains a local minimum at $y_0(\cdot) \in C^2[a, b]$ satisfying boundary conditions $y(a) = y_0(a)$ and $y(b) = y_0(b)$ and the classical strengthened Legendre condition $R(t, y_0(t), y_0'(t)) > 0$, $t \in [a, b]$, then its Lagrangian L has the form

$$\begin{aligned} L(t, y(t), y'(t)) &= P(t, y(t) - y_0(t)) \\ &+ (y'(t) - y_0'(t)) \left(C + \int_a^t P_y(\tau, -y_0(\tau))d\tau + q(t, y(t) - y_0(t)) - q(t, -y_0(t)) \right) \\ &+ \frac{1}{2} \left(p(t) + w(t, y(t) - y_0(t), y'(t) - y_0'(t)) - w(t, -y_0(t), -y_0'(t))\right) (y'(t) - y_0'(t))^2, \end{aligned}$$

where $C \in \mathbb{R}$.

Proof. Follows from Theorem 4.2 with $\mathbb{T} = \mathbb{R}$. □

Theorem 4.2 seems to be new for any time scale other than $\mathbb{T} = \mathbb{R}$. In the particular case of an isolated time scale, where $\mu(t) \neq 0$ for all $t \in \mathbb{T}$, we get the following corollary.

Corollary 4.4. *Let $\mathbb{T}$ be an isolated time scale. If functional* (4.4) *subject to the boundary conditions $y(a) = y(b) = 0$ attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition, then the Lagrangian L has the form*

$$\begin{aligned} L\left(t, y^\sigma(t), y^\Delta(t)\right) &= P\left(t, y^\sigma(t)\right) \\ &+ \left(C + \int_a^t P_y(\tau, 0)\Delta\tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t) \\ &+ \left(e_r(t, a)R_0 + \int_a^t e_r(t, \sigma(\tau))s(\tau)\Delta\tau + w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{(y^\Delta(t))^2}{2}, \end{aligned} \tag{4.13}$$

where $C, R_0 \in \mathbb{R}$ and $r(t)$ and $s(t)$ are given by

$$r(t) := -\frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger}, \quad s(t) := \frac{p(t) - \mu(t)[2Q_y(t, 0) + \mu(t)P_{yy}(t, 0)]}{\mu^2(t)(\mu^\sigma(t))^\dagger}, \tag{4.14}$$

with $\alpha^\dagger = \frac{1}{\alpha}$ if $\alpha \in \mathbb{R} \setminus \{0\}$ and $0^\dagger = 0$, and functions P, p, q, w satisfy assumptions of Theorem 4.1.

Proof. In the case of an isolated time scale $\mathbb{T}$, we may obtain the form of function Q in the same way as it is done in the proof of Theorem 4.1 (equation (4.9)). We derive a general form for R from Legendre's condition. By relation $f^\sigma = f + \mu f^\Delta$ (Theorem 1.9), one may write equation (4.11) as

$$R(t,0,0) + \mu(t)(\mu^\sigma(t))^\dagger \left(R(t,0,0) + \mu(t)R^\Delta(t,0,0)\right) \\ + \mu(t)\left\{2Q_y(t,0) + \mu(t)P_{yy}(t,0)\right\} - p(t) = 0.$$

Hence,

$$\mu^2(t)(\mu^\sigma(t))^\dagger R^\Delta(t,0,0) + \left[1 + \mu(t)(\mu^\sigma(t))^\dagger\right] R(t,0,0) \\ + \mu(t)[2Q_y(t,0) + \mu(t)P_{yy}(t,0)] - p(t) = 0. \quad (4.15)$$

For an isolated time scale $\mathbb{T}$, equation (4.15) is a first order delta dynamic equation of the following form:

$$R^\Delta(t,0,0) + \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu^2(t)(\mu^\sigma(t))^\dagger} R(t,0,0) + \frac{\mu(t)[2Q_y(t,0) + \mu(t)P_{yy}(t,0)] - p(t)}{\mu^2(t)(\mu^\sigma(t))^\dagger} = 0.$$

With notation (4.14), we have

$$R^\Delta(t,0,0) = r(t)R(t,0,0) + s(t). \quad (4.16)$$

Observe that $r(t)$ is regressive. Indeed, if $\mu(t) \neq 0$, then

$$1 + \mu(t)r(t) = 1 - \frac{1 + \mu(t)(\mu^\sigma(t))^\dagger}{\mu(t)(\mu^\sigma(t))^\dagger} = 1 - \frac{\mu^\sigma(t) + \mu(t)}{\mu(t)} = -\frac{\mu^\sigma(t)}{\mu(t)} \neq 0$$

for all $t \in [a,b]^\kappa$. Therefore, by Theorem 1.37, there is a unique solution to equation (4.16) with initial condition $R(a,0,0) = R_0 \in \mathbb{R}$:

$$R(t,0,0) = e_r(t,a)R_0 + \int_a^t e_r(t,\sigma(\tau))s(\tau)\Delta\tau. \quad (4.17)$$

Thus, a general form of the integrand L for functional (4.4) is given by (4.13). □

Remark 4.5. *Instead of* (4.17)*, we can use an alternative form for the solution of the initial value problem* (4.16) *subject to* $R(a,0,0) = R_0$ *(cf. Remark 1.38):*

$$R(t,0,0) = e_r(t,a)\left[R_0 + \int_a^t e_r(a,\sigma(\tau))s(\tau)\Delta\tau\right].$$

Then the Lagrangian L (4.13) can be written as

$$L\left(t, y^\sigma(t), y^\Delta(t)\right) = P\left(t, y^\sigma(t)\right)$$
$$+ \left(C + \int_a^t P_y(\tau, 0)\Delta\tau + q(t, y^\sigma(t)) - q(t, 0) \right) y^\Delta(t)$$
$$+ \left(e_r(t, a) \left[R_0 + \int_a^t e_r(a, \sigma(\tau))s(\tau)\Delta\tau \right] + w(t, y^\sigma(t), y^\Delta(t)) - w(t, 0, 0) \right) \frac{(y^\Delta(t))^2}{2}.$$

Based on Corollary 4.4, we present the form of Lagrangian L in the periodic time scale $\mathbb{T} = h\mathbb{Z}$.

Example 4.6. *Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and $a, b \in h\mathbb{Z}$ with $a < b$. Then $\mu(t) \equiv h$. We consider the variational functional*

$$\mathcal{L}[y] = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} L\left(kh, y(kh+h), \Delta_h y(kh)\right) \tag{4.18}$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(kh) \equiv 0$ under the strengthened Legendre condition

$$R(kh, 0, 0) + 2hQ_y(kh, 0) + h^2 P_{yy}(kh, 0) + R(kh+h, 0, 0) > 0,$$

$kh \in [a, b-2h] \cap h\mathbb{Z}$. Functions $r(t)$ and $s(t)$ (see (4.14)) have the following form:

$$r(t) = \frac{-2}{h} \in \mathcal{R}, \quad s(t) = \frac{p(t)}{h} - \left(2Q_y(t, 0) + hP_{yy}(t, 0)\right).$$

Hence,

$$\int_a^t P_y(\tau, 0)\Delta\tau = h \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} P_y(ih, 0),$$

$$\int_a^t e_r(t, \sigma(\tau))s(\tau)\Delta\tau = \sum_{i=\frac{a}{h}}^{\frac{t}{h}-1} (-1)^{\frac{t}{h}-i-1} \left(p(ih) - 2hQ_y(ih, 0) - h^2 P_{yy}(ih, 0)\right).$$

Therefore, the Lagrangian L of the variational functional (4.18) on $\mathbb{T} = h\mathbb{Z}$ has the form

$$L\left(kh, y(kh+h), \Delta_h y(kh)\right) = P\left(kh, y(kh+h)\right)$$
$$+ \left(C + \sum_{i=\frac{a}{h}}^{k-1} hP_y(ih, 0) + q(kh, y(kh+h)) - q(kh, 0) \right) \Delta_h y(kh)$$
$$+ \frac{1}{2}\Bigg((-1)^{k-\frac{a}{h}} R_0 + \sum_{i=\frac{a}{h}}^{k-1} (-1)^{k-i-1} \left(p(ih) - 2hQ_y(ih, 0) - h^2 P_{yy}(ih, 0)\right)$$
$$+ w(kh, y(kh+h), \Delta_h y(kh)) - w(kh, 0, 0) \Bigg) \left(\Delta_h y(kh)\right)^2,$$

where functions P, p, q, w are arbitrary but satisfy assumptions of Theorem 4.1.

Now we consider the q-scale $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. In order to present the form of Lagrangian L, we use Remark 4.5.

Example 4.7. *Let $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : q > 1, k \in \mathbb{N}_0\}$ and $a, b \in q^{\mathbb{N}_0}$ with $a < b$. We consider the variational functional*

$$\mathcal{L}[y] = (q-1) \sum_{t \in [a,b) \cap q^{\mathbb{N}_0}} tL\left(t, y(qt), \Delta_q y(t)\right) \tag{4.19}$$

subject to the boundary conditions $y(a) = y(b) = 0$, which attains a local minimum at $\hat{y}(t) \equiv 0$ under the strengthened Legendre condition

$$R(t,0,0) + (q-1)t\{2Q_y(t,0) + (q-1)tP_{yy}(t,0)\} + \frac{1}{q}R(qt,0,0) > 0$$

at the null extremal, $t \in \left[a, \frac{b}{q^2}\right] \cap q^{\mathbb{N}_0}$. Functions given by (4.14) may be written as

$$r(t) = \frac{q+1}{t(1-q)}, \quad s(t) = \frac{qp(t)}{t(q-1)} - 2qQ_y(t,0) - q(q-1)tP_{yy}(t,0).$$

Hence,

$$\int_a^t P_y(\tau,0)\Delta\tau = (q-1) \sum_{\tau \in [a,t) \cap q^{\mathbb{N}_0}} \tau P_y(\tau,0), \quad e_r(t,a) = \prod_{s \in [a,t) \cap q^{\mathbb{N}_0}} (-q),$$

$$\int_a^t e_r(a,\sigma(\tau))s(\tau)\Delta\tau = \sum_{\tau \in [a,t) \cap q^{\mathbb{N}_0}} \frac{(1-q)\tau}{q \prod\limits_{s \in [a,\tau) \cap q^{\mathbb{N}_0}} (-q)} \left[\frac{qp(\tau)}{\tau(q-1)} - 2qQ_y(\tau,0) - q(q-1)\tau P_{yy}(\tau,0)\right].$$

Therefore, the Lagrangian L of the variational functional (4.19) has the form

$$\begin{aligned} L(t, y(qt), \Delta_q y(t)) &= P(t, y(qt)) \\ &+ \left(C + (q-1) \sum_{\tau \in [a,t) \cap q^{\mathbb{N}_0}} \tau P_y(\tau,0) + q(t, y(qt)) - q(t,0)\right) \Delta_q y(t) + \Bigg\{ \prod_{s \in [a,t) \cap q^{\mathbb{N}_0}} (-q) \\ &\times \left[R_0 + \sum_{\tau \in [a,t) \cap q^{\mathbb{N}_0}} \frac{(1-q)\tau}{q \prod\limits_{s \in [a,\tau) \cap q^{\mathbb{N}_0}} (-q)} \left(\frac{qp(\tau)}{\tau(q-1)} - 2qQ_y(\tau,0) - q(q-1)\tau P_{yy}(\tau,0)\right)\right] \\ &+ w\left(t, y(qt), \Delta_q y(t)\right) - w(t,0,0) \Bigg\} \frac{(\Delta_q y(t))^2}{2}, \end{aligned}$$

where functions P, p, r, w are arbitrary but satisfy assumptions of Theorem 4.1.

4.2 Necessary condition for an Euler–Lagrange equation

This section provides a necessary condition for an integro-differential equation on an arbitrary time scale to be an Euler–Lagrange equation (Theorem 4.12). For that the notions of self-adjointness (Definition 4.8) and equation of variation (Definition 4.9) are essential.

Definition 4.8 (First order self-adjoint integro-differential equation)**.** *A first order integro-differential dynamic equation is said to be* self-adjoint *if it has the form*

$$Lu(t) = const, \ \ where\ Lu(t) = p(t)u^{\Delta}(t) + \int\limits_{t_0}^{t} [r(s)u^{\sigma}(s)]\, \Delta s, \tag{4.20}$$

with $p, r \in C_{rd}$, $p \neq 0$ for all $t \in \mathbb{T}$, and $t_0 \in \mathbb{T}$.

Let $\mathbb{D}$ be the set of all functions $y : \mathbb{T} \longrightarrow \mathbb{R}$ such that $y^{\Delta} : \mathbb{T}^{\kappa} \longrightarrow \mathbb{R}$ is continuous. A function $y \in \mathbb{D}$ is said to be a solution of (4.20) provided $Ly(t) = const$ holds for all $t \in \mathbb{T}^{\kappa}$. Along the text we use the operators $[\cdot]$ and $\langle \cdot \rangle$ defined as

$$[y](t) := (t, y^{\sigma}(t), y^{\Delta}(t)), \qquad \langle y \rangle(t) := (t, y^{\sigma}(t), y^{\Delta}(t), y^{\Delta\Delta}(t)), \tag{4.21}$$

and partial derivatives of function $(t, y, v, z) \longrightarrow L(t, y, v, z)$ are denoted by $\partial_2 L = L_y$, $\partial_3 L = L_v$, $\partial_4 L = L_z$.

Definition 4.9 (Equation of variation)**.** *Let*

$$H[y](t) + \int\limits_{t_0}^{t} G[y](s)\Delta s = const \tag{4.22}$$

be an integro-differential equation on time scales with $H_v \neq 0$, $t \longrightarrow F_y[y](t)$, $t \longrightarrow F_v[y](t) \in C_{rd}(\mathbb{T}, \mathbb{R})$ along every curve y, where $F \in \{G, H\}$. The equation of variation *associated with (4.22) is given by*

$$H_y[u](t)u^{\sigma}(t) + H_v[u](t)u^{\Delta}(t) + \int\limits_{t_0}^{t} G_y[u](s)u^{\sigma}(s) + G_v[u](s)u^{\Delta}(s)\Delta s = 0. \tag{4.23}$$

Remark 4.10. *The equation of variation (4.23) can be interpreted in the following way. Assuming $y = y(t, b)$, $b \in \mathbb{R}$, is a one-parameter solution of a given integro-differential equation (4.22), then*

$$H(t, y^{\sigma}(t, b), y^{\Delta}(t, b)) + \int\limits_{t_0}^{t} G(s, y^{\sigma}(s, b), y^{\Delta}(s, b))\Delta s = const. \tag{4.24}$$

Let $u(t)$ be a particular solution, that is, $u(t) = y(t, \bar{b})$ for a certain $\bar{b}$. Differentiating (4.24) with respect to the parameter b, and then putting $b = \bar{b}$, we obtain equation (4.23).

Lemma 4.11 (Sufficient condition of self-adjointness)**.** *Let* (4.22) *be a given integro-differential equation. If*

$$H_y[y](t) + G_v[y](t) = 0, \tag{4.25}$$

then its equation of variation (4.23) *is self-adjoint.*

Proof. Let us consider a given equation of variation (4.23). Using fourth item of Theorem 1.9 and sixth item of Theorem 1.19, we expand the two components of the given equation:

$$H_y[u](t)u^\sigma(t) = H_y[u](t)\left(u(t) + \mu(t)u^\Delta(t)\right),$$

$$\int_{t_0}^{t} G_v[u](s)u^\Delta(s)\Delta s = G_v[u](t)u(t) - G_v[u](t_0)u(t_0) - \int_{t_0}^{t} \left[G_v[u](s)\right]^\Delta u^\sigma(s)\Delta s.$$

Hence, equation of variation (4.23) can be written in the form

$$G_v[u](t_0)u(t_0) = u^\Delta(t)\left[\mu(t)H_y[u](t) + H_v[u](t)\right] + \int_{t_0}^{t} u^\sigma(s)\left[G_y[u](s) - (G_v[u](s))^\Delta\right]\Delta s + u(t)\left(H_y[u](t) + G_v[u](t)\right). \tag{4.26}$$

If (4.25) holds, then (4.26) is a particular case of (4.20) with

$$p(t) = \mu(t)H_y[u](t) + H_v[u](t),$$
$$r(t) = G_y[u](s) - (G_v[u](s))^\Delta,$$
$$G_v[u](t_0)u(t_0) = const.$$

This concludes the proof. □

Now we provide an answer to the general inverse problem of the calculus of variations on time scales.

Theorem 4.12 (Necessary condition for an Euler–Lagrange equation in integral form)**.** *Let* $\mathbb{T}$ *be an arbitrary time scale and*

$$H(t, y^\sigma(t), y^\Delta(t)) + \int_{t_0}^{t} G(s, y^\sigma(s), y^\Delta(s))\Delta s = const \tag{4.27}$$

be a given integro-differential equation. If (4.27) *is to be an Euler–Lagrange equation, then its equation of variation* (4.23) *is self-adjoint, in the sense of Definition 4.8.*

Proof. Assume (4.27) is the Euler–Lagrange equation of the variational functional

$$\mathcal{I}[y] = \int_{t_0}^{t_1} L(t, y^\sigma(t), y^\Delta(t))\Delta t, \tag{4.28}$$

where $L \in C^2$. Since the Euler–Lagrange equation in integral form of (4.28) is given by

$$L_v[y](t) + \int_{t_0}^{t} -L_y[y](s)\Delta s = const$$

(cf. [32, 43, 44]), we conclude that $H[y](t) = L_v[y](t)$ and $G[y](s) = -L_y[y](s)$. Having in mind that

$$\begin{aligned} H_y &= L_{vy}, \quad & H_v &= L_{vv}, \\ G_y &= -L_{yy} \quad & G_v &= -L_{yv}, \end{aligned}$$

it follows from the Schwarz theorem, $L_{vy} = L_{yv}$, that

$$H_y[y](t) + G_v[y](t) = 0.$$

We conclude from Lemma 4.11 that the equation of variation (4.27) is self-adjoint. □

Remark 4.13. *In practical terms, Theorem 4.12 is useful to identify equations which are not Euler–Lagrange: if the equation of variation* (4.23) *of a given dynamic equation* (4.22) *is not self-adjoint, then we conclude that* (4.22) *is not an Euler–Lagrange equation.*

Remark 4.14 (Self-adjointness for a second order differential equation)**.** *Let p be delta differentiable in Definition 4.8 and $u \in C^2_{rd}$. Then, by differentiating* (4.20)*, one obtains a second-order self-adjoint dynamic equation*

$$p^\sigma(t)u^{\Delta\Delta}(t) + p^\Delta(t)u^\Delta(t) + r(t)u^\sigma(t) = 0$$

or

$$p(t)u^{\Delta\Delta}(t) + p^\Delta(t)u^{\Delta\sigma}(t) + r(t)u^\sigma(t) = 0$$

with $r \in C_{rd}$ and $p \in C^1_{rd}$ and $p \neq 0$ for all $t \in \mathbb{T}$.

Now we present an example of a second order differential equation on time scales which is not an Euler–Lagrange equation.

Example 4.15. *Let us consider the following second order dynamic equation on an arbitrary time scale $\mathbb{T}$:*

$$y^{\Delta\Delta}(t) + y^\Delta(t) - t = 0. \tag{4.29}$$

We may write equation (4.29) *in integro-differential form* (4.22)*:*

$$y^{\Delta}(t) + \int_{t_0}^{t} \left(y^{\Delta}(s) - s\right) \Delta s = const, \tag{4.30}$$

where $H[y](t) = y^{\Delta}(t)$ *and* $G[y](t) = y^{\Delta}(t) - t$*. Because*

$$H_y[y](t) = G_y[y](t) = 0, \quad H_v[y](t) = G_v[y](t) = 1,$$

the equation of variation associated with (4.30) *is given by*

$$u^{\Delta}(t) + \int_{t_0}^{t} u^{\Delta}(s)\Delta s = 0 \iff u^{\Delta}(t) + u(t) = u(t_0). \tag{4.31}$$

We may notice that equation (4.31) *cannot be written in form* (4.20)*, hence, it is not self-adjoint. Indeed, notice that* (4.31) *is a first-order dynamic equation while from Remark 4.14 one obtains a second-order dynamic equation. Following Theorem 4.12 (see Remark 4.13) we conclude that equation* (4.29) *is not an Euler–Lagrange equation.*

Now we consider the particular case of Theorem 4.12 when $\mathbb{T} = \mathbb{R}$ and $y \in C^2([t_0, t_1]; \mathbb{R})$. In this case operator $[\cdot]$ of (4.21) has the form

$$[y](t) = (t, y(t), y'(t)) =: [y]_{\mathbb{R}}(t),$$

while condition (4.20) can be written as

$$p(t)u'(t) + \int_{t_0}^{t} r(s)u(s)ds = const. \tag{4.32}$$

Corollary 4.16. *If a given integro-differential equation*

$$H(t, y(t), y'(t)) + \int_{t_0}^{t} G(s, y(s), y'(s))ds = const$$

is to be the Euler–Lagrange equation of the variational problem

$$\mathcal{I}[y] = \int_{t_0}^{t_1} L(t, y(t), y'(t))dt$$

(cf., e.g., [90]), then its equation of variation

$$H_y[u]_{\mathbb{R}}(t)u(t) + H_v[u]_{\mathbb{R}}(t)u'(t) + \int_{t_0}^{t} G_y[u]_{\mathbb{R}}(s)u(s) + G_v[u]_{\mathbb{R}}(s)u'(s)ds = 0$$

must be self-adjoint, in the sense of Definition 4.8 with (4.20) *given by* (4.32)*.*

Proof. Follows from Theorem 4.12 with $\mathbb{T} = \mathbb{R}$. □

Now we consider the particular case of Theorem 4.12 when $\mathbb{T} = h\mathbb{Z}$, $h > 0$. In this case operator $[\cdot]$ of (4.21) has the form

$$[y](t) = (t, y(t+h), \Delta_h y(t)) =: [y]_h(t),$$

where

$$\Delta_h y(t) = \frac{y(t+h) - y(t)}{h}.$$

For $\mathbb{T} = h\mathbb{Z}$, $h > 0$, condition (4.20) can be written as

$$p(t)\Delta_h u(t) + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} hr(kh)u(kh+h) = const. \tag{4.33}$$

Corollary 4.17. *If a given difference equation*

$$H(t, y(t+h), \Delta_h y(t)) + \sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} hG(kh, y(kh+h), \Delta_h y(kh)) = const$$

is to be the Euler–Lagrange equation of the discrete variational problem

$$\mathcal{I}[y] = \sum_{k=\frac{t_0}{h}}^{\frac{t_1}{h}-1} hL(kh, y(kh+h), \Delta_h y(kh))$$

(cf., e.g., [15]), then its equation of variation

$$\begin{aligned} &H_y[u]_h(t)u(t+h) + H_v[u]_h(t)\Delta_h u(t) \\ &\qquad + h\sum_{k=\frac{t_0}{h}}^{\frac{t}{h}-1} \left(G_y[u]_h(kh)u(kh+h) + G_v[u]_h(kh)\Delta_h u(kh)\right) = 0 \end{aligned}$$

is self-adjoint, in the sense of Definition 4.8 with (4.20) *given by* (4.33).

Proof. Follows from Theorem 4.12 with $\mathbb{T} = h\mathbb{Z}$. □

Finally, let us consider the particular case of Theorem 4.12 when $\mathbb{T} = \overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\}$, where $q^{\mathbb{Z}} = \left\{q^k : k \in \mathbb{Z}, q > 1\right\}$. In this case operator $[\cdot]$ of (4.21) has the form

$$[y]_{\overline{q^{\mathbb{Z}}}}(t) = (t, y(qt), \Delta_q y(t)) =: [y]_q(t),$$

where

$$\Delta_q y(t) = \frac{y(qt) - y(t)}{(q-1)t}.$$

For $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, condition (4.20) can be written as (cf., e.g., [85]):

$$p(t)\Delta_q u(t) + (q-1) \sum_{s\in[t_0,t)\cap\mathbb{T}} sr(s)u(qs) = const. \tag{4.34}$$

Corollary 4.18. *If a given q-equation*

$$H(t, y(qt), \Delta_q y(t)) + (q-1) \sum_{s\in[t_0,t)\cap\mathbb{T}} sG(s, y(qs), \Delta_q y(s)) = const,$$

$q > 1$, is to be the Euler–Lagrange equation of the variational problem

$$\mathcal{I}[y] = (q-1) \sum_{t\in[t_0,t_1)\cap\mathbb{T}} tL(t, y(qt), \Delta_q y(t)),$$

$t_0, t_1 \in \overline{q^{\mathbb{Z}}}$, then its equation of variation

$$H_y[u]_q(t)u(qt) + H_v[u]_q(t)\Delta_q u(t) \\ + (q-1) \sum_{s\in[t_0,t)\cap\mathbb{T}} s\left(G_y[u]_q(s)u(qs) + G_v[u]_q(s)\Delta_q u(s)\right) = 0$$

is self-adjoint, in the sense of Definition 4.8 with (4.20) *given by* (4.34).

Proof. Choose $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 4.12. □

More information about the Euler–Lagrange equations for q-variational problems may be found in [44, 74, 78] and references therein.

4.3 Discussion

On an arbitrary time scale $\mathbb{T}$, we can easily show equivalence between the integro-differential equation (4.22) and the second order differential equation (4.35) below (Proposition 4.19). However, when we consider equations of variations of them, we notice that it is not possible to prove an equivalence between them on an arbitrary time scale. The main reason of this impossibility, even in the discrete time scale $\mathbb{Z}$, is the absence of a general chain rule on an arbitrary time scale (see Example 1.85 of [24]). However, on $\mathbb{T} = \mathbb{R}$ we can present this equivalence (Proposition 4.20).

Proposition 4.19. *The integro-differential equation* (4.22) *is equivalent to a second order delta differential equation*

$$W\left(t, y^{\sigma}(t), y^{\Delta}(t), y^{\Delta\Delta}(t)\right) = 0. \tag{4.35}$$

Proof. Let (4.35) be a given second order differential equation. We may write it as a sum of two components

$$W\langle y\rangle(t) = F\langle y\rangle(t) + G[y](t) = 0, \tag{4.36}$$

where operator $\langle\cdot\rangle$ is defined by $\langle y\rangle(t) := (t, y^\sigma(t), y^\Delta(t), y^{\Delta\Delta}(t))$. Let $F\langle y\rangle = H^\Delta[y]$. Then,

$$H^\Delta(t, y^\sigma(t), y^\Delta(t)) + G(t, y^\sigma(t), y^\Delta(t)) = 0, \tag{4.37}$$

where $H(t, y^\sigma(t), y^\Delta(t))$ is of class $C^1_{rd}(\mathbb{T}, \mathbb{R})$ for any admissible path y, $u \in C^1_{rd}(\mathbb{T}, \mathbb{R})$. Integrating both sides of equation (4.37) from t_0 to t, we obtain the integro-differential equation (4.22). □

Let $\mathbb{T}$ be a time scale such that μ is delta differentiable. The equation of variation of a second order differential equation (4.35) is given by

$$W_z\langle u\rangle(t)u^{\Delta\Delta}(t) + W_v\langle u\rangle(t)u^\Delta(t) + W_y\langle u\rangle(t)u^\sigma(t) = 0. \tag{4.38}$$

Equation (4.38) is obtained by using the method presented in Remark 4.10. On an arbitrary time scale it is impossible to prove the equivalence between the equation of variation (4.23) and (4.38). Indeed, after differentiating both sides of equation (4.23) and using the product rule given by Theorem 1.11, we have

$$H_y[u](t)u^{\sigma\Delta}(t) + H_y^\Delta[u](t)u^{\sigma\sigma}(t) + H_v[u](t)u^{\Delta\Delta}(t) + H_v^\Delta[u](t)u^{\Delta\sigma}(t) + G_y[u](t)u^\sigma(t) + G_v[u](t)u^\Delta(t) = 0. \tag{4.39}$$

The direct calculations

- $H_y[u](t)u^{\sigma\Delta}(t) = H_y[u](t)(u^\Delta(t) + \mu^\Delta(t)u^\Delta(t) + \mu^\sigma(t)u^{\Delta\Delta}(t))$,
- $H_y^\Delta[u](t)u^{\sigma\sigma}(t) = H_y^\Delta[u](t)(u^\sigma(t) + \mu^\sigma(t)u^\Delta(t) + \mu(t)\mu^\sigma(t)u^{\Delta\Delta}(t))$,
- $H_v^\Delta[u](t)u^{\Delta\sigma}(t) = H_v^\Delta[u](t)(u^\Delta(t) + \mu u^{\Delta\Delta}(t))$,

allow us to write the equation (4.39) in form

$$\begin{aligned} 0 = &\left[\mu^\sigma(t)H_y[u](t) + \mu(t)\mu^\sigma(t)H_y^\Delta[u](t) + H_v[u](t) + \mu(t)H_v^\Delta[u](t)\right]u^{\Delta\Delta}(t) \\ &+ \left[H_y[u](t) + (\mu(t)H_y[u](t))^\Delta + H_v^\Delta[u](t) + G_v[u](t)\right]u^\Delta(t) + \left[H_y^\Delta[u](t) + G_y[u](t)\right]u^\sigma(t), \end{aligned} \tag{4.40}$$

that is, using fourth item of Theorem 1.9,

$$\begin{aligned} u^{\Delta\Delta}(t)\,[\mu(t)H_y[u](t)+H_v[u](t)]^{\sigma} \\ +u^{\Delta}(t)\left[H_y[u](t)+(\mu(t)H_y[u](t))^{\Delta}+H_v^{\Delta}[u](t)+G_v[u](t)\right] \\ +u^{\sigma}(t)\left[H_y^{\Delta}[u](t)+G_y[u](t)\right]=0. \end{aligned} \quad (4.41)$$

We are not able to prove that the coefficients of equation (4.41) are the same as in (4.38), respectively. This is due to the fact that we cannot find the partial derivatives of (4.35), that is, $W_z\langle u\rangle(t)$, $W_v\langle u\rangle(t)$ and $W_y\langle u\rangle(t)$, from equation (4.37) because of lack of a general chain rule in an arbitrary time scale [20]. The equivalence, however, is true for $\mathbb{T}=\mathbb{R}$. In this case operator $\langle\cdot\rangle$ has the form $\langle y\rangle(t)=(t,y(t),y'(t),y''(t))=:\langle y\rangle_{\mathbb{R}}(t)$.

Proposition 4.20. *The equation of variation*

$$H_y[u]_{\mathbb{R}}(t)u(t)+H_v[u]_{\mathbb{R}}(t)u'(t)+\int_{t_0}^{t}G_y[u]_{\mathbb{R}}(s)u(s)+G_v[u]_{\mathbb{R}}(s)u'(s)ds=0 \quad (4.42)$$

is equivalent to the second order differential equation

$$W_z\langle u\rangle_{\mathbb{R}}(t)u''(t)+W_v\langle u\rangle_{\mathbb{R}}(t)u'(t)+W_y\langle u\rangle_{\mathbb{R}}(t)u(t)=0. \quad (4.43)$$

Proof. We show that coefficients of equations (4.42) and (4.43) are the same, respectively. Let $\mathbb{T}=\mathbb{R}$. From equation (4.36) and relation $F\langle u\rangle_{\mathbb{R}}=\frac{d}{dt}H[u]_{\mathbb{R}}$ we have

$$W(t,u(t),u'(t),u''(t))=\frac{d}{dt}H(t,u(t),u'(t))+G(t,u(t),u'(t)).$$

Using operators $[\cdot]$, $\langle\cdot\rangle$, and chain rule (which is valid for $\mathbb{T}=\mathbb{R}$), we can calculate the following partial derivatives:

- $W_y\langle u\rangle_{\mathbb{R}}(t)=\frac{d}{dt}H_y[u]_{\mathbb{R}}(t)+G_y[u]_{\mathbb{R}}(t)$,
- $W_v\langle u\rangle_{\mathbb{R}}(t)=H_y[u]_{\mathbb{R}}(t)+\frac{d}{dt}H_v[u]_{\mathbb{R}}(t)+G_v[u]_{\mathbb{R}}(t)$,
- $W_z\langle u\rangle_{\mathbb{R}}(t)=H_v[u]_{\mathbb{R}}(t)$.

After differentiation of both sides of (4.42) we obtain

$$\begin{aligned} H_v[u]_{\mathbb{R}}(t)u''(t)+\left(H_y[u]_{\mathbb{R}}(t)+\frac{d}{dt}H_v[u]_{\mathbb{R}}(t)+G_v[u]_{\mathbb{R}}(t)\right)u'(t) \\ +\left(\frac{d}{dt}H_y[u]_{\mathbb{R}}(t)+G_y[u]_{\mathbb{R}}(t)\right)u(t)=0. \end{aligned}$$

Hence, the intended equivalence is proved. □

Proposition 4.20 allows us to obtain the classical result of [34, Theorem II] as a corollary of our Theorem 4.12. The absence of a chain rule on an arbitrary time scale (even for $\mathbb{T} = \mathbb{Z}$) implies that the classical approach [34] fails on time scales. This is the reason why here we introduce a completely different approach to the subject based on the integro-differential form. The case $\mathbb{T} = \mathbb{Z}$ was recently investigated in [27]. However, similarly to [34], the approach of [27] is based on the differential form and cannot be extended to general time scales.

4.4 State of the art

The results of Section 4.1 are published in [36] and were presented by the author at PODE 2013 Progress on Difference Equations, July 21-26, 2013, Bialystok, Poland. The results from Section 4.2 were presented by the author at the XXX EURO mini-Conference on Optimization in the Natural Sciences, February 5-9, 2014, Aveiro, Portugal, in a contributed session entitled "Optimization in Dynamical Systems"; and at the 3rd International Conference on Dynamics, Games and Science, February 17-21, 2014, Porto, Portugal, in an invited session entitled "Dynamic Equations on Time Scales".

Chapter 5

Infinite Horizon Variational Problems on Time Scales

This chapter is devoted to infinite horizon problems of the calculus of variations on time scales. Infinite time horizon models have been considered in macroeconomics very early, see, e.g., the model of economic growth [12] or the Ramsey model [11]. In some cases, their importance is due to the fact that it is hard to predict a natural finite time and the consequences of investment are very long-lived [92]. In case of Ramsey's model, the infinite time horizon is connected with inheritance, which means that finitely lived people care about their offspring or because today's agents care about the value of their asset tomorrow. However, the infinite horizon assumption requires that people have some basic information about possible things that may happen many years from now. Moreover, they should be able to include these contingencies in their planning already today [86]. For infinite horizon variational problems in the discrete-time setting, we refer the reader to [17].

The infinite planning horizon entails, at least, two methodological complications: the convergence of the objective functional and the transversality conditions. In order to deal with the former problem we follow Brock's notion of optimality. Precisely, our optimality criterion (Definition 5.2) for the special case $\mathbb{T} = \mathbb{Z}$ coincides with Brock's notion of weak maximality [28, 68]. If $\mathbb{T} = \mathbb{R}$, then our definition of maximality coincides the extension of Brock's notion of weak maximality to the continuous situation [64, 68]. In this chapter, borrowing an idea from [77], we consider nabla infinite horizon problems of the calculus of variations that depend also on a nabla indefinite integral. We prove the Euler–Lagrange equation and the transversality condition.

5.1 Dubois–Reymond type lemma

In this section a Dubois–Reymond type lemma for infinite horizon nabla variational problems on time scales is presented (Lemma 5.3). This lemma is used in the proof of necessary optimality conditions (Section 5.2). Along this chapter, for simplicity, we use the following operators $\{\cdot\}$ and $\{\cdot,\cdot\}$:

$$\{y\}(t) := (t, y^\rho(t), y^\nabla(t)), \qquad \{y,z\}(t) := (t, y^\rho(t), y^\nabla(t), z(t)). \tag{5.1}$$

We assume that $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$ and $a, T, T' \in \mathbb{T}$ fulfill the inequalities $T > a$ and $T' > a$. Let us consider the following variational problem on $\mathbb{T}$:

$$\mathcal{L}[y] := \int_a^\infty L\{y,z\}(t)\nabla t = \int_a^\infty L\left(t, y^\rho(t), y^\nabla(t), z(t)\right)\nabla t \longrightarrow \max \tag{5.2}$$

subject to $y(a) = y_a$. For the definition of improper integrals on time scales we refer the reader to [69]. The variable z is the integral defined by

$$z(t) := \int_a^t g\{y\}(\tau)\nabla\tau = \int_a^t g\left(\tau, y^\rho(\tau), y^\nabla(\tau)\right)\nabla\tau.$$

We assume that $(t,y,v,w) \longrightarrow L(t,y,v,w)$, $(t,y,v) \longrightarrow g(t,y,v)$ have continuous partial derivatives with respect to y, v, w for all $t \in [a,b]$; $t \longrightarrow L(t, y^\rho(t), y^\nabla(t), z(t))$ belongs to the class $C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$ for any admissible function $y \in C^1_{ld}(\mathbb{T}, \mathbb{R}^n)$; $(y,v,w) \longrightarrow L(t,y,v,w)$ is a $C^1(\mathbb{R}^{3n}, \mathbb{R})$ function for all $t \in \mathbb{T}$; $L_v(t, y^\rho(t), y^\nabla(t)), g_v(t, y^\rho(t), y^\nabla(t)) \in C^1_{rd}(\mathbb{T}, \mathbb{R}^n)$ for any admissible y. By $L_y\{y,z\}(t)$, $L_v\{y,z\}(t)$, $L_z\{y,z\}(t)$ we denote, respectively, the partial derivatives of $L(\cdot,\cdot,\cdot,\cdot)$ with respect to its second, third and fourth argument, $g_y\{y\}(t)$ and $g_v\{y\}(t)$ are, respectively, the partial derivatives of $g(\cdot,\cdot,\cdot)$ with respect to its second and third argument.

Definition 5.1. *We say that y is an admissible path (function) for problem* (5.2) *if $y \in C^1_{ld}(\mathbb{T}; \mathbb{R}^n)$ and $y(a) = y_a$.*

Definition 5.2 (Cf. [28, 68])**.** *We say that $\hat{y}$ is a maximizer to problem* (5.2) *if $\hat{y}$ is an admissible path and, moreover,*

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} \left(L\{y,z\}(t) - L\{\hat{y},\hat{z}\}(t)\right)\nabla t \leq 0$$

for all admissible path y.

Lemma 5.3. *Let $g \in C_{ld}(\mathbb{T}; \mathbb{R})$. Then,*

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = 0$$

for all $\eta \in C_{ld}(\mathbb{T}; \mathbb{R})$ such that $\eta(a) = 0$ if and only if $g(t) = 0$ on $[a, +\infty)$.

Proof. The implication $\Leftarrow$ is obvious. We prove the latter implication $\Rightarrow$ by contradiction. Assume that $g(t) \not\equiv 0$. Let t_0 be a point on $[a, +\infty)$ such that $g(t_0) \neq 0$. Suppose, without loss of generality, that $g(t_0) > 0$. The proof falls naturally into two main parts: t_0 is left-dense (case I) or t_0 is left-scattered (case II). Case I: if t_0 is left-dense, then function g is positive on $[t_1, t_0]$ for $t_1 < t_0$. Define:

$$\eta(t) = \begin{cases} (t_0 - t)(t - t_1) \text{ for } t \in [t_1, t_0], \\ 0 \ \text{ otherwise.} \end{cases}$$

Then,

$$\eta(\rho(t)) = \begin{cases} (t_0 - \rho(t))\,(\rho(t) - t_1) \ \text{ for } \rho(t) \in [t_1, t_0], \\ 0 \ \text{ otherwise.} \end{cases}$$

If $\rho(t) \in [t_1, t_0]$, then $\eta(\rho(t)) = (t_0 - \rho(t))(\rho(t) - t_1) > 0$. Thus,

$$\lim_{T \to +\infty} \inf_{T' \geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_1}^{t_0} g(t)\eta(\rho(t))\nabla t > 0$$

and we obtain a contradiction. Case II: t_0 is left-scattered. Then two situations are possible: $\rho(t_0)$ is left-scattered or $\rho(t_0)$ is left-dense. If $\rho(t_0)$ is left-scattered, then $\rho(\rho(t_0)) < \rho(t_0) < t_0$. Let $t \in [\rho(t_0), t_0]$. Define

$$\eta(t) = \begin{cases} g(t_0) \ \text{ for } t = \rho(t_0), \\ 0 \text{ otherwise.} \end{cases}$$

Then, $\eta(\rho(t_0)) = g(t_0) > 0$ and from Theorem 1.28 we obtain

$$\begin{aligned} \lim_{T \to +\infty} \inf_{T' \geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t &= \int_{\rho(t_0)}^{t_0} g(t)\eta^\rho(t)\nabla t \\ &= g(t_0)\eta(\rho(t_0))\nu(t_0) = g(t_0)g(t_0)(t_0 - \rho(t_0)) > 0, \end{aligned}$$

which is a contradiction. The same conclusion can be drawn when $\rho(t_0)$ is left-dense. Hence, two cases are possible: $g(\rho(t_0)) \neq 0$ or $g(\rho(t_0)) = 0$. If $g(\rho(t_0)) \neq 0$, then we can assume that $g(\rho(t_0)) > 0$ and g is also positive in $[t_2, \rho(t_0)]$ for $t_2 < \rho(t_0)$. Define

$$\eta(t) = \begin{cases} (\rho(t_0) - t)\,(t - t_2) \text{ for } t \in [t_2, \rho(t_0)], \\ 0 \text{ otherwise.} \end{cases}$$

Then,

$$\eta(\rho(t)) = \begin{cases} (\rho(t_0) - \rho(t))\,(\rho(t) - t_2) \text{ for } \rho(t) \in [t_2, \rho(t_0)], \\ 0 \text{ otherwise.} \end{cases}$$

On the interval $[t_2, \rho(t_0)]$ the function $\eta(\rho(t))$ is greater than 0. Then,

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_2}^{\rho(t_0)} g(t)\eta^\rho(t)\nabla t > 0,$$

which is a contradiction. Suppose that $g(\rho(t_0)) = 0$. Here two situations may occur: (i) $g(t) = 0$ on $[t_3, \rho(t_0)]$ for some $t_3 < \rho(t_0)$ or (ii) for all $t_3 < \rho(t_0)$ there exists $t \in [t_3, \rho(t_0)]$ such that $g(t) \neq 0$. In case (i) $t_3 < \rho(t_0) < t_0$. Let us define

$$\eta(t) = \begin{cases} g(t_0) & \text{for } t = \rho(t_0), \\ \varphi(t) & \text{for } t \in [t_3, \rho(t_0)[, \\ 0 & \text{otherwise,} \end{cases}$$

for function φ such that $\varphi \in C_{ld}$, $\varphi(t_3) = 0$ and $\varphi(\rho(t_0)) = g(t_0)$. Then,

$$\eta(\rho(t)) = \begin{cases} g(t_0) & \text{for } \rho(t) = \rho(t_0), \\ \varphi(\rho(t)) & \text{for } \rho(t) \in [t_3, \rho(t_0)), \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 1.28 it follows that

$$\lim_{T\to+\infty} \inf_{T'\geq T} \int_a^{T'} g(t)\eta^\rho(t)\nabla t = \int_{t_3}^{t_0} g(t)\eta^\rho(t)\nabla t = \int_{\rho(t_0)}^{t_0} g(t)\eta^\rho(t)\nabla t$$
$$= \nu(t_0)g(t_0)\eta(\rho(t_0)) = (t_0 - \rho(t_0))g(t_0)\eta(\rho(t_0)) > 0,$$

which is a contradiction. In case (ii), $t_3 < \rho(t_0) < t_0$. When $\rho(t_0)$ is left-dense, then there exists a strictly increasing sequence $S = \{s_k : k \in \mathbb{N}\} \subseteq \mathbb{T}$ such that $\lim_{k\to\infty} s_k = \rho(t_0)$ and $g(s_k) \neq 0$ for all $k \in \mathbb{N}$. If there exists a left-dense s_k, then we have Case I with $t_0 := s_k$. If all points of the sequence S are left-scattered, then we have Case II with $t_0 := s_i$, $i \in \mathbb{N}$. Since $\rho(t_0)$ is a left-scattered point, we are in the first situation of Case II and we obtain a contradiction. Therefore, we conclude that $g \equiv 0$ on $[a, +\infty)$. □

Corollary 5.4. *Let $h \in C^1_{ld}(\mathbb{T}; \mathbb{R})$. Then,*

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} h(t)\eta^\nabla(t)\nabla t = 0 \tag{5.3}$$

for all $\eta \in C^1_{ld}(\mathbb{T}; \mathbb{R})$ such that $\eta(a) = 0$ if and only if $h(t) = c$, $c \in \mathbb{R}$, on $[a, +\infty)$.

Proof. Using integration by parts (sixth item of Theorem 1.29), we obtain

$$\int_a^{T'} h(t)\eta^\nabla(t)\nabla t = h(t)\eta(t)\Big|_{t=a}^{t=T'} - \int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t = h(T')\eta(T') - \int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t$$

for all $\eta \in C^1_{ld}(\mathbb{T};\mathbb{R})$. In particular, it holds for the subclass of η with $\eta(T')=0$. Therefore, (5.3) is equivalent to

$$\lim_{T\to\infty}\inf_{T'\geq T}\int_a^{T'} h^\nabla(t)\eta^\rho(t)\nabla t = 0.$$

From Lemma 5.3 it follows that $h^\nabla(t)=0$, i.e., $h(t)=c$, $c\in\mathbb{R}$, on $[a,+\infty)$. □

5.2 Euler–Lagrange equation and transversality condition

Now we recall a classical theorem of Analysis, which is used in the proof of Theorem 5.6 that provides Euler–Lagrange equations and a transversality condition to problem (5.2).

Theorem 5.5 (See, e.g., [66])**.** *Let S and T be subsets of a normed vector space. Let f be a map defined on $T\times S$, having values in some complete normed vector space. Let v be adherent to S and w adherent to T. Assume that*

1. $\lim_{x\to v} f(t,x)$ *exists for each $t\in T$;*
2. $\lim_{t\to w} f(t,x)$ *exists uniformly for $x\in S$.*

Then, $\lim_{t\to w}\lim_{x\to v} f(t,x)$, $\lim_{x\to v}\lim_{t\to w} f(t,x)$ and $\lim_{(t,x)\to(w,v)} f(t,x)$ all exist and are equal.

Now we are in a position to state the main theorem of this chapter.

Theorem 5.6. *Suppose that a maximizer to problem (5.2) exists and is given by $\hat{y}$. Let $p\in C^1_{ld}(\mathbb{T};\mathbb{R}^n)$ be such that $p(a)=0$. Define*

$$A(\varepsilon,T') := \int_a^{T'} \frac{L\left(t,\hat{y}^\rho(t)+\varepsilon p^\rho(t),\hat{y}^\nabla(t)+\varepsilon p^\nabla(t),\hat{z}(t,p)\right)-L\{\hat{y},\hat{z}\}(t)}{\varepsilon}\nabla t,$$

where

$$\hat{z}(t,p)=\int_a^t g\{\hat{y}+\varepsilon p\}(\tau)\nabla\tau, \qquad \hat{z}(t)=\int_a^t g\{\hat{y}\}(\tau)\nabla\tau,$$

and

$$V(\varepsilon,T) := \inf_{T'\geq T}\varepsilon A(\varepsilon,T'), \qquad V(\varepsilon) := \lim_{T\to\infty} V(\varepsilon,T).$$

Suppose that

1. $\lim_{\varepsilon\to 0}\frac{V(\varepsilon,T)}{\varepsilon}$ *exists for all T;*
2. $\lim_{T\to\infty}\frac{V(\varepsilon,T)}{\varepsilon}$ *exists uniformly for ε;*

3. *for every $T' > a$, $T > a$, $\varepsilon \in \mathbb{R} \setminus \{0\}$, there exists a sequence $(A(\varepsilon, T'_n))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} A(\varepsilon, T'_n) = \inf_{T'\geq T} A(\varepsilon, T')$ uniformly for ε.*

Then, $\hat{y}$ satisfies the Euler–Lagrange system of n equations

$$\lim_{T\to\infty} \inf_{T'\geq T} \left\{ g_y\{y\}(t) \int_{\rho(t)}^{T'} L_z\{y,z\}(\tau)\nabla\tau - \left(g_v\{y\}(t) \int_{\rho(t)}^{T'} L_z\{y,z\}(\tau)\nabla\tau \right)^{\nabla} \right\} + L_y\{y,z\}(t) - L_v^{\nabla}\{y,z\}(t) = 0 \quad (5.4)$$

for all $t \in [a, +\infty)$ and the transversality condition

$$\lim_{T\to\infty} \inf_{T'\geq T} \left(y(T') \cdot \left[L_v\{y,z\}(T') + g_v\{y\}(T')\nu(T')L_z\{y,z\}(T') \right] \right) = 0. \quad (5.5)$$

Proof. If $\hat{y}$ is optimal, in the sense of Definition 5.2, then $V(\varepsilon) \leq 0$ for any $\varepsilon \in \mathbb{R}$. Since $V(0) = 0$, then 0 is a maximizer of V. We prove that V is differentiable at 0, thus $V'(0) = 0$. From Theorem 5.5 and assumptions of Theorem 5.6 it follows that

$$\begin{aligned}
0 = V'(0) &= \lim_{\varepsilon\to 0} \frac{V(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \lim_{T\to\infty} \frac{V(\varepsilon,T)}{\varepsilon} = \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{V(\varepsilon,T)}{\varepsilon} \\
&= \lim_{T\to\infty} \lim_{\varepsilon\to 0} \inf_{T'\geq T} A(\varepsilon, T') = \lim_{T\to\infty} \lim_{\varepsilon\to 0} \lim_{n\to\infty} A(\varepsilon, T'_n) \\
&= \lim_{T\to\infty} \lim_{n\to\infty} \lim_{\varepsilon\to 0} A(\varepsilon, T'_n) = \lim_{T\to\infty} \inf_{T'\geq T} \lim_{\varepsilon\to 0} A(\varepsilon, T') \\
&= \lim_{T\to\infty} \inf_{T'\geq T} \lim_{\varepsilon\to 0} \int_a^{T'} \frac{L\left(t, \hat{y}^{\rho}(t) + \varepsilon p^{\rho}(t), \hat{y}^{\nabla}(t) + \varepsilon p^{\nabla}(t), \hat{z}(t,p)\right) - L\{\hat{y},\hat{z}\}(t)}{\varepsilon} \nabla t \\
&= \lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} \lim_{\varepsilon\to 0} \frac{L\left(t, \hat{y}^{\rho}(t) + \varepsilon p^{\rho}(t), \hat{y}^{\nabla}(t) + \varepsilon p^{\nabla}(t), \hat{z}(t,p)\right) - L\{\hat{y},\hat{z}\}(t)}{\varepsilon} \nabla t.
\end{aligned}$$

Hence,

$$\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} \Bigg[L_y\{\hat{y},\hat{z}\}(t) \cdot p^{\rho}(t) + L_v\{\hat{y},\hat{z}\}(t) \cdot p^{\nabla}(t) + L_z\{\hat{y},\hat{z}\}(t) \int_a^t \left(g_y\{\hat{y}\}(\tau) \cdot p^{\rho}(\tau) + g_v\{\hat{y}\}(\tau) \cdot p^{\nabla}(\tau) \right) \nabla\tau \Bigg] \nabla t = 0. \quad (5.6)$$

Using the integration by parts formula given by point 6 of Theorem 1.29, we obtain:

$$\begin{aligned}
\int_a^{T'} L_v\{\hat{y},\hat{z}\}(t) \cdot p^{\nabla}(t)\nabla t &= L_v\{\hat{y},\hat{z}\}(t) \cdot p(t) \Big|_{t=a}^{t=T'} - \int_a^{T'} L_v^{\nabla}\{\hat{y},\hat{z}\}(t) \cdot p^{\rho}(t)\nabla t \\
&= L_v\{\hat{y},\hat{z}\}(T') \cdot p(T') - \int_a^{T'} L_v^{\nabla}\{\hat{y},\hat{z}\}(t) \cdot p^{\rho}(t)\nabla t.
\end{aligned}$$

Next, we consider the second component of equation (5.6). First we use the third nabla differentiation formula of Theorem 1.23 and obtain

$$\left[\int_t^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\right]^\nabla$$
$$= \left(\int_t^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)^\nabla \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau$$
$$+ \left(\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\left(\int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\right)^\nabla$$
$$= -L_z\{\hat{y},\hat{z}\}(t)\int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau$$
$$+ \left(\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\left(g_y\{\hat{y}\}(t)\cdot p^\rho(t) + g_v\{\hat{y}\}(t)\cdot p^\nabla(t)\right).$$

Integrating both sides from $t = a$ to $t = T'$, yields

$$\int_a^{T'}\left[\int_t^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\right]^\nabla \nabla t$$
$$= -\int_a^{T'}\left[L_z\{\hat{y},\hat{z}\}(t)\int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\right]\nabla t$$
$$+ \int_a^{T'}\left[\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \left(g_y\{\hat{y}\}(t)\cdot p^\rho(t) + g_v\{\hat{y}\}(t)\cdot p^\nabla(t)\right)\right]\nabla t.$$

The left hand side of above equation is equal to zero:

$$\int_a^{T'}\left[\int_t^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\right]^\nabla \nabla t$$
$$= \int_t^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau\Bigg|_{t=a}^{t=T'} = 0,$$

and, therefore,

$$\begin{aligned}
&\int_a^{T'} \left[L_z\{\hat{y},\hat{z}\}(t) \int_a^t \left(g_y\{\hat{y}\}(\tau)\cdot p^\rho(\tau) + g_v\{\hat{y}\}(\tau)\cdot p^\nabla(\tau)\right)\nabla\tau \right]\nabla t \\
&\quad = \int_a^{T'} \left[\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \left(g_y\{\hat{y}\}(t)\cdot p^\rho(t) + g_v\{\hat{y}\}(t)\cdot p^\nabla(t)\right) \right]\nabla t \\
&\quad = \int_a^{T'} \left[g_y\{\hat{y}\}(t)\cdot p^\rho(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \right]\nabla t \\
&\quad + \int_a^{T'} \left[p^\nabla(t)\cdot g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \right]\nabla t.
\end{aligned} \tag{5.7}$$

Using point 6 of Theorem 1.29 and the fact that $p(a)=0$, we have

$$\begin{aligned}
&\int_a^{T'} \left[p^\nabla(t)\cdot g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \right]\nabla t = p(T')\cdot g_v\{\hat{y}\}(T') \int_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \\
&- \int_a^{T'} \left(g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \right)^\nabla \cdot p^\rho(t)\nabla t.
\end{aligned}$$

From (5.6) it follows that

$$\begin{aligned}
&\lim_{T\to\infty} \inf_{T'\geq T} \left\{ \int_a^{T'} L_y\{\hat{y},\hat{z}\}(t)\cdot p^\rho(t)\nabla t + L_v\{\hat{y},\hat{z}\}(T')\cdot p(T') - \int_a^{T'} L_v^\nabla\{\hat{y},\hat{z}\}(t)\cdot p^\rho(t)\nabla t \right. \\
&\left. + \int_a^{T'} \left(\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \left(g_y\{\hat{y}\}(t)\cdot p^\rho(t) + g_v\{\hat{y}\}(t)\cdot p^\nabla(t)\right) \right)\nabla t \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \inf_{T'\geq T} \Bigg\{ \int_a^{T'} L_y\{\hat{y},\hat{z}\}(t)\cdot p^\rho(t)\nabla t + L_v\{\hat{y},\hat{z}\}(T')\cdot p(T') \\
&- \int_a^{T'} \Bigg\{ L_v^\nabla\{\hat{y},\hat{z}\}(t)\cdot p^\rho(t) + \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau g_y\{\hat{y}\}(t)\cdot p^\rho(t) \Bigg\}\nabla t \\
&+ g_v\{\hat{y}\}(T') \int_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\cdot p(T') \\
&- \int_a^{T'} \Bigg(g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \Bigg)^\nabla \cdot p^\rho(t)\nabla t \Bigg\} \qquad (5.8) \\
&= \lim_{T\to\infty} \inf_{T'\geq T} \Bigg\{ \int_a^{T'} p^\rho(t)\cdot \Bigg[L_y\{\hat{y},\hat{z}\}(t) - L_v^\nabla\{\hat{y},\hat{z}\}(t) \\
&+ g_y\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau - \Bigg(g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \Bigg)^\nabla \Bigg]\nabla t \\
&+ L_v\{\hat{y},\hat{z}\}(T')\cdot p(T') + \Bigg(g_v\{\hat{y}\}(T') \int_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \Bigg)\cdot p(T') \Bigg\} = 0.
\end{aligned}
$$

The equation (5.8) holds for all $p \in C_{ld}^1$ such that $p(a) = 0$. Then, in particular, it also holds for the subclass of p with $p(T') = 0$. Therefore,

$$
\begin{aligned}
\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} p^\rho(t)\cdot \Bigg[L_y\{\hat{y},\hat{z}\}(t) - L_v^\nabla\{\hat{y},\hat{z}\}(t) + g_y\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \\
- \Bigg(g_v\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \Bigg)^\nabla \Bigg]\nabla t = 0.
\end{aligned}
$$

Choosing $p = (p_1, \ldots, p_n)$ such that $p_2 \equiv \cdots \equiv p_n \equiv 0$, yields

$$
\begin{aligned}
\lim_{T\to\infty} \inf_{T'\geq T} \int_a^{T'} p_1^\rho(t) \Bigg[L_{y_1}\{\hat{y},\hat{z}\}(t) + g_{y_1}\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \\
- L_{v_1}^\nabla\{\hat{y},\hat{z}\}(t) - \Bigg(g_{v_1}\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau \Bigg)^\nabla \Bigg]\nabla t = 0.
\end{aligned}
$$

From Lemma 5.3 it follows that

$$g_{y_1}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau - \left(g_{v_1}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)^{\nabla} + L_{y_1}\{\hat{y},\hat{z}\}(t) - L_{v_1}^{\nabla}\{\hat{y},\hat{z}\}(t) = 0$$

holds for all $t \in [a,+\infty)$ and all $T' \geq t$. The same procedure may be done for other coordinates. For all $i = 1,\ldots,n$ we obtain the equation

$$g_{y_i}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau - \left(g_{v_i}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)^{\nabla} + L_{y_i}\{\hat{y},\hat{z}\}(t) - L_{v_i}^{\nabla}\{\hat{y},\hat{z}\}(t) = 0$$

for all $t \in [a,+\infty)$ and all $T' \geq t$. These n conditions can be written in vector form as

$$g_{y}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau - \left(g_{v}\{\hat{y}\}(t)\int\limits_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)^{\nabla} + L_{y}\{\hat{y},\hat{z}\}(t) - L_{v}^{\nabla}\{\hat{y},\hat{z}\}(t) = 0 \quad (5.9)$$

for all $t \in [a,+\infty)$ and all $T' \geq t$, which implies the Euler–Lagrange system of n equations (5.4). From equation (5.8) and the system of equations (5.9), we conclude that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\left(L_v\{\hat{y},\hat{z}\}(T') + g_v\{\hat{y}\}(T')\int\limits_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\cdot p(T')\right\} = 0. \quad (5.10)$$

Next, we define a special curve p: for all $t \in [a,\infty)$

$$p(t) = \alpha(t)\hat{y}(t)\,, \quad (5.11)$$

where $\alpha : [a,\infty) \to \mathbb{R}$ is a C_{ld}^1 function satisfying $\alpha(a) = 0$ and for which there exists $T_0 \in \mathbb{T}$ such that $\alpha(t) = \beta \in \mathbb{R}\setminus\{0\}$ for all $t > T_0$. Substituting $p(T') = \alpha(T')\hat{y}(T')$ into (5.10), we conclude that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{L_v\{\hat{y},\hat{z}\}(T')\cdot\beta\hat{y}(T') + g_v\{\hat{y}\}(T')\int\limits_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\cdot\beta\hat{y}(T')\right\}$$

vanishes and, therefore,

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\hat{y}(T')\cdot\left[L_v\{\hat{y},\hat{z}\}(T') + g_v\{\hat{y}\}(T')\int\limits_{\rho(T')}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right]\right\} = 0.$$

From Theorem 1.28 it follows that $\hat{y}$ satisfies the transversality condition (5.5). □

In contrast to Theorem 5.6, the following theorem is proved by manipulating equation (5.6) differently: using integration by parts and nabla differentiation formulas, the composition p^ρ is transformed into the nabla derivative p^∇. Therefore, we apply Corollary 5.4 instead of Lemma 5.3 in order to obtain the intended conclusions.

Theorem 5.7. *Under assumptions of Theorem 5.6, the Euler–Lagrange system of* n *equations*

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\int_t^{T'} g_y\{\hat{y}\}(\tau)\int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\nabla\tau + g_v\{\hat{y}\}(t)\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\} + L_v\{\hat{y},\hat{z}\}(t) - \int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau = c \quad (5.12)$$

holds for all $t\in[a,\infty)$, $c\in\mathbb{R}^n$, *together with the transversality condition*

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\hat{y}(T')\cdot\int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\} = 0. \quad (5.13)$$

Proof. Our proof starts with the necessary optimality condition (5.6) computed in the proof of Theorem 5.6. Using point 3 of Theorem 1.23, we have

$$\begin{aligned}\left[p(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right]^\nabla &= p^\nabla(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau + p^\rho(t)\cdot\left[\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right]^\nabla\\ &= p^\nabla(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau + p^\rho(t)\cdot L_y\{\hat{y},\hat{z}\}(t).\end{aligned}$$

Then, integrating both sides from $t=a$ to $t=T'$, yields

$$\int_a^{T'}\left[p(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right]^\nabla\nabla t = \int_a^{T'}\left(p^\nabla(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\nabla t + \int_a^{T'} p^\rho(t)\cdot L_y\{\hat{y},\hat{z}\}(t)\nabla t.$$

Therefore,

$$p(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\Bigg|_{t=a}^{t=T'} = \int_a^{T'}\left(p^\nabla(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\nabla t + \int_a^{T'} p^\rho(t)\cdot L_y\{\hat{y},\hat{z}\}(t)\nabla t.$$

Since $p(a) = 0$, we have

$$\int_a^{T'} p^\rho(t) \cdot L_y\{\hat{y},\hat{z}\}(t)\nabla t$$
$$= -\int_a^{T'} p^\nabla(t) \cdot \left(\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\nabla t + p(T') \cdot \int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau.$$

We obtain (5.7) in the same manner as in the proof of Theorem 5.6. Using again point 3 of Theorem 1.23, we have

$$\left[p(t)\cdot \int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\right]^\nabla$$
$$= p^\nabla(t) \cdot \int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau$$
$$+ p^\rho(t) \cdot \left[\int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\right]^\nabla$$
$$= p^\nabla(t) \cdot \int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau - p^\rho(t)\cdot g_y\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau.$$

Integrating both sides from $t = a$ to $t = T'$ and using point 7 of Theorem 1.29 and condition $p(a) = 0$, we have

$$\int_a^{T'} \left[p(t) \cdot \int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\right]^\nabla \nabla t$$
$$= p(t) \cdot \int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau \Bigg|_{t=a}^{t=T'} = 0.$$

Then,

$$\int_a^{T'} p^\rho(t) \cdot \left(g_y\{\hat{y}\}(t) \int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla t$$
$$= \int_a^{T'} p^\nabla(t) \cdot \left[\int_t^{T'} \left(g_y\{\hat{y}\}(\tau) \int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\right]\nabla t.$$

From (5.7) and previous relations, we write (5.6) in the following way:

$$\begin{aligned}
&\lim_{T\to\infty}\inf_{T'\geq T}\left\{-\int_a^{T'} p^{\nabla}(t)\cdot\int_a^t L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\nabla t + p(T')\cdot\int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right.\\
&+\int_a^{T'} L_v\{\hat{y},\hat{z}\}(t)\cdot p^{\nabla}(t)\nabla t + \int_a^{T'} p^{\nabla}(t)\cdot\int_t^{T'}\left(g_y\{\hat{y}\}(\tau)\int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\nabla t\\
&\left.+\int_a^{T'} g_v\{\hat{y}\}(t)\cdot\left(p^{\nabla}(t)\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right)\nabla t\right\}\\
&=\lim_{T\to\infty}\inf_{T'\geq T}\left\{\int_a^{T'} p^{\nabla}(t)\cdot\left[\int_a^t -L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau + L_v\{\hat{y},\hat{z}\}(t)\right.\right.\\
&+\int_t^{T'}\left(g_y\{\hat{y}\}(\tau)\int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau\\
&\left.\left.+g_v\{\hat{y}\}(t)\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right]\nabla t + p(T')\cdot\int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\} = 0.
\end{aligned}\tag{5.14}$$

Since (5.14) holds for all $p \in C_{ld}^1$ with $p(a) = 0$, in particular it also holds in the subclass of functions $p \in C_{ld}^1$ with $p(a) = p(T') = 0$. Let $i \in \{1, \dots, n\}$. Choosing $p = (p_1, \dots, p_n)$ such that all $p_j \equiv 0$, $j \neq i$, and $p_i \in C_{ld}^1$ with $p_i(a) = p_i(T') = 0$, we conclude that

$$\begin{aligned}
\lim_{T\to\infty}\inf_{T'\geq T}\int_a^{T'} p_i^{\nabla}(t)&\left\{\int_a^t -L_{y_i}\{\hat{y},\hat{z}\}(\tau)\nabla\tau + L_{v_i}\{\hat{y},\hat{z}\}(t)\right.\\
&\left.+\int_t^{T'} g_{y_i}\{\hat{y}\}(\tau)\int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\nabla\tau + g_{v_i}\{\hat{y}\}(t)\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\}\nabla t = 0.
\end{aligned}$$

From Corollary 5.4 it follows that

$$\begin{aligned}
L_{v_i}\{\hat{y},\hat{z}\}(t) - \int_a^t L_{y_i}\{\hat{y},\hat{z}\}(\tau)\nabla\tau + \int_t^{T'}\left(g_{y_i}\{\hat{y}\}(\tau)\int_{\rho(\tau)}^{T'} L_z\{\hat{y},\hat{z}\}(s)\nabla s\right)\nabla\tau&\\
+ g_{v_i}\{\hat{y}\}(t)\int_{\rho(t)}^{T'} L_z\{\hat{y},\hat{z}\}(\tau)\nabla\tau = c_i,&
\end{aligned}\tag{5.15}$$

$c_i \in \mathbb{R}$, $i = 1, \dots, n$, for all $t \in [a, +\infty)$ and all $T' \geq t$. These n conditions imply the

Euler–Lagrange system of equations (5.12). From (5.14) and (5.15), we conclude that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{p(T')\cdot\int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\}=0. \tag{5.16}$$

Using the special curve p defined by (5.11) and equation (5.16), we obtain that

$$\lim_{T\to\infty}\inf_{T'\geq T}\left\{\beta\hat{y}(T')\cdot\int_a^{T'} L_y\{\hat{y},\hat{z}\}(\tau)\nabla\tau\right\}=0.$$

Therefore, $\hat{y}$ satisfies the transversality condition (5.13). □

5.3 State of the art

The results of this chapter are published in [37].

Chapter 6

The Delta-nabla Calculus of Variations for Composition Functionals

This chapter is devoted to problems of the time-scale calculus of variations for a functional that is the composition of a certain scalar function with the delta and nabla integrals of a vector valued field. We begin by proving general Euler–Lagrange equations in integral form (Theorem 6.3). Then we consider cases when initial or terminal boundary conditions are not specified, obtaining corresponding transversality conditions (Theorems 6.5 and 6.6). Furthermore, we prove necessary optimality conditions for general isoperimetric problems given by the composition of delta-nabla integrals (Theorem 6.10). Finally, some illustrating examples are presented (Section 6.4).

6.1 The Euler–Lagrange equations

This section starts with the definition of the class of functions $C^1_{k,n}([a,b];\mathbb{R})$, which contains delta and nabla differentiable functions. Next, a necessary optimality condition (in integral form) and an illustrative example for an irregular time scale are provided.

Definition 6.1. *By $C^1_{k,n}([a,b];\mathbb{R})$, $k,n \in \mathbb{N}$, we denote the class of functions $y : [a,b] \to \mathbb{R}$ such that: if $k \neq 0$ and $n \neq 0$, then y^Δ is continuous on $[a,b]^\kappa_\kappa$ and y^∇ is continuous on $[a,b]^\kappa_\kappa$, where $[a,b]^\kappa_\kappa := [a,b]^\kappa \cap [a,b]_\kappa$; if $n = 0$, then y^Δ is continuous on $[a,b]^\kappa$; if $k = 0$, then y^∇ is continuous on $[a,b]_\kappa$.*

We consider the following variational problem:

$$\mathcal{L}[y] = H\left(\int_a^b f_1(t, y^\sigma(t), y^\Delta(t))\Delta t, \ldots, \int_a^b f_k(t, y^\sigma(t), y^\Delta(t))\Delta t, \right.$$
$$\left. \int_a^b f_{k+1}(t, y^\rho(t), y^\nabla(t))\nabla t, \ldots, \int_a^b f_{k+n}(t, y^\rho(t), y^\nabla(t))\nabla t\right) \longrightarrow \text{extr}, \quad (6.1)$$

$$(y(a) = y_a), \quad (y(b) = y_b), \qquad (6.2)$$

in the class of functions $y \in C^1_{k,n}$, where "extr" means "minimize" or "maximize". The parentheses in (6.2), around the end-point conditions, means that those conditions may or may not occur (it is possible that both $y(a)$ and $y(b)$ are free). A function $y \in C^1_{k,n}$ is said to be admissible provided it satisfies the boundary conditions (6.2) (if any is given). For $k = 0$ problem (6.1) becomes a nabla problem (neither delta integral nor delta derivative is present); for $n = 0$ problem (6.1) reduces to a delta problem (neither nabla integral nor nabla derivative is present). For simplicity, along the text we introduce the operators $[\cdot]$ and $\{\cdot\}$ by

$$[y](t) := (t, y^\sigma(t), y^\Delta(t)), \quad \{y\}(t) := (t, y^\rho(t), y^\nabla(t)). \qquad (6.3)$$

Along the chapter, c denotes constants that are generic and may change at each occurrence. We assume that:

1. the function $H : \mathbb{R}^{n+k} \to \mathbb{R}$ has continuous partial derivatives with respect to its arguments, which we denote by $H_i^{'}$, $i = 1, \ldots, n+k$;
2. functions $(t, y, v) \to f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, $i = 1, \ldots, n+k$, have continuous partial derivatives with respect to y and v uniformly in $t \in [a, b]$, which we denote by f_{iy} and f_{iv};
3. f_i, f_{iy}, f_{iv} are rd-continuous on $[a, b]^\kappa$, $i = 1, \ldots, k$, and ld-continuous on $[a, b]_\kappa$, $i = k+1, \ldots, k+n$, for all $y \in C^1_{k,n}$.

Definition 6.2 (Cf. [70]). *We say that an admissible function $\hat{y} \in C^1_{k,n}([a, b]; \mathbb{R})$ is a local minimizer (respectively, local maximizer) to the problem* (6.1)–(6.2)*, if there exists $\delta > 0$ such that $\mathcal{L}[\hat{y}] \leq \mathcal{L}[y]$ (respectively, $\mathcal{L}[\hat{y}] \geq \mathcal{L}[y]$) for all admissible functions $y \in C^1_{k,n}([a, b]; \mathbb{R})$ satisfying the inequality $||y - \hat{y}||_{1,\infty} < \delta$, where*

$$||y||_{1,\infty} := ||y^\sigma||_\infty + ||y^\Delta||_\infty + ||y^\rho||_\infty + ||y^\nabla||_\infty \qquad (6.4)$$

with $||y||_\infty := \sup_{t \in [a,b]^\kappa_\kappa} |y(t)|$.

Depending on the given boundary conditions, we can distinguish four different problems. The first one is the problem (P_{ab}), where the two boundary conditions are specified. To solve this problem we need an Euler–Lagrange necessary optimality condition, which is given by Theorem 6.3 below. Next two problems — denoted by (P_a) and (P_b) — occur when $y(a)$ is given and $y(b)$ is free (problem (P_a)) and when $y(a)$ is free and $y(b)$ is specified (problem (P_b)). To solve both of them we need an Euler–Lagrange equation and one proper transversality condition. The last problem — denoted by (P) — occurs when both boundary conditions are not present. To find a solution for such a problem we need to use an Euler–Lagrange equation and two transversality conditions (one at each time a and b).

For brevity, in what follows we omit the arguments of $H_i^{'}$. Precisely,

$$H_i^{'} := \frac{\partial H}{\partial \mathcal{F}_i}(\mathcal{F}_1(y), \ldots, \mathcal{F}_{k+n}(y)),$$

$i = 1, \ldots, n+k$, where

$$\mathcal{F}_i(y) = \int_a^b f_i[y](t)\Delta t, \text{ for } i = 1, \ldots, k, \quad \mathcal{F}_i(y) = \int_a^b f_i\{y\}(t)\nabla t, \text{ for } i = k+1, \ldots, k+n.$$

Theorem 6.3 (The Euler–Lagrange equations in integral form)**.** *If $\hat{y}$ is a local solution to problem* (6.1)–(6.2)*, then the Euler–Lagrange equations (in integral form)*

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iv}[\hat{y}](\rho(t)) - \int_a^{\rho(t)} f_{iy}[\hat{y}](\tau)\Delta\tau \right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iv}\{\hat{y}\}(t) - \int_a^{t} f_{iy}\{\hat{y}\}(\tau)\nabla\tau \right) = c, \quad t \in \mathbb{T}_\kappa, \quad (6.5)$$

and

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iv}[\hat{y}](t) - \int_a^{t} f_{iy}[\hat{y}](\tau)\Delta\tau \right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iv}\{\hat{y}\}(\sigma(t)) - \int_a^{\sigma(t)} f_{iy}\{\hat{y}\}(\tau)\nabla\tau \right) = c, \quad t \in \mathbb{T}^\kappa, \quad (6.6)$$

hold.

Proof. Suppose that $\mathcal{L}[y]$ has a local extremum at $\hat{y}$. Consider a variation $h \in C_{k,n}^1$ of $\hat{y}$ for which we define the function $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\hat{y} + \varepsilon h]$. A necessary condition for $\hat{y}$ to be an extremizer for $\mathcal{L}[y]$ is given by $\phi'(\varepsilon) = 0$ for $\varepsilon = 0$. Using the chain rule, we obtain that

$$0 = \phi'(0) = \sum_{i=1}^{k} H_i' \cdot \int_a^b \left(f_{iy}[\hat{y}](t) h^{\sigma}(t) + f_{iv}[\hat{y}](t) h^{\Delta}(t) \right) \Delta t + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^b \left(f_{iy}\{\hat{y}\}(t) h^{\rho}(t) + f_{iv}\{\hat{y}\}(t) h^{\nabla}(t) \right) \nabla t.$$

Integration by parts of the first terms of both integrals gives

$$\int_a^b f_{iy}[\hat{y}](t) h^{\sigma}(t) \Delta t = \int_a^t f_{iy}[\hat{y}](\tau) \Delta\tau h(t) \Bigg|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{y}](\tau) \Delta\tau \right) h^{\Delta}(t) \Delta t,$$

$$\int_a^b f_{iy}\{\hat{y}\}(t) h^{\rho}(t) \nabla t = \int_a^t f_{iy}\{\hat{y}\}(\tau) \nabla\tau h(t) \Bigg|_a^b - \int_a^b \left(\int_a^t f_{iy}\{\hat{y}\}(\tau) \nabla\tau \right) h^{\nabla}(t) \nabla t.$$

Thus, the necessary condition $\phi'(0) = 0$ can be written as

$$\sum_{i=1}^{k} H_i' \cdot \left[\int_a^t f_{iy}[\hat{y}](\tau) \Delta\tau h(t) \Bigg|_a^b - \int_a^b \left(\int_a^t f_{iy}[\hat{y}](\tau) \Delta\tau \right) h^{\Delta}(t) \Delta t + \int_a^b f_{iv}[\hat{y}](t) h^{\Delta}(t) \Delta t \right]$$
$$+ \sum_{i=k+1}^{k+n} H_i' \cdot \left[\int_a^t f_{iy}\{\hat{y}\}(\tau) \nabla\tau h(t) \Bigg|_a^b - \int_a^b \left(\int_a^t f_{iy}\{\hat{y}\}(\tau) \nabla\tau \right) h^{\nabla}(t) \nabla t + \int_a^b f_{iv}\{\hat{y}\}(t) h^{\nabla}(t) \nabla t \right] = 0. \quad (6.7)$$

In particular, condition (6.7) holds for all variations that are zero at both ends: $h(a) = 0$ and $h(b) = 0$. Then, we obtain:

$$\int_a^b \sum_{i=1}^{k} H_i' \cdot h^{\Delta}(t) \left(f_{iv}[\hat{y}](t) - \int_a^t f_{iy}[\hat{y}](\tau) \Delta\tau \right) \Delta t + \int_a^b \sum_{i=k+1}^{k+n} H_i' \cdot h^{\nabla}(t) \left(f_{iv}\{\hat{y}\}(t) - \int_a^t f_{iy}\{\hat{y}\}(\tau) \nabla\tau \right) \nabla t = 0.$$

Introducing ξ and χ by

$$\xi(t) := \sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iv}[\hat{y}](t) - \int_a^t f_{iy}[\hat{y}](\tau)\Delta\tau \right) \tag{6.8}$$

and

$$\chi(t) := \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iv}\{\hat{y}\}(t) - \int_a^t f_{iy}\{\hat{y}\}(\tau)\nabla\tau \right), \tag{6.9}$$

we obtain the following relation:

$$\int_a^b h^{\Delta}(t)\xi(t)\Delta t + \int_a^b h^{\nabla}(t)\chi(t)\nabla t = 0. \tag{6.10}$$

The further part of the proof follows naturally into two fragments. (i) In the former part, we change the first integral of (6.10) and we obtain two nabla-integrals and, subsequently, the equation (6.5). (ii) In the latter case, we change the second integral of (6.10) and obtain two delta-integrals, which leads us to (6.6).

(i) Using relation (1.6) of Theorem 1.32, we have:

$$\int_a^b \left(h^{\Delta}(t)\right)^{\rho} \xi^{\rho}(t)\nabla t + \int_a^b h^{\nabla}(t)\chi(t)\nabla t = 0.$$

From (1.4) of Theorem 1.31 it follows that

$$\int_a^b h^{\nabla}(t)\left(\xi^{\rho}(t) + \chi(t)\right)\nabla t = 0.$$

From the Dubois–Reymond Lemma 3.12 we conclude that

$$\xi^{\rho}(t) + \chi(t) = const, \tag{6.11}$$

hence, we obtain (6.5).

(ii) From (6.10), and using relation (1.7) of Theorem 1.32, we have

$$\int_a^b h^{\Delta}(t)\xi(t)\Delta t + \int_a^b (h^{\nabla}(t))^{\sigma}\chi^{\sigma}(t)\Delta t = 0.$$

Using (1.5) of Theorem 1.31, we obtain:

$$\int_a^b h^{\Delta}(t)(\xi(t) + \chi^{\sigma}(t))\Delta t = 0.$$

From the Dubois–Reymond Lemma 3.5, it follows that $\xi(t)+\chi^{\sigma}(t) = const$. Hence, we obtain the Euler–Lagrange equation (6.6). □

For regular time scales (Definition 1.6), the Euler–Lagrange equations (6.5) and (6.6) coincide; on a general time scale, they are different. Such a difference is illustrated in Example 6.4.

Example 6.4. *Let us consider the irregular time scale* $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$. *We show that for this time scale there is a difference between the Euler–Lagrange equations* (6.5) *and* (6.6). *The forward and backward jump operators are given by*

$$\sigma(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} [2k, 2k+1), \\ t+1, & t \in \bigcup_{k=0}^{\infty} \{2k+1\}, \end{cases} \qquad \rho(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} (2k, 2k+1], \\ t-1, & t \in \bigcup_{k=1}^{\infty} \{2k\}, \\ 0, & t = 0. \end{cases}$$

For $t = 0$ *and* $t \in \bigcup_{k=0}^{\infty} (2k, 2k+1)$, *equations* (6.5) *and* (6.6) *coincide. We can distinguish between them for* $t \in \bigcup_{k=0}^{\infty} \{2k+1\}$ *and* $t \in \bigcup_{k=1}^{\infty} \{2k\}$. *In what follows we use the notations* (6.8) *and* (6.9). *If* $t \in \bigcup_{k=0}^{\infty} \{2k+1\}$, *then we obtain from* (6.5) *and* (6.6) *the Euler–Lagrange equations* $\xi(t) + \chi(t) = c$ *and* $\xi(t) + \chi(t+1) = c$, *respectively. If* $t \in \bigcup_{k=1}^{\infty} \{2k\}$, *then the Euler–Lagrange equation* (6.5) *has the form* $\xi(t-1) + \chi(t) = c$ *while* (6.6) *takes the form* $\xi(t) + \chi(t) = c$.

6.2 Natural boundary conditions

In this section we minimize or maximize the variational functional (6.1), but initial and/or terminal boundary condition $y(a)$ and/or $y(b)$ are not specified. In what follows we obtain corresponding transversality conditions.

Theorem 6.5 (Transversality condition at the initial time $t = a$)**.** *Let* $\mathbb{T}$ *be a time scale for which* $\rho(\sigma(a)) = a$. *If* $\hat{y}$ *is a local extremizer to* (6.1) *with* $y(a)$ *not specified, then*

$$\sum_{i=1}^{k} H_i' \cdot f_{iv}[\hat{y}](a) + \sum_{i=k+1}^{k+n} H_i' \cdot \left(f_{iv}\{\hat{y}\}(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}\{\hat{y}\}(t)\nabla t \right) = 0 \tag{6.12}$$

holds together with the Euler–Lagrange equations (6.5) *and* (6.6).

Proof. From (6.7), what has already been proved, and (6.11), we have

$$\sum_{i=1}^{k} H_i' \cdot \int_a^t f_{iy}[\hat{y}](\tau)\Delta\tau h(t) \Bigg|_a^b + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^t f_{iy}\{\hat{y}\}(\tau)\nabla\tau h(t) \Bigg|_a^b + \int_a^b h^{\nabla}(t) \cdot c\nabla t = 0.$$

It follows that

$$\sum_{i=1}^{k} H_i' \cdot \int_a^t f_{iy}[\hat{y}](\tau)\Delta\tau h(t)\Bigg|_a^b + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^t f_{iy}\{\hat{y}\}(\tau)\nabla\tau h(t)\Bigg|_a^b + h(t)\cdot c|_a^b = 0.$$

Next, we conclude that

$$\begin{aligned} h(b)&\left[\sum_{i=1}^{k} H_i' \cdot \int_a^b f_{iy}[\hat{y}](\tau)\Delta\tau + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^b f_{iy}\{\hat{y}\}(\tau)\nabla\tau + c\right] \\ &- h(a)\left[\sum_{i=1}^{k} H_i' \cdot \int_a^a f_{iy}[\hat{y}](\tau)\Delta\tau + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^a f_{iy}\{\hat{y}\}(\tau)\nabla\tau + c\right] = 0, \end{aligned} \tag{6.13}$$

where

$$c = \xi(\rho(t)) + \chi(t). \tag{6.14}$$

The Euler–Lagrange equation (6.5) of Theorem 6.3 (or (6.14)) is given at $t = \sigma(a)$ as

$$\begin{aligned} \sum_{i=1}^{k} H_i' \cdot &\left(f_{iv}[\hat{y}](\rho(\sigma(a))) - \int_a^{\rho(\sigma(a))} f_{iy}[\hat{y}](\tau)\Delta\tau\right) \\ &+ \sum_{i=k+1}^{k+n} H_i' \cdot \left(f_{iv}\{\hat{y}\}(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}\{\hat{y}\}(\tau)\nabla\tau\right) = c. \end{aligned}$$

We obtain that

$$\sum_{i=1}^{k} H_i' \cdot f_{iv}[\hat{y}](a) + \sum_{i=k+1}^{k+n} H_i' \cdot \left(f_{iv}\{\hat{y}\}(\sigma(a)) - \int_a^{\sigma(a)} f_{iy}\{\hat{y}\}(\tau)\nabla\tau\right) = c.$$

Restricting the variations h to those such that $h(b) = 0$, it follows from (6.13) that $h(a)\cdot c = 0$. From the arbitrariness of h, we conclude that $c = 0$. Hence, we obtain (6.12). □

Theorem 6.6 (Transversality condition at the terminal time $t = b$). *Let $\mathbb{T}$ be a time scale for which $\sigma(\rho(b)) = b$. If $\hat{y}$ is a local extremizer to* (6.1) *with $y(b)$ not specified, then*

$$\sum_{i=1}^{k} H_i' \cdot \left(f_{iv}[\hat{y}](\rho(b)) + \int_{\rho(b)}^{b} f_{iy}[\hat{y}](t)\Delta t\right) + \sum_{i=k+1}^{k+n} H_i' \cdot f_{iv}\{\hat{y}\}(b) = 0 \tag{6.15}$$

holds together with the Euler–Lagrange equations (6.5) *and* (6.6).

Proof. The calculations in the proof of Theorem 6.5 give us (6.13). When $h(a) = 0$, the Euler–Lagrange equation (6.6) of Theorem 6.3 has the following form at $t = \rho(b)$:

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iv}[\hat{y}](\rho(b)) - \int\limits_{a}^{\rho(b)} f_{iy}[\hat{y}](\tau)\Delta\tau \right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iv}\{\hat{y}\}(\sigma(\rho(b))) - \int\limits_{a}^{\sigma(\rho(b))} f_{iy}\{\hat{y}\}(t)\nabla\tau \right) = c.$$

Then,

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iv}[\hat{y}](\rho(b)) - \int\limits_{a}^{\rho(b)} f_{iy}[\hat{y}](\tau)\Delta\tau \right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iv}\{\hat{y}\}(b) - \int\limits_{a}^{b} f_{iy}\{\hat{y}\}(t)\nabla\tau \right) = c. \quad (6.16)$$

We obtain (6.15) from (6.13) and (6.16). □

Several new interesting results can be immediately obtained from Theorems 6.3, 6.5 and 6.6. An example of such results is given by Corollary 6.7.

Corollary 6.7. *If $\hat{y}$ is a solution to the problem*

$$\mathcal{L}[y] = \frac{\int\limits_a^b f_1(t, y^\sigma(t), y^\Delta(t))\Delta t}{\int\limits_a^b f_2(t, y^\rho(t), y^\nabla(t))\nabla t} \longrightarrow extr,$$

$$(y(a) = y_a), \quad (y(b) = y_b),$$

then the Euler–Lagrange equations

$$\frac{1}{\mathcal{F}_2}\left(f_{1v}[\hat{y}](\rho(t)) - \int\limits_a^{\rho(t)} f_{1y}[\hat{y}](\tau)\Delta\tau \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2}\left(f_{2v}\{\hat{y}\}(t) - \int\limits_a^{t} f_{2y}\{\hat{y}\}(\tau)\nabla\tau \right) = c, \quad t \in \mathbb{T}_\kappa$$

and

$$\frac{1}{\mathcal{F}_2}\left(f_{1v}[\hat{y}](t) - \int\limits_a^{t} f_{1y}[\hat{y}](\tau)\Delta\tau \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2}\left(f_{2v}\{\hat{y}\}(\sigma(t)) - \int\limits_a^{\sigma(t)} f_{2y}\{\hat{y}\}(\tau)\nabla\tau \right) = c, \quad t \in \mathbb{T}^\kappa$$

hold, where

$$\mathcal{F}_1 := \int_a^b f_1(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))\Delta t \quad \text{and} \quad \mathcal{F}_2 := \int_a^b f_2(t, \hat{y}^\rho(t), \hat{y}^\nabla(t))\nabla t.$$

Moreover, if $y(a)$ is free and $\rho(\sigma(a)) = a$, then

$$\frac{1}{\mathcal{F}_2} f_{1v}[\hat{y}](a) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2}\left(f_{2v}\{\hat{y}\}(\sigma(a)) - \int_a^{\sigma(a)} f_{2y}\{\hat{y}\}(t)\nabla t \right) = 0;$$

if $y(b)$ is free and $\sigma(\rho(b)) = b$, then

$$\frac{1}{\mathcal{F}_2}\left(f_{1v}[\hat{y}](\rho(b)) + \int_{\rho(b)}^b f_{1y}[\hat{y}](t)\Delta t \right) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2} f_{2v}\{\hat{y}\}(b) = 0.$$

6.3 Isoperimetric problems

Let us consider the general delta–nabla composition isoperimetric problem on time scales subject to given boundary conditions. The problem consists of minimizing or maximizing

$$\mathcal{L}[y] = H\left(\int_a^b f_1(t, y^\sigma(t), y^\Delta(t))\Delta t, \ldots, \int_a^b f_k(t, y^\sigma(t), y^\Delta(t))\Delta t, \right.$$
$$\left. \int_a^b f_{k+1}(t, y^\rho(t), y^\nabla(t))\nabla t, \ldots, \int_a^b f_{k+n}(t, y^\rho(t), y^\nabla(t))\nabla t \right) \quad (6.17)$$

in the class of functions $y \in C^1_{k+m,n+p}$ satisfying given boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad (6.18)$$

and a generalized isoperimetric constraint

$$\mathcal{K}[y] = P\left(\int_a^b g_1(t, y^\sigma(t), y^\Delta(t))\Delta t, \ldots, \int_a^b g_m(t, y^\sigma(t), y^\Delta(t))\Delta t, \right.$$
$$\left. \int_a^b g_{m+1}(t, y^\rho(t), y^\nabla(t))\nabla t, \ldots, \int_a^b g_{m+p}(t, y^\rho(t), y^\nabla(t))\nabla t \right) = d, \quad (6.19)$$

where $y_a, y_b, d \in \mathbb{R}$. We assume that:

1. the functions $H : \mathbb{R}^{n+k} \to \mathbb{R}$ and $P : \mathbb{R}^{m+p} \to \mathbb{R}$ have continuous partial derivatives with respect to all their arguments, which we denote by H_i', $i = 1, \ldots, n+k$, and P_i', $i = 1, \ldots, m+p$;

2. functions $(t, y, v) \to f_i(t, y, v)$, $i = 1, \ldots, n+k$, and $(t, y, v) \to g_j(t, y, v)$, $j = 1, \ldots, m+p$, from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, have continuous partial derivatives with respect to y and v uniformly in $t \in [a, b]$, which we denote by f_{iy}, f_{iv}, and g_{jy}, g_{jv};

3. for all $y \in C^1_{k+m,n+p}$, f_i, f_{iy}, f_{iv} and g_j, g_{jy}, g_{jv} are rd-continuous in $t \in [a, b]^\kappa$, $i = 1, \ldots, k$, $j = 1, \ldots, m$, and ld-continuous in $t \in [a, b]_\kappa$, $i = k+1, \ldots, k+n$, $j = m+1, \ldots, m+p$.

A function $y \in C^1_{k+m,n+p}$ is said to be admissible provided it satisfies the boundary conditions (6.18) and the isoperimetric constraint (6.19).

Definition 6.8. *We say that an admissible function $\hat{y}$ is a local minimizer (respectively, a local maximizer) to the isoperimetric problem* (6.17)–(6.19), *if there exists a $\delta > 0$ such that $\mathcal{L}[\hat{y}] \leqslant \mathcal{L}[y]$ (respectively, $\mathcal{L}[\hat{y}] \geqslant \mathcal{L}[y]$) for all admissible functions $y \in C^1_{k+m,n+p}$ satisfying the inequality $||y - \hat{y}||_{1,\infty} < \delta$.*

For brevity, we omit the argument of P_i': $P_i' := \frac{\partial P}{\partial \mathcal{G}_i}(\mathcal{G}_1(\hat{y}), \ldots, \mathcal{G}_{m+p}(\hat{y}))$ for $i = 1, \ldots, m+p$, with $\mathcal{G}_i(\hat{y}) = \int_a^b g_i(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))\Delta t$, $i = 1, \ldots, m$, and $\mathcal{G}_i(\hat{y}) = \int_a^b g_i(t, \hat{y}^\rho(t), \hat{y}^\nabla(t))\nabla t$, $i = m+1, \ldots, m+p$. Let us define u and w by

$$u(t) := \sum_{i=1}^{m} P_i' \cdot \left(g_{iv}[\hat{y}](t) - \int_a^t g_{iy}[\hat{y}](\tau)\Delta\tau \right) \tag{6.20}$$

and

$$w(t) := \sum_{i=m+1}^{m+p} P_i' \cdot \left(g_{iv}\{\hat{y}\}(t) - \int_a^t g_{iy}\{\hat{y}\}(\tau)\nabla\tau \right). \tag{6.21}$$

Definition 6.9. *An admissible function $\hat{y}$ is said to be an extremal for $\mathcal{K}$ if $u(t) + w(\sigma(t)) =$ const and $u(\rho(t)) + w(t) =$ const for all $t \in [a, b]^\kappa_\kappa$. An extremizer (i.e., a local minimizer or a local maximizer) to problem* (6.17)–(6.19) *that is not an extremal for $\mathcal{K}$ is said to be a normal extremizer; otherwise (i.e., if it is an extremal for $\mathcal{K}$), the extremizer is said to be abnormal.*

Theorem 6.10 (Optimality condition to the isoperimetric problem (6.17)–(6.19))**.** *Let ξ and χ be given as in* (6.8) *and* (6.9), *and u and w be given as in* (6.20) *and* (6.21). *If $\hat{y}$ is a normal extremizer to the isoperimetric problem* (6.17)–(6.19), *then there exists a real number λ such that*

1. $\xi^\rho(t) + \chi(t) - \lambda\left(u^\rho(t) + w(t)\right) = const;$

2. $\xi(t) + \chi^\sigma(t) - \lambda\left(u^\rho(t) + w(t)\right) = const;$

3. $\xi^\rho(t) + \chi(t) - \lambda\left(u(t) + w^\sigma(t)\right) = const;$

4. $\xi(t)+\chi^{\sigma}(t)-\lambda\left(u(t)+w^{\sigma}(t)\right)=const;$

for all $t\in[a,b]^{\kappa}_{\kappa}$.

Proof. We prove the first item of Theorem 6.10. The other items are proved in a similar way. Consider a variation of $\hat{y}$ such that $\overline{y}=\hat{y}+\varepsilon_1 h_1+\varepsilon_2 h_2$, where $h_i\in C^1_{k+m,n+p}$ and $h_i(a)=h_i(b)=0$, $i=1,2$, and parameters ε_1 and ε_2 are such that $||\overline{y}-\hat{y}||_{1,\infty}<\delta$ for some $\delta>0$. Function h_1 is arbitrary and h_2 is chosen later. Define

$$\overline{\mathcal{K}}(\varepsilon_1,\varepsilon_2)=\mathcal{K}[\overline{y}]=P\left(\int_a^b g_1(t,\overline{y}^{\sigma}(t),\overline{y}^{\Delta}(t))\Delta t,\ldots,\int_a^b g_m(t,\overline{y}^{\sigma}(t),\overline{y}^{\Delta}(t))\Delta t,\right.$$
$$\left.\int_a^b g_{m+1}(t,\overline{y}^{\rho}(t),\overline{y}^{\nabla}(t))\nabla t,\ldots,\int_a^b g_{m+p}(t,\overline{y}^{\rho}(t),\overline{y}^{\nabla}(t))\nabla t\right)-d.$$

A direct calculation gives

$$\left.\frac{\partial\overline{\mathcal{K}}}{\partial\varepsilon_2}\right|_{(0,0)}=\sum_{i=1}^{m}P_i^{'}\cdot\int_a^b\left(g_{iy}[\hat{y}](t)h_2^{\sigma}(t)+g_{iv}[\hat{y}](t)h_2^{\Delta}(t)\right)\Delta t$$
$$+\sum_{i=m+1}^{m+p}P_i^{'}\cdot\int_a^b\left(g_{iy}\{\hat{y}\}(t)h_2^{\rho}(t)+g_{iv}\{\hat{y}\}(t)h_2^{\nabla}(t)\right)\nabla t.$$

Integration by parts of the first terms of both integrals yields:

$$\sum_{i=1}^{m}P_i^{'}\cdot\left[\left.\int_a^t g_{iy}[\hat{y}](\tau)\Delta\tau h_2(t)\right|_a^b-\int_a^b\left(\int_a^t g_{iy}[\hat{y}](\tau)\Delta\tau\right)h_2^{\Delta}(t)\Delta t\right.$$
$$\left.+\int_a^b g_{iv}[\hat{y}](t)h_2^{\Delta}(t)\Delta t\right]$$
$$+\sum_{i=m+1}^{m+p}P_i^{'}\cdot\left[\left.\int_a^t g_{iy}\{\hat{y}\}(\tau)\nabla\tau h_2(t)\right|_a^b-\int_a^b\left(\int_a^t g_{iy}\{\hat{y}\}(\tau)\nabla\tau\right)h_2^{\nabla}(t)\nabla t\right.$$
$$\left.+\int_a^b g_{iv}\{\hat{y}\}(t)h_2^{\nabla}(t)\nabla t\right].$$

Since $h_2(a) = h_2(b) = 0$, we have

$$\int_a^b \sum_{i=1}^m P_i' h_2^\Delta(t) \left(g_{iv}[\hat{y}](t) - \int_a^t g_{iy}[\hat{y}](\tau)\Delta\tau \right) \Delta t + \int_a^b \sum_{i=m+1}^{m+p} P_i' h_2^\nabla(t) \left(g_{iv}\{\hat{y}\}(t) - \int_a^t g_{iy}\{\hat{y}\}(\tau)\nabla\tau \right) \nabla t.$$

Therefore,

$$\left.\frac{\partial \overline{\mathcal{K}}}{\partial \varepsilon_2}\right|_{(0,0)} = \int_a^b h_2^\Delta(t) u(t) \Delta t + \int_a^b h_2^\nabla(t) w(t) \nabla t.$$

Using relation (1.4) of Theorem 1.31, we obtain that

$$\int_a^b \left(h_2^\Delta\right)^\rho(t) u^\rho(t) \nabla t + \int_a^b h_2^\nabla(t) w(t) \nabla t = \int_a^b h_2^\nabla(t) \left(u^\rho(t) + w(t)\right) \nabla t.$$

By the Dubois–Reymond Lemma 3.12, there exists a function h_2 such that $\left.\frac{\partial \overline{\mathcal{K}}}{\partial \varepsilon_2}\right|_{(0,0)} \neq 0$. Since $\overline{\mathcal{K}}(0,0) = 0$, there exists a function ε_2, defined in the neighborhood of zero, such that $\overline{\mathcal{K}}(\varepsilon_1, \varepsilon_2(\varepsilon_1)) = 0$, i.e., we may choose a subset of variations $\hat{y}$ satisfying the isoperimetric constraint. Let us consider the real function

$$\overline{\mathcal{L}}(\varepsilon_1, \varepsilon_2) = \mathcal{L}[\overline{y}] = H \left(\int_a^b f_1(t, \overline{y}^\sigma(t), \overline{y}^\Delta(t)) \Delta t, \ldots, \int_a^b f_k(t, \overline{y}^\sigma(t), \overline{y}^\Delta(t)) \Delta t, \int_a^b f_{k+1}(t, \overline{y}^\rho(t), \overline{y}^\nabla(t)) \nabla t, \ldots, \int_a^b f_{k+n}(t, \overline{y}^\rho(t), \overline{y}^\nabla(t)) \nabla t \right).$$

The point $(0,0)$ is an extremal of $\overline{\mathcal{L}}$ subject to the constraint $\overline{\mathcal{K}} = 0$ and $\nabla \overline{\mathcal{K}}(0,0) \neq 0$. By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that $\nabla \left(\overline{\mathcal{L}}(0,0) - \lambda \overline{\mathcal{K}}(0,0)\right) = 0$. Due to $h_1(a) = h_2(b) = 0$, we have

$$\left.\frac{\partial \overline{\mathcal{L}}}{\partial \varepsilon_1}\right|_{(0,0)} = \sum_{i=1}^k H_i' \cdot \int_a^b \left(f_{iy}[\hat{y}](t) h_1^\sigma(t) + f_{iv}[\hat{y}](t) h_1^\Delta(t) \right) \Delta t + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^b \left(f_{iy}\{\hat{y}\}(t) h_1^\rho(t) + f_{iv}\{\hat{y}\}(t) h_1^\nabla(t) \right) \nabla t.$$

Integrating by parts, and using $h_1(a) = h_1(b) = 0$, gives

$$\left.\frac{\partial \overline{\mathcal{L}}}{\partial \varepsilon_1}\right|_{(0,0)} = \int_a^b h_1^\Delta(t) \xi(t) \Delta t + \int_a^b h_1^\nabla(t) \chi(t) \nabla t.$$

Using (1.6) of Theorem 1.32 and (1.4) of Theorem 1.31, we obtain that

$$\left.\frac{\partial\overline{\mathcal{L}}}{\partial\varepsilon_1}\right|_{(0,0)} = \int_a^b \left(h_1^{\Delta}\right)^{\rho}(t)\xi^{\rho}(t)\nabla t + \int_a^b h_1^{\nabla}(t)\chi(t)\nabla t = \int_a^b h_1^{\nabla}(t)\left(\xi^{\rho}(t)+\chi(t)\right)\nabla t$$

and

$$\left.\frac{\partial\overline{\mathcal{K}}}{\partial\varepsilon_1}\right|_{(0,0)} = \sum_{i=1}^{m} P_i^{'}\cdot\int_a^b \left(g_{iy}[\hat{y}](t)h_1^{\sigma}(t)+g_{iv}[\hat{y}](t)h_1^{\Delta}(t)\right)\Delta t + \sum_{i=m+1}^{m+p} P_i^{'}\cdot\int_a^b \left(g_{iy}\{\hat{y}\}(t)h_1^{\rho}(t)+g_{iv}\{\hat{y}\}(t)h_1^{\nabla}(t)\right)\nabla t.$$

Integrating by parts, and recalling that $h_1(a)=h_1(b)=0$,

$$\left.\frac{\partial\overline{\mathcal{K}}}{\partial\varepsilon_1}\right|_{(0,0)} = \int_a^b h_1^{\Delta}(t)u(t)\Delta t + \int_a^b h_1^{\nabla}(t)w(t)\nabla t.$$

Using relation (1.6) of Theorem 1.32 and relation (1.4) of Theorem 1.31, we obtain that

$$\left.\frac{\partial\overline{\mathcal{K}}}{\partial\varepsilon_1}\right|_{(0,0)} = \int_a^b \left(h_1^{\Delta}\right)^{\rho}(t)u^{\rho}(t)\nabla t + \int_a^b h_1^{\nabla}(t)w(t)\nabla t = \int_a^b h_1^{\nabla}(t)\left(u^{\rho}(t)+w(t)\right)\nabla t.$$

Since $\left.\frac{\partial\overline{\mathcal{L}}}{\partial\varepsilon_1}\right|_{(0,0)} - \lambda\left.\frac{\partial\overline{\mathcal{K}}}{\partial\varepsilon_1}\right|_{(0,0)} = 0$, we have

$$\int_a^b h_1^{\nabla}(t)\left[\xi^{\rho}(t)+\chi(t)-\lambda\left(u^{\rho}(t)+w(t)\right)\right]\nabla t = 0$$

for any $h_1\in C_{k+m,n+p}$. Therefore, by the Dubois–Reymond Lemma 3.12, one has $\xi^{\rho}(t)+\chi(t)-\lambda\left(u^{\rho}(t)+w(t)\right)=c$, where $c\in\mathbb{R}$. □

Remark 6.11. *One can easily cover both normal and abnormal extremizers with Theorem 6.10, if in the proof we use the abnormal Lagrange multiplier rule [90].*

6.4 Illustrative examples

In this section we consider four examples which illustrate the results obtained in Theorem 6.3 and Theorem 6.10. We begin with a nonautonomous problem.

Example 6.12. *Consider the problem*

$$\mathcal{L}[y] = \frac{\int_0^1 ty^{\Delta}(t)\Delta t}{\int_0^1 (y^{\nabla}(t))^2 \nabla t} \longrightarrow \min, \tag{6.22}$$

$$y(0) = 0, \quad y(1) = 1.$$

If y is a local minimizer to problem (6.22)*, then the Euler–Lagrange equations of Corollary 6.7 must hold, i.e.,*

$$\frac{1}{\mathcal{F}_2}\rho(t) - 2\frac{\mathcal{F}_1}{\mathcal{F}_2^2} y^{\nabla}(t) = c, \quad t \in \mathbb{T}_{\kappa}, \qquad \text{and} \qquad \frac{1}{\mathcal{F}_2} t - 2\frac{\mathcal{F}_1}{\mathcal{F}_2^2} y^{\nabla}(\sigma(t)) = c, \quad t \in \mathbb{T}^{\kappa},$$

where $\mathcal{F}_1 := \mathcal{F}_1(y) = \int_0^1 ty^{\Delta}(t)\Delta t$ *and* $\mathcal{F}_2 := \mathcal{F}_2(y) = \int_0^1 (y^{\nabla}(t))^2 \nabla t$*. Let us consider the second equation. Using* (1.5) *of Theorem 1.31, it can be written as*

$$\frac{1}{\mathcal{F}_2} t - 2\frac{\mathcal{F}_1}{\mathcal{F}_2^2} y^{\Delta}(t) = c, \quad t \in \mathbb{T}^{\kappa}. \tag{6.23}$$

Solving equation (6.23) *and using the boundary conditions* $y(0) = 0$ *and* $y(1) = 1$*, gives*

$$y(t) = \frac{1}{2Q} \int_0^t \tau \Delta\tau - t \left(\frac{1}{2Q} \int_0^1 \tau \Delta\tau - 1 \right), \quad t \in \mathbb{T}^{\kappa}, \tag{6.24}$$

where $Q := \frac{\mathcal{F}_1}{\mathcal{F}_2}$*. Therefore, the solution depends on the time scale. Let us consider two examples:* $\mathbb{T} = \mathbb{R}$ *and* $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$*. On* $\mathbb{T} = \mathbb{R}$*, from* (6.24) *we obtain*

$$y(t) = \frac{1}{4Q} t^2 + \frac{4Q-1}{4Q} t, \qquad y^{\Delta}(t) = y^{\nabla}(t) = y'(t) = \frac{1}{2Q} t + \frac{4Q-1}{4Q}, \tag{6.25}$$

as solution of (6.23)*. Substituting* (6.25) *into* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *gives* $\mathcal{F}_1 = \frac{12Q+1}{24Q}$ *and* $\mathcal{F}_2 = \frac{48Q^2+1}{48Q^2}$*, that is,*

$$Q = \frac{2Q(12Q+1)}{48Q^2+1}. \tag{6.26}$$

Solving equation (6.26) *we get* $Q \in \left\{\frac{3-2\sqrt{3}}{12}, \frac{3+2\sqrt{3}}{12}\right\}$*. Because* (6.22) *is a minimizing problem, we select* $Q = \frac{3-2\sqrt{3}}{12}$ *and we get the extremal*

$$y(t) = -(3+2\sqrt{3})t^2 + (4+2\sqrt{3})t. \tag{6.27}$$

If $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$*, then from* (6.24) *we obtain* $y(t) = \frac{1}{8Q} \sum_{k=0}^{2t-1} k + \frac{8Q-1}{8Q} t$*, that is,*

$$y(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{8Q-1}{16Q}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases}$$

Direct calculations show that

$$\begin{aligned} y^{\Delta}(0) &= \frac{y(\frac{1}{2}) - y(0)}{\frac{1}{2}} = \frac{8Q-1}{8Q}, \quad y^{\Delta}\left(\frac{1}{2}\right) = \frac{y(1) - y(\frac{1}{2})}{\frac{1}{2}} = \frac{8Q+1}{8Q}, \\ y^{\nabla}\left(\frac{1}{2}\right) &= \frac{y(\frac{1}{2}) - y(0)}{\frac{1}{2}} = \frac{8Q-1}{8Q}, \quad y^{\nabla}(1) = \frac{y(1) - y(\frac{1}{2})}{\frac{1}{2}} = \frac{8Q+1}{8Q}. \end{aligned} \tag{6.28}$$

Substituting (6.28) *into the integrals* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *gives*

$$\mathcal{F}_1 = \frac{8Q+1}{32Q}, \quad \mathcal{F}_2 = \frac{64Q^2+1}{64Q^2}, \quad Q = \frac{\mathcal{F}_1}{\mathcal{F}_2} = \frac{2Q(8Q+1)}{64Q^2+1}.$$

Thus, we obtain the equation $64Q^2 - 16Q - 1 = 0$. *The solutions to this equation are:* $Q \in \left\{\frac{1-\sqrt{2}}{8}, \frac{1+\sqrt{2}}{8}\right\}$. *We are interested in the minimum value* Q, *so we select* $Q = \frac{1+\sqrt{2}}{8}$ *to get the extremal*

$$y(t) = \begin{cases} 0, & \textit{if } t = 0, \\ 1 - \frac{\sqrt{2}}{2}, & \textit{if } t = \frac{1}{2}, \\ 1, & \textit{if } t = 1. \end{cases} \tag{6.29}$$

Note that the extremals (6.27) *and* (6.29) *are different: for* (6.27) *one has* $x(1/2) = \frac{5}{4} + \frac{\sqrt{3}}{2}$.

We now present a problem where, in contrast with Example 6.12, the extremal does not depend on the time scale $\mathbb{T}$.

Example 6.13. *Consider the autonomous problem*

$$\mathcal{L}[y] = \frac{\int_0^2 \left(y^{\Delta}(t)\right)^2 \Delta t}{\int_0^2 \left[y^{\nabla}(t) + (y^{\nabla}(t))^2\right] \nabla t} \longrightarrow \min, \qquad y(0) = 0, \quad y(2) = 4. \tag{6.30}$$

If y *is a local minimizer to* (6.30), *then the Euler–Lagrange equations of Corollary 6.7 must hold, i.e.,*

$$\frac{2}{\mathcal{F}_2} y^{\nabla}(t) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2}(2y^{\nabla}(t)+1) = c, \quad t \in \mathbb{T}_{\kappa}, \qquad \textit{and} \qquad \frac{2}{\mathcal{F}_2} y^{\Delta}(t) - \frac{\mathcal{F}_1}{\mathcal{F}_2^2}(2y^{\Delta}(t)+1) = c, \quad t \in \mathbb{T}^{\kappa}, \tag{6.31}$$

where $\mathcal{F}_1 := \mathcal{F}_1(y) = \int_0^2 \left(y^{\Delta}(t)\right)^2 \Delta t$ *and* $\mathcal{F}_2 := \mathcal{F}_2(y) = \int_0^2 \left[y^{\nabla}(t) + \left(y^{\nabla}(t)\right)^2\right] \nabla t$. *Choosing one of the equations of* (6.31), *for example the first one, we get*

$$y^{\nabla}(t) = \left(c + \frac{\mathcal{F}_1}{\mathcal{F}_2^2}\right) \frac{\mathcal{F}_2^2}{2\mathcal{F}_2 - 2\mathcal{F}_1}, \quad t \in \mathbb{T}^{\kappa}. \tag{6.32}$$

Using (6.32) *with boundary conditions* $y(0) = 0$ *and* $y(2) = 4$, *we obtain, for any given time scale* $\mathbb{T}$, *the extremal* $y(t) = 2t$.

In the previous two examples, the variational functional is given by the ratio of a delta and a nabla integral. We now discuss a variational problem where the composition is expressed by the product of three time-scale integrals.

Example 6.14. *Consider the problem*

$$\mathcal{L}[y] = \left(\int_0^1 t y^{\Delta}(t)\Delta t\right)\left(\int_0^1 y^{\Delta}(t)\,(1+t)\,\Delta t\right)\left(\int_0^1 \left(y^{\nabla}(t)\right)^2 \nabla t\right) \longrightarrow \min, \qquad y(0)=0, \quad y(1)=1. \tag{6.33}$$

If y is a local minimizer to problem (6.33)*, then the Euler–Lagrange equations must hold, and we can write that*

$$(\mathcal{F}_1\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_3)\,t + \mathcal{F}_1\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2 y^{\nabla}(\sigma(t)) = c, \quad t \in \mathbb{T}^{\kappa}, \tag{6.34}$$

where c is a constant, $\mathcal{F}_1 := \mathcal{F}_1(y) = \int_0^1 t y^{\Delta}(t)\Delta t$, $\mathcal{F}_2 := \mathcal{F}_2(y) = \int_0^1 y^{\Delta}(t)\,(1+t)\,\Delta t$, and $\mathcal{F}_3 := \mathcal{F}_3(y) = \int_0^1 \left(y^{\nabla}(t)\right)^2 \nabla t$. Using relation (1.5)*, we can write* (6.34) *as*

$$(\mathcal{F}_1\mathcal{F}_3 + \mathcal{F}_2\mathcal{F}_3)\,t + \mathcal{F}_1\mathcal{F}_3 + 2\mathcal{F}_1\mathcal{F}_2 y^{\Delta}(t) = c, \quad t \in \mathbb{T}^{\kappa}. \tag{6.35}$$

Using the boundary conditions $y(0)=0$ and $y(1)=1$, from (6.35) *we get that*

$$y(t) = \left(1 + Q\int_0^1 \tau\Delta\tau\right) t - Q\int_0^t \tau\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{6.36}$$

where $Q = \frac{\mathcal{F}_1\mathcal{F}_3+\mathcal{F}_2\mathcal{F}_3}{2\mathcal{F}_1\mathcal{F}_2}$. Therefore, the solution depends on the time scale. Let us consider $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\left\{0, \frac{1}{2}, 1\right\}$. On $\mathbb{T}=\mathbb{R}$, expression (6.36) *gives*

$$y(t) = \left(\frac{2+Q}{2}\right) t - \frac{Q}{2}t^2, \quad y^{\Delta}(t) = y^{\nabla}(t) = y'(t) = \frac{2+Q}{2} - Qt \tag{6.37}$$

as solution of (6.35)*. Substituting* (6.37) *into $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ gives:*

$$\mathcal{F}_1 = \frac{6-Q}{12}, \quad \mathcal{F}_2 = \frac{18-Q}{12}, \quad \mathcal{F}_3 = \frac{Q^2+12}{12}.$$

One can proceed by solving the equation $Q^3 - 18Q^2 + 60Q - 72 = 0$, to find the extremal $y(t) = \left(\frac{2+Q}{2}\right)t - \frac{Q}{2}t^2$ with $Q = 2\sqrt[3]{9+\sqrt{17}} + \frac{9-\sqrt{17}}{8}\sqrt[3]{(9+\sqrt{17})^2} + 6$.

Let us consider now the time scale $\mathbb{T}=\left\{0, \frac{1}{2}, 1\right\}$. From (6.36)*, we obtain*

$$y(t) = \left(\frac{4+Q}{4}\right) t - \frac{Q}{4}\sum_{k=0}^{2t-1} k = \begin{cases} 0, & \text{if } t=0, \\ \frac{4+Q}{8}, & \text{if } t=\frac{1}{2}, \\ 1, & \text{if } t=1, \end{cases} \tag{6.38}$$

as solution of (6.35)*. Substituting* (6.38) *into* $\mathcal{F}_1$*,* $\mathcal{F}_2$ *and* $\mathcal{F}_3$*, we obtain*

$$\mathcal{F}_1 = \frac{4-Q}{16}, \quad \mathcal{F}_2 = \frac{20-Q}{16}, \quad \mathcal{F}_3 = \frac{Q^2+16}{16}.$$

Solving equation $Q^3 - 18Q^2 + 48Q - 96 = 0$*, we find the extremal*

$$y(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{5+\sqrt[3]{5}+\sqrt[3]{25}}{4}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1, \end{cases}$$

for problem (6.33)*.*

Finally, we apply the results of Section 6.3 to an isoperimetric variational problem.

Example 6.15. *Let us consider the problem of extremizing*

$$\mathcal{L}[y] = \frac{\int_0^1 (y^\Delta(t))^2 \Delta t}{\int_0^1 t y^\nabla(t) \nabla t}$$

subject to the boundary conditions $y(0) = 0$ *and* $y(1) = 1$*, and the constraint*

$$\mathcal{K}[y] = \int_0^1 t y^\nabla(t) \nabla t = 1.$$

Applying Theorem 6.10, we get the nabla differential equation

$$\frac{2}{\mathcal{F}_2} y^\nabla(t) - \left(\lambda + \frac{\mathcal{F}_1}{(\mathcal{F}_2)^2}\right) t = c, \quad t \in \mathbb{T}_\kappa^\kappa. \tag{6.39}$$

Solving this equation, we obtain

$$y(t) = \left(1 - Q \int_0^1 \tau \nabla \tau\right) t + Q \int_0^t \tau \nabla \tau, \tag{6.40}$$

where $Q = \frac{\mathcal{F}_2}{2}\left(\frac{\mathcal{F}_1}{(\mathcal{F}_2)^2} + \lambda\right)$*. Therefore, the solution of equation* (6.39) *depends on the time scale. As before, let us consider* $\mathbb{T} = \mathbb{R}$ *and* $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$*.*

On $\mathbb{T} = \mathbb{R}$*, from* (6.40) *we obtain that* $y(t) = \frac{2-Q}{2}t + \frac{Q}{2}t^2$*. Substituting this expression for* y *into the integrals* $\mathcal{F}_1$ *and* $\mathcal{F}_2$*, gives* $\mathcal{F}_1 = \frac{Q^2+12}{12}$ *and* $\mathcal{F}_2 = \frac{Q+6}{12}$*. Using the given isoperimetric constraint, we obtain* $Q = 6$*,* $\lambda = 8$*, and* $y(t) = 3t^2 - 2t$*.*

Let us consider now the time scale $\mathbb{T} = \left\{0, \frac{1}{2}, 1\right\}$*. From* (6.40)*, we have*

$$y(t) = \frac{4-3Q}{4}t + Q\sum_{k=1}^{2t} \frac{k}{4} = \begin{cases} 0, & \text{if } t = 0, \\ \frac{4-Q}{8}, & \text{if } t = \frac{1}{2}, \\ 1, & \text{if } t = 1. \end{cases}$$

Simple calculations show that

$$\mathcal{F}_1 = \sum_{k=0}^{1} \frac{1}{2}\left(y^{\Delta}\left(\frac{k}{2}\right)\right)^2 = \frac{1}{2}\left(y^{\Delta}(0)\right)^2 + \frac{1}{2}\left(y^{\Delta}\left(\frac{1}{2}\right)\right)^2 = \frac{Q^2+16}{16},$$
$$\mathcal{F}_2 = \sum_{k=1}^{2} \frac{1}{4} k y^{\nabla}\left(\frac{k}{2}\right) = \frac{1}{4} y^{\nabla}\left(\frac{1}{2}\right) + \frac{1}{2} y^{\nabla}(1) = \frac{Q+12}{16}$$

and $\mathcal{K}(y) = \frac{Q+12}{16} = 1$*. Therefore,* $Q = 4$*,* $\lambda = 6$*, and we have the extremal*

$$y(t) = \begin{cases} 0, & \textit{if } t \in \left\{0, \frac{1}{2}\right\}, \\ 1, & \textit{if } t = 1. \end{cases}$$

6.5 State of the art

The results of this chapter are published in [38].

Chapter 7

Applications to Economics

This chapter is divided into two parts (Sections 7.1 and 7.2) and at each of them an economic model is presented. In Section 7.1 we study a general nonclassical problem of the calculus of variations on time scales. More precisely, we consider problems of minimizing or maximizing a composition of delta and nabla integral functionals. We prove general necessary optimality conditions of Euler–Lagrange type in differential form (Theorem 7.1), which are then applied to the particular time scales $\mathbb{T} = \mathbb{R}$ (Corollary 7.2) and $\mathbb{T} = \mathbb{Z}$ (Corollary 7.3). Next we consider an economic problem describing a firm that wants to program its production and investment policies to reach a given production rate and to maximize its future market competitiveness. The continuous case, denoted by (P), was discussed in [30]; here we focus our attention on four different discretizations of problem (P), in particular to two mixed delta-nabla discretizations that we call $(P_{\Delta\nabla})$ and $(P_{\nabla\Delta})$. For these discrete problems the direct discretization of the Euler–Lagrange equation for (P) does not lead to the solution of the problems: the results found by applying our Corollary 7.3 to $(P_{\Delta\nabla})$ and $(P_{\nabla\Delta})$ are shown to be better. The comparison is done in Section 7.1.4.

In Section 7.2 we present a relation between inflation and unemployment which both inflict social losses. When a Phillips tradeoff exists between them, what would be the best combination of inflation and unemployment? A well-known approach in economics to address this question consists to write the social loss function as a function of the rate of inflation p and the rate of unemployment u, with different weights; then, using relations between p, u and the expected rate of inflation π, to rewrite the social loss function as a function of π; finally, to apply the theory of the calculus of variations in order to find an optimal path π that minimizes the total social loss over a certain time interval $[0, T]$. Economists dealing with this question implement the above approach using both continuous and discrete models [31, 87]. Here we propose a new, more general, time-scale model. We derive necessary (Theorem 7.6 and Corollary 7.9) and sufficient (Theorem 7.12) optimality conditions for the variational problem that models the economical situation. For the time scale $\mathbb{T} = h\mathbb{Z}$ with appropriate values of $h > 0$, we obtain an explicit solution for the global minimizer of the total social loss problem (Theorem 7.13).

7.1 A general delta-nabla problem of the calculus of variations on time scales

Let $\mathbb{T}$ be a given time scale with at least three points, and let $a, b \in \mathbb{T}$. We consider the following general problem of the calculus of variations on time scales.

Problem. *Find a function y that extremizes, that is, minimizes or maximizes, the functional*

$$\mathcal{L}[y] = H\left(\int_a^b f_1(t, y^\sigma(t), y^\Delta(t))\Delta t, \ldots, \int_a^b f_k(t, y^\sigma(t), y^\Delta(t))\Delta t,\right.$$
$$\left.\int_a^b f_{k+1}(t, y^\rho(t), y^\nabla(t))\nabla t, \ldots, \int_a^b f_{k+n}(t, y^\rho(t), y^\nabla(t))\nabla t\right) \quad (7.1)$$

subject to the boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \quad (7.2)$$

where $y \in C^1_{k,n}([a, b], \mathbb{R})$ (Definition 6.1), $k, n \in \mathbb{N}$.

For brevity, we use the operators $[\cdot]$ and $\{\cdot\}$ defined by (6.3). We assume that

1. function $H : \mathbb{R}^{n+k} \to \mathbb{R}$ has continuous partial derivatives with respect to its arguments, which we denote by H_i', $i = 1, \ldots, n + k$;

2. functions $(t, y, v) \to f_i(t, y, v)$ from $[a, b] \times \mathbb{R}^2$ to $\mathbb{R}$, $i = 1, \ldots, n + k$, have continuous partial derivatives with respect to y and v uniformly in $t \in [a, b]$, which we denote by f_{iy} and f_{iv}, respectively;

3. functions f_i, f_{iy}, f_{iv} are rd-continuous in $t \in [a, b]^\kappa$, $i = 1, \ldots, k$, and ld-continuous in $t \in [a, b]_\kappa$, $i = k + 1, \ldots, k + n$, for all $y \in C^1_{k,n}([a, b]; \mathbb{R})$.

A function $y \in C^1_{k,n}([a, b]; \mathbb{R})$ is said to be *admissible* provided it satisfies the boundary conditions (7.2). A local minimizer $\hat{y}$ (respectively, maximizer) to problem (7.1)–(7.2) is given by Definition 6.2 for all admissible functions $y \in C^1_{k,n}([a, b]; \mathbb{R})$.

7.1.1 The Euler–Lagrange equations

Now Euler–Lagrange type optimality conditions in differential form are obtained, which are different than the ones presented in [38] and Chapter 6. If one considers the particular case where function H in problem (7.1)–(7.2) does not depend on nabla operators, then one obtains exactly the delta problem studied in [71]. In this case, the assumptions we are considering for problem (7.1)–(7.2) coincide with the ones of [71]. However, it should be noted that when it is written $\frac{\Delta}{\Delta t}$ or $\frac{\nabla}{\nabla t}$ for some given expression, this is formal and does not mean that one can really expand the delta (or nabla) derivative. Such formal expressions are common in the literature of calculus of variations (see, e.g., [48, Theorem 1 of Section 4], [83, Corollary 2 to Theorem 2.3] or [89, Section 6.1]). All our expressions are valid in integral form (see Chapter 6).

For brevity, in what follows we omit the arguments of $H_i^{'}$, that is,

$$H_i^{'} := \frac{\partial H}{\partial \mathcal{F}_i}(\mathcal{F}_1(y), \ldots, \mathcal{F}_{k+n}(y)),$$

$i = 1, \ldots, n+k$, where

$$\mathcal{F}_i(y) = \int_a^b f_i[y](t)\Delta t, \text{ for } i = 1, \ldots, k, \qquad \mathcal{F}_i(y) = \int_a^b f_i\{y\}(t)\nabla t, \text{ for } i = k+1, \ldots, k+n.$$

Theorem 7.1 (The delta-nabla Euler–Lagrange equations)**.** *Let $\mathbb{T}$ be a time scale having at least three points and let $a, b \in \mathbb{T}$. If $\hat{y}$ is a solution to problem* (7.1)–(7.2)*, then the delta-nabla Euler–Lagrange equations*

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iy}[\hat{y}](t) - f_{iv}^{\Delta}[\hat{y}](t)\right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iy}\{\hat{y}\}(\sigma(t)) - f_{iv}^{\Delta}\{\hat{y}\}(t)\right)$$
$$+ \frac{\Delta}{\Delta t}\left[\sum_{i=k+1}^{k+n} H_i^{'} \cdot \nu(t) \cdot \left(f_{iy}\{\hat{y}\}(t) - f_{iv}^{\nabla}\{\hat{y}\}(t)\right)\right]^{\sigma}(t) = 0 \quad (7.3)$$

and

$$\sum_{i=1}^{k} H_i^{'} \cdot \left(f_{iy}[\hat{y}](\rho(t)) - f_{iv}^{\nabla}[\hat{y}](t)\right) + \sum_{i=k+1}^{k+n} H_i^{'} \cdot \left(f_{iy}\{\hat{y}\}(t) - f_{iv}^{\nabla}\{\hat{y}\}(t)\right)$$
$$- \frac{\nabla}{\nabla t}\left[\sum_{i=1}^{k} H_i^{'} \cdot \mu(t) \cdot \left(f_{iy}[\hat{y}](t) - f_{iv}^{\Delta}[\hat{y}](t)\right)\right]^{\rho}(t) = 0 \quad (7.4)$$

hold for all $t \in \mathbb{T}_{\kappa}^{\kappa}$.

Proof. Suppose that $\mathcal{L}[y]$ has a local extremum at $\hat{y}$. Consider a variation $h \in C_{k,n}^1([a,b], \mathbb{R})$ of $\hat{y}$ for which we define the function $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\varepsilon) = \mathcal{L}[\hat{y} + \varepsilon h]$. A necessary condition for $\hat{y}$ to be an extremizer for $\mathcal{L}[y]$ is given by $\phi'(\varepsilon) = 0$ for $\varepsilon = 0$. Using the chain rule, we obtain that

$$\phi'(0) = \sum_{i=1}^{k} H_i' \cdot \int_a^b \left(f_{iy}[\hat{y}](t)h^\sigma(t) + f_{iv}[\hat{y}](t)h^\Delta(t)\right)\Delta t + \sum_{i=k+1}^{k+n} H_i' \cdot \int_a^b \left(f_{iy}\{\hat{y}\}(t)h^\rho(t) + f_{iv}\{\hat{y}\}(t)h^\nabla(t)\right)\nabla t = 0.$$

Using delta and nabla product rules (Theorems 1.11 and 1.23) we have

$$[f_{iv}[\hat{y}](t)h(t)]^\Delta = f_{iv}[\hat{y}](t)h^\Delta(t) + (f_{iv}[\hat{y}](t))^\Delta h^\sigma(t)$$

and

$$[f_{iv}\{\hat{y}\}(t)h(t)]^\nabla = f_{iv}\{\hat{y}\}(t)h^\nabla(t) + (f_{iv}\{\hat{y}\}(t))^\nabla h^\rho(t).$$

Integrating both sides from $t = a$ to $t = b$ and having in mind that from (7.2) one has $h(a) = h(b) = 0$, we obtain that

$$\int_a^b \sum_{i=1}^{k} H_i' \cdot \left(f_{iy}[\hat{y}](t) - (f_{iv}[\hat{y}](t))^\Delta\right) h^\sigma(t)\Delta t + \int_a^b \sum_{i=k+1}^{k+n} H_i' \cdot \left(f_{iy}\{\hat{y}\}(t) - (f_{iv}\{\hat{y}\}(t))^\nabla\right) h^\rho(t)\nabla t = 0.$$

Let us denote

$$s(t) := \sum_{i=1}^{k} H_i' \cdot \left(f_{iy}[\hat{y}](t) - (f_{iv}[\hat{y}](t))^\Delta\right),$$
$$r(t) := \sum_{i=k+1}^{k+n} H_i' \cdot \left(f_{iy}\{\hat{y}\}(t) - (f_{iv}\{\hat{y}\}(t))^\nabla\right).$$

Then,

$$\int_a^b s(t)h^\sigma(t)\Delta t + \int_a^b r(t)h^\rho(t)\nabla t = 0.$$

The proof is divided into two parts. First we use (1.7) of Theorem 1.32 and (1.5) of Theorem 1.31 in order to obtain the Euler–Lagrange equation (7.3). In the latter case we apply (1.6) of Theorem 1.32 and (1.4) of Theorem 1.31 to receive the Euler–Lagrange equation (7.4).

(i) Since h is nabla differentiable, we have that $h^\rho(t) = h(t) - \nu(t)h^\nabla(t)$ (cf. item (iv) of [5, Theorem 3.2]) and thus

$$\int_a^b s(t)h^\sigma(t)\Delta t + \int_a^b \left[r(t)h(t) - r(t)\nu(t)h^\nabla(t)\right]\nabla t = 0.$$

Using equation (1.7) of Theorem 1.32, it follows that

$$\int_a^b s(t)h^\sigma(t)\Delta t + \int_a^b \left[(rh)^\sigma(t) - (r\nu)^\sigma(t)(h^\nabla)^\sigma(t)\right]\Delta t = 0.$$

Therefore, from equation (1.5) of Theorem 1.31, we obtain

$$\int_a^b s(t)h^\sigma(t)\Delta t + \int_a^b \left[(rh)^\sigma(t) - (r\nu)^\sigma(t)h^\Delta(t)\right]\Delta t = 0.$$

Integrating the second part of the latter integral, gives

$$\int_a^b (r\nu)^\sigma(t)h^\Delta(t)\Delta t = (r\nu)^\sigma(t)h(t)\Big|_a^b - \int_a^b h^\sigma(t)\frac{\Delta}{\Delta t}(r\nu)^\sigma(t)\Delta t,$$

and it follows that

$$\int_a^b \left[s(t)h^\sigma(t) + r^\sigma(t)h(t)^\sigma + h^\sigma(t)\frac{\Delta}{\Delta t}(r\nu)^\sigma(t)\right]\Delta t = 0.$$

Thus,

$$\int_a^b \left[s(t) + r^\sigma(t) + \frac{\Delta}{\Delta t}(r\nu)^\sigma(t)\right]h^\sigma(t)\Delta t = 0.$$

From the fundamental lemma of the delta calculus of variations (cf. [2, Lemma 8] and [45, Lemma 3.2]), we get the Euler–Lagrange equation

$$s(t) + r^\sigma(t) + \frac{\Delta}{\Delta t}(r\nu)^\sigma(t) = 0$$

and therefore equation (7.3).

(ii) Since h is delta differentiable, the following relation holds (cf. item (iv) of [23, Theorem 1.3]):

$$h^\sigma(t) = h(t) + \mu(t)h^\Delta(t).$$

Then we obtain that

$$\int_a^b s(t)h(t) + s(t)\mu(t)h^\Delta(t)\Delta t + \int_a^b r(t)h^\rho(t)\nabla t = 0.$$

Using equation (1.6) of Theorem 1.32, we have

$$\int_a^b \left[s^\rho(t)h^\rho(t) + (s\mu)^\rho(t)(h^\Delta)^\rho(t) + r(t)h^\rho(t)\right]\nabla t = 0.$$

From equation (1.4) of Theorem 1.31 it follows that

$$\int_a^b \left[s^\rho(t)h^\rho(t) + (s\mu)^\rho(t)h^\nabla(t) + r(t)h^\rho(t)\right]\nabla t = 0.$$

Integrating the second item of the above integral,

$$\int_a^b (s\mu)^\rho(t)h^\nabla(t)\nabla t = (s\mu)^\rho(t)h(t)\Big|_a^b - \int_a^b \frac{\nabla}{\nabla t}(s\mu)^\rho(t)h^\rho(t)\nabla t,$$

yields

$$\int_a^b \left[s^\rho(t)h^\rho(t) + r(t)h^\rho(t) - h^\rho(t)\frac{\nabla}{\nabla t}(s\mu)^\rho(t)\right]\nabla t = 0$$

and then

$$\int_a^b \left[s^\rho(t) + r(t) - \frac{\nabla}{\nabla t}(s\mu)^\rho(t)\right]h^\rho(t)\nabla t = 0.$$

From the fundamental lemma of the nabla calculus of variations (cf. [75, Lemma 15]), we get the Euler–Lagrange equation

$$s^\rho(t) + r(t) - \frac{\nabla}{\nabla t}(s\mu)^\rho(t) = 0$$

and therefore equation (7.4). □

Corollary 7.2 (See Theorem 3 of [30])**.** *Let $a, b \in \mathbb{R}$ with $a < b$. If $\hat{y}$ is solution to problem*

$$\mathcal{L}[y] = H\left(\int_a^b f_1(t, y(t), y'(t))dt, \int_a^b f_2(t, y(t), y'(t))dt\right) \longrightarrow extr \qquad (7.5)$$

$$y(a) = y_a, \quad y(b) = y_b,$$

then the Euler–Lagrange differential equation

$$H_1'(\mathcal{F}_1, \mathcal{F}_2) \cdot \left(f_{1y}(t, \hat{y}(t), \hat{y}'(t)) - \frac{d}{dt}f_{1v}(t, \hat{y}(t), \hat{y}'(t))\right) + H_2'(\mathcal{F}_1, \mathcal{F}_2) \cdot \left(f_{2y}(t, \hat{y}(t), \hat{y}'(t)) - \frac{d}{dt}f_{2v}(t, \hat{y}(t), \hat{y}'(t))\right) = 0 \qquad (7.6)$$

holds for all $t \in [a, b]$, where

$$\mathcal{F}_i = \int_a^b f_i(t, \hat{y}(t), \hat{y}'(t))dt, \quad i = 1, 2.$$

Proof. Let $\mathbb{T} = \mathbb{R}$ and $k = n = 1$. The result follows from Theorem 7.1. □

Corollary 7.3. *Let $a, b \in \mathbb{N}$ with $b - a > 1$ and denote by $\Delta y(t)$ and $\nabla y(t)$ the standard forward and backward differences operators, that is, $\Delta y(t) := y(t+1) - y(t)$ and $\nabla y(t) := y(t) - y(t-1)$. If $\hat{y}$ is solution to problem*

$$\begin{gathered}\mathcal{L}[y] = H\left(\sum_{t=a}^{b-1} f_1(t, y(t+1), \Delta y(t)), \sum_{t=a+1}^{b} f_2(t, y(t-1), \nabla y(t))\right) \longrightarrow extr \\ y(a) = y_a, \quad y(b) = y_b,\end{gathered} \tag{7.7}$$

then both Euler–Lagrange difference equations

$$\begin{aligned}&H_1^{'}(\mathcal{F}_1, \mathcal{F}_2) \cdot [f_{1y}(t, \hat{y}(t+1), \Delta\hat{y}) - \Delta f_{1v}(t, \hat{y}(t+1), \Delta\hat{y})] \\ &+ H_2^{\prime}(\mathcal{F}_1, \mathcal{F}_2) \cdot [f_{2y}(t+1, \hat{y}(t), \nabla\hat{y}(t+1)) - \Delta f_{2v}(t, \hat{y}(t-1), \nabla\hat{y}(t))] \\ &+ H_2^{\prime}(\mathcal{F}_1, \mathcal{F}_2) \cdot \Delta\left[f_{2y}(t+1, \hat{y}(t), \nabla\hat{y}(t+1)) - \nabla f_{2v}(t+1, \hat{y}(t), \nabla\hat{y}(t+1))\right] = 0\end{aligned} \tag{7.8}$$

and

$$\begin{aligned}&H_1^{'}(\mathcal{F}_1, \mathcal{F}_2) \cdot [f_{1y}(t-1, \hat{y}(t), \Delta\hat{y}(t-1)) - \nabla f_{1v}(t, \hat{y}(t+1), \Delta\hat{y}(t))] \\ &- H_1^{'}(\mathcal{F}_1, \mathcal{F}_2) \cdot \nabla\left[f_{1y}(t-1, \hat{y}(t), \Delta\hat{y}(t-1)) - \Delta f_{1v}(t-1, \hat{y}(t), \Delta\hat{y}(t-1))\right] \\ &+ H_2^{'}(\mathcal{F}_1, \mathcal{F}_2) \cdot [f_{2y}(t, \hat{y}(t-1), \nabla\hat{y}(t)) - \nabla f_{2v}(t, \hat{y}(t-1), \nabla\hat{y}(t))] = 0\end{aligned} \tag{7.9}$$

hold for $t \in \{a+1, \dots, b-1\}$, where

$$\mathcal{F}_1 := \sum_{t=a}^{b-1} f_1(t, \hat{y}(t+1), \Delta\hat{y}(t)), \quad \mathcal{F}_2 := \sum_{t=a+1}^{b} f_2(t, \hat{y}(t-1), \nabla\hat{y}(t)).$$

Proof. The result is a direct consequence of Theorem 7.1 with $\mathbb{T} = \mathbb{Z}$ and $k = n = 1$. □

7.1.2 Economic model and its direct discretizations

In this section we introduce an economic problem that is considered in continuous (Example 7.4 below) and discrete (Example 7.5 below) cases. The first example is made under the assumptions from Section 6 of [30]. The latter example corresponds to discretizations of the problem of Example 7.4, for which one can discretize the Euler–Lagrange equation (7.6). In what follows,

$$\Delta y(t) := y^{\sigma}(t) - y(t), \quad \nabla y(t) := y(t) - y^{\rho}(t).$$

In particular, if $\mathbb{T}$ has a maximum M, then $\Delta y(M) = 0$; if $\mathbb{T}$ has a minimum m, then $\nabla y(m) = 0$.

Example 7.4 (A continuous problem of the calculus of variations – see Section 6 of [30])**.** *Consider the following problem, denoted in the sequel by (P):*

$$\max_{y(t)} f(k(T), a(T)) = \min_{y(t)} \left[-k(T)a(T)\right] = \min_{y(t)} K(T)a(T),$$

where

$$K(T) = -k(T) = \int_0^T e^{-\rho(T-t)} \left[c_0 + c_1 y(t) + c_2 y'^2(t) - y(t)p(t)\right] dt,$$

$$a(T) = \int_0^T e^{-\rho(T-t)} \left[\lambda y(t) + \beta\sqrt{y'(t) + b}\right] dt$$

with ρ *the discount rate (not to be confused with the backward jump operator* $\rho(t)$ *of time scales). For this problem the Euler–Lagrange equation* (7.6) *in the differential form can be written as*

$$\begin{aligned} a(T) \cdot e^{-\rho(T-t)} & \left[c_1 - p(t) - 2c_2(\rho y'(t) + y''(t))\right] \\ & + K(T) \cdot e^{-\rho(T-t)} \left[\lambda - \frac{\beta}{2}\left(\frac{\rho}{\sqrt{y'(t)+b}} - \frac{y''(t)}{2\sqrt{(y'(t)+b)^3}}\right)\right] = 0. \end{aligned} \quad (7.10)$$

The solution of the continuous problem (P) *is found by solving the Euler–Lagrange equation* (7.10)*. It turns out that this is a highly nonlinear differential equation of second order, for which no analytical solution is known. In other words, to solve the continuous problem one needs to apply a suitable discretization. This is exactly one of the main motivations of our study: to provide an appropriate theory of discretization.*

A discretization can always be done in two different ways: using the delta or the nabla approach. In the next example we consider four different discretizations for the problem (P) of Example 7.4 and the corresponding four discretizations of the Euler–Lagrange equation (7.10).

Example 7.5. *Consider a firm that wants to program its production and investment policies to reach a given production rate* $k(T)$*,* $T \in \mathbb{N}$*, and to maximize its future market competitiveness at time horizon* T*. Economic models, leading to the maximization of a variational functional, are presented below and are based on the following assumptions:*

1. *The firm competitiveness is measured by the function* $f(k(T), a(T))$*, which depends on the accumulated capital* $k(T)$ *and on the accumulated technology* $a(T)$*, both at time horizon* $t = T$*. Here, the function to measure the firm market competitiveness is assumed to be of form*

$$f(k(T), a(T)) = k(T)^{\gamma_1} a(T)^{\gamma_2} \quad (7.11)$$

   with given constants γ_1 *and* γ_2 *that measure the absolute and relative importance of capital and technology competitiveness, respectively.*

2. *The acquisition technology rate is given by the function* $g(y(t_{k+1}), \Delta y(t_k))$ *(delta version) or* $g(y(t_{k-1}), \nabla y(t_k))$ *(nabla version), where* $y(t_k)$ *is the sales rate at time* t_k*, which we assume equal to the actual production rate at the same point of time, that is,* $\Delta y(t_k)$ *(delta version) or* $\nabla y(t_k)$ *(nabla version) are the actual production rate change.*

3. *The firm starts operating at point* $t_0 = 0$ *and accumulates capital as*

$$K_\Delta(T) = \sum_{t_k=0}^{T-1} (1+\rho)^{t_k - T} (c_0 + c_1 y_{k+1} + c_2 (\Delta y_k)^2 - y_{k+1} p_{k+1}) \tag{7.12}$$

(delta version) or

$$K_\nabla(T) = \sum_{t_k=1}^{T} (1-\rho)^{T - t_k} (c_0 + c_1 y_{k-1} + c_2 (\nabla y_k)^2 - y_{k-1} p_{k-1}) \tag{7.13}$$

(nabla version), where ρ *is the discount rate,* $p_k = p(t_k)$ *is the unit product price,* $y_k = y(t_k)$ *is the sales rate at time* t_k*, and* $c(y_{k+1}, \Delta y_k)$ *(delta) or* $c(y_{k-1}, \nabla y_k)$ *(nabla) is the cost of producing* y_{k+1} *(delta) or* y_{k-1} *(nabla) units of product at time* t_{k+1} *(delta) or* t_{k-1} *(nabla) plus technology increases.*

4. *The accumulate technology is given by*

$$a_\Delta(T) = \sum_{t_k=0}^{T-1} (1+\rho)^{t_k - T} \left(\lambda y_{k+1} + \beta\sqrt{\Delta y_k + b}\right) \tag{7.14}$$

(delta version) or

$$a_\nabla(T) = \sum_{t_k=1}^{T} (1-\rho)^{T - t_k} \left(\lambda y_{k-1} + \beta\sqrt{\nabla y_k + b}\right) \tag{7.15}$$

(nabla version).

5. *The price-sales relationship regulating the market is given by the equation*

$$h(y_{k+1}, p_{k+1}) = (y_{k+1} - y_0)(p_{k+1} - p_0) - B = 0 \tag{7.16}$$

(delta version) or by the equation

$$h(y_{k-1}, p_{k-1}) = (y_{k-1} - y_0)(p_{k-1} - p_0) - B = 0 \tag{7.17}$$

(nabla version). There is an upper bound b *for the size of production rate change, so that* $|\Delta y_k| \leq b$ *(delta) or* $|\nabla y_k| \leq b$ *(nabla).*

6. *Two boundary conditions are given:*

$$y(0) = y_0, \quad y(T) = y_T, \tag{7.18}$$

which are the initial sales rate at point $t_0 = 0$ *and the target sales rate at the terminal point of time* $t_k = T$.

Then, the firm problem is stated as:

$$\max_{y_k} k(T)^{\gamma_1} a(T)^{\gamma_2}$$

subject to the hypotheses (7.11)–(7.18). *For illustrative purposes and to be coherent with Example 7.4 borrowed from [30], we assume* $\gamma_1 = \gamma_2 = 1$ *and transform the maximization problem into an equivalent minimization process:*

$$\min_{y_k}(-k(T))a(T) = \min_{y_k} K(T)a(T).$$

Each component of the objective functional $f(K(T), a(T))$ *may be discretized in two ways (using the delta or the nabla approach). Due to this reason, we obtain four different discrete problems of the calculus of variations:*

1. *Problem* $(P_{\Delta\nabla})$ *with cost functional* $\min_{y_k} K_\Delta(T)a_\nabla(T)$;
2. *Problem* $(P_{\nabla\Delta})$ *with cost functional* $\min_{y_k} K_\nabla(T)a_\Delta(T)$;
3. *Problem* $(P_{\Delta\Delta})$ *with cost functional* $\min_{y_k} K_\Delta(T)a_\Delta(T)$;
4. *Problem* $(P_{\nabla\nabla})$ *with cost functional* $\min_{y_k} K_\nabla(T)a_\nabla(T)$;

where $K_{\mathcal{D}}(T)$ *and* $a_{\mathcal{D}}(T)$, $\mathcal{D} \in \{\Delta, \nabla\}$, *are defined as in* (7.12)–(7.15). *With the notation of Section 7.1, such functionals consist of the following integrands:*

$$\begin{aligned}
f_{1\Delta} &= (1+\rho)^{t_k - T}(c_0 + c_1 y_{k+1} + c_2 (\Delta y_k)^2 - y_{k+1} p_{k+1}),\\
f_{1\nabla} &= (1-\rho)^{T - t_k}(c_0 + c_1 y_{k-1} + c_2 (\nabla y_k)^2 - y_{k-1} p_{k-1}),\\
f_{2\Delta} &= (1+\rho)^{t_k - T}\left(\lambda y_{k+1} + \beta\sqrt{\Delta y_k + b}\right),\\
f_{2\nabla} &= (1-\rho)^{T - t_k}\left(\lambda y_{k-1} + \beta\sqrt{\nabla y_k + b}\right),
\end{aligned}$$

where $f_{i\Delta} = f_{i\Delta}(t_k, y_{k+1}, \Delta y_k)$, $f_{i\nabla} = f_{i\Delta}(t_k, y_{k-1}, \nabla y_k)$, $i = 1, 2$, *and function* $f_{1\mathcal{D}}$ *is associated with functional* $K_{\mathcal{D}}(T)$ *and function* $f_{2\mathcal{D}}$ *is associated with functional* $a_{\mathcal{D}}(T)$, $\mathcal{D} \in \{\Delta, \nabla\}$. *Using the same discretization as the one from* (P) *to* $(P_{\Delta\nabla})$, *the Euler–Lagrange equation* (7.6) *is discretized into*

$$a_\nabla(T) \cdot \left(\frac{\partial f_{1\Delta}}{\partial y_{k+1}} - \Delta \frac{\partial f_{1\Delta}}{\partial \Delta y_k}\right) + K_\Delta(T) \cdot \left(\frac{\partial f_{2\nabla}}{\partial y_{k-1}} - \nabla \frac{\partial f_{2\nabla}}{\partial \nabla y_k}\right) = 0, \tag{7.19}$$

which for our economic problem (P) *takes the form*

$$a_{\nabla}(T)(1+\rho)^{t_k-T}\left[c_1-p_0+\frac{By_0}{(y_{k+1}-y_0)^2}-2c_2\left(\rho\Delta y_k+(1+\rho)\Delta^2 y_k\right)\right]$$
$$+K_{\Delta}(T)(1-\rho)^{T-t_k}\left[\lambda-\frac{\beta\left(\rho\sqrt{\nabla y_k+b}-\nabla\sqrt{\nabla y_k+b}\right)}{2\sqrt{\nabla y_k+b}\sqrt{\nabla y_{k-1}+b}}\right]=0, \quad ((EL_P)_{\Delta\nabla})$$

valid for $t_k\in\mathbb{T}_{\kappa}^{\kappa}$*. Note that we start with a given value of sales (or production) rate* y_0 *that the firm wants to improve (increase) in order to generate a profit. For this reason, the next values* y_k*,* $k>0$*, are assumed to be greater than the initial value* y_0*. This economic assumption, makes valid the Euler–Lagrange equation* $((EL_P)_{\Delta\nabla})$*. Indeed, it is known a priori, from economic insight, that* $y(t)$ *is an increasing function [30]. Similarly, the discretization from* (P) *into* $(P_{\nabla\Delta})$ *gives the discretized Euler–Lagrange equation*

$$a_{\Delta}(T)\cdot\left(\frac{\partial f_{1\nabla}}{\partial y_{k-1}}-\nabla\frac{\partial f_{1\nabla}}{\partial\nabla y_k}\right)+K_{\nabla}(T)\cdot\left(\frac{\partial f_{2\Delta}}{\partial y_{k+1}}-\Delta\frac{\partial f_{2\Delta}}{\partial\Delta y_k}\right)=0 \tag{7.20}$$

that, for our example, reads

$$a_{\Delta}(T)(1-\rho)^{T-t_k}\left[c_1-p_0+\frac{By_0}{(y_{k-1}-y_0)^2}-2c_2\left(\rho\nabla y_k+(1-\rho)\nabla^2 y_k\right)\right]$$
$$+K_{\nabla}(T)(1+\rho)^{t_k-T}\left[\lambda-\frac{\beta\left(\rho\sqrt{\Delta y_k+b}-\Delta\sqrt{\Delta y_k+b}\right)}{2\sqrt{\Delta y_k+b}\sqrt{\Delta y_{k+1}+b}}\right]=0, \quad ((EL_P)_{\nabla\Delta})$$

for $t_k\in\mathbb{T}_{\kappa}^{\kappa}$*; the discretization from* (P) *into* $(P_{\Delta\Delta})$ *leads to the discretized Euler–Lagrange equation*

$$a_{\Delta}(T)\cdot\left(\frac{\partial f_{1\Delta}}{\partial y_{k+1}}-\Delta\frac{\partial f_{1\Delta}}{\partial\Delta y_k}\right)+K_{\Delta}(T)\cdot\left(\frac{\partial f_{2\Delta}}{\partial y_{k+1}}-\Delta\frac{\partial f_{2\Delta}}{\partial\Delta y_k}\right)=0 \tag{7.21}$$

and to

$$a_{\Delta}(T)(1+\rho)^{t_k-T}\left[c_1-p_0+\frac{By_0}{(y_{k+1}-y_0)^2}-2c_2\left(\rho\Delta y_k+(1+\rho)\Delta^2 y_k\right)\right]$$
$$+K_{\Delta}(T)(1+\rho)^{t_k-T}\left[\lambda-\frac{\beta\left(\rho\sqrt{\Delta y_k+b}-\Delta\sqrt{\Delta y_k+b}\right)}{2\sqrt{\Delta y_k+b}\sqrt{\Delta y_{k+1}+b}}\right]=0, \quad ((EL_P)_{\Delta\Delta})$$

for $t_k\in\mathbb{T}^{\kappa^2}$*; while the discretization from* (P) *into problem* $(P_{\nabla\nabla})$ *gives*

$$a_{\nabla}(T)\cdot\left(\frac{\partial f_{1\nabla}}{\partial y_{k-1}}-\nabla\frac{\partial f_{1\nabla}}{\partial\nabla y_k}\right)+K_{\nabla}(T)\cdot\left(\frac{\partial f_{2\nabla}}{\partial y_{k-1}}-\nabla\frac{\partial f_{2\nabla}}{\partial\nabla y_k}\right)=0 \tag{7.22}$$

that reduces in our case to

$$a_{\nabla}(T)(1-\rho)^{T-t_k}\left[c_1-p_0+\frac{By_0}{(y_{k-1}-y_0)^2}-2c_2\left(\rho\nabla y_k+(1-\rho)\nabla^2 y_k\right)\right]$$
$$+K_{\nabla}(T)(1-\rho)^{T-t_k}\left[\lambda-\frac{\beta\left(\rho\sqrt{\nabla y_k+b}-\nabla\sqrt{\nabla y_k+b}\right)}{2\sqrt{\nabla y_k+b}\sqrt{\nabla y_{k-1}+b}}\right]=0, \quad ((EL_P)_{\nabla\nabla})$$

valid for $t_k \in \mathbb{T}_{\kappa^2}$. *As can be easily noticed, all the four discretizations of the continuous Euler–Lagrange equation* (7.6) *are different but consist of the same items. For this reason, we define:*

$$\gamma_{1\Delta} := \left(\frac{\partial f_{1\Delta}}{\partial y_{k+1}} - \Delta \frac{\partial f_{1\Delta}}{\partial \Delta y_k}\right), \quad \gamma_{1\nabla} := \left(\frac{\partial f_{1\nabla}}{\partial y_{k-1}} - \nabla \frac{\partial f_{1\nabla}}{\partial \nabla y_k}\right),$$

$$\gamma_{2\Delta} := \left(\frac{\partial f_{2\Delta}}{\partial y_{k+1}} - \Delta \frac{\partial f_{2\Delta}}{\partial \Delta y_k}\right), \quad \gamma_{2\nabla} := \left(\frac{\partial f_{2\nabla}}{\partial y_{k-1}} - \nabla \frac{\partial f_{2\nabla}}{\partial \nabla y_k}\right).$$

With such notations, the discretizations of the Euler–Lagrange equation (7.6) *are conveniently written in the following way:*

1. *equation* (7.19) *is equivalently written as*

$$a_\nabla(T)\gamma_{1\Delta} + K_\Delta(T)\gamma_{2\nabla} = 0, \quad t_k \in \mathbb{T}_\kappa^\kappa; \tag{7.23}$$

2. *equation* (7.20) *is equivalently written as*

$$a_\Delta(T)\gamma_{1\nabla} + K_\nabla(T)\gamma_{2\Delta} = 0, \quad t_k \in \mathbb{T}_\kappa^\kappa; \tag{7.24}$$

3. *equation* (7.21) *is equivalently written as*

$$a_\Delta(T)\gamma_{1\Delta} + K_\Delta(T)\gamma_{2\Delta} = 0, \quad t_k \in \mathbb{T}^{\kappa^2}; \tag{7.25}$$

4. *and equation* (7.22) *is equivalently written as*

$$a_\nabla(T)\gamma_{1\nabla} + K_\nabla(T)\gamma_{2\nabla} = 0, \quad t_k \in \mathbb{T}_{\kappa^2}. \tag{7.26}$$

7.1.3 Time-scale Euler–Lagrange equations in discrete time scales

The equation (7.25) coincides with the time-scale Euler–Lagrange delta equation given by [71, Corollary 3.4] while equation (7.26) coincides with the time-scale Euler–Lagrange equation given by [72, Corollary 3.4]. From our Corollary 7.3 it follows that such coincidence, between the direct discretization of the continuous Euler–Lagrange equation (7.6) and the discrete Euler–Lagrange equations (7.8)–(7.9) obtained from the calculus of variations on time scales, does not hold for mixed delta-nabla discretizations: neither (7.23) is a time-scale Euler–Lagrange equation (7.8) or (7.9) nor (7.24) is a time-scale Euler–Lagrange equation (7.8) or (7.9).

For the economic problem $(P_{\Delta\nabla})$ the Euler–Lagrange equations have the following form: the Euler–Lagrange equation (7.8) takes the form

$$\begin{aligned} &a_\nabla(T)\,(1+\rho)^{t_k-T}\left[c_1-p_0+\frac{By_0}{(y_{k+1}-y_0)^2}-2c_2\left(\rho\Delta y_k+(1+\rho)\Delta^2 y_k\right)\right]\\ &+K_\Delta(T)(1-\rho)^{T-t_k}\left(\lambda-\frac{\beta\left(\rho\sqrt{\nabla y_k+b}-(1-\rho)\Delta\sqrt{\nabla y_k+b}\right)}{2\sqrt{\nabla y_k+b}\sqrt{\nabla y_{k+1}+b}}\right) \qquad (EL^1_{P_{\Delta\nabla}})\\ &+\Delta\left[K_\Delta(T)(1-\rho)^{T-t_k}\left(\lambda-\frac{\beta\left(\rho\sqrt{\nabla y_k+b}-\nabla\sqrt{\nabla y_k+b}\right)}{2\sqrt{\nabla y_k+b}\sqrt{\nabla y_{k-1}+b}}\right)\right](t_{k+1})=0 \end{aligned}$$

for $t_k\in\mathbb{T}^\kappa_\kappa$, while the Euler–Lagrange equation (7.9) gives

$$\begin{aligned} &a_\nabla(T)\,(1+\rho)^{t_{k-1}-T}\left[c_1-p_0+\frac{By_0}{(y_k-y_0)^2}-2c_2\left(\rho\Delta y_k+\nabla\left(\Delta y_k\right)\right)\right]\\ &+K_\Delta(T)(1-\rho)^{T-t_k}\left(\lambda-\frac{\beta\left(\rho\sqrt{\nabla y_{k1}+b}-\nabla\sqrt{\nabla y_k+b}\right)}{2\sqrt{\nabla y_k+b}\sqrt{\nabla y_{k-1}+b}}\right)\\ &-\nabla\left[a_\nabla(T)(1+\rho)^{t_k-T}\left(c_1-p_0+\frac{By_0}{(y_k-y_0)^2}-2c_2\left(\rho\Delta y_k+(1+\rho)\Delta^2 y_k\right)\right)\right](t_{k-1})=0\\ &\qquad (EL^2_{P_{\Delta\nabla}}) \end{aligned}$$

for $t_k\in\mathbb{T}^\kappa_\kappa$.

For problem $(P_{\nabla\Delta})$ the Euler–Lagrange equations take the following form: the Euler–Lagrange equation (7.8) gives

$$\begin{aligned} &a_\Delta(T)\,(1-\rho)^{T-t_k-1}\left[c_1-p_0+\frac{By_0}{(y_k-y_0)^2}-2c_2\left(\rho\nabla y_k+\Delta(\nabla y_k)\right)\right]\\ &+K_\nabla(T)(1+\rho)^{t_k-T}\left[\lambda-\frac{\beta\left(\rho\sqrt{\Delta y_k+b}-\Delta\sqrt{\Delta y_k+b}\right)}{2\sqrt{\Delta y_k+b}\sqrt{\Delta y_{k+1}+b}}\right]\\ &+\Delta\left[a_\Delta(T)(1-\rho)^{T-t_k}\left[c_1-p_0+\frac{By_0}{(y_{k-1}-y_0)^2}-2c_2\left(\rho\nabla y_k+(1-\rho)\nabla^2 y_k\right)\right]\right](t_{k+1})=0\\ &\qquad (EL^1_{P_{\nabla\Delta}}) \end{aligned}$$

for $t_k\in\mathbb{T}^\kappa_\kappa$, and (7.9) gives

$$\begin{aligned} &a_\Delta(T)\,(1-\rho)^{T-t_k}\left[c_1-p_0+\frac{By_0}{(y_{k-1}-y_0)^2}-2c_2\left(\rho\nabla y_k+(1-\rho)\nabla^2 y_k\right)\right]\\ &+K_\nabla(T)(1+\rho)^{t_{k-1}-T}\left[\lambda-\frac{\beta\left(\rho\sqrt{\Delta y_k+b}-(1+\rho)\nabla\sqrt{\Delta y_k+b}\right)}{2\sqrt{\Delta y_k+b}\sqrt{\nabla y_k+b}}\right] \qquad (EL^2_{P_{\nabla\Delta}})\\ &-\nabla\left[K_\nabla(T)(1+\rho)^{t_k-T}\left[\lambda-\frac{\beta\left(\rho\sqrt{\Delta y_k+b}-\Delta\sqrt{\Delta y_k+b}\right)}{2\sqrt{\Delta y_k+b}\sqrt{\Delta y_{k+1}+b}}\right]\right](t_{k-1})=0 \end{aligned}$$

for $t_k\in\mathbb{T}^\kappa_\kappa$. Then the Euler–Lagrange equations $(EL^1_{P_{\Delta\nabla}})$ and $(EL^2_{P_{\Delta\nabla}})$ for $(P_{\Delta\nabla})$ are

$$a_\nabla(T)\gamma_{1\Delta}+K_\Delta(T)\left(\frac{\partial f_{2\nabla}}{\partial y_{k-1}}\circ\sigma-\Delta\frac{\partial f_{2\nabla}}{\partial\nabla y_k}\right)+\Delta\left[K_\Delta(T)\gamma_{2\nabla}\right]\circ\sigma=0,\quad t_k\in\mathbb{T}^\kappa_\kappa, \tag{7.27}$$

and

$$a_\nabla(T)\left(\frac{\partial f_{1\Delta}}{\partial y_{k+1}}\circ\rho-\nabla\frac{\partial f_{1\Delta}}{\partial\Delta y_k}\right)+K_\Delta(T)\gamma_{2\nabla}-\nabla\left[a_\nabla(T)\gamma_{1\Delta}\right]\circ\rho=0,\quad t_k\in\mathbb{T}_\kappa^\kappa, \tag{7.28}$$

respectively, and the Euler–Lagrange equations $(EL^1_{P_{\nabla\Delta}})$ and $(EL^2_{P_{\nabla\Delta}})$ for $(P_{\nabla\Delta})$ are

$$a_\Delta(T)\left(\frac{\partial f_{1\nabla}}{\partial y_{k-1}}\circ\sigma-\Delta\frac{\partial f_{1\nabla}}{\partial\nabla y_k}\right)+K_\nabla(T)\gamma_{2\Delta}+\Delta\left[a_\Delta(T)\gamma_{1\nabla}\right]\circ\sigma=0,\quad t_k\in\mathbb{T}_\kappa^\kappa, \tag{7.29}$$

and

$$a_\Delta(T)\gamma_{1\nabla}+K_\nabla(T)\left(\frac{\partial f_{2\Delta}}{\partial y_{k+1}}\circ\rho-\nabla\frac{\partial f_{2\Delta}}{\partial\Delta y_k}\right)-\nabla\left[K_\nabla(T)\gamma_{2\Delta}\right]\circ\rho=0,\quad t_k\in\mathbb{T}_\kappa^\kappa, \tag{7.30}$$

respectively.

For the convenience of the reader, we recall the introduced notations:

- P – the continuous economic problem describing a market policy of a firm, presented in Section 7.1.2;
- EL_P – the continuous Euler–Lagrange equation (7.6) associated to problem P (see (7.10));
- P_D – a discretization of problem P, in four possible forms: $D\in\{\Delta\Delta,\nabla\nabla,\Delta\nabla,\nabla\Delta\}$;
- $(EL_P)_D$ – a discretization of the Euler–Lagrange equation EL_P, in four different forms: $D\in\{\Delta\Delta,\nabla\nabla,\Delta\nabla,\nabla\Delta\}$;
- EL_{P_D} – discrete Euler–Lagrange equations associated to problem P_D, obtained from the calculus of variations on time scales (see Corollary 7.3).

7.1.4 Standard versus time-scale discretizations: $(EL_P)_D$ vs (EL_{P_D})

The discrepancy between direct discretization of the classical optimality conditions and the time-scale approach to the calculus of variations was discussed, from an embedding point of view, in [32]. Here we compare the results obtained from direct and time-scale discretizations for the more general problem (7.1)–(7.2), in concrete for the economic problem (P) discussed in Section 7.1.2. For illustrative purposes, the following values have been selected (borrowed from [30]):

$$\rho=0.05,\quad c_0=3,\quad c_1=0.5,\quad c_2=3,\quad T=3,$$
$$b=4,\quad \lambda=\frac{1}{2},\quad \beta=\frac{1}{4},\quad B=2,\quad y_0=2,\quad y_T=3.$$

Moreover, we fixed the time scale to be $\mathbb{T}=\{0,1,2,3\}$. In what follows we compare the candidates for solutions of the variational problems $(P_{\Delta\nabla})$, $(P_{\nabla\Delta})$, $(P_{\Delta\Delta})$, and $(P_{\nabla\nabla})$, obtained from the direct discretizations of the continuous Euler–Lagrange equation (Section 7.1.2) and the discrete time-scale Euler–Lagrange equations (Section 7.1.3). All calculations were done using the Computer Algebra System Maple, version 10 (see Appendix A). For problems $(P_{\Delta\Delta})$ and $(P_{\nabla\nabla})$ the discretization of the continuous Euler–Lagrange equation and the discrete time-scale Euler–Lagrange equations coincide. The Euler–Lagrange equation for problem $(P_{\Delta\Delta})$ is defined on $\mathbb{T}^{\kappa^2} = \{0, 1\}$ and we obtain a system of two equations with two unknowns y_1 and y_2 that leads to $y_1 = 2.322251304$ and $y_2 = 2.679109437$ with the cost functional value $K_\Delta(T)a_\Delta(T) = -16.97843026$. Similarly, the Euler–Lagrange equation for problem $(P_{\nabla\nabla})$ is defined on $\mathbb{T}_{\kappa^2} = \{2, 3\}$ and we obtain a system of two equations with two unknowns y_1 and y_2 that leads to $y_1 = 1.495415602$ and $y_2 = 2.228040364$ with the cost functional value $K_\nabla(T)a_\nabla(T) = -13.20842214$. As we show next, for hybrid delta-nabla discrete problems of the calculus of variations, the time-scale results seem superior.

Problem $(P_{\Delta\nabla})$

The Euler–Lagrange equations for problem $(P_{\Delta\nabla})$ are defined on $\mathbb{T}^\kappa_\kappa = \{1, 2\}$. Therefore, we obtain a system of equations with two unknowns y_1 and y_2. The discretized Euler–Lagrange equation $(EL_P)_{\Delta\nabla}$ gives

$$y_1 = 2.910488556, \quad y_2 = 2.970017180$$

with value of cost functional

$$K_\Delta(T)a_\nabla(T) = -10.11399047.$$

A better result is obtained using the discrete time-scale Euler–Lagrange equation $EL^1_{P_{\Delta\nabla}}$:

$$y_1 = 2.901851949, \quad y_2 = 2.967442285$$

with cost

$$K_\Delta(T)a_\nabla(T) = -10.30544712.$$

Problem $(P_{\nabla\Delta})$

The Euler–Lagrange equations for problem $(P_{\nabla\Delta})$ are also defined on $\mathbb{T}^\kappa_\kappa = \{1, 2\}$ and also lead to a system of two equations with the two unknowns y_1 and y_2. The discretized Euler–Lagrange equation $(EL_P)_{\nabla\Delta}$ gives

$$y_1 = 2.183517532, \quad y_2 = 2.446990272$$

with cost

$$K_\nabla(T)a_\Delta(T) = -19.09167089.$$

Our time-scale Euler–Lagrange equation $EL^2_{P_{\nabla\Delta}}$ gives better results:

$$y_1 = 2.186742579, \quad y_2 = 2.457402400$$

with cost

$$K_{\nabla}(T)a_{\Delta}(T) = -19.17699675.$$

The results are gathered in Table 7.1.

D	The value of the functional of (P_D), $\rho = 0.05$, for candidates to minimizers obtained from:		
	$(EL_P)_D$	$EL^1_{P_D}$	$EL^2_{P_D}$
$\Delta\nabla$	-10.11399047	-10.30544712	$-0.1537986252 \times 10^{-5}$
$\nabla\Delta$	-19.09167089	1020.105142	-19.17699675
$\Delta\Delta$	-16.97843026		
$\nabla\nabla$	-13.20842214		

Table 7.1: The value of the functional associated to problem P_D, $D \in \{\Delta\nabla, \nabla\Delta, \Delta\Delta, \nabla\nabla\}$, with $\rho = 0.05$, calculated using: (i) the direct discretization of the continuous Euler–Lagrange equation, that is, $(EL_P)_D$; (ii) discrete Euler–Lagrange equations EL_{P_D}, obtained from the calculus of variations on time scales with $\mathbb{T} = \mathbb{Z}$.

For comparison purposes, we have used the same values for the parameters as the ones available in [30]. We have, however, done simulations with other values of the parameters and the conclusion persists: in almost all cases the results obtained from our time-scale approach are better; hardly ever, they coincide with the classical method; never are worse. In particular, we changed the value of the discount rate, ρ, in the set $\{0.01, 0.02, 0.03, \ldots, 0.1\}$. This is motivated by the fact that this value depends much on the economic and politic situation. The case where the time-scale advantage is more visible is given in Table 7.2, which corresponds to a discount rate of 2% ($\rho = 0.02$). The interested reader can easily do his/her own simulations using the Maple code found in Appendix A.

7.2 The inflation and unemployment tradeoff

In this section we briefly recall the economic problem discussed in Chapter 2. The problem describes a strict relation between rate of inflation, p, and rate of unemployment, u, which entails a social loss. The Phillips tradeoff between p and u is defined as $p := -\beta u + \pi$, $\beta > 0$, where π is the expected rate of inflation. The government loss function, λ, is specified by $\lambda = u^2 + \alpha p^2$, $\alpha > 0$. The problem is to find the optimal function π that minimizes the total

D	The value of the functional of (P_D), $\rho = 0.02$, for candidates to minimizers obtained from:		
	$(EL_P)_D$	$EL^1_{P_D}$	$EL^2_{P_D}$
$\Delta\nabla$	-10.62044023	-10.70908681	0.00001078869584
$\nabla\Delta$	-21.05128963	$3.014255571 \times 10^{-8}$	-264.5250742
$\Delta\Delta$	-19.03571446		
$\nabla\nabla$	-14.19294557		

Table 7.2: The value of the functional associated to problem P_D, $D \in \{\Delta\nabla, \nabla\Delta, \Delta\Delta, \nabla\nabla\}$, with $\rho = 0.02$, calculated using: (i) the direct discretization of the continuous Euler–Lagrange equation, that is, $(EL_P)_D$; (ii) discrete Euler–Lagrange equations EL_{P_D}, obtained from the calculus of variations on time scales with $\mathbb{T} = \mathbb{Z}$.

social loss, in a finite time horizon T, subject to given boundary conditions $\pi(0) = \pi_0$ and $\pi(T) = \pi_T$, $\pi_0, \pi_T > 0$. For more details we invite the reader to see Chapter 2.

In the literature two types of inflation and unemployment models are available: the *continuous model*

$$\Lambda_C(\pi) = \int_0^T \lambda(\pi(t), \pi'(t)) e^{-\delta t} dt \longrightarrow \min \tag{7.31}$$

subject to given boundary conditions

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T, \tag{7.32}$$

and the *discrete model*

$$\Lambda_D(\pi) = \sum_{t=0}^{T-1} \lambda(\pi(t), \Delta\pi(t))(1+\delta)^{-t} \longrightarrow \min, \tag{7.33}$$

also subject to the boundary conditions (7.32). In both cases, (7.31) and (7.33),

$$\lambda(t, \pi, \upsilon) := \left(\frac{\upsilon}{\beta j}\right)^2 + \alpha \left(\frac{\upsilon}{j} + \pi\right)^2. \tag{7.34}$$

Here we propose the more general *time-scale model*

$$\Lambda_{\mathbb{T}}(\pi) = \int_0^T \lambda(t, \pi(t), \pi^\Delta(t)) e_{\ominus\delta}(t, 0) \Delta t \longrightarrow \min \tag{7.35}$$

subject to boundary conditions (7.32) and with λ defined by (7.34). Clearly, the time-scale model includes both the discrete and continuous models as special cases: our time-scale functional (7.35) reduces to (7.31) when $\mathbb{T} = \mathbb{R}$ and to (7.33) when $\mathbb{T} = \mathbb{Z}$.

Let us consider the problem

$$\mathcal{L}[\pi] = \int_0^T L(t, \pi(t), \pi^{\Delta}(t))\Delta t \longrightarrow \min \tag{7.36}$$

in the class of functions $\pi \in C^1_{rd}([0,T])$ subject to boundary conditions

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T. \tag{7.37}$$

We are particularly interested in the situation where

$$L(t, \pi(t), \pi^{\Delta}(t)) = \left[\left(\frac{\pi^{\Delta}(t)}{\beta j}\right)^2 + \alpha\left(\frac{\pi^{\Delta}(t)}{j} + \pi(t)\right)^2\right] e_{\ominus\delta}(t,0). \tag{7.38}$$

For simplicity, along this section we use the notation $[\pi](t) := (t, \pi(t), \pi^{\Delta}(t))$.

Theorem 7.6. *If $\hat{\pi} \in C^2_{rd}([0,T])$ is a local minimizer to problem (7.36)–(7.37) and the graininess function μ is a delta differentiable function on $[0,T]^{\kappa}_{\mathbb{T}}$, then $\hat{\pi}$ satisfies the Euler–Lagrange equation*

$$[L_v[\pi](t)]^{\Delta} = \left(1 + \mu^{\Delta}(t)\right) L_y[\pi](t) + \mu^{\sigma}(t)\left[L_y[\pi](t)\right]^{\Delta} \tag{7.39}$$

for all $t \in [0,T]^{\kappa^2}_{\mathbb{T}}$.

Proof. If $\hat{\pi}$ is a local minimizer to (7.36)–(7.37), then, by Theorem 3.8, $\hat{\pi}$ satisfies the following equation:

$$L_v[\pi](t) = \int_0^{\sigma(t)} L_y[\pi](\tau)\Delta\tau + c.$$

Using the properties of the delta integral (Theorem 1.17), we can write that $\hat{\pi}$ satisfies

$$L_v[\pi](t) = \int_0^t L_y[\pi](\tau)\Delta\tau + \mu(t)L_y[\pi](t) + c. \tag{7.40}$$

Taking the delta derivative to both sides of (7.40), we obtain equation (7.39). □

Using Theorem 7.6, we can immediately write the classical Euler–Lagrange equations for the continuous (7.31) and the discrete (7.33) models.

Example 7.7. *Let $\mathbb{T} = \mathbb{R}$. Then, $\mu \equiv 0$ and (7.39) with the Lagrangian (7.38) reduces to*

$$\left(1 + \alpha\beta^2\right)\pi''(t) - \delta\left(1 + \alpha\beta^2\right)\pi'(t) - \alpha j\beta^2\left(\delta + j\right) = 0. \tag{7.41}$$

This is the Euler–Lagrange equation for the continuous model (7.31).

Example 7.8. *Let $\mathbb{T} = \mathbb{Z}$. Then, $\mu \equiv 1$ and* (7.39) *with the Lagrangian* (7.38) *reduces to*

$$\left(\alpha j \beta^2 - \alpha\beta^2 - 1\right) \Delta^2\pi(t) + \left(\alpha j^2\beta^2 + \delta\alpha\beta + \delta\right) \Delta\pi(t) + \alpha j \beta^2 \left(\delta + j\right) \pi(t) = 0. \tag{7.42}$$

This is the Euler–Lagrange equation for the discrete model (7.33).

Corollary 7.9. *Let $\mathbb{T} = h\mathbb{Z}$, $h > 0$, $\pi_0, \pi_T \in \mathbb{R}$, and $T = Nh$ for a certain integer $N > 2h$. The difference forward operator in $\mathbb{T} = h\mathbb{Z}$ is defined as $\Delta_h f(t) = \frac{f(t+h)-f(t)}{h}$. If $\hat{\pi}$ is a solution to the problem*

$$\Lambda_h(\pi) = \sum_{t=0}^{T-h} L(t, \pi(t), \Delta_h\pi(t))h \longrightarrow \min,$$

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T,$$

then $\hat{\pi}$ satisfies the Euler–Lagrange equation

$$\Delta_h L_v[\pi](t) = L_y[\pi](t) + h \cdot \Delta_h L_y[\pi](t) \tag{7.43}$$

for all $t \in \{0, \ldots, T - 2h\}$.

Proof. Follows from Theorem 7.6 by choosing $\mathbb{T}$ to be the periodic time scale $h\mathbb{Z}$, $h > 0$. □

Example 7.10. *The Euler–Lagrange equation for problem* (7.35) *with the Lagrangian* (7.38) *on $\mathbb{T} = h\mathbb{Z}$, $h > 0$, is given by* (7.43)*:*

$$(1 + \alpha\beta^2 - \alpha\beta^2 jh)\Delta_h^2\pi + (-\delta - \alpha\beta^2\delta - \alpha\beta^2 j^2 h)\Delta_h\pi + (-\alpha\beta^2\delta j - \alpha\beta^2 j^2)\pi = 0. \tag{7.44}$$

Assume that $1 + \alpha\beta^2 - \alpha\beta^2 jh \neq 0$. Then equation (7.44) *is regressive and we can use the theorems in the theory of dynamic equations on time scales (see Section 1.4), in order to find its general solution. Introducing the quantities*

$$\Omega := 1 + \alpha\beta^2 - \alpha\beta^2 jh, \quad A := -\left(\delta + \alpha\beta^2\delta + \alpha\beta^2 j^2 h\right), \quad B := \alpha\beta^2 j(\delta + j), \tag{7.45}$$

we rewrite equation (7.44) *as*

$$\Delta_h^2\pi + \frac{A}{\Omega}\Delta_h\pi - \frac{B}{\Omega}\pi = 0. \tag{7.46}$$

The characteristic equation for (7.46) *is*

$$\varphi(\lambda) = \lambda^2 + \frac{A}{\Omega}\lambda - \frac{B}{\Omega} = 0$$

with determinant

$$\zeta = \frac{A^2 + 4B\Omega}{\Omega^2}. \tag{7.47}$$

In general, we have three different cases depending on the sign of the determinant ζ: $\zeta > 0$, $\zeta = 0$ and $\zeta < 0$. However, because $\pi : \mathbb{T} \to \mathbb{R}$, the last case cannot occur. The two possible cases are:

1. *If $\zeta > 0$, then we have two different characteristic roots:*

$$\lambda_1 = \frac{-A + \sqrt{A^2 + 4B\Omega}}{2\Omega} > 0 \text{ and } \lambda_2 = \frac{-A - \sqrt{A^2 + 4B\Omega}}{2\Omega} < 0,$$

and by Theorems 1.42 and 1.41, and by using (1.8) *we get that*

$$\pi(t) = C_1 (1 + \lambda_1 h)^{\frac{t}{h}} + C_2 (1 + \lambda_2 h)^{\frac{t}{h}}$$

is the general solution to (7.46)*, where C_1 and C_2 are constants determined by using the boundary conditions* (7.37)*.*

2. *If $\zeta = 0$ and $2\Omega \neq Ah$ (or $A+2hB \neq 0$), then by Theorems 1.46 and 1.41, Example 1.18 and* (1.8)*, we get that*

$$\pi(t) = K_1 \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} + K_2 \left(1 - \frac{A}{2\Omega}h\right)^{\frac{t}{h}} \frac{2\Omega t}{2\Omega - Ah}$$

is the general solution to (7.46)*, where K_1 and K_2 are constants, determined by using the boundary conditions* (7.37)*.*

In certain cases one can show that the Euler–Lagrange extremals are indeed minimizers. In particular, this is true for the Lagrangian (7.38) under study. We recall the notion of jointly convex function (see, e.g., [73, Definition 1.6]).

Definition 7.11. *Function $(t, y, v) \mapsto L(t, y, v) \in C^1\left([a, b]_{\mathbb{T}} \times \mathbb{R}^2; \mathbb{R}\right)$ is jointly convex in (y, v) if*

$$L(t, y + y_0, v + v_0) - L(t, y, v) \geq L_y(t, y, v)y_0 + L_v(t, y, v)v_0$$

for all (t, y, v), $(t, y + y^0, v + v^0) \in [a, b]_{\mathbb{T}} \times \mathbb{R}^2$.

Theorem 7.12. *Let $(t, y, v) \mapsto L(t, y, v)$ be jointly convex with respect to (y, v) for all $t \in [a, b]_{\mathbb{T}}$. If $\hat{y}$ is a solution to the Euler–Lagrange equation* (3.8)*, then $\hat{y}$ is a global minimizer to* (3.6)*–*(3.7)*.*

Proof. Since L is jointly convex with respect to (y, v) for all $t \in [a, b]_{\mathbb{T}}$,

$$\begin{aligned}\mathcal{L}[y] - \mathcal{L}[\hat{y}] &= \int_a^b [L(t, y(t), y^{\Delta}(t)) - L(t, \hat{y}(t), \hat{y}^{\Delta}(t))]\Delta t \\ &\geq \int_a^b \left[L_y(t, \hat{y}(t), \hat{y}^{\Delta}(t)) \cdot (y(t) - \hat{y}(t)) + L_v(t, \hat{y}(t), \hat{y}^{\Delta}(t)) \cdot (y^{\Delta}(t) - \hat{y}^{\Delta}(t))\right] \Delta t\end{aligned}$$

for any admissible path y. Let $h(t) := y(t) - \hat{y}(t)$. Using boundary conditions (3.7), we obtain that

$$\mathcal{L}[y] - \mathcal{L}[\hat{y}] \geq \int_a^b h^\Delta(t) \left[-\int_a^{\sigma(t)} L_y(\tau, \hat{y}(\tau), \hat{y}^\Delta(\tau))\Delta\tau + L_v(t, \hat{y}(t), \hat{y}^\Delta(t)) \right] \Delta t$$

$$+ h(t) \int_a^b L_2(t, \hat{y}(t), \hat{y}^\Delta(t))\Delta t \Big|_a^b$$

$$= \int_a^b h^\Delta(t) \left[-\int_a^{\sigma(t)} L_y(\tau, \hat{y}(\tau), \hat{y}^\Delta(\tau))\Delta\tau + L_v(t, \hat{y}(t), \hat{y}^\Delta(t)) \right] \Delta t.$$

From (3.8) it follows that

$$\mathcal{L}[y] - \mathcal{L}[\hat{y}] \geq \int_a^b h^\Delta(t) c \Delta t = 0$$

for some $c \in \mathbb{R}$. Hence, $\mathcal{L}[y] - \mathcal{L}[\hat{y}] \geq 0$. □

Theorem 7.13 (Solution to the total social loss problem of the calculus of variations in the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$)**.** *Let us consider our economic problem*

$$\Lambda_h(\pi) = \sum_{t=0}^{T-h} \left[\left(\frac{\Delta_h \pi(t)}{\beta j} \right)^2 + \alpha \left(\frac{\Delta_h \pi(t)}{j} + \pi(t) \right)^2 \right] \left(1 - \frac{h\delta}{1 + h\delta} \right)^{\frac{t}{h}} h \longrightarrow \min, \tag{7.48}$$

$$\pi(0) = \pi_0, \quad \pi(T) = \pi_T,$$

with $\mathbb{T} = h\mathbb{Z}$, $h > 0$, and the delta derivative given by (1.1). More precisely, let $T = Nh$ for a certain integer $N > 2h$, $\alpha, \beta, \delta, \pi_0, \pi_T \in \mathbb{R}^+$, and $0 < j \leq 1$ be such that $h > 0$ and $1 + \alpha\beta^2 - \alpha\beta^2 jh \neq 0$. Let Ω, A and B be given as in (7.45).

1. *If $A^2 + 4B\Omega > 0$, then the solution $\hat{\pi}$ to problem (7.48) is given by*

$$\hat{\pi}(t) = C \left(1 - \frac{A - \sqrt{A^2 + 4B\Omega}}{2\Omega} h \right)^{\frac{t}{h}} + (\pi_0 - C) \left(1 - \frac{A + \sqrt{A^2 + 4B\Omega}}{2\Omega} h \right)^{\frac{t}{h}}, \tag{7.49}$$

$t \in \{0, \ldots, T - 2h\}$, where

$$C := \frac{\pi_T - \pi_0 \left(\frac{2\Omega - hA - h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}}}{\left(\frac{2\Omega - hA + h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}} - \left(\frac{2\Omega - hA - h\sqrt{A^2 + 4B\Omega}}{2\Omega} \right)^{\frac{T}{h}}}.$$

2. *If $A^2 + 4B\Omega = 0$ and $2\Omega \neq Ah$ (or $A + 2hB \neq 0$), then the solution $\hat{\pi}$ to problem (7.48) is given by*

$$\hat{\pi}(t) = \left(1 - \frac{A}{2\Omega} h \right)^{\frac{t}{h}} \pi_0 + \left(1 - \frac{A}{2\Omega} h \right)^{\frac{t}{h}} \left[\pi_T \left(\frac{2\Omega}{2\Omega - Ah} \right)^{\frac{T}{h}} - \pi_0 \right] \frac{t}{T}, \tag{7.50}$$

$t \in \{0, \ldots, T-2h\}$.

Proof. From Example 7.10, $\hat{\pi}$ satisfies the Euler–Lagrange equation for problem (7.48). Moreover, the Lagrangian of functional Λ_h of (7.48) is a convex function because it is the sum of convex functions. Hence, by Theorem 7.12, $\hat{\pi}$ is a global minimizer. □

7.3 State of the art

The results of Section 7.1 are submitted [40] and the results of Section 7.2 are published in [35]. The original results of the paper [35] were presented in The International Conference on Pure and Applied Mathematics, ICPAM'12, May 28-30, 2012, Guelma, Algeria; and at the 5th Podlasie Conference on Mathematics, June 25-28, 2012, in a contributed session entitled "Applications of Mathematics in Economy and Finance".

Conclusions and Future Work

This Ph.D. thesis had two major objectives: to develop the calculus of variations on an arbitrary time scale and to present some applications of this theory in economics. We claim that economics is an excellent area where the time-scale calculus can be useful.

We started with two inverse problems of the calculus of variations, which have not been studied before in the time-scale framework. First we derive a general form of a variational functional having an extremum at a given function y_0 under the assumption of Euler–Lagrange and strengthened Legendre conditions (Theorem 4.2). Next we considered a new approach to the inverse problem of the calculus of variations using an integral perspective instead of the classical differential point of view. In order to deal with this problem, we introduced new definitions of self-adjointness of an integro-differential equation and its equation of variation. We proved a necessary condition for an integro-differential equation to be an Euler–Lagrange equation on an arbitrary time scale $\mathbb{T}$ (Theorem 4.12). Next we turn to the nabla approach of the calculus of variations, and we proved an Euler–Lagrange type equation and a transversality condition for generalized infinite horizon problems (Theorem 5.6). The Lagrangian depends on the independent variable, an unknown function and its nabla derivative, as well as a nabla indefinite integral that depends on the unknown function. Next we develop the calculus of variations for a functional that is a composition of a certain scalar function with the delta and nabla integrals of a vector valued field. For this problem we obtain delta-nabla Euler–Lagrange equations in integral (Theorem 6.3) and differential (Theorem 7.1) forms, and necessary optimality conditions for isoperimetric problems (Theorem 6.10).

With respect to applications to economics, we investigate the process of discretization of economic models. Firstly, we work on a model that describes the market policy of a firm. We consider two discrete minimization delta-nabla problems ($P_{\Delta\nabla}$ and $P_{\nabla\Delta}$) for which the time-scale approach leads to better results (smaller values for the respective objective functional) than the ones obtained by a direct discretization of the continuous necessary optimality condition. It might be concluded that the time-scale theory of the calculus of variations leads to more precise results than the standard methods of discretization. The latter economic problem describes the relation between inflation and unemployment and its inflict to social loss. This tradeoff is also presented by Phillips curve.

Some of the possible directions of future research are:

- We would like to generalize our mixed delta-nabla results, in particular Theorem 7.1, for infinite horizon variational problems on time scales.

- We can consider an economic model with infinite time horizon and compare the values of the functional in different time scales (including the classical ones, i.e., $\mathbb{R}$ and $\mathbb{Z}$).

- With respect to our time-scale model describing the tradeoff between inflation and unemployment, it is interesting to work on a set of real data and check whether it is possible, or not, to find a time scale for which our functional approximates reality sufficiently well.

We end this Ph.D. thesis with a list of author's publications done during the Ph.D. studies: [35–40]. We are grateful to the Awarding Committee of the Symposium on Differential Equations and Difference Equations (SDEDE 2014), Homburg/Germany, 5th-8th September 2014, for awarding us with the Bernd Aulbach Prize 2014 for students.

Appendix A

Appendix: Maple Code

We provide here all the definitions and computations done in Maple for the problems considered in Section 7.1.4. The definitions follow closely the notations introduced along Section 7.1, and should be clear even for readers not familiar with the Computer Algebra System Maple.

```
> restart:
> rho := 5/100:
> c0 := 3:
> lambda := 1/2:
> c1 := 1/2:
> c2 := 3:
> p0 := 1:
> y0 := 1:
> b := 4:
> beta := 1/4:
> B := 2:
> T := 3:
> y(0) := 2:
> y(T) := 3:
> TimeScale := [seq(i,i=0..T)];
```

$$TimeScale := [0, 1, 2, 3]$$

```
> Sigma := t-> piecewise(t < T, t+1, t):
> Rho := t -> piecewise(t > 0, t-1, t):
> Delta := f -> f@Sigma-f:
> Nabla := f -> f-f@Rho:
> KDelta := sum((1+rho)^(t-T)*(c0+c1*(y@Sigma)(t)+c2*(Delta(y)(t))^2
    -(y@Sigma)(t)*p0-(B*(y@Sigma)(t))/((y@Sigma)(t)-y0)),t=0..T-1):
> KNabla := sum((1-rho)^(T-t)*(c0+c1*(y@Rho)(t)+c2*(Nabla(y)(t))^2
```

```
    -(y@Rho)(t)*p0-(B*(y@Rho)(t))/((y@Rho)(t)-y0)),t=1..T):
> aDelta := sum((1+rho)^(t-T)*(lambda*(y@Sigma)(t)
    +beta*sqrt(Delta(y)(t)+b)),t=0..T-1):
> aNabla := sum((1-rho)^(T-t)*(lambda*(y@Rho)(t)
    +beta*sqrt(Nabla(y)(t)+b)),t=1..T):
> Functional_PDN := subs({y(1)=y1,y(2)=y2},KDelta*aNabla):
> Functional_PND := subs({y(1)=y1,y(2)=y2},KNabla*aDelta):
> Functional_PDD := subs({y(1)=y1,y(2)=y2},KDelta*aDelta):
> Functional_PNN := subs({y(1)=y1,y(2)=y2},KNabla*aNabla):
> gamma1delta := t -> (1+rho)^(t-T)*(((c1-p0+(B*y0)/(((y@Sigma)(t)-y0)^2)))
    -2*c2*(rho*Delta(y)(t)+(1+rho)*Delta(Delta(y))(t))):
> gamma1nabla := t -> (1-rho)^(T-t)*((c1-p0+(B*y0)/(((y@Rho)(t)-y0)^2))
    -2*c2*(rho*Nabla(y)(t)+(1-rho)*Nabla(Nabla(y))(t))):
> gamma2delta := t -> (1+rho)^(t-T)*(lambda-(beta*(rho*sqrt(Delta(y)(t)+b)
    -(Delta(unapply(sqrt(Delta(y)(s)+b),s))(t))))/(2*sqrt(Delta(y)(t)+b)
    *sqrt((Delta(y)@Sigma)(t)+b))):
> gamma2nabla := t -> (1-rho)^(T-t)*(lambda
    -(beta*(rho*sqrt(Nabla(y)(t)+b)-Nabla(unapply(sqrt(Nabla(y)(s)+b),s))(t)))
    /(2*sqrt(Nabla(y)(t)+b)*sqrt((Nabla(y)@Rho)(t)+b))):
> # now we define the 4 problems that are considered in the paper
> # discretization of the continuous E-L equations
> # Problem Delta Nabla PDN
> # domain T_{kappa}^{kappa}
> PDN := t -> aNabla*gamma1delta(t)+KDelta*gamma2nabla(t):
> # Problem  Nabla Delta PND
> # domain T_{kappa}^{kappa}
> PND := t -> aDelta*gamma1nabla(t)+KNabla*gamma2delta(t):
> # Problem Delta Delta  PDD
> # domain T^{kappa^2}
> PDD := t -> aDelta*gamma1delta(t)+KDelta*gamma2delta(t):
> # Problem  Nabla Nabla PNN
> # domain T_{kappa^2}
> PNN := t -> aNabla*gamma1nabla(t)+KNabla*gamma2nabla(t):
> eqPDN := subs({y(1)=y1,y(2)=y2},{PDN(1)=0,PDN(2)=0}):
> SolutionPDN := fsolve(eqPDN,{y1,y2});
```

$$SolutionPDN := \{y1 = 2.910488556, y2 = 2.970017180\}$$

```
> subs(SolutionPDN,Functional_PDN);
```

$$-10.11399047$$

```
> eqPND := subs({y(1)=y1,y(2)=y2},{PND(1)=0,PND(2)=0}):
```

```
> SolutionPND := fsolve(eqPND,{y1,y2});
```

$$SolutionPND := \{y1 = 2.183517532, y2 = 2.446990272\}$$

```
subs(SolutionPND,Functional_PND);
```

$$-19.09167089$$

```
> eqPDD := subs({y(1)=y1,y(2)=y2},{PDD(0)=0,PDD(1)=0}):
> SolutionPDD := fsolve(eqPDD,{y1,y2});
```

$$SolutionPDD := \{y1 = 2.322251304, y2 = 2.679109437\}$$

```
> subs(SolutionPDD,Functional_PDD);
```

$$-16.97843026$$

```
> eqPNN := subs({y(1)=y1,y(2)=y2},{PNN(2)=0,PNN(3)=0}):
> SolutionPNN := fsolve(eqPNN,{y1,y2});
```

$$SolutionPNN := \{y1 = 1.495415602, y2 = 2.228040364\}$$

```
> subs(SolutionPNN,Functional_PNN);
```

$$-13.20842214$$

```
> # discretization of the time scale Euler-Lagrange equations
> # domain T_{kappa}^{kappa}
> part1 := t -> lambda*(1-rho)^(T-Sigma(t)):
> part2 := t ->(beta*(1-rho)^(T-Sigma(t))*((rho*sqrt(Nabla(y)(t)+b))
    -(1-rho)*(Delta(unapply(sqrt(Nabla(y)(s)+b),s))(t))))
    /(2*sqrt(Nabla(y)(t)+b)*sqrt(Delta(y)(t)+b)):
> part3 := t -> (1+rho)^(Rho(t)-T)*(c1-p0+(B*y0)/((y(t)-y0)^2)):
> part4 := t -> 2*c2*(1+rho)^(Rho(t)-T)
    *(rho*Delta(y)(t)+(y@Sigma)(t)-2*y(t)+(y@Rho)(t)):
> partDelta := Delta(unapply(KDelta*gamma2nabla(t),t))@Sigma:
> partNabla := Nabla(unapply(aNabla*gamma1delta(t),t))@Rho:
```

```
> # E-L equation (7.8) for Problem Delta Nabla
```

```
> EL_delta := t -> aNabla*gamma1delta(t)+KDelta*(part1(t)-part2(t))+partDelta(t):
```

```
> # E-L equation (7.9) for Problem Delta Nabla
```

```
> EL_nabla := t -> aNabla*(part3(t)-part4(t))+KDelta*gamma2nabla(t)-partNabla(t):
> # systems of E-L equations for Problem Delta Nabla
> EL_delta_system := subs({y(1)=y1,y(2)=y2},{EL_delta(1)=0,EL_delta(2)=0}):
> Solution_EL_eqs_system_delta_version := fsolve(EL_delta_system,{y1,y2});
```

$$Solution_{E}L_{e}qs_{s}ystem_{d}elta_{v}ersion := \{y1 = 2.901851949, y2 = 2.967442285\}$$

```
> subs(Solution_EL_eqs_system_delta_version,Functional_PDN);
```

$$-10.30544712$$

```
> EL_nabla_system := subs({y(1)=y1,y(2)=y2},{EL_nabla(1)=0,EL_nabla(2)=0}):
> Solution_EL_eqs_system_nabla_version := fsolve(EL_nabla_system,{y1,y2});
```

$$\{y1 = 0.5930298703, y2 = 1.090438395\}$$

```
subs(Solution_EL_eqs_system_nabla_version,Functional_PDN);
```

$$-0.000001537986252$$

```
> # E-L equations for Problem Nabla Delta
> part5 := t -> (1-rho)^(T-Sigma(t))*(c1-p0+(B*y0)/((y(t)-y0)^2)):
> part6 := t -> 2*c2*(1-rho)^(T-Sigma(t))*(rho*(Nabla(y)(t))+(Delta(Nabla(y))(t))):
> part7 := t -> lambda*(1+rho)^(Rho(t)-T):
> part8 := t -> (1+rho)^(Rho(t)-T)*((beta*(rho*sqrt(Delta(y)(t)+b)
    -(1+rho)*Nabla(unapply(sqrt(Delta(y)(s)+b),s))(t)))
    /(2*sqrt(Delta(y)(t)+b)*sqrt(Nabla(y)(t)+b))):
> partDelta2 := Delta(unapply(aDelta*gamma1nabla(t),t))@Sigma:
> partNabla2 := Nabla(unapply(KNabla*gamma2delta(t),t))@Rho:
```

```
> # E-L equation (7.8) for Problem Nabla Delta
```

```
> EL_delta2 := t -> KNabla*gamma2delta(t)+aDelta*(part5(t)-part6(t))+partDelta2(t):
```

```
> # E-L equation (7.9) for Problem Nabla Delta
```

```
> EL_nabla2 := t -> KNabla*(part7(t)-part8(t))+aDelta*gamma1nabla(t)-partNabla2(t):
> # systems of E-L equations for Problem Nabla Delta
> EL_delta2_system := subs({y(1)=y1,y(2)=y2},{EL_delta2(1)=0,EL_delta2(2)=0}):
> Solution_EL_eqs_system_delta2_version := fsolve(EL_delta2_system,{y1,y2});
```

$$Solution_{E}L_{e}qs_{s}ystem_{d}elta2_{v}ersion := \{y1 = 7.879260741, y2 = 4.775003718\}$$

```
> subs(Solution_EL_eqs_system_delta2_version,Functional_PND);
```

$$1020.105142$$

```
> EL_nabla2_system := subs({y(1)=y1,y(2)=y2},{EL_nabla2(1)=0,EL_nabla2(2)=0}):
> Solution_EL_eqs_system_nabla2_version := fsolve(EL_nabla2_system,{y1,y2});
```

$$Solution_{E}L_{e}qs_{s}ystem_{n}abla2_{v}ersion := \{y1 = 2.186742579, y2 = 2.457402400\}$$

```
> subs(Solution_EL_eqs_system_nabla2_version,Functional_PND);
```

$$-19.17699675$$

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