**Universidade de Aveiro** Departamento de Matemática,
**2014**

**Lirong Huang**

# Resultados de multiplicidade para sistemas do tipo Schrödinger-Poisson

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Dissertação apresentada à Universidade de Aveiro e à Universidade do Minho para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programma Doutoral em Matemática e Aplicações – PDMA 2010-2014 – da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação científica do Doutor Eugénio Alexandre Miguel Rocha, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro.

Thesis submitted to the University of Aveiro and University of Minho in fulfilment of the requirements for the degree of *Doctor of Philosophy* in Mathematics, Doctoral Program in Mathematics and Applications – PDMA 2010-2014 – of the University of Aveiro and University of Minho, under the supervision of Professor Doctor Eugénio Alexandre Miguel Rocha.

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**palavras-chave**

Sistemas de Schrödinger-Poisson; Métodos variacionais; Existência e multiciplidade de soluções; Soluções positivas e com mudança de sinal.

**resumo**

Nesta tese, estudamos a existência e a multiplicidade de soluções da seguinte classe de sistemas denominada de Schrödinger-Poisson:

$$\begin{cases} -\Delta u + u + l(x)\phi u = \kappa(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

onde $l \in L^2(\mathbb{R}^3)$ ou $l \in L^\infty(\mathbb{R}^3)$. Consideram-se não-linearidades $\kappa$ que satisfazem um dos seguintes casos:

**(i)** potências que envolvem um expoente sub-crítico, da forma $\kappa(x,u) = k(x)|u|^{p-2}u + \mu h(x)u$, $(4 \leq p < 2^*)$, sendo $k$ uma função com sinal indefinido e $h$ uma função positiva;

**(ii)** caso geral de uma não-linearidade indefinida, da forma $\kappa(x,u) = k(x)g(u) + \mu h(x)u$, sendo $k$ uma função com sinal indefinido e $h$ uma função positiva;

**(iii)** potências que envolvem o expoente crítico, da forma $\kappa(x,u) = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u$ $(2 \leq q < 2^*)$.

Convém salientar que esta tese tem três principais inovações, as quais ultrapassam dificuldades geradas pela natureza dos problemas estudados. Primeiro, como um relator anónimo referiu, este é o primeiro trabalho em que se trata a existência de várias soluções de sistemas de Schrödinger-Poisson com não-linearidade indefinida.

Segundo, neste estudo encontrou-se um fenómeno interessante, ver Capítulos 2 e 3, nomeadamente, não ser necessária a condição $\int_{\mathbb{R}^3} k(x)e_1^p dx < 0$ no caso indefinido e não-coercivo, sendo $e_1$ a função associada ao primeiro valor próprio de $-\Delta + id$ em $H^1(\mathbb{R}^3)$ com peso $h$. Note-se que foi demonstrado que uma condição semelhante é condição necessária e suficiente na existência de soluções positivas para equações elíticas semilineares com não-linearidades indefinidas em domínios limitados (ver e.g. Alama-Tarantello, *Calc. Var. PDE* **1** (1993), 439–475), ou ser uma condição suficiente na existência de soluções positivas para equações elíticas semilineares com não-linearidades indefinidas em $\mathbb{R}^N$ (see e.g. Costa-Tehrani, *Calc. Var. PDE* **13** (2001), 159–189). Adicionalmente, o método utilizado pode ser utilizado para estudar outros aspetos dos sistemas de Schrödinger-Poisson, permite também estudar sistemas de Kirchhoff e sistemas de Schrödinger quasilineares.

Por fim, para obter soluções com mudança de sinal no Cap. 5, segue-se a ideia de Hirano-Shioji, *Proc. Roy. Soc. Edinburgh Sect. A* **137** *(2007), 333*, mas o método utilizado é uma versão simplificada do método apresentado no artigo referido.

**keywords** Non-autonomous Schrödinger-Poisson systems; Variational methods; Existence and multiplicity of solutions; Positive and sign changing solutions.

**abstract** In this thesis, we study the existence and multiplicity of solutions of the following class of Schrödinger-Poisson systems:

$$\begin{cases} -\Delta u + u + l(x)\phi u = \kappa(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $l \in L^2(\mathbb{R}^3)$ or $l \in L^\infty(\mathbb{R}^3)$. And we consider that the nonlinearity $\kappa$ satisfies the following three kinds of cases:

**(i)** a subcritical exponent with $\kappa(x,u) = k(x)|u|^{p-2}u + \mu h(x)u$ $(4 \le p < 2^*)$ under an indefinite case;

**(ii)** a general indefinite nonlinearity with $\kappa(x,u) = k(x)g(u) + \mu h(x)u$;

**(iii)** a critical growth exponent with $\kappa(x,u) = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u$ $(2 \le q < 2^*)$.

It is worth mentioning that the thesis contains three main innovations except overcoming several difficulties, which are generated by the systems themselves. First, as an unknown referee said in his report, we are the first authors concerning the existence of multiple positive solutions for Schrödinger-Poisson systems with an indefinite nonlinearity.

Second, we find an interesting phenomenon in Chapter 2 and Chapter 3 that we do not need the condition $\int_{\mathbb{R}^3} k(x)e_1^p dx < 0$ with an indefinite non-coercive case, where $e_1$ is the first eigenfunction of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$. A similar condition has been shown to be a sufficient and necessary condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearity for a bounded domain (see e.g. Alama-Tarantello, *Calc. Var. PDE* **1** (1993), 439–475), or to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearity in $\mathbb{R}^N$ (see e.g. Costa-Tehrani, *Calc. Var. PDE* **13** (2001), 159–189). Moreover, the process used in this case can be applied to study other aspects of the Schrödinger-Poisson systems and it gives a way to study the Kirchhoff system and quasilinear Schrödinger system.

Finally, to get sign changing solutions in Chapter 5, we follow the spirit of Hirano-Shioji, *Proc. Roy. Soc. Edinburgh Sect. A* **137** *(2007), 333*, but the procedure is simpler than that they have proposed in their paper.

# Contents

# List of symbols

| | |
|---|---|
| $\mathbb{R}^N$ | the Euclidean space with a dimension $N$ |
| $\Omega$ | an open domain of $\mathbb{R}^N$ |
| $C_0^\infty(\Omega)$ | space of smooth functions on $\Omega$ with compact support |
| $L^p(\Omega)$ | space of Lebesgue-measurable functions with the norm $\|u\|_p = \int_\Omega |u|^p dx$ $(1 \leq p < +\infty)$ |
| $H^1(\mathbb{R}^3)$ | the Hilbert space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2\right) dx$ and the standard inner product |
| $D^{1,2}(\mathbb{R}^3)$ | the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$ |
| $L^\infty(\mathbb{R}^3)$ | space of all essentially bounded functions with the norm $\|u\|_\infty = \operatorname{ess\,sup} |u|$ |
| $H^{-1}(\mathbb{R}^3)$ | the dual space of $H^1(\mathbb{R}^3)$ |
| $\liminf$ | infimum limit |
| $\limsup$ | supremum limit |
| $B_\rho(x)$ | the ball of radius $\rho$ centered at $x$ |
| $C$ | general positive constant |
| $C_i$ | various positive constants for $(i = 1, 2, ...)$ |
| $\rightarrow$ | the strong convergence |
| $\rightharpoonup$ | the weak convergence |
| $2^*$ | the Sobolev critical exponent, i.e. $2^* = \frac{2N}{N-2}$ as $N \geq 3$ |

# Introduction

Variational methods and critical point theory are powerful tools in studying nonlinear differential equations (see Ambrosetti-Malchiodi [5], Ambrosetti-Rabinowitz [6], Costa [33], Ekeland [48], Mawhin-Willem [80], Rabinowitz [85], Struwe [96], Willem [103]), in particular, in Hamiltonian systems (see also for instance Rabinowitz [84]) and elliptic equations (see for example Costa [34], Costa-Magalhaes [35], Figueiredo-Felmer [51], Yang-Yu [104]). In the last years, there have been a great number of works dealing with equations arising in Quantum Mechanics studied by means of variational tools. Indeed, this was the author's aim of study during the Doctoral Program in Mathematics and Applications for the period 2010-2014 through the PhD scholarship SFRH/BD/51162/2010 by Portuguese science foundation "Fundação para a Ciencia e Tecnodogia" (FCT). In the first year (curriculum year), students in this program have a broad scientific preparation including the broad band mandatory courses, optional courses, "Research Laboratory" and thesis project preparation as well. While taking one of the Research Lab courses "Nonlinear Analysis", an optional course "Functional Space" and the thesis project preparation in the second semester, the author was not only going over the knowledge related to critical point theory, which the author has already studied during attaining the Master's degree, but also further studying the related tools and theory, where the author started a paper about the Lorentz space tutored by Professor Eugénio A.M. Rocha, see Huang-Murillo-Rocha [58]. In [58], the authors studied a nonlinear elliptic Dirichlet problem involving a Leray-Lions type differential operator and proved the existence of solutions in a Lorentz space as well as gave an apriori estimate for the solutions. It was at that time that Professor Rocha, who became the author's supervisor on the second year of the PhD program, gave the author such a deep impression about his knowledge in this scientific area and was relevant in the decision for the subject of investigation of this thesis.

In this thesis, we will focus on the following class of elliptic equations

$$\begin{cases} -\Delta u + u + l(x)\phi u = \kappa(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{SP}$}$$

where $\kappa(x,u)$ satisfies the variational structure and the solutions $(u,\phi)$ will be searched in $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$. In $(\mathcal{SP})$, if $\kappa(x,u)$ is nonlinear with respect to $u$, then the first equation is a nonlinear Schrödinger equation in which the potential $\phi$ satisfies a nonlinear Poisson equation. For this reason, $(\mathcal{SP})$ is referred to as a nonlinear Schrödinger-Poisson (also called Schrödinger-Maxwell ) system, which arises from several interesting physical contexts. It is well known that $(\mathcal{SP})$ has a strong physical meaning since it appears in quantum mechanics models (see Benguria-Brézis-Lieb [19], Catto-Lions [25], Lieb [74] and Lieb-Simon [75]) and in semiconductor theory (see Benci-Fortunato [16], Benci-Fortunato [17], Lions [77] and Markowich [79]). As Arriola-Soler wrote in their paper [8], recent advances in technology design, in particular, the progressive tendency to fabricate semiconductor devices with extremely small sizes, have obliged to account for quantum-mechanical and numerical methods in order to describe quantum effects such as tunneling, size quantization or quantum interference. In this direction, Schrödinger-Poisson system constitutes, since the early eighties, a quite extended mathematical framework to understand and analyze mathematical aspects, which may prove relevant for the study of semiconductor heterostructures modeling.

From the point view of quantum mechanics, system $(\mathcal{SP})$ describes the mutual interactions of many particles (see Ruiz [88] and Vaira [99]). Indeed, if the term $\kappa(x,u)$ is replaced with 0, then problem $(\mathcal{SP})$ becomes the Schrödinger-Poisson system. This type of system appears in semiconductor theory and it describes the behavior of a single particle. In some recent works (see e.g. [3, 10, 26, 39, 59, 94, 101, 109]), different nonlinearities $\kappa(x,u)$ have been added to the Schrödinger-Poisson system, giving rise to the so-called nonlinear Schrödinger-Poisson systems. These nonlinear terms have been traditionally used in the Schrödinger equation to model the interaction among particles.

If $l \equiv 0$, system $(\mathcal{SP})$ becomes the standard Schrödinger equation

$$-\Delta u + u = \kappa(x,u), \tag{$\mathcal{NLS}$}$$

which has been broadly investigated in the past several decades, see for example, Bartsch-Wang [14], Benci-Cerami [18], Berestyeki-Lions [20], Del Pino-Felmer [44], Ding-Ni [46] and the references therein. We wonder how the results obtained on $(\mathcal{NLS})$ can be extended to system $(\mathcal{SP})$. To answer this question, let us take a look at the following results obtained from the nonlinear Schrödinger-Poisson systems.

The literature on the nonlinear Schrödinger-Poisson systems in the presence of a pure nonlinearity has been exhaustedly investigated. We mention [3, 7, 11, 26, 31, 38, 39, 40, 42, 43, 52, 57, 64, 65, 66, 70, 87, 88, 99, 101, 108] and the references therein. Also in [30, 32] and [94, 95, 102, 110], the linear and the asymptotically linear cases have been studied, respectively, whereas in [82, 83, 89] the problems have been studied in a bounded domain. Recently, different general nonlinearities on the Schrödinger-Poisson systems as on the Schrödinger equations are well studied, such as [10, 92] in the presence of Berestycki-Lions type nonlinearity, [2, 29, 72, 106] involving the nonlinear terms with or without Ambrosetti-Rabinoviz condition under additional assumptions, [81] in the presence of so called "positive potentials".

Among the above works listed on the Schrödinger-Poisson system, there are lots of works in the literature not only on the subcritical cases such as [7, 26, 31, 38, 41, 42, 43, 68, 70, 71, 81, 88, 94, 99, 102, 108], and on the supercritical cases like [11], but also on the critical cases like [11, 39, 54, 107, 109].

But when it comes to autonomous cases, i.e. the weight function $l \equiv 1$ in system $(\mathcal{SP})$, or non-autonomous cases, we find works such as [7, 10, 11, 31, 38, 39, 42, 43, 55, 68, 88] concerning the autonomous case, [3, 40, 70, 87, 101, 108] with the non-autonomous case, the search of the so-called semi-classical states, and [26, 59, 72, 81, 93, 94, 99, 106, 109] about other non-autonomous cases.

Above we have just briefly described the works in the literature. We now give further details for some works concerning the non-autonomous cases.

Cerami and Vaira [26] study system $(\mathcal{SP})$ in the case of $\kappa(x,u) = a(x)|u|^{p-2}u$ with $4 < p < 6$ and $a(x)$ being non-negative. They establish a global compactness lemma to overcome the lack of compactness of embedding $H^1(\mathbb{R}^3)$ into the Lebesgue space $L^p(\mathbb{R}^3)$, $p \in [2,6)$, which prevents the use of variational techniques in a standard way. They prove the existence of positive ground state and bound state solutions by minimizing the associated functional restricted to the Nehari manifold, where they assume that $l \in L^2(\mathbb{R}^3)$ and $a : \mathbb{R}^3 \to \mathbb{R}$ are non-negative functions satisfying $\lim_{|x|\to+\infty} l(x) = 0$, $l \geq 0$ for all $x \in \mathbb{R}^3$, $l \not\equiv 0$, $\lim_{|x|\to+\infty} a(x) = a_\infty > 0$ and $a(x) - a_\infty \in L^{6/(6-p)}(\mathbb{R}^3)$.

In Sun-Chen-Nieto [94], the authors consider another instance of $k$ in system $(\mathcal{SP})$ $\kappa(x,u) = a(x)f(u)$, where $f$ is asymptotically linear at infinity, i.e., $f(s)/s \to c$ as $s \to +\infty$ with a suitable constant $c$. They establish a compactness lemma different from that in Cerami-Vaira [26] and prove the existence of ground state solutions. The conditions on the coefficient $l$ in Sun-Chen- Nieto [94] are

$$l \in L^2(\mathbb{R}^3), \quad l(x) \geq 0 \ \text{ for any } x \in \mathbb{R}^3 \text{ and } l \not\equiv 0.$$

And in Zhao-Zhao [109] the authors study the case that $l \equiv 1$ in system $(\mathcal{SP})$ with the following form

$$\begin{cases} -\Delta u + u + \phi u = K(x)|u|^{2^*-2}u + \mu Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $K, Q \in C^1(\mathbb{R}^3, \mathbb{R})$ satisfy some suitable conditions. They obtain that when $4 < q < 6$ the problem has at least a positive solution for each $\mu > 0$ and when $q = 4$ the problem has at least a positive solution for each $\mu$ sufficiently large. They also get a positive radial solution for $\mu$ sufficiently large when $3 \leq q < 4$ under further assumptions. Note that there was no information about the case $q = 2$.

The main aim in this thesis is to generalize some of the above results or attain some results that have not been discussed in the previous works in the literature. Here, we study the existence and multiplicity of solutions for system $(\mathcal{SP})$ involving the following nonlinearities

**(i)** a subcritical exponent in the form $\kappa(x,u) = k(x)|u|^{p-2}u + \mu h(x)u$ $(4 \leq p < 2^*)$, where

$k$ is sign changing in $\mathbb{R}^3$;

**(ii)** a general indefinite nonlinearity with $\kappa(x,u) = k(x)g(u) + \mu h(x)u$, where $k$ is sign changing in $\mathbb{R}^3$;

**(iii)** a critical growth exponent with $\kappa(x,u) = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u$ $(2 \leq q < 2^*)$,

where the weight functions $l, k$ and $h$ satisfy some suitable conditions like $l \in L^2(\mathbb{R}^3)$ or $l \in L^\infty(\mathbb{R}^3)$.

We are concerned with the existence of positive solutions and sign changing solutions as well. As far as we know, we did not see any information of system $(\mathcal{SP})$ with the above three types of nonlinearities, which we will study in this work. It is worth mentioning that, as an unknown referee said in his report, we are the first authors concerning the existence of multiple positive solutions for Schrödinger-Poisson system with an indefinite nonlinearity. Moreover, concerning sign changing solutions, there is little information in the literature even for different classes of Schrödinger-Poisson systems. In the following, we will mention other innovations of our work. But first let us give the reduced form of system $(\mathcal{SP})$.

As we shall see in Chapter 1, system $(\mathcal{SP})$ can be reduced into a nonlinear Schrödinger equation with a nonlocal term. Briefly, the Poisson equation is solved by using the Lax-Milgram theorem, so, for every $u$ in $H^1(\mathbb{R}^3)$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained such that $-\Delta\phi = l(x)u^2$ and that, inserted into the first equation of $(\mathcal{SP})$, gives

$$-\Delta u + u + l(x)\phi_u u = \kappa(x,u) \quad \text{in } \mathbb{R}^3. \tag{$\mathcal{NSN}$}$$

Since $(\mathcal{NSN})$ is variational in nature, there is a one to one correspondence between the solutions of $(\mathcal{NSN})$ and the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2 dx - \int_{\mathbb{R}^3} \mathcal{K}(x,u)dx,$$

where $\mathcal{K}(x,u) = \int_0^u \kappa(x,s)ds$. Hence if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I$ on $H^1(\mathbb{R}^3)$, then $(u,\phi_u)$ is a solution of system $(\mathcal{SP})$.

Let us summarize the techniques used throughout this thesis as well as how it is organized. The main common difficulties in this thesis lie in the following two aspects. On one hand, the equation $(\mathcal{NSN})$ is considered in the whole space $\mathbb{R}^3$, and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \leq s \leq 6)$ is not compact any more. Due to different results, we use three different methods to restore the compactness in our work. We manage to apply some techniques, motivated by Willem [103], to get the compactness "directly" in some sense under suitable assumptions on the weight functions in Chapter 2 and Chapter 4. We also try to apply concentration-compactness lemma of Lions in Chapter 3, motivated by Costa-Tehrani [36]. However, in Chapter 5, we use a completely different method to attain our goal, which involves neither the Palais-Smale sequence nor the Ekeland variational principle, following the idea of Hirano-Shioji [56], but our procedure is simpler than that in Hirano-Shioji [56].

On the other hand, equation ($\mathcal{NSN}$) with a nonlocal term usually gives more difficulties than the standard Schrödinger equation ($\mathcal{NLS}$), because of the presence of nonlocal term in ($\mathcal{NSN}$) compared with ($\mathcal{NLS}$), which is the common difficulty while studying the nonlinear Schrödinger-Poisson system. The details to overcome them can be seen in above references on the Schrödinger-Poisson systems, and in our Chapter 2 to Chapter 5.

It seems that all the works in the literature on the Schrödinger-Poisson system show the disadvantages of the involvement of Poisson equation, namely the nonlocal term.

But to our surprise, when we study system ($\mathcal{SP}$) with an indefinite nonlinearity, i.e.

$$\kappa(x,u) = k(x)|u|^{p-2}u + \mu h(x)u \ \text{ with } \ (4 \le p < 2^*)$$

and with a more general indefinite nonlinearity $g$ behaving like $|u|^{p-2}u$ near zero in the form

$$\kappa(x,u) = k(x)g(u) + \mu h(x)u,$$

an interesting phenomenon is that we succeed in making use of the nonlocal term to technically help to deal with the key difficulty that the indefinite nonlinearity has created, and we do not need the condition

$$\int_{\mathbb{R}^3} k(x)e_1^p dx < 0, \tag{$*$}$$

where $e_1 > 0$ is the eigenfunction of the eigenvalue problem

$$-\Delta u + u = \mu_1 h(x)u \ \text{ in } \ \mathbb{R}^3,$$

with $\mu_1$ the associated first eigenvalue of $-\Delta + id$ in $\mathbb{R}^3$ with weight function $h$. We will show the procedure why we do not need the condition ($*$) in Chapter 2, also in Chapter 3.

Condition ($*$) has been shown to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities with a bounded or an unbounded domain, or even for the problem with critical exponent. Let us give some examples. In [1], for a given bounded open set $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, Alama and Tarantello seek positive solutions for

$$\begin{cases} -\Delta u - \tilde{\lambda} u = W(x)f(u) & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where $W$ changes sign in $\Omega$ and $f$ behaves like $|u|^{p-2}u$ near zero. They prove that, for $\tilde{\lambda}$ in a small right neighborhood of $\tilde{\lambda}_1$, that is, the first eigenvalue, the condition

$$\int_{\mathbb{R}^3} W(x)\tilde{e}_1^p dx < 0$$

is sufficient for existence of a positive solution, which is shown to be necessary for homogenous $f$, where $\tilde{e}_1$ is the first eigenfunction of the problem

$$\begin{cases} -\Delta \tilde{e} = \tilde{\lambda}\tilde{e} & \text{in } \Omega, \\ \tilde{e} = 0 & \text{in } \partial\Omega. \end{cases}$$

For an unbounded domain $\mathbb{R}^N$, Costa and Tehrani [36] establish a similar result, where they prove that

$$\int_{\mathbb{R}^3} a(x)\bar{e}_1^p dx < 0$$

is sufficient condition for existence of a positive solution of the problem

$$-\Delta u - \bar{\lambda}\bar{h}(x)u = a(x)f(u),$$

where $a$ is sign changing in $\mathbb{R}^N$ and $f$ behaves like $|u|^{p-2}u$ near zero. Here $\bar{e}_1$ is the first eigenfunction of the problem $-\Delta u = \bar{\lambda}_1\bar{h}(x)u$, where $\bar{\lambda}_1$ is the first eigenvalue of $-\Delta + id$ with weight function $\bar{h}$. For the critical case like in Drábek-Huang [47], the authors consider the problem

$$-\Delta_p u = \hat{\lambda}\hat{h}(x)|u|^{p-2}u + w(x)|u|^{p^*-2}u \text{ in } \mathbb{R}^N,$$

where $w$ is sign changing in $\mathbb{R}^N$ and $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent. Let $\hat{\lambda}_1$ be the principal eigenvalue of

$$-\Delta_p u = \hat{\lambda}\hat{h}(x)|u|^{p-2}u$$

with $\hat{e}_1$ the associated eigenfunction. To get a positive solution in the right neighborhood of $\hat{\lambda}_1$, the authors also need to use the sufficient condition

$$\int_{\mathbb{R}^3} w(x)\hat{e}_1^{p^*} dx < 0.$$

In fact, we also find that similar condition $(*)$ was used earlier by a corresponding necessary condition derived in Bandle-Pozio-Tesei [12] for a Neumann problem. It appears, however, that in the context of Neumann problem, conditions of the same type were already introduced by Kazdan and Warner [67] as an obstruction to the solvability of the prescribed scalar curvature problem for compact Riemannian manifolds (see also Escobar-Schoen [49]).

This thesis is organized as follows. In Chapter 1, we provide some preliminaries fundamentally to our work. Chapter 2 is devoted to the kind of indefinite nonlinearity $(i)$ mentioned above, which is a combination of homogeneous indefinite nonlinearity and a linear term. Furthermore, in Chapter 3, we generalize this nonlinearity to the form

$$\kappa(x, u) = k(x)g(u) + \mu h(x)u,$$

where $k$ is sign changing in $\mathbb{R}^3$ and $g$ behaves like a power at zero and infinity. We mainly prove the existence of at least two positive solutions in the case that $\mu > \mu_1$ and near $\mu_1$, where $\mu_1$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$ in these two chapters. Except the common difficulties, for the indefinite nonlinearity, it is still necessary to overcome another obstacle when we consider the "non-coercive" case $\mu > \mu_1$, that is, the linear part of $(\mathcal{NSN})$ is not coercive. Because in this situation we can not use the standard methods to prove the boundedness of $(PS)$-sequence directly. Even if in these two chapters the topics are both concerned with the indefinite problems and the results we will seek are almost similar, the methods to be used to get the results are completely different. A relatively "direct" method will be used to restore the compactness for the former. In Chapter 3, we will apply concentration-compactness lemma of Lions to get the compactness. Moreover, we extend the weight function $l \in L^2(\mathbb{R}^3)$ in Chapter 2 to $l \in L^\infty(\mathbb{R}^3)$ in Chapter 3.

Meanwhile, Chapter 4 and Chapter 5 concern the critical case with a constant sign nonlinearity. In Chapter 4 we prove the existence of at least one positive solution. In comparison to Chapter 4, we use a completely different method in Chapter 5 mentioned above to mainly seek sign changing solutions.

We finish the thesis by discussing some considerations and potential problems to be studied in the future.

# Chapter 1

# Variational approaches for elliptic equations

In this chapter we will present some useful preliminaries related to variational methods, which we will use throughout this thesis. The literature on variational methods is quite extensive, but here we will just recall several fundamental theorems on this subject such as Ekeland's variational principle, one type of Mountain Pass Theorem and strong maximum theorem. For more information, we refer the readers for instance to Ambrosetti-Malchiodi [5], Costa [33], Rabinowitz [85] and Struwe [96]. Here we emphasize our preparation on variational methods to be shared in the other parts in this thesis.

## 1.1 Some fundamental theorems

Let us start with the following particular case of Ekeland's variational principle, which was used in 1984 by Aubin and Ekeland in the construction of Palais-Smale sequences. This principle has been a very useful tool in studying optimization problems in Control Theory, Differential Geometry and Differential Equations.

### 1.1.1 Ekeland's variational principle

**Definition 1.1.1.** *Let $(M, d)$ be a complete metric space and $\psi : M \to \mathbb{R} \cup \{+\infty\}$. We say that $\psi$ is a lower semicontinuous functional, if any sequence $(v_m)_{m\in\mathbb{N}} \subset M$ such that $v_m \to v$ for some $v \in M$ satisfies*

$$\psi(v) \leq \liminf_{m} \psi(v_m).$$

**Theorem 1.1.2.** *(see Aubin-Ekeland [9].) Let $(M, d)$ be a complete metric space and $\psi : M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functional which is bounded from below. Suppose $\varepsilon > 0$ and $u \in M$ are such that*

$$\psi(u) \leq \inf_{M} \psi + \varepsilon.$$

*Then, given any $\lambda > 0$, there exists $v \in M$ such that*

**(i)** $\psi(v) \leq \psi(u)$;

**(ii)** $d(u, v) \leq \lambda$;

**(iii)** $\psi(v) \leq \psi(w) + \frac{\varepsilon}{\lambda} d(v, w)$ *for any* $v \neq w$.

Let $E$ be a Banach space, $E^{'}$ be the dual of $E$, and $\psi : E \to \mathbb{R}$ be a differentiable functional bounded from below. Set

$$c := \inf_{E} \psi.$$

Ekeland's variational principle implies the existence of a sequence $(u_n)_{n \in \mathbb{N}} \subset E$ such that

$$\psi(u_n) \to c, \quad \psi'(u_n) \to 0 \quad \text{in } E^{'}.$$

Such a sequence is called a Palais-Smale sequence at the level $c$. Let us see what the definition is used for.

**Definition 1.1.3.** *Let $\psi : U \to \mathbb{R}$ where $U$ is an open subset of a Banach space $E$. The functional $\psi$ is called Fréchet differentiable at $u \in U$ if there exists a bounded linear operator $f \in E'$ such that*

$$\lim_{v \to 0} \frac{1}{\|v\|_E} [\psi(u + v) - \psi(u) - \langle f, v \rangle] = 0 \ \ \textit{for any} \ \ v \in E.$$

*If the limit exists, we write $\psi'(u) = f$ and call it the Fréchet derivative of $\psi$ at $u$.*

*The functional $\psi$ belongs to $C^1(U, \mathbb{R})$ if the Fréchet derivative of $\psi$ exists and continuous on $U$.*

**Definition 1.1.4.** *Let $c \in \mathbb{R}$, $E$ be a Banach space, and $\psi \in C^1(E, \mathbb{R})$. A Palais-Smale sequence $(u_n)_{n \in \mathbb{N}}$ for the functional $\psi$ at the level $c$ ($(PS)_c$-sequence for short) is referred to a sequence $(u_n)_{n \in \mathbb{N}}$ such that $\psi(u_n) \to c$ and $\psi'(u_n) \to 0$ in $E'$. The functional $\psi$ is said to satisfy Palais-Smale condition at the level $c$ ($(PS)_c$-condition for short) if any $(PS)_c$-sequence possesses a convergent subsequence in $E$, and to satisfy $(PS)$-condition if $\psi$ satisfies $(PS)_c$-condition for every $c \in \mathbb{R}$.*

**Definition 1.1.5.** *Let $E$ be a Banach space and $\psi \in C^1(E, \mathbb{R})$. If the Fréchet derivative $\psi'(u) = 0$ for $u \in E$, then we say that $u \in E$ is a critical point of $\psi$. We call $c \in \mathbb{R}$ a critical value, if there exists a critical point $u \in E$ such that $\psi(u) = c$.*

If $\psi \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the $(PS)_c$-condition at the level $c := \inf_E \psi$, then $c$ is a critical value of $\psi$. Finding a critical point or a critical value of a differentiable functional, which is usually the associated functional to some given equation, is the main aim in the so-called variational methods. But in fact in the application the associated functional is not always bounded from below. The following version of the

Mountain Pass Theorem gives us an opportunity to overcome this problem, which is a major contribution to variational methods and one of the most useful minimax theorems.

### 1.1.2 Mountain Pass Theorem

**Theorem 1.1.6.** *(see Willem [103].) Let $E$ be a real Banach space, and the functional $I \in C^1(X, \mathbb{R})$. Suppose*

$(I_0)$ $I(0) = 0$;

$(I_1)$ *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$; and*

$(I_2)$ *there is $\bar{u} \in E \setminus \bar{B}_\rho$ such that $I(\bar{u}) < 0$.*

*Then, for each $\varepsilon > 0$, there exists $u \in E$ such that*

*(a)* $c - 2\varepsilon \leq I(u) \leq c + 2\varepsilon$ *and*

*(b)* $\|I'(u)\|_{E'} < 2\varepsilon$,

*where*

$$c = \inf_{g \in \Gamma} \max_{u \in g[0,1]} I(u) \tag{1.1}$$

*with*

$$\Gamma = \{g \in C([0,1], E) \ : \ g(0) = 0, g(1) = \bar{u}\}.$$

**Remark 1.1.7.** *If a functional has the geometric conditions $(I_0)$, $(I_1)$ and $(I_2)$, we say that the functional satisfies the "mountain pass geometry". In general, the minimax level $c$ defined by (1.1) may be not a critical level. Let us see the following example, which was given by Brézis-Nirenberger.*

**Example.** Set $F(x, y) = x^2 + (1 - x^3)y^2$ for $(x, y) \in \mathbb{R}^2$. Then $F$ has a unique critical point, which is $(0, 0)$ and its value is equal to 0. In fact, we can prove that

$(I_0)$ $F(0, 0) = 0$;

$(I_1)$ $(0, 0)$ is a strict local minimizer; and

$(I_2)$ $F$ is not bounded from below, i.e. $F(s, s) = s^2 + (1 - s^3)s^2 \to -\infty$ as $s \to \infty$.

The problem here is that we know that $F$ satisfies mountain pass geometry, and then, by Theorem 1.1.6, we can just get a $(PS)$-sequence, which has not any convergent subsequence. In sum, it does not satisfy $(PS)$-condition. From the following Mountain Pass Theorem, we can find that $(PS)$-condition plays an important role in order to get a critical value. In other words, to find a critical point, the key step is to prove the compactness of the given functional. And usually this is the most difficult procedure to achieve, in particular, for the subject with unbounded domain or critical Sobolev exponent.

**Theorem 1.1.8.** *(see Ambrosetti-Rabinowitz [6].) Under the assumptions of Theorem 1.1.6, if $I$ satisfies the $(PS)_c$-condition, then $I$ possesses a critical value $c \geq \alpha$.*

## 1.2 Maximum principle

The maximum principle presents a property of solutions to certain partial differential equations. In the following, we give a version of the maximum principle from Struwe [96], which is due to Walter [100].

**Theorem 1.2.1.** *Suppose $L$ is of elliptic type*

$$Lu = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}u\right) + cu$$

*on a domain $\Omega$ of $\mathbb{R}^N$ with bounded coefficients $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$, $c \in C^{\alpha}(\overline{\Omega})$, and $a_{ij} = a_{ji}$ satisfying the ellipticity condition*

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

*with a uniform constant $\lambda > 0$, for all $\xi \in \mathbb{R}^N$. And suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies*

$$Lu \geq 0 \text{ in } \Omega, \quad \text{and } u \geq 0 \text{ on } \Omega.$$

*Then either $u > 0$ in $\Omega$, or $u \equiv 0$ in $\Omega$,*

## 1.3 Sobolev embedding and the best Sobolev constant

To apply variational methods, usually an important step is to establish that the object functional has compactness. To achieve this goal, the common way is to apply the Sobolev embedding theorem. Before presenting the theorem, let us first give some notation of Sobolev spaces from Struwe [96].

Let $\Omega \subset \mathbb{R}^N$ be an open set. For $u \in L^1_{loc}(\Omega)$ and any multi-index $\alpha = \sum_{i=1}^N \alpha_i$, with $|\alpha| = \alpha_1 + ... + \alpha_N$, define the distributional derivative

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_1^{\alpha_N}} u$$

by letting

$$\langle \varphi, D^\alpha u \rangle = \int_\Omega (-1)^{|\alpha|} u D^\alpha \varphi dx,$$

for all $\varphi \in C_0^\infty(\Omega)$. We say $D^\alpha u \in L^p(\Omega)$, if there is a function $g_\alpha \in L^p(\Omega)$ satisfying

$$\langle \varphi, D^\alpha u \rangle = \langle \varphi, g_\alpha \rangle = \int_\Omega \varphi g_\alpha dx,$$

for all $\varphi \in C_0^\infty(\Omega)$. In this case, we identify $D^\alpha u$ with $g_\alpha \in L^p(\Omega)$.

Hence, for $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$, we may define the space

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \ : \ D^\alpha u \in L^p(\Omega) \text{ for all } \alpha : |\alpha| \leq k\}$$

with norm

$$\|u\|_{W^{k,p}}^p = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p}^p, \text{ if } 1 \le p < \infty,$$

respectively, with norm

$$\|u\|_{W^{k,\infty}} = \max_{|\alpha|\le k} \|D^\alpha u\|_{L^\infty}.$$

And $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. When $k=1$ and $p=2$, as usual $W_0^{k,p}(\Omega)$ is denoted by $H_0^1(\Omega)$.

### 1.3.1 Sobolev embedding theorem

**Theorem 1.3.1.** *(see Struwe [96].) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$, $1 \le p \le \infty$. Then the following holds:*

*(i) If $kp < N$, we have $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \le q \le \frac{Np}{N-kp}$; the embedding is compact, if $1 \le q < \frac{Np}{N-kp}$;*

*(ii) If $0 \le m < k - \frac{N}{p} < m+1$, we have $W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}\left(\overline{\Omega}\right)$ for $0 \le \alpha \le k - m - \frac{N}{p}$; the embedding is compact for $0 \le \alpha < k - m - \frac{N}{p}$.*

**Remark 1.3.2.** *(see Struwe [96].) Theorem 1.3.1 is valid for $W_0^{k,p}(\Omega)$ space on arbitrary bounded domain $\Omega$.*

Note that the domain $\Omega$ is a bounded domain in Theorem 1.3.1. If the domain $\Omega$ is unbounded, then the embedding in Theorem 1.3.1 is only continuous but not compact any more. We can see, for example, the following theorem.

**Theorem 1.3.3.** *(see Willem [103].) The following embeddings are continuous:*

$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \ \ for\ \ 2 \le p < \infty,\ N = 1, 2;$$

$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N) \ \ for\ \ 2 \le p \le 2^*,\ N \ge 3$$

$$D^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N) \ \ for\ \ N \ge 3.$$

According to the Sobolev embedding theorem, if the domain is bounded and the object one needs to study is subcritical, then one usually can directly apply the theorem to get the compactness. But when it comes to unbounded domain or critical problem, one can not apply the compactness of the Sobolev embedding theorem and should need to seek other methods to restore the compactness. As we mentioned in the introduction, we use three different methods to restore the compactness throughout the thesis, since the domain we study is $\mathbb{R}^3$. Of course, besides the methods we use here, there are other variant versions of concentration compactness method, for example, concentration compactness at infinity in Bianchi-Chabrowski-Szulkin [21] and Chabrowski [27] to achieve the goal. During the study of the restoration of compactness, we still use the following best Sobolev constant.

### 1.3.2 The best Sobolev constant

**Theorem 1.3.4.** *(see Willem [103].)*

**(i)** *Let $N \geq 3$. The optimal constant in the Sobolev inequality is given by*

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^N)\ \|u\|_{2^*}=1} \|\nabla u\|_2^2 > 0.$$

**(ii)** *For every open subset $\Omega$ of $\mathbb{R}^N$,*

$$S(\Omega) := \inf_{u \in D_0^{1,2}(\Omega)\ \|u\|_{2^*}=1} \|\nabla u\|_2^2 = S$$

*and $S(\Omega)$ is never achieved except when $\Omega = \mathbb{R}^N$.*

**(iii)** *Let $N \geq 2$ and $2 < p < 2^*$. The Sobolev theorem impies that*

$$S_p := \inf_{u \in H^1(\mathbb{R}^N)\ \|u\|_p=1} \|u\|^2 > 0.$$

*The infimum $S_p$ is achieved by a positive, radially symmetric function in $H^1(\mathbb{R}^N)$.*

## 1.4 Some preparation shared in all chapters of our original work

### 1.4.1 Definition of solutions of system $(\mathcal{SP})$

We search solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ for problem $(\mathcal{SP})$. Since $H^1(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$ are both Sobolev spaces, solutions of this kind are often referred to as *weak solutions* with the following meaning.

**Definition 1.4.1.** *The pair $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is called a solution of problem $(\mathcal{SP})$ if for any $(v, \psi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ there holds*

$$\begin{cases} \int_{\mathbb{R}^3} (\nabla u \nabla v + uv + l(x)\phi uv)\, dx = \int_{\mathbb{R}^3} \kappa(x, u) v dx, \\ \int_{\mathbb{R}^3} \nabla \phi \nabla \psi dx = \int_{\mathbb{R}^3} l(x) u^2 \psi dx. \end{cases}$$

*And we say that a solution $(u, \phi)$ is positive if $u > 0$, $\phi > 0$ for any $x \in \mathbb{R}^3$ and sign changing if $u$ is sign changing since $\phi$ is always nonnegative.*

### 1.4.2 Variational setting of problem $(\mathcal{SP})$

In this section, we will construct an energy functional associated to system $(\mathcal{SP})$. This is a very basic but fundamental process to apply variational methods. But first let us start with the following well-known Lax-Milgram theorem. Usually, the basic existence and uniqueness result for general elliptic differential equations is based on this theorem.

Let $X$ be a real Hilbert space and $X'$ its dual space. Denote by $\langle \cdot\, , \cdot \rangle$ the dual product between $X$ and $X'$.

**Definition 1.4.2.** *Let $a(u,v)$ be a bilinear form on the Hilbert space $X$,*

*(i) $a(u,v)$ is said to be bounded, if there exists $M>0$ such that*

$$|a(u,v)| \leq M\|u\|_X\|v\|_X, \quad \text{for any } u,v \in X.$$

*(ii) $a(u,v)$ is said to be coercive, if there exists $\delta>0$ such that*

$$|a(u,u)| \geq \delta\|u\|_X^2, \quad \text{for any } u \in X.$$

**Theorem 1.4.3.** *(see Lax-Milgram [69].) Let $a(u,v)$ be a bounded, coercive bilinear form on $X$. Then for any $f \in X'$, there exists a unique $u \in X$ such that*

$$a(u,v) = \langle f, v\rangle, \quad \text{for any } v \in X,$$

*and*

$$\|u\|_X \leq \frac{1}{\delta}\|f\|_{X'},$$

*where $\delta>0$ is shown in the Definition 1.4.2.*

**Example.** The following elliptic partial differential equation in $D^{1,2}(\mathbb{R}^3)$

$$-\Delta\phi = l(x)u^2 \quad \text{in } \mathbb{R}^3$$

for every fixed $u \in H^1(\mathbb{R}^3)$, which is the second equation in system $(\mathcal{SP})$, where $l \in L^\infty(\mathbb{R}^3)$ or $l \in L^2(\mathbb{R}^3)$, may be solved using the above result.

In fact, for any fixed $u \in H^1(\mathbb{R}^3)$, let

$$a(\phi,v) = \int_{\mathbb{R}^3} \nabla\phi\nabla v dx$$

and

$$f(v) = \int_{\mathbb{R}^3} l(x)u^2 v dx.$$

We know that $f$ is a bounded linear functional on $D^{1,2}(\mathbb{R}^3)$. And it is obvious that $a$ is a bilinear functional on $D^{1,2}(\mathbb{R}^3)$. Moreover one has that

$$|a(\phi,v)| \leq \|\phi\|_{D^{1,2}}\|v\|_{D^{1,2}}$$

and

$$|a(\phi,\phi)| = \|\phi\|^2_{D^{1,2}}, \quad \text{for any } \phi \in D^{1,2}(\mathbb{R}^3).$$

Therefore the problem satisfies the conditions of Lax-Milgram theorem and then there exists, for every fixed $u \in H^1(\mathbb{R}^3)$, a unique $\phi_u$ satisfying

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = \int_{\mathbb{R}^3} l(x) u^2 v dx, \quad \text{for every } v \in D^{1,2}(\mathbb{R}^3),$$

that is, the problem is solved.

As we have pointed out in the introduction, system $(\mathcal{SP})$ can be reduced to a single equation with a non-local term. Here we assume that $l \in L^\infty(\mathbb{R}^3)$ or $l \in L^2(\mathbb{R}^3)$, which will be used in our work. Actually, considering for any $u \in H^1(\mathbb{R}^3)$, denote $L_u(v)$ the linear functional in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} l(x) u^2 v dx.$$

If $l \in L^\infty(\mathbb{R}^3)$, one may deduce from Hölder and Sobolev inequalities that

$$|L_u(v)| \leq \|l\|_\infty \|u\|^2_{12/5} \|v\|_6 \leq C\|l\|_\infty \|u\|^2_{12/5} \|v\|_{D^{1,2}}. \tag{1.2}$$

And one may get a similar result with $l \in L^2(\mathbb{R}^3)$ that

$$|L_u(v)| \leq \|l\|_2 \|u^2\|_3 \|v\|_6 \leq C\|l\|_2 \|u\|^2_6 \|v\|_{D^{1,2}}. \tag{1.3}$$

Hence, for any $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v = \int_{\mathbb{R}^3} l(x) u^2 v dx \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3),$$

i.e.,

$$\phi_u \ \text{ is the unique weak solution of } \ -\Delta\phi = l(x)u^2. \tag{1.4}$$

We will prove the representation of $\phi_u$. By taking $u \in H^1(\mathbb{R}^3)$, we set

$$\phi = \frac{1}{4\pi}\left(\frac{1}{|x|} * (lu^2)\right) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy.$$

Then

$$\frac{\partial}{\partial x_i}\phi = -\frac{1}{4\pi}\left(\frac{x_i}{|x|^3} * (lu^2)\right) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x_i - y_i)l(y)u^2(y)}{|x-y|^3} dy$$

and so

$$\left|\frac{\partial}{\partial x_i}\phi\right| \leq \frac{1}{4\pi}\left|\left(\frac{1}{|x|^2} * (lu^2)\right)\right| = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|l(y)|u^2(y)}{|x-y|^2} dy.$$

It follows from the Hardy-Littlewood-Sobolev inequality (see Lieb-Loss [76]) that if $f \in L^p(\mathbb{R}^3)$ and $\lambda > 0$, then

$$\left\| \frac{1}{|x|^\lambda} * f \right\|_r \leq C\|f\|_p, \tag{1.5}$$

where

$$\frac{1}{p} + \frac{\lambda}{3} = 1 + \frac{1}{r}.$$

Thus, if $l \in L^2(\mathbb{R}^3)$, for $\lambda = 1$ and $\lambda = 2$, by (1.5) we have

$$\left\| \frac{1}{|x|} * lu^2 \right\|_6 \leq C\|lu^2\|_{\frac{6}{5}} \leq C\|l\|_2\|u\|_6^2 \leq C\|l\|_2\|u\|^2$$

and

$$\left\| \frac{1}{|x|^2} * lu^2 \right\|_2 \leq C\|lu^2\|_{\frac{6}{5}} \leq C\|l\|_2\|u\|_6^2 \leq C\|l\|_2\|u\|^2,$$

respectively. If $l \in L^\infty(\mathbb{R}^3)$, for $\lambda = 1$ and $\lambda = 2$, by (1.5) we have

$$\left\| \frac{1}{|x|} * lu^2 \right\|_6 \leq C\|lu^2\|_{\frac{6}{5}} \leq C\|l\|_\infty\|u^2\|_{\frac{6}{5}} \leq C\|l\|_\infty\|u\|^2$$

and

$$\left\| \frac{1}{|x|^2} * lu^2 \right\|_2 \leq C\|lu^2\|_{\frac{6}{5}} \leq C\|l\|_\infty\|u^2\|_{\frac{6}{5}}^2 \leq C\|l\|_\infty\|u\|^2,$$

respectively. Then, for $l \in L^\infty(\mathbb{R}^3)$ or $l \in L^2(\mathbb{R}^3)$, we obtain that $\phi \in L^6(\mathbb{R}^3)$ and $\frac{\partial}{\partial x_i}\phi \in L^2(\mathbb{R}^3)$ and so

$$\phi \in D^{1,2}(\mathbb{R}^3). \tag{1.6}$$

Moreover, from Evans' book [50], we know that

$$\phi = \frac{1}{4\pi}\left(\frac{1}{|x|} * (lu^2)\right)$$

solves

$$-\Delta\phi = l(x)u^2$$

in the sense of distributions

$$\frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * (lu^2)\right)\Delta v = \int_{\mathbb{R}^3} l(x)u^2 v dx \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3), \tag{1.7}$$

Combining (1.6) and (1.7), we get

$$\phi = \frac{1}{4\pi}\left(\frac{1}{|x|} * (lu^2)\right) \text{ is the solution of } -\Delta\phi = l(x)u^2. \tag{1.8}$$

Therefore, by (1.4) and (1.8), we know that for every fixed $u \in H^1(\mathbb{R}^3)$ the unique solution

$\phi_u$ has the following representation

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{l(y)u^2(y)}{|x-y|}dy.$$

Clearly $\phi_u(x) \geq 0$ for any $x \in \mathbb{R}^3$ if $l$ is a nonnegative function. We also, in particular, have that

$$\|\phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3}|\nabla\phi_u|^2dx = \int_{\mathbb{R}^3}l(x)\phi_u u^2dx. \tag{1.9}$$

Using (1.2) (or (1.3)), (1.9) and Sobolev inequalities, we obtain that

$$\|\phi_u\|_6 \leq C\|\phi_u\|_{D^{1,2}} \leq C\|u\|^2_{12/5} \leq C\|u\|^2 \ \text{ if } l \in L^\infty(\mathbb{R}^3) \tag{1.10}$$

or

$$\|\phi_u\|_6 \leq C\|\phi_u\|_{D^{1,2}} \leq C\|u\|^2_6 \leq C\|u\|^2 \ \text{ if } l \in L^2(\mathbb{R}^3).$$

Then we arrive at

$$\int_{\mathbb{R}^3}l(x)\phi_u(x)u^2(x)dx \leq C\|u\|^4. \tag{1.11}$$

Thus $F : H^1(\mathbb{R}^3) \to \mathbb{R}$ is well defined with

$$F(u) = \int_{\mathbb{R}^3}l(x)\phi_u(x)u^2(x)dx. \tag{1.12}$$

We know from (1.10) and (1.11) that the functional $F \in C^2(H^1(\mathbb{R}^3), \mathbb{R})$ (see for instance [26]). In fact, in our work, we only need that $F \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, and we give a detailed proof of this result for $l \in L^\infty(\mathbb{R}^3)$ by Lemma 4.1.2.

Let us introduce the following Euler functional of problem $(\mathcal{NSN})$ as $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3}\mathcal{K}(x,u)dx, \tag{1.13}$$

where $\mathcal{K}(x,u) = \int_0^u \kappa(x,s)ds$. Hence the functional $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ under the assumption that $\kappa(x,u)$ has the variational structure. Moreover,

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla v + uv)\,dx + \int_{\mathbb{R}^3}l(x)\phi_u uvdx - \int_{\mathbb{R}^3}\kappa(x,u)vdx \tag{1.14}$$

for any $v \in H^1(\mathbb{R}^3)$. Hence if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I$ on $H^1(\mathbb{R}^3)$, then $(u, \phi_u)$ is a solution of system $(\mathcal{SP})$. Noting that $\phi_u$ is always nonnegative if the weight function $l \geq 0$ and $l \not\equiv 0$ in $\mathbb{R}^3$, in particular, if $u > 0$, then $\phi_u > 0$. Therefore, to find the positive and sign changing solutions of system $(\mathcal{SP})$, it suffices to study the positive and sign changing critical points of $I$ in $H^1(\mathbb{R}^3)$, respectively.

### 1.4.3 One weakly continuous condition

To end this chapter, we prove a basic lemma, which will be used throughout the following four chapters in our work.

**Definition 1.4.4.** *Let $E$ be a Banach space. We call a functional $\psi : E \to \mathbb{R}$ weakly continuous if, for any sequence $(u_n)_{n\in\mathbb{N}}$ such that $u_n \rightharpoonup u$ in $E$, there holds*

$$\psi(u_n) \to \psi(u).$$

**Lemma 1.4.5.** *(see Willem [103].) Let $\Omega$ be a open subset of $\mathbb{R}^N$. Denote $D_0^{1,2}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $D^{1,2}(\mathbb{R}^N)$. If $N \geq 3$ and $a \in L^{N/2}(\Omega)$, then the functional*

$$\psi : D_0^{1,2}(\Omega) \to \mathbb{R} : u \mapsto \int_\Omega a(x)u^2 dx$$

*is weakly continuous.*

From a similar proof as Lemma 1.4.5, we obtain the following result. For the convenience, we give the proof as following.

**Lemma 1.4.6.** *If $h \in L^{6/(6-q)}(\mathbb{R}^3)$ and $2 \leq q < 6$, then the functional*

$$\psi_h : H^1(\mathbb{R}^3) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^3} h(x)|u|^q dx$$

*is weakly continuous. And if $h \in L^{3/2}(\mathbb{R}^3)$, for each $v \in H^1(\mathbb{R}^3)$, the functional*

$$\Psi_h : H^1(\mathbb{R}^3) \to \mathbb{R} : u \mapsto \int_{\mathbb{R}^3} h(x)uv dx$$

*is weakly continuous.*

**Proof.** It follows from the Sobolev and Hölder inequalities that

$$|\psi_h(u)| = \left|\int_{\mathbb{R}^3} h(x)|u|^q dx\right| \leq \|h\|_{\frac{6}{6-q}} \|u\|_6^q \leq C\|h\|_{\frac{6}{6-q}} \|u\|^q,$$

and

$$|\Psi_h(u)| = \left|\int_{\mathbb{R}^3} h(x)uv dx\right| \leq \|h\|_{\frac{3}{2}} \|u\|_6 \|v\|_6 \leq C\|h\|_{\frac{3}{2}} \|u\|\|v\|,$$

which imply that the functionals $\psi_h$ and $\Psi_h$ are well defined, respectively. Assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence (still denoted by $(u_n)_{n\in\mathbb{N}}$), we may assume that

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3 \quad \text{and} \quad |u_n| \to |u| \quad \text{a.e. in } \mathbb{R}^3.$$

Since we get, by Sobolev inequality, that $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^6(\mathbb{R}^3)$, $(|u_n|^q)_{n\in\mathbb{N}}$ is bounded in $L^{6/q}(\mathbb{R}^3)$, and we have

$$|u_n|^q \rightharpoonup |u|^q \text{ in } L^{6/q}(\mathbb{R}^3)$$

and

$$u_n \rightharpoonup u \text{ in } L^6(\mathbb{R}^3).$$

Thus by $h \in L^{6/(6-q)}(\mathbb{R}^3)$ we arrive at

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx \to \int_{\mathbb{R}^3} h(x)|u|^q dx.$$

Moreover, for every $v \in H^1(\mathbb{R}^3)$, we deduce that $h(x)v \in L^{6/5}(\mathbb{R}^3)$ from

$$\int_{\mathbb{R}^3} |h(x)v|^{6/5} dx \leq \|h\|_{\frac{3}{2}}^{\frac{6}{5}} \|v\|^{\frac{6}{5}},$$

since $h \in L^{3/2}(\mathbb{R}^3)$. Then we conclude that

$$\int_{\mathbb{R}^3} h(x)u_n v dx \to \int_{\mathbb{R}^3} h(x)uv dx$$

because $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$, which finishes the proof of this lemma. □

# Chapter 2

# Two positive solutions of a class of Schrödinger-Poisson systems with an indefinite nonlinearity

In this chapter, we consider the case that $\kappa(x,u) = k(x)|u|^{p-2}u + \mu h(x)u$, that is, the following non-autonomous nonlinear Schrödinger-Poisson system with the form

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{p-2}u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{$\mathcal{P}_\mu$}$$

where $4 < p < 6$ and $\mu > 0$. We are interested in the case that $k$ is sign changing in $\mathbb{R}^3$ and $\lim_{|x|\to\infty} k(x) = k_\infty < 0$, which is why we call it an indefinite nonlinearity. We are concerned with the two situations: $(a)$ $0 < \mu < \mu_1$ and $(b)$ $\mu \geq \mu_1$ but near $\mu_1$, where $\mu_1$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$. It is worth mentioning that, in the case when $\mu$ is contained in a small right neighborhood of $\mu_1$, the question of existence of positive solutions to the problem with indefinite nonlinearity is even more interesting. We will explain it in detail later on in this section.

Assume the following hypotheses $(H)$:

$(H_h)$ $h \in L^{3/2}(\mathbb{R}^3)$, $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;

$(H_{k_1})$ $k \in C(\mathbb{R}^3)$ and $k$ changes sign in $\mathbb{R}^3$;

$(H_{k_2})$ $\lim_{|x|\to\infty} k(x) = k_\infty < 0$;

$(H_{l_1})$ $l \in L^2(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;

$(H_{l_2})$ $l = 0$ a.e. in $\Omega^0$, where $\Omega^0 = \{x \in \mathbb{R}^3 : k(x) = 0\}$ and $\Omega_0$ coincides with the closure of its interior.

Then with the above assumptions in this chapter we mainly prove the following result.

**Theorem 2.0.7.** *Assume the hypotheses $(H)$ hold and $4 < p < 6$. Then*

*(1) for* $0 < \mu \le \mu_1$ *problem* $(\mathcal{P}_\mu)$ *has at least one positive solution in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$*;*

*(2) there exists* $\bar{\epsilon} > 0$ *such that, for* $\mu_1 < \mu < \mu_1 + \bar{\epsilon}$*, problem* $(\mathcal{P}_\mu)$ *has at least two positive solutions in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$*.*

To study Theorem 2.0.7, we use the variational method. For system $(\mathcal{P}_\mu)$, the equation $(\mathcal{NSN})$ becomes

$$-\Delta u + u + l(x)\phi_u u = k(x)|u|^{p-2}u + \mu h(x)u \quad \text{in } \mathbb{R}^3. \tag{2.1}$$

In the case $0 < \mu < \mu_1$, the linear part of (2.1) is *coercive*, i.e.

$$\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 - \mu h(x)u^2\right) dx \to \infty \ \text{ as } \ \|u\| \to \infty$$

and one may use standard variational techniques to get a positive solution, provided that the $(PS)$-condition is satisfied. It should be pointed out, however, that the $(PS)$-condition is a difficult issue here, since the system is considered in the whole space $\mathbb{R}^3$ and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \le s \le 6)$ is not compact any more. We manage to restore the compactness by some techniques, motivated by Willem [103], under some or under the above assumptions on $l, k$ and $h$. For the case that $\mu \ge \mu_1$ and near $\mu_1$, in which the linear part of (2.1) is *not coercive* any more, the situation is more delicate. We need to face two more difficulties than in the case $0 < \mu < \mu_1$. One is to prove the boundedness of the $(PS)$-sequence, since in this situation with indefinite nonlinearity, the standard method of getting bounded $(PS)$-sequence is not applicable. We have to analyze the $(PS)$-sequence carefully and prove the boundedness of $(PS)$-sequence indirectly. The other is to verify the mountain pass geometry. Usually one may find that zero is a local minimizer of the associated functional to the equation (2.1) and then use the Mountain Pass Theorem to find a nontrivial solution. However, for the case $\mu \ge \mu_1$, the principal part of the associated functional to the equation (2.1) is non-coercive which makes it difficult to prove the geometry. To explain our strategy and new phenomenon of dealing with $(\mathcal{P}_\mu)$ in the case $\mu \ge \mu_1$, we recall some known results for semilinear elliptic equations.

Costa and Tehrani [36] studied the existence of positive solutions to the following elliptic equation with indefinite nonlinearity

$$-\triangle u = a(x)g(u) + \bar{\lambda}\bar{h}(x)u, \qquad u \in D^{1,2}(\mathbb{R}^N),$$

where $a$ is sign changing in $\mathbb{R}^N$, $\lim_{|x|\to\infty} a(x) = a_\infty < 0$ and $g(s) \sim O(s^{p-1})$ as $|s| \to 0$ with suitable assumptions on $p$. Besides some other assumptions, they assume the condition $\int_{\mathbb{R}^N} a(x)\bar{e}_1^p dx < 0$ and $\bar{\lambda}$ is contained in a small right neighborhood of $\bar{\lambda}_1$, where $\bar{\lambda}_1$ is the first eigenvalue of the eigenvalue problem $-\triangle u = \bar{\lambda}\bar{h}(x)u$ in $D^{1,2}(\mathbb{R}^N)$ and

$\bar{e}_1 \in D^{1,2}(\mathbb{R}^N)$ is the eigenfunction corresponding to $\bar{\lambda}_1$. In [36] the condition

$$\int_{\mathbb{R}^N} a(x)\bar{e}_1^p dx < 0 \tag{2.2}$$

is a sufficient condition to get the existence of multiple positive solutions. Alama and Tarantello [1] studied the existence of multiple positive solutions of

$$-\triangle u - \tilde{\lambda} u = W(x) f(u), \qquad u \in H_0^1(\Omega)$$

with $\Omega$ being a smooth bounded domain of $\mathbb{R}^N$, where $W(x) \in C(\bar{\Omega})$ is sign changing. Denoted by $\tilde{\lambda}_1$ the first eigenvalue of the eigenvalue problem $-\triangle u = \tilde{\lambda} u$, $H_0^1(\Omega)$. The corresponding eigenfunction is denoted by $\tilde{e}_1$. Alama and Tarantello have shown that, for $f$ behaving like $|u|^{q-2}u$ near zero with suitable assumptions on $q$,

$$\int_{\Omega} W(x)\tilde{e}_1^q dx < 0 \tag{2.3}$$

is a sufficient condition to the existence of multiple positive solutions of the equation, which is also shown to be necessary for homogenous $f$. To our best knowledge, we know that, for the semilinear elliptic equations with indefinite nonlinearity, it needs a similar condition (2.2) to get positive solutions, also see e.g. Drábek-Huang [47]. However, in the present chapter we show a new phenomenon that, for the Schrödinger-Poisson system with indefinite nonlinearity, this kind of condition is *not* necessary.

We emphasize here that the condition (2.2) in [36] or the condition (2.3) in [1] is technically used to overcome the obstacle of verifying the mountain pass geometry in the non-coercive case with sign changing nonlinearity. But in our case, we delicately analyze the behavior of the non-local term and find that, in the competing of the non-local term with the indefinite nonlinear term, the former may dominate the situation, which implies that it is not necessary to involve a similar condition (2.2) any more.

This chapter is organized as follows. In Section 1, we mainly prove that the $(PS)_c$-condition holds at any level $c$ for the associated functional to the equation (2.1), where the assumptions $(H_{l_1})$, $(H_{l_2})$ and $(H_{k_2})$ play an important role. Section 2 is devoted to the proof of Theorem 2.0.7, where we get two positive solutions of the problem. One is a mountain-pass type solution, the other is a local minimizer. Hence we get two positive solutions of $(\mathcal{P}_\mu)$.

The results of this chapter are published in [60].

## 2.1 The proof of Palais-Smale condition

In this section, our main objective is to prove the $(PS)$-condition. But first let us introduce some notations. Define the sets $\Omega^+ = \{x \in \mathbb{R}^3 : k(x) > 0\}$, $\Omega^- = \{x \in \mathbb{R}^3 : k(x) < 0\}$ and $\Omega^0 = \{x \in \mathbb{R}^3 : k(x) = 0\}$. Let $\sigma(-\Delta + id, \Omega^0, h)$ denote the collection of eigenvalues of $-\Delta + id$ in $H_0^1(\Omega^0)$ with the weight function $h$. If the Lebesgue measure of $\Omega^0$ is zero, i.e., $|\Omega^0| = 0$, then $\sigma(-\Delta + id, \Omega^0, h(x)) = \emptyset$.

Denote the functional $I$ defined by (1.13) corresponding to the equation (2.1) in this chapter as $I_\mu$ with

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx - \int_{\mathbb{R}^3}\left(\frac{1}{p}k(x)|u|^p + \frac{\mu}{2}h(x)u^2\right)dx.$$

Therefore one has that the functional $I_\mu$ is of class $C^2$ in $H^1(\mathbb{R}^3)$, moreover,

$$\langle I'_\mu(u), \varphi\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla\varphi + u\varphi)\,dx + \int_{\mathbb{R}^3} l(x)\phi_u u\varphi dx - \int_{\mathbb{R}^3}\left(k(x)|u|^{p-2}u\varphi + \mu h(x)u\varphi\right)dx$$

for any $\varphi \in H^1(\mathbb{R}^3)$.

**Lemma 2.1.1.** *(see [26, Lemma 2.1].) Let the operator $\Phi : H^1(\mathbb{R}^3) \to D^{1,2}(\mathbb{R}^3)$ be defined by $\Phi(u) := \phi_u$, that is, the solution in $D^{1,2}(\mathbb{R}^3)$ of $-\Delta\phi = l(x)u^2$. If the hypothesis $(H_{l_1})$ holds and a sequence $(u_n)_{n\in\mathbb{N}}$ satisfies $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\Phi(u_n) \rightharpoonup \Phi(u)$ in $D^{1,2}(\mathbb{R}^3)$.*

Now we are in a position to prove $(PS)$-condition for the functional $I_\mu$.

**Lemma 2.1.2.** *Suppose that the hypotheses $(H_{l_1})$, $(H_{l_2})$, $(H_{k_1})$ and $(H_h)$ hold, and $4 < p < 6$. If $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, then for every $c \in \mathbb{R}$, the $(PS)_c$-sequence is bounded in $H^1(\mathbb{R}^3)$.*

**Proof.** Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_c$-sequence for $I_\mu$ at the level $c$, i.e.,

$$I_\mu(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}F(u_n) - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u_n|^p dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)|u_n|^2 dx \to c \tag{2.4}$$

and

$$\begin{aligned}\langle I'_\mu(u_n), \varphi\rangle &= \int_{\mathbb{R}^3}(\nabla u_n\nabla\varphi + u_n\varphi)\,dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n\varphi \\ &\quad - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n\varphi dx - \mu\int_{\mathbb{R}^3} h(x)u_n\varphi dx \to 0\end{aligned} \tag{2.5}$$

for any $\varphi \in H^1(\mathbb{R}^3)$ as $n \to \infty$. Arguing by contradiction, we assume that $t_n := \|u_n\|$ and $t_n \to \infty$ as $n \to \infty$. Denote $v_n := u_n/t_n$. Then we have that

$$\|v_n\| = \frac{1}{t_n}\|u_n\| = 1$$

for each $n \in \mathbb{N}$. Going if necessary to a subsequence, we may assume that there is $v \in H^1(\mathbb{R}^3)$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{aligned}&v_n \rightharpoonup v && \text{in } H^1(\mathbb{R}^3),\\ &v_n(x) \to v(x) && \text{a.e. in } \mathbb{R}^3,\\ &v_n \to v && \text{in } L^s(\Omega) \text{ for } 2 < s < 6,\\ &|v_n(x)| \le w_\Omega(x) && \text{for some } w_\Omega \in L^s(\Omega).\end{aligned} \tag{2.6}$$

Hence, for any $\varphi \in H^1(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3} (\nabla v_n \nabla \varphi + v_n \varphi)\, dx \to \int_{\mathbb{R}^3} (\nabla v \nabla \varphi + v\varphi)\, dx. \tag{2.7}$$

In the first place, we claim that $v(x) = 0$ a.e. in $\mathbb{R}^3$. In fact, since $u_n = t_n v_n$, (2.5) becomes

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla v_n \nabla \varphi + v_n \varphi)\, dx + t_n^2 \int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n \varphi dx \\ &-t_n^{p-2} \int_{\mathbb{R}^3} k(x)|v_n|^{p-2} v_n \varphi dx - \mu \int_{\mathbb{R}^3} h(x) v_n \varphi dx \to 0 \end{aligned} \tag{2.8}$$

as $n \to \infty$. In the following we will prove the claim for $x$ contained in $\Omega^+, \Omega^-$ and $\Omega^0$, respectively. Hypothesis $(H_{k_1})$ implies that $\Omega^+ \neq \emptyset$ and $\Omega^- \neq \emptyset$. First, we consider the case of $x \in \Omega^+$. Since $k \in C(\mathbb{R}^3)$, there exists $\delta > 0$ such that

$$k(y) > 0 \text{ for any } y \in B_\delta(x). \tag{2.9}$$

Define $\zeta_m \in C^1(\mathbb{R}^3)$ $(m > 2)$ such that $\zeta_m(y) \geq 0$ for any $y \in \mathbb{R}^3$ and

$$\zeta_m(y) = \begin{cases} 1, & y \in B_{\left(\frac{1}{2} - \frac{1}{m^2}\right)\delta}(x), \\ 0, & y \in \mathbb{R}^3 \setminus B_{\delta/2}(x). \end{cases}$$

Taking $\varphi = v\zeta_m$ in (2.8), we know that $\text{supp}\varphi \subset B_{\delta/2}(x)$ for any $m \in \mathbb{N}$ and $m > 2$. It is deduced from (2.6) that

$$k(y)|v_n(y)|^{p-2} v_n(y)\varphi(y) \to k(y)|v(y)|^{p-2} v(y)\varphi(y), \text{ for } y \in B_{\delta/2}(x),$$

and

$$|k(y)|v_n(y)|^{p-2} v_n(y)\varphi(y)| \leq C|w_\Omega(y)|^{p-1}|\varphi(y)| \in L^1(B_{\delta/2}(x)).$$

Therefore, by the Lebesgue dominated convergent theorem, we achieve that

$$\int_{B_{\delta/2}(x)} k(y)|v_n|^{p-2} v_n \varphi dy \to \int_{B_{\delta/2}(x)} k(y)|v|^{p-2} v\varphi dy. \tag{2.10}$$

Dividing (2.8) by $t^{p-2}$ and passing to the limit as $n \to \infty$, by the boundedness of $v_n$ and (1.10), we get that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\mathbb{R}^3} k(y)|v_n|^{p-2} v_n \varphi dy \\ &= \int_{B_{\delta/2}(x)} k(y)|v|^{p-2} v\varphi dy \\ &= \int_{B_{\left(\frac{1}{2} - \frac{1}{m^2}\right)\delta}(x)} k(y)|v|^p dy + \int_{B_{\frac{\delta}{2}}(x) \setminus B_{\left(\frac{1}{2} - \frac{1}{m^2}\right)\delta}(x)} k(y)|v|^p \zeta_m dy \end{aligned} \tag{2.11}$$

for any $m \in \mathbb{N}$ and $m > 2$. Passing to the limit in (2.11) as $m \to \infty$, we obtain

$$\int_{B_{\delta/2}(x)} k(y)|v|^p dy = 0,$$

which, with (2.9), implies that $v = 0$ a.e. in $B_{\delta/2}(x)$. Since $x \in \Omega^+$ is chosen arbitrarily, we get that $v = 0$ a.e. in $\Omega^+$. In a similar way, we can get that $v = 0$ a.e. in $\Omega^-$. Next, to finish the proof of the claim, it is sufficient to prove that $v = 0$ a.e. in $\Omega^0$. If $|\Omega^0| = 0$, we finish the proof of the claim. If $|\Omega^0| \neq 0$, take $\varphi \in C^1(\mathbb{R}^3)$ with $\mathrm{supp}\varphi \subseteq \Omega^0$ in (2.8). Hence, by the definition of $\Omega^0$ and the assumption that $l = 0$ a.e. in $\Omega^0$ respectively, we obtain that

$$\int_{\mathbb{R}^3} k(y)|v_n|^{p-2} v_n \varphi dy = \int_{\mathrm{supp}\varphi} k(y)|v_n|^{p-2} v_n \varphi dy = 0 \tag{2.12}$$

and

$$\int_{\mathbb{R}^3} l(y)\phi_{v_n} v_n \varphi dy = \int_{\mathrm{supp}\varphi} l(y)\phi_{v_n} v_n \varphi dy = 0 \tag{2.13}$$

for any $n \in \mathbb{N}$. Using (2.7), (2.12), (2.13), Lemma 1.4.6 and passing to the limit in (2.8) as $n \to \infty$, we arrive at

$$\int_{\mathbb{R}^3} (\nabla v \nabla \varphi + v\varphi)\, dx = \mu \int_{\mathbb{R}^3} h(x) v \varphi dx.$$

Combining this with the fact that $v = 0$ a.e. in $\Omega^+ \cup \Omega^-$, we deduce that

$$\int_{\Omega^0} (\nabla v \nabla \varphi + v\varphi)\, dx = \mu \int_{\Omega^0} h(x) v \varphi dx.$$

Since $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, one obtains that $v = 0$ a.e. in $\Omega^0$. The proof of the claim is complete. Hence, $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Lemma 1.4.6 implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h(x) v_n^2 dx = 0. \tag{2.14}$$

In the second place, choosing $\varphi = v_n$ in (2.5), dividing (2.4) by $t_n^2 = \|u_n\|^2$ and dividing (2.5) by $t_n = \|u_n\|$, we get that

$$\frac{1}{2} + \frac{t_n^2}{4} F(v_n) - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x) v_n^2 dx \to 0 \tag{2.15}$$

and

$$1 + t_n^2 F(v_n) - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx - \mu \int_{\mathbb{R}^3} h(x) v_n^2 dx \to 0 \tag{2.16}$$

as $n \to \infty$. With the help of (2.14), we can obtain from (2.15) that as $n \to \infty$,

$$\frac{1}{2} + \frac{t_n^2}{4} F(v_n) - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx \to 0. \tag{2.17}$$

Similarly, we deduce from (2.16) that as $n \to \infty$,

$$1 + t_n^2 F(v_n) - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx \to 0. \tag{2.18}$$

Combining (2.17)–(2.18) with the assumption of $4 < p < 6$, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx = \frac{p}{4-p} < 0.$$

On the other hand, (2.17) implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n|^{p-2} v_n^2 dx > 0,$$

which is a contradiction. Hence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. □

**Lemma 2.1.3.** *Suppose that the hypotheses* $(H)$ *hold and* $4 < p < 6$. *If* $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, *then the functional* $I_\mu$ *satisfies* $(PS)_c$*-condition for any* $c \in \mathbb{R}$.

**Proof.** We have to prove that any sequence $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ for which $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ contains a convergent subsequence. According to Lemma 2.1.2, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence (still denoted by $(u_n)_{n\in\mathbb{N}}$), we may assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3),$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3,$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^3)$$

and

$$u_n \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^3).$$

Define $w_n = k(x)|u_n|^{p-2}u_n$ and $w = k(x)|u|^{p-2}u$. Then $w_n \to w$ a.e. in $\mathbb{R}^3$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^3)$ for $4 < p < 6$ and $k$ is bounded in $\mathbb{R}^3$, then $w_n$ is bounded in $L^{p/(p-1)}(\mathbb{R}^3)$ and so $w_n \rightharpoonup w$ in $L^{p/(p-1)}(\mathbb{R}^3)$ with $4 < p < 6$. Note that, for any $\psi \in H^1(\mathbb{R}^3)$, one has that

$$\int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n\psi dx \to \int_{\mathbb{R}^3} k(x)|u|^{p-2}u\psi dx \tag{2.19}$$

and

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla\psi + u_n\psi)\, dx \to \int_{\mathbb{R}^3} (\nabla u \nabla\psi + u\psi)\, dx \tag{2.20}$$

as $n \to \infty$. By Lemma 1.4.6, we also have that

$$\int_{\mathbb{R}^3} h(x)u_n\psi dx \to \int_{\mathbb{R}^3} h(x)u\psi dx. \tag{2.21}$$

Moreover from the hypothesis $(H_{l_1})$ one deduces that

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2(x)dx = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx + o(1) \tag{2.22}$$

for $n$ large and that for all $\psi \in H^1(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n(x)\psi(x)dx = \int_{\mathbb{R}^3} l(x)\phi_u(x)u(x)\psi(x)dx + o(1). \tag{2.23}$$

For the proof of (2.22) and (2.23), we borrow the strategy from Cerami-Vaira [26]. In fact, in view of the Sobolev embedding theorems and Lemma 2.1.1, one obtains from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ that

$$\begin{array}{ll} \text{(a) } u_n \rightharpoonup u & \text{in } L^6(\mathbb{R}^3), \\ \text{(b) } u_n^2 \to u^2 & \text{in } L^3_{loc}(\mathbb{R}^3), \\ \text{(c) } \phi_{u_n} \rightharpoonup \phi_u & \text{in } D^{1,2}(\mathbb{R}^3), \\ \text{(d) } \phi_{u_n} \to \phi_u & \text{in } L^3_{loc}(\mathbb{R}^3). \end{array} \tag{2.24}$$

Thus, given $\varepsilon > 0$, using (2.24) (c), we have that, for large $n$

$$\left| \int_{\mathbb{R}^3} l(x)u^2(x)(\phi_{u_n} - \phi_u)(x)dx \right| \leq \varepsilon \tag{2.25}$$

and, for any fixed $\psi$, using (2.24) (a),

$$\left| \int_{\mathbb{R}^3} l(x)\phi_u(x)\psi(x)(u_n - u)(x)dx \right| < \varepsilon. \tag{2.26}$$

Furthermore, considering (2.24) (b) and (2.24) (d) respectively, we can assert that for any choice of $\varepsilon$ and $\rho > 0$, the relations

$$\left( \int_{B_\rho(0)} |u_n^2 - u^2|^3 dx \right)^{\frac{1}{3}} < \varepsilon \tag{2.27}$$

and

$$\left( \int_{B_\rho(0)} |\phi_{u_n} - \phi_u|^6 dx \right)^{\frac{1}{6}} < \varepsilon \tag{2.28}$$

hold true for large $n$. On the other hand, being $(u_n)_{n\in\mathbb{N}}$ bounded in $H^1(\mathbb{R}^3)$, $(\phi_{u_n})_{n\in\mathbb{N}}$ is bounded in $D^{1,2}(\mathbb{R}^3)$ and in $L^6(\mathbb{R}^3)$, because of (1.10) and the continuity of the Sobolev embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$. Moreover, $l \in L^2(\mathbb{R}^3)$ implies that $lu_n^2$ and $lu^2$ belong to $L^{\frac{6}{5}}(\mathbb{R}^3)$ and that to any $\varepsilon > 0$ there corresponds $\bar{\rho} = \bar{\rho}(\varepsilon)$ such that

$$\left( \int_{\mathbb{R}^3 \setminus B_\rho(0)} |l(x)|^2 \, dx \right)^{\frac{1}{2}} < \varepsilon \quad \text{for } \rho \geq \bar{\rho}. \tag{2.29}$$

Hence, by using (2.25), (2.27), and (2.29), we deduce that for large $n$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n^2(x)dx - \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx\right| \\
&\le \left|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)\left(u_n^2-u^2\right)(x)dx\right| + \left|\int_{\mathbb{R}^3} l(x)\left(\phi_{u_n}(x)-\phi_u(x)\right)u^2(x)dx\right| \\
&\le \|\phi_{u_n}\|_6 \cdot \left(\int_{\mathbb{R}^3} |l(x)\left(u_n^2-u^2\right)(x)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} + \varepsilon \\
&\le C\left(\int_{\mathbb{R}^3\setminus B_\rho(0)} |l(x)\left(u_n^2-u^2\right)(x)|^{\frac{6}{5}}dx + \int_{B_\rho(0)} |l(x)\left(u_n^2-u^2\right)(x)|^{\frac{6}{5}}dx\right)^{\frac{5}{6}} + \varepsilon \\
&\le C\left(\left(\int_{\mathbb{R}^3\setminus B_\rho(0)} |l(x)|^2dx\right)^{\frac{3}{5}} \left|u_n^2-u^2\right|_3^{\frac{6}{5}} + |l|_2^{\frac{6}{5}}\left(\int_{B_\rho(0)} \left|u_n^2-u^2\right|^3dx\right)^{\frac{2}{5}}\right)^{\frac{5}{6}} + \varepsilon \\
&\le C\varepsilon,
\end{aligned}
$$

proving (2.22). Analogously, by using (2.26), (2.28), and (2.29), we infer for large $n$

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3} l(x)\phi_{u_n}(x)u_n(x)\psi(x)dx - \int_{\mathbb{R}^3} l(x)\phi_u(x)u(x)\psi(x)dx\right| \\
&\le \left|\int_{\mathbb{R}^3} l(x)\phi_u(x)\left(u_n(x)-u(x)\right)\psi(x)dx\right| + \left|\int_{\mathbb{R}^3} l(x)\left(\phi_{u_n}(x)-\phi_u(x)\right)u_n(x)\psi(x)dx\right| \\
&\le \|u_n\|_6\|\psi\|_6 \cdot \left(\int_{\mathbb{R}^3} |l(x)\left(\phi_{u_n}-\phi_u\right)(x)|^{\frac{3}{2}}dx\right)^{\frac{2}{3}} + \varepsilon \\
&\le C\varepsilon
\end{aligned}
$$

proving (2.23). Combining (2.19)–(2.21) with (2.22), we obtain that

$$
\begin{aligned}
\langle I'_\mu(u_n),\psi\rangle &= \int_{\mathbb{R}^3}(\nabla u_n\nabla\psi + u_n\psi)dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n\psi dx \\
&\quad - \int_{\mathbb{R}^3} k(x)|u_n|^{p-2}u_n\psi dx - \mu\int_{\mathbb{R}^3} h(x)u_n\psi dx \\
&\to \int_{\mathbb{R}^3}(\nabla u\nabla\psi + u\psi)dx + \int_{\mathbb{R}^3} l(x)\phi_u u\psi dx \\
&\quad - \int_{\mathbb{R}^3} k(x)|u|^{p-2}u\psi dx - \mu\int_{\mathbb{R}^3} h(x)u\psi dx \\
&= \langle I'_\mu(u),\psi\rangle.
\end{aligned}
$$

On the other hand, from $I'_\mu(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$, we get that $\langle I'_\mu(u_n),\psi\rangle \to 0$ for any $\psi \in H^1(\mathbb{R}^3)$. Therefore $\langle I'_\mu(u),\psi\rangle = 0$ for any $\psi \in H^1(\mathbb{R}^3)$. In particular,

$$\langle I'_\mu(u), u\rangle = 0. \tag{2.30}$$

Denote $v_n = u_n - u$. Then $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, which implies, by (2.22) and Lemma 1.4.6, that

$$\lim_{n\to\infty} F(v_n) = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} h(x)v_n^2(x)dx = 0. \tag{2.31}$$

Furthermore, it is deduced from the Brézis-Lieb lemma [22] and Lemma 1.4.6 respectively that, for $n$ large,

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} k(x)|u_n|^p dx = \int_{\mathbb{R}^3} k(x)|v_n|^p dx + \int_{\mathbb{R}^3} k(x)|u|^p dx + o(1)$$

and

$$\int_{\mathbb{R}^3} h(x)u_n^2 dx = \int_{\mathbb{R}^3} h(x)v_n^2 dx + \int_{\mathbb{R}^3} h(x)u^2 dx + o(1).$$

Therefore, with the help of (2.22), we get that

$$\begin{aligned}\langle I'_\mu(u_n), u_n\rangle &= \langle I'_\mu(u), u\rangle + \|v_n\|^2 + F(v_n) \\ &\quad - \int_{\mathbb{R}^3} k(x)|v_n|^p dx - \mu \int_{\mathbb{R}^3} h(x)v_n^2 dx + o(1),\end{aligned}$$

which, together with (2.30) and (2.31), implies

$$\lim_{n\to\infty}\left(\|u_n - u\|^2 - \int_{\mathbb{R}^3} k(x)|u_n - u|^p dx\right) = 0 \tag{2.32}$$

since $I'_\mu(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ and $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$.

Next, without loss of generality, we may assume that $k_\infty < -1$. Then $(H_{k_2})$ implies that there is $R_0 > 0$ such that

$$k(x) < -1 \text{ if } |x| > R_0. \tag{2.33}$$

Moreover, since $k \in C(\mathbb{R}^3)$ and $4 < p < 6$, we arrive at

$$\int_{|x|\le R_0} k(x)|u_n - u|^p dx \to 0 \tag{2.34}$$

as $n \to \infty$. It is now deduced from (2.32)–(2.34) that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \|u_n - u\|^2 \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n - u|^p dx \\ &\le \lim_{n\to\infty} \int_{|x|\le R_0} k(x)|u_n - u|^p dx = 0.\end{aligned}$$

This proves that $u_n \to u$ in $H^1(\mathbb{R}^3)$. □

## 2.2 Existence of two positive solutions

In this section, we will prove the existence and multiplicity of positive critical points of $I_\mu$ on $H^1(\mathbb{R}^3)$. Our main strategy is to study suitable minimization problem and minimax procedure. We emphasize that, with the help of Lemma 2.1.3, an important thing is to study the geometrical structure of $I_\mu$. Let us start with the following well-known lemma.

**Lemma 2.2.1.** *Assume $h \in L^{3/2}(\mathbb{R}^3)$ and $h(x) \geq 0$. Then for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $w \in H^1(\mathbb{R}^3)$ such that*

$$-\Delta w + w = h(x)u.$$

*Moreover, the operator $K_h : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ defined by $K_h(u) = w$ is compact.*

Using the spectral theory of compact symmetric operators on Hilbert space, the above lemma implies the existence of a sequence of eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ of

$$-\Delta u + u = \mu h(x)u, \quad u \in H^1(\mathbb{R}^3)$$

with $\mu_1 < \mu_2 \leq \cdots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \cdots$ with $\|e_i\| = 1$, $i = 1, 2, \cdots$. Moreover, since $K_h$ is a positive operator, one has $\mu_1 > 0$ with a positive eigenfunction $e_1 > 0$ in $\mathbb{R}^3$. In addition, we have the following variational characterization of $\mu_n$:

$$\mu_1 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2 dx}, \quad \mu_n = \inf_{u\in S_{n-1}^{\perp}\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2 dx}, \tag{2.35}$$

where $S_{n-1}^{\perp} = \{span\{e_1, e_2, \cdots, e_{n-1}\}\}^{\perp}$. Let $\bar{\mu}_1$ be the first eigenvalue of

$$-\Delta u + u = \mu h(x)u, \text{ in } H_0^1(\Omega^0).$$

Then clearly $\mu_1 < \bar{\mu}_1$ and we have that $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$ for any $\mu < \bar{\mu}_1$.

In the following, we prove the mountain pass geometry for the functional $I_\mu$ for $\mu$ less than $\mu_1$ and $\mu$ in the right neighborhood of $\mu_1$, respectively.

**Lemma 2.2.2.** *Assume that the hypotheses $(H)$ hold and $4 < p < 6$.*

- $(I_1)$ *If $0 < \mu < \mu_1$, then $u = 0$ is the local minimum of $I_\mu$;*
- $(I_1')$ *There are positive constants $\bar{\delta}, \rho$ and $\alpha$ such that, for any $\mu \in [\mu_1, \mu_1+\bar{\delta})$, $I_\mu|_{\partial B_\rho} \geq \alpha$; And*
- $(I_2)$ *there is $\bar{u} \in H^1(\mathbb{R}^3)$ with $\|\bar{u}\| > \rho$ such that $I_\mu(\bar{u}) < 0$ for any $\mu > 0$.*

**Proof.** (i) *Proof of $(I_1)$:* By $(H_{l_1})$ one has that $F(u) \geq 0$. $(H_{k_1})$ and $(H_{k_2})$ imply that $k$ is bounded in $\mathbb{R}^3$. Thus, by (2.35) and the continuity of the Sobolev embedding of $H^1(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$, we deduce that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2 dx \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^p - \frac{\mu}{2\mu_1}\|u\|^2 \\ &= \|u\|^2\left(\frac{1}{2} - \frac{\mu}{2\mu_1} - C\|u\|^{p-2}\right). \end{aligned}$$

Choosing $\rho = \|u\|$ small enough such that

$$C\rho^{p-2} \leq \frac{1}{4}\left(1 - \frac{\mu}{\mu_1}\right),$$

we obtain that

$$I(u) \geq \frac{1}{4}\left(1 - \frac{\mu}{\mu_1}\right)\rho^2. \tag{2.36}$$

Therefore the conclusion $(I_1)$ follows.

(ii) *Proof of* $(I_1')$*:* For any $u \in H^1(\mathbb{R}^3)$, there exist $t \in \mathbb{R}$ and $v \in S_1^{\perp}$ such that

$$u = te_1 + v, \text{ where } \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + ve_1)\, dx = 0. \tag{2.37}$$

Hence we get from direct computation that

$$\|u\| = \left(\|\nabla(te_1 + v)\|_2^2 + \|te_1 + v\|_2^2\right)^{\frac{1}{2}} = \left(t^2 + \|v\|^2\right)^{\frac{1}{2}}, \tag{2.38}$$

$$\mu_2 \int_{\mathbb{R}^3} h(x)v^2 dx \leq \|v\|^2, \tag{2.39}$$

$$\mu_1 \int_{\mathbb{R}^3} h(x)e_1^2 dx = \|e_1\|^2 = 1 \tag{2.40}$$

and

$$\mu_1 \int_{\mathbb{R}^3} h(x)e_1 v dx = \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + ve_1)\, dx = 0. \tag{2.41}$$

Using the mean value theorem, we know that there exists $\vartheta$ with $0 < \vartheta < 1$ such that

$$\begin{aligned} |F(te_1 + v) - F(te_1)| &= 4\left|\int_{\mathbb{R}^3} l(x)\phi_{te_1+\vartheta v}(te_1 + \vartheta v)v dx\right| \\ &\leq 4\,|l|_2\,|\phi_{te_1+\vartheta v}|_6\,|te_1 + \vartheta v|_6\,|v| \\ &\leq C\|l\|_2^2\|te_1 + \vartheta v\|^3\|v\| \\ &\leq C_0\left(|t|^3\|v\| + \vartheta^3\|v\|^4\right). \end{aligned} \tag{2.42}$$

We first consider the case that $\mu = \mu_1$ and estimate the value of $I_{\mu_1}$ for $u$ not too small. Denoting $\theta := \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right) > 0$, by (2.38)–(2.42) and the boundedness of the function $k$, one has that

$$\begin{aligned} I_{\mu_1}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)u^2dx - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p dx \\ &= \frac{1}{2}\left(t^2 + \|v\|^2\right) + \frac{1}{4}F(te_1) + \frac{1}{4}F(te_1+v) - \frac{1}{4}F(te_1) \\ &\quad -\frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)(te_1+v)^2dx - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p dx \\ &\geq \frac{1}{2}\|v\|^2 - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)v^2dx + \frac{1}{4}t^4F(e_1) \\ &\quad -C_0\left(|t|^3\|v\| + \vartheta^3\|v\|^4\right) - \frac{1}{p}\int_{\mathbb{R}^3} k(x)|u|^p dx \\ &\geq \frac{1}{2}\|v\|^2 - \frac{\mu_1}{2\mu_2}\|v\|^2 + \frac{1}{4}t^4F(e_1) - C_0|t|^3\|v\| - C_0\vartheta^3\|v\|^4 - C_2|t|^p - C_5\|v\|^p \\ &\geq \theta\|v\|^2 + C_1|t|^4 - C_2|t|^p - C_3|t|^3\|v\| - C_4\|v\|^4 - C_5\|v\|^p. \end{aligned}$$

Note that, for $\|v\| \leq |t|^{p-3}$, one obtains that

$$I_{\mu_1}(u) \geq \theta\|v\|^2 + C_1|t|^4 - (C_2+C_3)\,|t|^p - C_4\|v\|^4 - C_5\|v\|^p.$$

Since there is $C_6 > 0$ such that

$$C_1|t|^4 - (C_2+C_3)\,|t|^p \geq C_6t^2$$

for some $t$ with

$$\left(\frac{2C_6}{C_1}\right)^{\frac{1}{2}} \leq |t| < \left(\frac{C_1}{2(C_2+C_3)}\right)^{\frac{1}{p-4}}. \tag{2.43}$$

On the other hand, for $\|v\| \leq 1$, one may deduce that

$$\theta\|v\|^2 - C_4\|v\|^4 - C_5\|v\|^p \geq \theta\|v\|^2 - (C_4+C_5)\,\|v\|^4 \geq \frac{\theta}{2}\|v\|^2,$$

as long as $\|v\|^2 \leq \frac{\theta}{2(C_4+C_5)}$. In sum, there is $C_6 > 0$ such that for $t$ satisfying (2.43) and $v$ with

$$\|v\| \leq \min\left\{|t|^{p-3},\ 1,\ \left(\frac{\theta}{2(C_4+C_5)}\right)^{\frac{1}{2}}\right\}, \tag{2.44}$$

one deduce by some computations that

$$I_{\mu_1}(u) \geq \frac{\theta}{2}\|v\|^2 + C_6t^2 \geq \min\left\{\frac{\theta}{2}, C_6\right\}\left(\|v\|^2 + t^2\right) = \min\left\{\frac{\theta}{2}, C_6\right\}\|u\|^2. \tag{2.45}$$

Let $\bar{\delta} = \mu_1 \min\left\{\frac{\theta}{2}, C_6\right\} > 0$. For any $\mu \in [\mu_1, \mu_1+\bar{\delta})$, by (2.45), we obtain

$$\begin{aligned} I_\mu(u) &= I_{\mu_1}(u) + \frac{1}{2}(\mu_1-\mu)\int_{\mathbb{R}^3} h(x)u^2dx \\ &\geq \min\left\{\frac{\theta}{2}, C_6\right\}\|u\|^2 - \frac{\mu-\mu_1}{2\mu_1}\|u\|^2 \\ &\geq \frac{1}{2}\min\left\{\frac{\theta}{2}, C_6\right\}\|u\|^2. \end{aligned}$$

Take

$$\rho := \|u\| = \left(t^2 + \|v\|^2\right)^{\frac{1}{2}}$$

with $t$ satisfying (2.43) and $v$ satisfying (2.44). Hence $(I_1')$ follows by choosing $\alpha := \frac{1}{2}\min\left\{\frac{\theta}{2}, C_6\right\}\rho^2$.

(iii) *Proof of* $(I_2)$*:* Choose $\varphi \in H^1(\mathbb{R}^3)$ with supp $\varphi \subset \Omega^+$ such that $\varphi(x) \geq 0$ for all $x \in \Omega^+$ and $\varphi = t_0 e_1 + v$ with $t_0 \neq 0$. Then for any $s > 0$, we have that

$$I_\mu(s\varphi) = \frac{s^2}{2}\int_{\mathbb{R}^3}\left(|\nabla\varphi|^2 + \varphi^2\right)dx + \frac{s^4}{4}F(\varphi) - \frac{\mu s^2}{2}\int_{\mathbb{R}^3} h(x)\varphi^2 dx - \frac{s^p}{p}\int_{\Omega^+} k(x)|\varphi|^p dx.$$

From the choice of $\varphi$ we know that $I_\mu(s\varphi) < 0$ for $s$ with $s|t_0| > \sqrt{\frac{2C_6}{C_1}}$ sufficiently large. Thus the conclusion of $(I_2)$ follows by taking $\bar{u} = s\varphi$. □

**Remark 2.2.3.** *Set* $\bar{\epsilon} = \min\{\bar{\mu}_1 - \mu_1, \bar{\delta}\}$*. If* $0 < \mu < \mu_1 + \bar{\epsilon}$*, then* $\mu < \min\{\bar{\mu}_1, \mu_1 + \bar{\delta}\}$*. Hence, when* $0 < \mu < \mu_1 + \bar{\epsilon}$*, Lemma 2.2.2 implies that the functional* $I_\mu$ *has the mountain pass geometry and Lemma 2.1.3 means that the functional* $I_\mu$ *satisfies* $(PS)$*-condition.*

We are in a position to prove Theorem 2.0.7.

**Proposition 2.2.4.** *Assume that the hypotheses* $(H)$ *hold and* $4 < p < 6$*. Then problem (2.1) has a positive solution* $u_\mu$ *with* $I_\mu(u_\mu) > 0$ *for* $0 < \mu < \mu_1 + \bar{\epsilon}$*.*

**Proof.** We denote

$$c_{1,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \ : \ \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Remark 2.2.3, the Mountain Pass Theorem implies that $c_{1,\mu}$ is a critical value of $I_\mu$ and $c_{1,\mu} > 0$. The proof of positivity for at least one of the corresponding (nontrivial) critical point is inspired by the idea of Alama-Tarantello [1]. In fact, since $I_\mu(u) = I_\mu(|u|)$ in $H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$c_{1,\mu} \leq \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < c_{1,\mu} + \frac{1}{n}. \tag{2.46}$$

Consequently, by means of Ekeland's variational principle, there exists $\gamma_n^* \in \Gamma$ with the following properties:

$$\begin{cases} c_{1,\mu} \leq \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)) \leq \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < c_{1,\mu} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t) - \gamma_n^*(t)\| < \frac{1}{\sqrt{n}}; \\ \text{ there exists } t_n \in [0,1] \text{ such that } z_n = \gamma_n^*(t_n) \text{ satisfies :} \\ I_\mu(z_n) = \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)), \text{ and } \|I_\mu'(z_n)\| \leq \frac{1}{\sqrt{n}}. \end{cases} \tag{2.47}$$

In particular, we get a $(PS)_{c_{1,\mu}}$-sequence $(z_n)_{n\in\mathbb{N}}$. By Lemma 2.1.3 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). Let $z_n \to z$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$. On the other hand, by (2.47), we also arrive at $\gamma_n(t_n)\to z$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$. Since $\gamma_n(t)\geq 0$, we conclude that $z\geq 0$ a.e. in $\mathbb{R}^3$ with $I_\mu(z)>0$ and it is a solution of problem (2.1). The strong maximum principle implies that $z>0$ in $\mathbb{R}^3$. The conclusion of this proposition follows from choosing $u_\mu := z$. □

**Proposition 2.2.5.** *Assume that the hypotheses* $(H)$ *hold and* $4<p<6$. *Then, for any* $\mu_1<\mu<\mu_1+\bar{\epsilon}$, *problem (2.1) has a positive solution* $\omega_\mu$ *with* $I_\mu(\omega_\mu)<0$.

**Proof.** Let $B_\rho$ denote the closed ball $B_\rho=\{u\in H^1(\mathbb{R}^3):\|u\|\leq\rho\}$ with $\rho$ as in Lemma 2.2.2. Set

$$c_{2,\mu} := \inf_{\|u\|\leq\rho} I_\mu(u). \tag{2.48}$$

It is clear that $c_{2,\mu}>-\infty$. We claim that $c_{2,\mu}<0$. In fact, given $R>0$, define $\eta_R\in C_0^\infty(\mathbb{R}^3)$ with $0\leq\eta_R(x)\leq 1$ and $|\nabla\eta_R(x)|\leq\frac{2}{R}$ for any $x\in\mathbb{R}^3$ and

$$\eta_R(x)=\begin{cases}1, & |x|\leq R,\\ 0, & |x|\geq 2R.\end{cases}$$

Then $\eta_R e_1\in H^1(\mathbb{R}^3)$. To complete the proof of the claim, it suffices to show that $I_\mu(t\eta_R e_1)<0$ for all $t>0$ small. First we have that

$$\begin{aligned}
I_\mu(t\eta_R e_1) &= \frac{t^2}{2}\|\eta_R e_1\|^2+\frac{t^4}{4}F(\eta_R e_1)\\
&-\frac{\mu t^2}{2}\int_{\mathbb{R}^3}h(x)(\eta_R e_1)^2dx-\frac{t^p}{p}\int_{\mathbb{R}^3}k(x)|\eta_R e_1|^p dx\\
&=\frac{t^2}{2}\int_{\mathbb{R}^3}\eta_R^2|\nabla e_1|^2dx+\frac{t^2}{2}\int_{\mathbb{R}^3}\eta_R^2e_1^2dx\\
&+t^2\int_{\mathbb{R}^3}\eta_R e_1\nabla\eta_R\nabla e_1dx+\frac{t^4}{4}F(\eta_R e_1)+\frac{t^2}{2}\int_{\mathbb{R}^3}e_1^2|\nabla\eta_R|^2dx\\
&-\frac{\mu t^2}{2}\int_{\mathbb{R}^3}h(x)\eta_R^2e_1^2dx-\frac{t^p}{p}\int_{\mathbb{R}^3}k(x)\eta_R^pe_1^pdx.
\end{aligned}\tag{2.49}$$

On the other hand, multiplying both sides of the equation

$$-\Delta e_1+e_1=\mu_1h(x)e_1$$

by $\eta_R^2e_1$ and integrating by parts, one obtains that

$$2\int_{\mathbb{R}^3}\eta_R e_1\nabla\eta_R\nabla e_1dx+\int_{\mathbb{R}^3}\eta_R^2e_1^2dx+\int_{\mathbb{R}^3}\eta_R^2|\nabla e_1|^2dx=\mu_1\int_{\mathbb{R}^3}h(x)\eta_R^2e_1^2dx. \tag{2.50}$$

Inserting (2.50) into (2.49), we get that

$$\begin{aligned} I_\mu(t\eta_R e_1) &= (\mu_1-\mu)\frac{t^2}{2}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx \\ &\quad + \frac{t^4}{4}F(\eta_R e_1) - \frac{t^p}{p}\int_{\mathbb{R}^3} k(x)\eta_R^p e_1^p dx. \end{aligned} \tag{2.51}$$

By the definition of $\eta_R$, the Hölder inequality and the Sobolev inequality, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx &= \int_{R\le|x|\le 2R} e_1^2|\nabla\eta_R|^2 dx \\ &\le \left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \left(\int_{R\le|x|\le 2R} |\nabla\eta_R|^3 dx\right)^{\frac{2}{3}} \\ &\le \left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \left(\left(\frac{2}{R}\right)^3 \int_{R\le|x|\le 2R} dx\right)^{\frac{2}{3}} \\ &\le C\left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \\ &\to 0, \text{ as } R\to\infty, \end{aligned} \tag{2.52}$$

since $\|e_1\| = 1$. Meanwhile, multiplying both sides of the equation

$$-\Delta e_1 + e_1 = \mu_1 h(x) e_1$$

by $e_1$ and integrating by parts, we get that

$$\mu_1 \int_{\mathbb{R}^3} h(x)e_1^2 dx = \|e_1\|^2 = 1. \tag{2.53}$$

Moreover, by choosing $R$ sufficiently large, we obtain that

$$\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx \ge \int_{|x|\le R} h(x)\eta_R^2 e_1^2 dx = \int_{|x|\le R} h(x)e_1^2 dx \ge \frac{1}{2\mu_1}, \tag{2.54}$$

and then choosing $R_2 > 0$ sufficiently large with $R \ge R_2$, we deduce from (2.52)–(2.54) that

$$\int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx \le \frac{\mu-\mu_1}{2}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx \tag{2.55}$$

for all $R > R_2$. From (2.51) and (2.55), we deduce that

$$\begin{aligned} I_\mu(t\eta_R e_1) &\le (\mu_1-\mu)\frac{t^2}{4}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx + \frac{t^4}{4}F(\eta_R e_1) - \frac{t^p}{p}\int_{\mathbb{R}^3} k(x)\eta_R^p e_1^p dx \\ &\le -C_7 t^2 + C_8 t^4 + C_9 t^p, \end{aligned}$$

for all $R > R_2$, which means $I_\mu(t\eta_R e_1) < 0$ for $t > 0$ small enough. Thus $c_{2,\mu} < 0$ and the proof of the claim is complete.

In addition, since $I_\mu(u) = I_\mu(|u|)$, given $n$, by (2.48), there exists $w_n^* \ge 0$ with $\|w_n^*\| \le \rho$

such that

$$c_{2,\mu} \le I_\mu(w_n^*) < c_{2,\mu} + \frac{1}{n}.$$

Then, according to the Ekeland's variational principle, there is a sequence $(w_n)_{n\in\mathbb{N}}$ with $\|w_n\| \le \rho$ satisfying

$$c_{2,\mu} \le I_\mu(w_n) \le I_\mu(w_n^*) < c_{2,\mu} + \frac{1}{n},$$

$$\|w_n - w_n^*\| \le \frac{1}{\sqrt{n}} \text{ and } \|I'_\mu(w_n)\| \le \frac{1}{n}. \tag{2.56}$$

As $n \to \infty$, the sequence $(w_n)_{n\in\mathbb{N}}$ satisfies $I_\mu(w_n) \to c_{2,\mu}$ and $I'_\mu(w_n) \to 0$. Then Lemma 2.1.3 implies the existence of a minimizer $w \in B_\rho$ for the functional $I_\mu$ and $w_n \to w$ in $H^1(\mathbb{R}^3)$. Hence, by (2.56), $w_n^* \to w$. Since $w_n^* \ge 0$, we get that $w \ge 0$ a.e. in $\mathbb{R}^3$ with $I_\mu(w) < 0$ and it is a solution of problem (2.1). The maximum principle implies that $w > 0$ in $\mathbb{R}^3$. The conclusion of this proposition follows from choosing $\omega_\mu := w$. The proof of Proposition 2.2.5 is complete. □

**Remark 2.2.6.** *In fact, for the case of* $0 < \mu < \mu_1$*, to get a positive solution, it is not necessary to involve the condition* $(H_{l_2})$*. Since this condition is used to get the boundedness of* $(PS)$*-sequence, for this case, one may use standard variational methods.*

**Remark 2.2.7.** *Since* $\phi_u$ *is always positive for every* $u \in H^1(\mathbb{R}^3)$ *and* $u \ne 0$*, we get that* $(u_\mu, \phi_{u_\mu})$ *and* $(\omega_\mu, \phi_{\omega_\mu})$ *are positive solutions of problem* $(\mathcal{P}_\mu)$ *in* $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ *by Proposition 2.2.4 and Proposition 2.2.5, respectively. Hence we finish the proof of Theorem 2.0.7.*

**Remark 2.2.8.** *Theorem 2.0.7 shows the existence of multiple positive solutions of* $(\mathcal{P}_\mu)$ *for* $4 < p < 6$*. It would be very interesting to study the existence/nonexistence of positive solutions of* $(\mathcal{P}_\mu)$ *for* $2 < p \le 4$*, which will be an issue for further studies. We thank an unknown referee for pointing out this remark.*

# Chapter 3

# Schrödinger-Poisson system with a general indefinite nonlinearity

In this chapter, we still consider the indefinite nonlinearity, that is, this chapter is a continuation of Chapter 2, in which we studied the existence of multiple positive solutions to the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{p-2}u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1}$$

where $4 < p < 6$, $k \in C(\mathbb{R}^3)$ changes sign in $\mathbb{R}^3$ and $\lim_{|x|\to\infty} k(x) = k_\infty < 0$. There we mainly proved that the system (3.1) has at least two positive solutions for $\mu > \mu_1$ (but not far from $\mu_1$), where $\mu_1$ is the first eigenvalue of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$, whose corresponding eigenfunction is denoted by $e_1$. An interesting phenomenon there is that we have succeeded in making use of the nonlocal term to technically help deal with the key difficulty that the indefinite nonlinearity has created, and we do not need the condition

$$\int_{\mathbb{R}^3} k(x)e_1^p dx < 0, \tag{$*$}$$

which has been shown to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities with a bounded or an unbounded domain, like

$$-\Delta_m u = \mu a(x)|u|^{m-2}u + f(x,u),\ m \geq 2,$$

where $\Delta_m u = div(|\nabla u|^{m-2}\nabla u)$, see [1, 36, 47] and their references. In this work, instead of considering the homogeneous nonlinearity $k(x)|u|^{p-2}u$, we are concerned with a more general nonlinearity $k(x)g(u)$, where $g$ is a nonlinear function with superquadratic growth both at zero and at infinity. Surprisingly, we find that, for this general case, it is still not necessary to involve the condition (*) either. Moreover, here we extend the weight function $l \in L^2(\mathbb{R}^3)$ to $l \in L^\infty(\mathbb{R}^3)$.

More precisely, in the present chapter, we study the existence and multiplicity of positive solutions to the following problem

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)g(u) + \mu h(x)u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{3.2}$$

where for the continuous nonlinearity $g \in C(\mathbb{R}, \mathbb{R})$, we assume the hypotheses $(G)$:

$(G_1)$ there is $q \in \mathbb{R}$ with $4 < q < 6$ such that $\lim_{s \to 0} \frac{g(s)}{|s|^{q-2}s} = 1$;

$(G_2)$ there is $p \in \mathbb{R}$ with $4 < p < 6$ such that $\lim_{|s| \to \infty} \frac{g(s)}{|s|^{p-2}s} = 1$;

$(G_3)$ $g(s) > 0$ for all $s > 0$.

Since we just aim to find the positive solutions, it is only necessary to consider all $u > 0$ for problem (3.2), and throughout this chapter we assume, without loss of generality, that $g$ is defined in $\mathbb{R}$ as an odd function. And for the weight functions we consider the following hypotheses $(H)$:

$(H_h)$ $h \in L^{3/2}(\mathbb{R}^3)$, $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;

$(H_{k_1})$ $k \in C(\mathbb{R}^3)$ and $k$ changes sign in $\mathbb{R}^3$;

$(H_{k_2})$ $\lim_{|x| \to \infty} k(x) = k_\infty < 0$;

$(H_{l_1})$ $l \in L^\infty(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;

$(H_{l_2})$ $l = 0$ a.e. in $\Omega^0$, where $\Omega^0 = \{x \in \mathbb{R}^3 \ : \ k(x) = 0\}$, and $\Omega^0$ coincides the closure of its interior .

Our main result is as following

**Theorem 3.0.9.** *Suppose the hypotheses $(G)$ and $(H)$. In addition, if one of the following conditions holds:*

*(i) $g(s)$ satisfies the stronger form $(G_2')$ of $(G_2)$ given by $g(s) = |s|^{p-2}s + O\left(|s|^\beta\right)$ as $|s| \to \infty$ for some $0 \leq \beta < 1$;*

*(ii) the weight function $k$ has a thick zero set $\Omega^0$ in the sense that $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$, where $\Omega^+ = \{x \in \mathbb{R}^3 \ : \ k(x) > 0\}$ and $\Omega^- = \{x \in \mathbb{R}^3 \ : \ k(x) < 0\}$,*

*then*

*(1) for $0 < \mu \leq \mu_1$, problem (3.2) has at least one positive solution in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$;*

*(2) there exists $\bar{\epsilon} > 0$ such that, for $\mu_1 < \mu < \mu_1 + \bar{\epsilon}$, problem (3.2) has at least two positive solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

The approach here is variational. The Mountain Pass Theorem is used to get one of the two positive solutions. To apply this theorem, one needs to ensure that the associated functional should satisfy the mountain pass geometry, which is not easy, in particular, for the case $\mu_1 < \mu < \mu_1 + \bar{\epsilon}$. Here we use the same procedure as in our preceding work by making use of the nonlocal term to compensate the technical condition (*) mentioned above with $p$ replacing by $q$ under the assumptions of $(G)$. Moreover, one also needs to restore the compactness, since the domain $\mathbb{R}^3$ is unbounded and the compactness of Sobolev embedding does not hold. The main different thing from the preceding result is that here we use a different method to restore the compactness. The concentration-compactness principle of Lions [78] is used to overcome the difficulty of the lack of compactness, in which we follow the idea of Costa-Tehrani [36], which is on the Schrödinger equation and in which a similar general nonlinearity was used by the authors. But the situation here becomes more delicate, because of the involvement of the Poisson equation in our case, namely, with additionally non-local term to compare with the Schrödinger equation (see details in the second section). Also comparing with the results in the previous chapter, we allow $l \in L^\infty(\mathbb{R}^3)$. This also extends the results in the previous chapter because when $l \in L^2(\mathbb{R}^3)$, the functional $\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx$ is weakly continuous; while for $l \in L^\infty(\mathbb{R}^3)$, the functional $\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx$ may be not weakly continuous. The second solution is obtained by the Ekeland variational principle and it is a local minimizer.

The results of the present chapter are contained in [62] and the author presented them in the fourth annual workshop of Functional Analysis and Applications Group, University of Aveiro, 8 June 2013.

## 3.1 The proof of Palais-Smale condition

As Chapter 1 mentioned, system (3.2) can be reduced into

$$-\Delta u + u + l(x)\phi_u u = k(x)g(u) + \mu h(x)u, \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

Denote $G(u) = \int_0^u g(s)ds$. With $F$ denoted by (1.12) as $F(u) = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx$, we have the following associated functional to (3.3)

$$I_\mu(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3} k(x)G(u)dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2dx.$$

Hypotheses $(G)$ imply that

$$|g(s)| \le b_1|s|^{q-1} + b_2|s|^{p-1}, \quad |G(s)| \le b_1|s|^q + b_2|s|^p, \text{ for all } s \in \mathbb{R}, \tag{3.4}$$

and

$$\begin{aligned} b_3|s|^q \le G(s), \quad b_3|s|^q \le g(s)s, \quad \text{if } |s| \le \delta_0, \\ b_4|s|^p \le G(s), \quad b_4|s|^p \le g(s)s, \quad \text{if } |s| \ge \delta_0, \end{aligned} \tag{3.5}$$

for some $b_1, b_2, b_3, b_4, \delta_0 > 0$. Therefore the functional $I_\mu$ is of class $C^1(H^1(\mathbb{R}^3), \mathbb{R})$, and

$$\langle I'_\mu(u), \varphi\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + u\varphi)\, dx + \int_{\mathbb{R}^3} l(x)\phi_u u\varphi dx - \int_{\mathbb{R}^3} (k(x)g(u)\varphi + \mu h(x)u\varphi)\, dx$$

for any $\varphi \in H^1(\mathbb{R}^3)$. Moreover, there is a one to one corresponding between the solutions of (3.3) and the critical points of $I_\mu$. Then if $u \in H^1(\mathbb{R}^3)$ is a critical point of $I_\mu$ on $H^1(\mathbb{R}^3)$, then $(u, \phi_u)$ is a solution of system (3.2). Hence, to solve (3.2), it suffices to study positive critical points of the functional $I_\mu$ on $H^1(\mathbb{R}^3)$.

Now we are in a position to prove $(PS)$-condition for the functional $I_\mu$. Denote $\sigma(-\Delta + id, \Omega^0, h)$ the collection of eigenvalues of $-\Delta + id$ in $H_0^1(\Omega^0)$ with the weight function $h$. If the Lebesgue measure of $\Omega^0$ is zero, i.e., $|\Omega^0| = 0$, then $\sigma(-\Delta + id, \Omega^0, h) = \emptyset$.

**Lemma 3.1.1.** *Suppose that the hypotheses (G) and (H) hold. If $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, then for $c \in \mathbb{R}$, any $(PS)_c$-sequence is bounded in $H^1(\mathbb{R}^3)$, provided either of the following conditions holds:*

*(a) there is $\beta$ with $0 \leq \beta < 1$ such that*

$$g(s) = |s|^{p-2}s + O\left(|s|^\beta\right) \text{ as } |s| \to \infty;$$

*(b) $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$.*

**Proof.** Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_c$-sequence for $I_\mu$ at the level $c$, i.e.,

$$I_\mu(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}F(u_n) - \int_{\mathbb{R}^3} k(x)G(u_n)dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u_n^2 dx = c + o(1), \tag{3.6}$$

$$\begin{aligned} \langle I'_\mu(u_n), \varphi\rangle &= \int_{\mathbb{R}^3} (\nabla u_n \nabla\varphi + u_n\varphi)\, dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n\varphi dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(u_n)\varphi dx - \mu\int_{\mathbb{R}^3} h(x)u_n\varphi dx \\ &= o(1)\|\varphi\| \end{aligned} \tag{3.7}$$

for any $\varphi \in H^1(\mathbb{R}^3)$. We assume, by contradiction, that $t_n := \|u_n\|$ and $t_n \to \infty$ as $n \to \infty$. Denote $v_n := u_n/t_n$. Then we have that

$$\|v_n\| = \frac{1}{t_n}\|u_n\| = 1 \tag{3.8}$$

for each $n \in \mathbb{N}$. Going if necessary to a subsequence, we may assume that there is $v \in H^1(\mathbb{R}^3)$ such that for each bounded domain $\Omega \subset \mathbb{R}^3$,

$$\begin{array}{ll} v_n \rightharpoonup v & \text{in } H^1(\mathbb{R}^3), \\ v_n(x) \to v(x) & \text{a.e. in } \mathbb{R}^3, \\ v_n \to v \text{ in } L^t(\Omega) & \text{for } 2 < t < 6, \\ |v_n(x)| \leq w_\Omega(x) & \text{for some } w_\Omega \in L^t(\Omega). \end{array} \tag{3.9}$$

Hence, for any $\varphi \in H^1(\mathbb{R}^3)$, we have that

$$\int_{\mathbb{R}^3} (\nabla v_n \nabla \varphi + v_n \varphi)\, dx \to \int_{\mathbb{R}^3} (\nabla v \nabla \varphi + v\varphi)\, dx. \tag{3.10}$$

We first claim that $v(x) = 0$ a.e. in $\mathbb{R}^3$. In fact, by (3.8) one may deduce that $\int_{\mathbb{R}^3} (\nabla v_n \nabla \varphi + v_n \varphi)\, dx$, $\int_{\mathbb{R}^3} l(x)\phi_{v_n} v_n \varphi dx$ and $\int_{\mathbb{R}^3} h(x) v_n \varphi dx$ are all bounded with the given $\varphi$. Since $u_n = t_n v_n$ and $p > 4$, dividing (3.7) by $t_n^{p-1}$, one obtains that

$$\int_{\mathbb{R}^3} T_n(x) dx = o(1) \tag{3.11}$$

as $n \to \infty$, where $T_n(x) = \frac{k(x)g(t_n v_n(x))\varphi(x)}{t_n^{p-1}}$. We prove the claim in the three parts $\Omega^-$, $\Omega^+$ and $\Omega^0$, respectively. For each $y \in \Omega^+$, since $k \in C(\mathbb{R}^3)$, there exists $\delta_1 > 0$ such that

$$k(x) > 0 \text{ for all } x \in B_{\delta_1}(y). \tag{3.12}$$

Define $\zeta_m \in C^1(\mathbb{R}^3)$ $(m > 2)$ such that $\zeta_m(x) \geq 0$ for all $x \in \mathbb{R}^3$ and

$$\zeta_m(x) = \begin{cases} 1, & x \in B_{\left(\frac{1}{2} - \frac{1}{m^2}\right)\delta_1}(y), \\ 0, & x \in \mathbb{R}^3 \setminus B_{\delta_1/2}(y). \end{cases}$$

Let $\varphi = v\zeta_m$ in (3.7) and then $\text{supp}\varphi \subset B_{\delta_1/2}(y)$ for all $m \in \mathbb{N}$ and $m > 2$. If there is $N_0$ such that, for every $n > N_0$, $v_n(x) = 0$, then $v(x) = 0$; If, for some large $n$, $v_n(x) \neq 0$, then $|t_n v_n(x)| \to +\infty$ as $n \to \infty$. Hence, by $(G_2)$, one arrives at

$$\begin{aligned} T_n(x) &= \frac{k(x)|v_n(x)|^{p-2} v_n(x) v\zeta_m g(t_n v_n(x))}{|t_n v_n(x)|^{p-2} t_n v_n(x)} \\ &= k(x)|v_n(x)|^{p-2} v_n(x) v\zeta_m + o(1), \end{aligned}$$

which goes to $k(x)|v(x)|^p$ as $n \to \infty$ and $m \to \infty$. And it follows from $(G_2)$ that

$$g(s) \leq C_3(1 + |s|^{p-1})$$

for some $C_3 > 0$ and then

$$\begin{aligned} |T_n(x)| &\leq \frac{C_3|k(x)||v\zeta_m|\left(1 + t_n^{p-1}|v_n(x)|^{p-1}\right)}{t_n^{p-1}} \\ &\leq \frac{C\left(1 + t_n^{p-1}|v_n(x)|^{p-1}\right)}{t_n^{p-1}} \\ &\leq C\left(1 + |w_\Omega(x)|^{p-1}\right) \in L^1(B_{\delta_1/2}(y)). \end{aligned} \tag{3.13}$$

Thus, by Lebesgue dominated convergence theorem, (3.11) becomes

$$0 = \int_{B_{\delta_1/2}(y)} \lim_{n\to\infty} k(x)|v_n(x)|^{p-2} v_n(x) v(x) dx = \int_{B_{\delta_1/2}(y)} k(x)|v(x)|^p dx,$$

which, together with (3.12), implies that $v = 0$ a.e. in $B_{\delta_1/2}(y)$, and then $v = 0$ a.e. in $\Omega^+$. We reach the claim for $x \in \Omega^-$ in a similar way. Furthermore, if $|\Omega^0| = 0$, we finish the proof the claim. If $|\Omega^0| \neq 0$, take $\varphi \in C^1(\mathbb{R}^3)$ with $\mathrm{supp}\varphi \subseteq \Omega^0$ in (3.7). By the notation of $\Omega^0$ and the assumption $(H_{l_2})$, we deduce respectively

$$\int_{\mathbb{R}^3} \frac{k(x)g(t_n v_n(x))\varphi(x)}{t_n} dx = \int_{\mathrm{supp}\varphi} \frac{k(x)g(t_n v_n(x))\varphi(x)}{t_n} dx = 0 \tag{3.14}$$

and

$$\int_{\mathbb{R}^3} l(y)\phi_{t_n v_n} v_n \varphi dy = \int_{\mathrm{supp}\varphi} l(y)\phi_{t_n v_n} v_n \varphi dy = 0 \tag{3.15}$$

for any $n \in \mathbb{N}$. Inserting (3.14) and (3.15) into (3.7), and using (3.10) and Lemma 1.4.6, one arrives at

$$\int_{\mathbb{R}^3} (\nabla v \nabla \varphi + v\varphi)\, dx = \mu \int_{\mathbb{R}^3} h(x) v \varphi dx.$$

Combining this equality with the fact that $v = 0$ a.e. in $\Omega^+ \cup \Omega^-$, we obtain that

$$\int_{\Omega^0} (\nabla v \nabla \varphi + v\varphi)\, dx = \mu \int_{\Omega^0} h(x) v \varphi dx.$$

Since $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, one obtains that $v = 0$ a.e. in $\Omega^0$. Therefore, we prove the claim, that is, $v(x) = 0$ a.e. in $\mathbb{R}^3$ and then $v_n \rightharpoonup 0$, which implies, by Lemma 1.4.6, that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h(x) v_n^2 dx = 0. \tag{3.16}$$

Next, dividing (3.6) by $t_n^2$ and combining it with (3.16), one gets that

$$\frac{1}{2} + \frac{1}{4t_n^2} F(u_n) - \frac{1}{t_n^2} \int_{\mathbb{R}^3} k(x) G(u_n) dx = o(1). \tag{3.17}$$

Take $\varphi = v_n \zeta$ in (3.7) with $\zeta \in C_0^\infty(\mathbb{R}^3)$ and divide it by $t_n$ to get

$$\int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + v_n^2\right) \zeta dx + \frac{1}{t_n^2} \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 \zeta dx - \frac{1}{t_n^2} \int_{\mathbb{R}^3} k(x) g(u_n) u_n \zeta dx = o(1), \tag{3.18}$$

where, in fact, the right hand side is equal to

$$\mu \int_{\mathbb{R}^3} h(x) v_n^2 \zeta dx - \int_{\mathbb{R}^3} v_n \nabla v_n \nabla \zeta dx + o(1) \frac{\|u_n \zeta\|}{t_n^2}.$$

In the following, we consider the two cases (a) and (b), respectively.

Case (a). If the condition (a) holds, then

$$\int_{\mathbb{R}^3} \frac{pk(x)G(u_n)\zeta}{t_n^2} dx = \int_{\mathbb{R}^3} \frac{k(x)g(u_n)u_n\zeta}{t_n^2} dx + o(1). \tag{3.19}$$

In fact, the condition $(a)$ implies that there exist $M_0 > 0$ and $\delta_2 > 0$ such that, for all

$|s| > M_0$, one has that

$$|pG(s) - g(s)s| \leq \delta_2 |s|^{\beta+1}$$

with $1 \leq \beta + 1 < 2$ and then

$$\begin{aligned}
&\int_{|u_n|>M_0} \frac{|k(x)||pG(u_n) - g(u_n)u_n||\zeta|}{t_n^2} dx \\
&\leq C \int_{\text{supp}\zeta} \frac{\delta_2 |u_n|^{\beta+1}}{t_n^2} dx \\
&\leq C \int_{\text{supp}\zeta} \frac{|v_n|^{\beta+1}}{t_n^{1-\beta}} dx \\
&\to 0
\end{aligned}$$

as $n \to \infty$. And since

$$|pG(s) - g(s)s| \leq C$$

for all $|s| \leq M_0$, we also have

$$\int_{|u_n|\leq M_0} \frac{|k(x)||pG(u_n) - g(u_n)u_n||\zeta|}{t_n^2} dx = o(1).$$

Without loss of generality, let us assume that $k_\infty < -1$. Then $(H_{k_2})$ implies that there is $R_0 > 0$ with $R_0 > \delta_0$ such that

$$k(x) < -1 \quad \text{for all} \quad |x| > R_0. \tag{3.20}$$

Then choosing $\zeta \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \zeta \leq 1$ and $\zeta(x) = 1$ if $|x| \leq R_0$, one deduces, by (3.17)-(3.19), that

$$\begin{aligned}
&\lim_{n\to\infty} \sup \int_{\mathbb{R}^3} \frac{k(x)G(u_n)(1-\zeta)}{t_n^2} dx \\
&= \frac{1}{2} + \lim_{n\to\infty} \sup \frac{F(u_n)}{4t_n^2} - \lim_{n\to\infty} \inf \frac{1}{p} \int_{\mathbb{R}^3} \frac{k(x)g(u_n)u_n}{t_n^2} \zeta dx \\
&= \frac{1}{2} + \lim_{n\to\infty} \sup \frac{F(u_n)}{4t_n^2} - \lim_{n\to\infty} \inf \frac{1}{p} \left( \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + v_n^2\right) \zeta dx + \frac{1}{t_n^2} \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 \zeta dx \right) \\
&\geq \frac{1}{2} + \lim_{n\to\infty} \sup \frac{F(u_n)}{4t_n^2} - \lim_{n\to\infty} \inf \frac{1}{p} \left( \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + v_n^2\right) dx + \frac{1}{t_n^2} \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 dx \right) \\
&\geq \frac{1}{2} - \frac{1}{p} + \left(\frac{1}{4} - \frac{1}{p}\right) \lim_{n\to\infty} \sup \frac{F(u_n)}{t^2} \\
&> 0.
\end{aligned} \tag{3.21}$$

On the other hand, by the choice of $R_0$, we obtain that

$$\int_{\mathbb{R}^3} \frac{k(x)G(u_n)(1-\zeta)}{t_n^2} dx \leq 0,$$

which contradicts (3.21). We prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$ in the case (a).

Case (b). It follows from $(G_2)$ that there are some $r > 4$ and $M_1 > R_0$ with $R_0 > 0$ as in (3.20) such that

$$0 < rG(s) \leq g(s)s, \tag{3.22}$$

for all $|s| \geq M_1$. If the condition (b) holds, then by choosing $\zeta \in C_0^\infty(\mathbb{R}^3)$ in (3.18) such that $0 \leq \zeta \leq 1$ with $\zeta(x) = 1$ if $x \in \Omega^+$ and $\zeta(x) = 0$ if $x \in \Omega^-$, and using (3.18) and (3.22), one deduces that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{k(x)G(u_n)\zeta}{t_n^2} dx \\
&= \int_{\{x:\, |u_n| \geq M_1\}} \frac{k(x)G(u_n)\zeta}{t_n^2} dx + \int_{\{x:\, |u_n| \leq M_1\}} \frac{k(x)G(u_n)\zeta}{t_n^2} dx \\
&\leq \frac{1}{r} \int_{\{x:\, |u_n| \geq M_1\}} \frac{k(x)g(u_n)u_n\zeta}{t_n^2} dx + o(1) \\
&= \frac{1}{r} \int_{\mathbb{R}^3} \left(|\nabla v_n|^2 + v_n^2\right) \zeta dx + \frac{1}{rt_n^2} \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n^2 \zeta dx + o(1) \\
&\leq \frac{1}{r} + \frac{1}{rt_n^2} F(u_n) + o(1).
\end{aligned} \tag{3.23}$$

However, (3.17) yields that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{k(x)G(u_n)\zeta}{t_n^2} dx &= \int_{\Omega^+} \frac{k(x)G(u_n)}{t_n^2} dx \\
&\geq \int_{\mathbb{R}^3} \frac{k(x)G(u_n)}{t_n^2} dx \\
&\geq \frac{1}{2} + \frac{1}{4t_n^2} F(u_n).
\end{aligned}$$

This is a contradiction with (3.23) and so we finish the proof of the case (b). This proves Lemma 3.1.1. □

To end this part, by Lemma 3.1.1, it remains to prove that $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence. To achieve this goal, we recall the following known concentration-compactness lemma of Lions.

**Lemma 3.1.2.** *(see Lions [78].) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^3)$ satisfying $\rho_n \geq 0$ and $\int \rho_n dx \to \bar{\lambda} > 0$. Then there exists a subsequence, still denoted by $(\rho_n)_{n\in\mathbb{N}}$, for which one of the three possibilities holds:*

*Vanishing:* $\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_R(y)} \rho_n(x)dx = 0$ *for all* $R > 0$;

*Dichotomy: There exits $0 < \alpha < \bar{\lambda}$ such that, for any given $\varepsilon > 0$, there are a sequence $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^3$, a number $R > 0$ and a sequence $(R_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$, with $R < R_1$, $R_n < R_{n+1} \to +\infty$, such that, if we set $\rho_n^1 = \rho_n \chi_{[|x-y_n|\leq R]}$ and $\rho_n^2 = \rho_n \chi_{[|x-y_n|\geq R_n]}$, then we have*

$$\|\rho_n - \rho_n^1 - \rho_n^2\|_1 \leq \varepsilon, \quad \left|\int \rho_n^1 dx - \alpha\right| \leq \varepsilon, \quad \left|\int \rho_n^2 dx - (\bar{\lambda} - \alpha)\right| \leq \varepsilon;$$

*Compactness: There exists $y_n \in \mathbb{R}^{\mathbb{N}}$ such that $\rho_n(\cdot + y_n)$ is tight, i.e.*

$$\forall \varepsilon > 0 \; \exists R > 0 \quad \textit{such that} \quad \int_{B_R(y_n)} \rho_n(x)dx \geq \bar{\lambda} - \varepsilon.$$

And we will also use the following lemma.

**Lemma 3.1.3.** *(see Lions [78].) Let $r > 0$ and $2 \leq q < 2^*$. If $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,r)} |u_n|^q dx = 0,$$

*as $n \to \infty$. Then*

$$u_n \to 0 \;\; \textit{in} \;\; L^t(\mathbb{R}^{\mathbb{N}}) \;\; \textit{for} \;\; t \in (2, 2^*).$$

Now we are ready to prove the last but fundamental lemma in this part.

**Lemma 3.1.4.** *Suppose that the hypotheses $(G)$ and $(H)$ hold. And we assume that either*

$$\textit{there is } \beta \textit{ with } 0 \leq \beta < 1 \textit{ such that } g(s) = |s|^{p-2}s + O\left(|s|^{\beta}\right) \;\; \textit{as} \; |s| \to \infty,$$

*or*

$$\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset.$$

*If $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$, then the functional $I_\mu$ satisfies $(PS)_c$-condition for each $c \in \mathbb{R}$.*

**Proof.** Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a $(PS)_c$-sequence for $I_\mu$, i.e., $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$. It follows from Lemma 3.1.1 that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Setting

$$\rho_n := |\nabla u_n|^2 + u_n^2$$

and then we get that $(\rho_n)_{n\in\mathbb{N}}$ is bounded in $L^1(\mathbb{R}^3)$. Passing if necessary to a subsequence, we may assume that for some $\bar{\lambda} \geq 0$

$$\|\rho_n\|_1 \to \bar{\lambda} \;\; \text{as } n \to \infty.$$

Clearly, we may assume that $\bar{\lambda} > 0$. In the following, we shall apply the Concentration-Compactness Lemma 3.1.2 to get the compactness by ruling out the vanishing and dichotomy.

First, if there is a subsequence, still denoting $(\rho_n)_{n\in\mathbb{N}}$, vanishing, then $(u_n)_{n\in\mathbb{N}}$ also vanishes, and so there exists $R_1 > 0$ satisfying

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_{R_1}(y)} u_n^2(x)dx = 0,$$

which implies, by Lemma 3.1.3, that $u_n \to 0$ in $L^t(\mathbb{R}^3)$, $t \in (2, 6)$. We get that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and then $\lim\limits_{n\to\infty} \int_{\mathbb{R}^3} h(x)u_n^2(x)dx = 0$ by Lemma 1.4.6. Moreover, $F(u_n) \leq$

$C\|u_n\|_{\frac{12}{5}}^4 = o(1)$ and by (3.4)

$$\int_{\mathbb{R}^3} k(x)g(u_n)u_n dx \le \|k\|_\infty \int_{\mathbb{R}^3} \left(b_1|u_n|^p + b_2|u_n|^q\right) dx = o(1).$$

Thus it follows from $I'_\mu(u_n) \to 0$ that $\|u_n\| \to 0$, which contradicts $\bar{\lambda} > 0$. Hence vanishing does not occur.

Second, we show that dichotomy does not happen. If dichotomy occurs, there exists $\alpha \in (0, \bar{\lambda})$ such that, for each given $\varepsilon > 0$, there are sequences $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^3$, $(R_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\hat{R} > 0$ with $R_0 < \hat{R} < \frac{R_1}{2}$, $R_n \le R_{n+1} \to +\infty$ such that

$$\alpha - \varepsilon \le \int_{|x-y_n|\le \frac{\hat{R}}{2}} \rho_n dx \ \text{ and } \ \bar{\lambda} - \alpha - \varepsilon \le \int_{|x-y_n|\ge 3R_n} \rho_n dx. \tag{3.24}$$

Therefore, from $\|\rho_n\|_1 \to \bar{\lambda}$, for n large, we obtain that

$$\int_{\frac{\hat{R}}{2}\le|x-y_n|\le 3R_n} \rho_n dx < \bar{\lambda} + \varepsilon - \int_{|x-y_n|\le \frac{\hat{R}}{2}} \rho_n dx - \int_{|x-y_n|\ge 3R_n} \rho_n dx = 3\varepsilon. \tag{3.25}$$

Note that we also have

$$\int_{\hat{R}\le|x-y_n|\le 2R_n} |u_n|^6 dx \le C\varepsilon^3. \tag{3.26}$$

Indeed, take $\eta_n \in C_0^\infty(\mathbb{R}^3)$ such that $\eta_n(x) = 0$ if $|x - y_n| \le \frac{\hat{R}}{2}$ or $|x - y_n| \ge 3R_n$; $\eta_n(x) = 1$, if $\hat{R} \le |x - y_n| \le 2R_n$; $|\eta_n(x)| \le 1$, and $|\nabla\eta_n(x)| \le \frac{1}{R_n - \hat{R}}$ for each $x \in \mathbb{R}^3$. Then $\eta_n u_n \in H^1(\mathbb{R}^3)$. It follows from the Sobolev inequality and (3.25) that

$$\begin{aligned}
&\left(\int_{\hat{R}\le|x-y_n|\le 2R_n} |u_n|^6 dx\right)^{\frac{1}{6}} \\
&\le C\left(\int_{\mathbb{R}^3} \left(|\nabla(\eta_n u_n)|^2 + |\eta_n u_n|^2\right) dx\right)^{\frac{1}{2}} \\
&\le C\left(2\int_{\mathbb{R}^3} \left(|u_n\nabla\eta_n|^2 + |\eta_n\nabla u_n|^2 + |\eta_n u_n|^2\right) dx\right)^{\frac{1}{2}} \\
&\le C\left(\int_{\frac{\hat{R}}{2}\le|x-y_n|\le 3R_n} \left(\frac{1 + (R_n - \hat{R})^2}{(R_n - \hat{R})^2} u_n^2 + |\nabla u_n|^2\right) dx\right)^{\frac{1}{2}} \\
&\le C\frac{\left(1 + (R_n - \hat{R})^2\right)^{\frac{1}{2}}}{|R_n - \hat{R}|}\left(\int_{\frac{\hat{R}}{2}\le|x-y_n|\le 3R_n} \rho_n dx\right)^{\frac{1}{2}} \\
&\le C\left(1 + \frac{1}{\hat{R}^2}\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

This proves (3.26). Next, let $\eta$ be another cut-off function such that $\eta(s) = 1$ for $s \le 1$,

$\eta(s) = 0$ for $s \geq 2$, $|\eta(s)| \leq 1$ for $s \in \mathbb{R}$, and $|\eta'(s)| \leq 2$ for $1 \leq s \leq 2$. Define

$$\bar{w}_n(x) := \eta\left(\frac{|x - y_n|}{\hat{R}}\right) u_n(x)$$

and

$$\bar{v}_n(x) := \left(1 - \eta\left(\frac{|x - y_n|}{R_n}\right)\right) u_n(x).$$

In addition, let us assume first that the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded. Then from (3.26) and the fact that $R_n \geq 2\hat{R}$ one deduces that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} h(x)\bar{v}_n\left(\bar{v}_n - u_n\right)dx\right| &= \left|\int_{R_n\leq|x-y_n|\leq 2R_n} h(x)\bar{v}_n\left(\bar{v}_n - u_n\right)dx\right| \\ &\leq 2\int_{R_n\leq|x-y_n|\leq 2R_n} |h(x)||u_n|^2 dx \\ &\leq 2\|h\|_{\frac{3}{2}}\left(\int_{R_n\leq|x-y_n|\leq 2R_n} |u|^6 dx\right)^{\frac{1}{3}} \\ &\leq C\|h\|_{\frac{3}{2}}\varepsilon,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} h(x)\bar{v}_n u_n dx = \int_{\mathbb{R}^3} h(x)\bar{v}_n^2 dx + \mu_1(\varepsilon), \tag{3.27}$$

where $\mu_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. One also deduces from (3.25), the fact that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$ and the Sobolev inequality that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} l(x)\bar{v}_n\left(\phi_{u_n}u_n - \phi_{\bar{v}_n}\bar{v}_n\right)dx\right| \\ &= \left|\int_{R_n\leq|x-y_n|\leq 2R_n} l(x)\bar{v}_n\left(\phi_{u_n}u_n - \phi_{\bar{v}_n}\bar{v}_n\right)dx\right| \\ &\leq 18\int_{R_n\leq|x-y_n|\leq 2R_n} |l(x)||\phi_{u_n}||u_n|^2 dx \\ &\leq 18\|l\|_\infty\left(\int_{R_n\leq|x-y_n|\leq 2R_n} |\phi_{u_n}|^6 dx\right)^{\frac{1}{6}}\left(\int_{R_n\leq|x-y_n|\leq 2R_n} |u_n|^{\frac{12}{5}}\right)^{\frac{5}{6}} dx \\ &\leq C\left(\int_{R_n\leq|x-y_n|\leq 2R_n} \rho_n dx\right)^2 \\ &\leq C\varepsilon^2,\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n\bar{v}_n dx = \int_{\mathbb{R}^3} l(x)\phi_{\bar{v}_n}\bar{v}_n^2 dx + \mu_2\left(\varepsilon\right), \tag{3.28}$$

where $\mu_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, one gets that

$$\int_{\mathbb{R}^3}\left(|\nabla\bar{v}_n|^2 + |\bar{v}_n|^2\right)dx = \int_{\mathbb{R}^3}\left(\nabla\bar{v}_n\nabla u_n + \bar{v}_n u_n\right)dx + \mu_3(\varepsilon), \tag{3.29}$$

and

$$\int_{\mathbb{R}^3} k(x)g(u_n)\bar{v}_n dx = \int_{\mathbb{R}^3} k(x)g(\bar{v}_n)\bar{v}_n dx + \mu_4(\varepsilon), \tag{3.30}$$

where $\mu_3(\varepsilon) \to 0$ and $\mu_4(\varepsilon) \to 0$ as $\varepsilon \to 0$. In fact, (3.30) comes from

$$\begin{aligned} \left| \int_{\mathbb{R}^3} k(x)\left(g(u_n) - g(\bar{v}_n)\right)\bar{v}_n dx \right| &= \left| \int_{R_n \le |x-y_n| \le 2R_n} k(x)\left(g(u_n) - g(\bar{v}_n)\right)\bar{v}_n dx \right| \\ &\le C \int_{R_n \le |x-y_n| \le 2R_n} \left(|u_n|^p + |u_n|^q\right) dx \\ &\le C\left(\varepsilon^{\frac{p}{2}} + \varepsilon^{\frac{q}{2}}\right). \end{aligned}$$

Using (3.27)–(3.30), one obtains

$$\begin{aligned} o(1) &= \langle I'_\mu(u_n), \bar{v}_n \rangle \\ &= \int_{\mathbb{R}^3} \left(\nabla \bar{v}_n \nabla u_n + \bar{v}_n u_n\right) dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n \bar{v}_n dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(u_n)\bar{v}_n dx - \mu \int_{\mathbb{R}^3} h(x)\bar{v}_n u_n dx \\ &= \int_{\mathbb{R}^3} \left(|\nabla \bar{v}_n|^2 + |\bar{v}_n|^2\right) dx + \int_{\mathbb{R}^3} l(x)\phi_{\bar{v}_n} \bar{v}_n^2 dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(\bar{v}_n)\bar{v}_n dx - \mu \int_{\mathbb{R}^3} h(x)\bar{v}_n^2 dx + \beta(\varepsilon), \end{aligned} \tag{3.31}$$

where $\beta(\varepsilon)$ goes to zero as $\varepsilon$ goes to zero. From the assumption that $(y_n)_{n\in\mathbb{N}}$ is bounded, for each $x \in \mathbb{R}^3$, there exists $N_x > 0$ such that, for all $n > N_x$, $|x - y_n| \le R_n$, since $R_n \to \infty$. Then for $n$ large and every fixed $x$ one has that $\bar{v}_n(x) = 0$ and so

$$\bar{v}_n \rightharpoonup 0 \;\text{ in }\; H^1(\mathbb{R}^3).$$

Therefore, it is deduced from Lemma 1.4.6 that $\int_{\mathbb{R}^3} h(x)\bar{v}_n^2 dx = o(1)$, which makes (3.31) become into

$$\|\bar{v}_n\|^2 + F(\bar{v}_n) - \int_{\mathbb{R}^3} k(x)g(\bar{v}_n)\bar{v}_n dx = o(1) + \beta(\varepsilon).$$

Moreover, for $M_1 > 0$ as in (3.22), one deduces that

$$-\int_{\mathbb{R}^3} k(x)g(\bar{v}_n)\bar{v}_n dx = o(1) - \int_{|x|\ge M_1} k(x)g(\bar{v}_n)\bar{v}_n dx$$

with

$$\int_{|x|\ge M_1} k(x)g(\bar{v}_n)\bar{v}_n dx < 0.$$

Therefore it follows that

$$\|\bar{v}_n\|^2 = o(1) + \beta(\varepsilon). \tag{3.32}$$

On the other hand, since $(y_n)_{n\in\mathbb{N}}$ is bounded, one calculates by (3.24) that for large $n$

$$\begin{aligned}\|\bar{v}_n\|^2 &= \int_{|x-y_n|\geq R_n} \left(|\nabla \bar{v}_n|^2 + |\bar{v}_n|^2\right) dx \\ &\geq \int_{|x-y_n|\geq 3R_n} \left(|\nabla \bar{v}_n|^2 + |\bar{v}_n|^2\right) dx \\ &= \int_{|x-y_n|\geq 3R_n} \left(|\nabla u_n|^2 + u_n^2\right) dx \\ &\geq \bar{\lambda} - \alpha - \varepsilon.\end{aligned} \tag{3.33}$$

Clearly (3.32) and (3.33) are a contradiction with each other for the case that $(y_n)_{n\in\mathbb{N}}$ is bounded.

Now, let us consider the case that $(y_n)_{n\in\mathbb{N}}$ is not bounded. We apply the similar argument with the case that $(y_n)_{n\in\mathbb{N}}$ is bounded to $(\bar{w}_n)_{n\in\mathbb{N}}$ to get a contradiction. For the convenience, we give details in the following. In fact, from (3.26) one deduces that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} h(x)\bar{w}_n\left(\bar{w}_n - u_n\right) dx\right| &= \left|\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} h(x)\bar{w}_n\left(\bar{w}_n - u_n\right) dx\right| \\ &\leq 2\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} |h(x)||u_n|^2 dx \\ &\leq 2\|h\|_{\frac{3}{2}}\left(\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} |u|^6 dx\right)^{\frac{1}{3}} \\ &\leq C\|h\|_{\frac{3}{2}}\varepsilon,\end{aligned}$$

and then one concludes that

$$\int_{\mathbb{R}^3} h(x)u_n\bar{w}_n dx = \int_{\mathbb{R}^3} h(x)\bar{w}_n^2 dx + \mu_1'(\varepsilon), \tag{3.34}$$

where $\mu_1'(\varepsilon) \to 0$ as $\varepsilon \to 0$. One also deduces from (3.25), the fact that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$ and the Sobolev inequality that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3} l(x)\bar{w}_n\left(\phi_{u_n}u_n - \phi_{\bar{w}_n}\bar{w}_n\right) dx\right| \\ &= \left|\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} l(x)\bar{w}_n\left(\phi_{u_n}u_n - \phi_{\bar{w}_n}\bar{w}_n\right) dx\right| \\ &\leq 2\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} |l(x)||\phi_{u_n}||u_n|^2 dx \\ &\leq 2\|l\|_\infty \left(\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} |\phi_{u_n}|^6 dx\right)^{\frac{1}{6}} \left(\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} |u_n|^{\frac{12}{5}}\right)^{\frac{5}{6}} dx \\ &\leq C\left(\int_{\hat{R}\leq|x-y_n|\leq 2\hat{R}} \rho_n dx\right)^2 \\ &\leq C\varepsilon^2,\end{aligned}$$

which means

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n \bar{w}_n dx = \int_{\mathbb{R}^3} l(x)\phi_{\bar{w}_n} \bar{w}_n^2 dx + \mu_2'(\varepsilon), \tag{3.35}$$

where $\mu_2'(\varepsilon) \to 0$ as $\varepsilon \to 0$. And it follows from (3.4) that

$$\begin{aligned}\left|\int_{\mathbb{R}^3} k(x)\left(g(u_n)-g(\bar{w}_n)\right)\bar{w}_n dx\right| &= \left|\int_{\hat{R}\le|x-y_n|\le 2\hat{R}} k(x)\left(g(u_n)-g(\bar{w}_n)\right)\bar{w}_n dx\right| \\ &\le 2\|k\|_\infty \int_{\hat{R}\le|x-y_n|\le 2\hat{R}} \left(b_2|u_n|^p + b_1|u_n|^q\right) dx \\ &\le C\int_{\hat{R}\le|x-y_n|\le 2\hat{R}} \left(|u_n|^p+|u_n|^q\right) dx \\ &\le C\left(\varepsilon^{\frac{p}{2}}+\varepsilon^{\frac{q}{2}}\right),\end{aligned}$$

and so

$$\int_{\mathbb{R}^3} k(x)g(u_n)\bar{w}_n dx = \int_{\mathbb{R}^3} k(x)g(\bar{w}_n)\bar{w}_n dx + \mu_3'(\varepsilon), \tag{3.36}$$

where $\mu_3'(\varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, one gets that

$$\int_{\mathbb{R}^3} \left(\nabla \bar{w}_n \nabla u_n + \bar{w}_n u_n\right) dx = \int_{\mathbb{R}^3} \left(|\nabla \bar{w}_n|^2 + |\bar{w}_n|^2\right) dx + \mu_4'(\varepsilon), \tag{3.37}$$

where $\mu_4'(\varepsilon) \to 0$ as $\varepsilon \to 0$. Therefore one can calculate by (3.34)–(3.37) that

$$\begin{aligned} o(1) &= \langle I_\mu'(u_n), \bar{w}_n\rangle \\ &= \int_{\mathbb{R}^3} \left(\nabla \bar{w}_n \nabla u_n + \bar{w}_n u_n\right) dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n \bar{w}_n dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(u_n)\bar{w}_n dx - \mu\int_{\mathbb{R}^3} h(x)\bar{w}_n u_n dx \\ &= \int_{\mathbb{R}^3} \left(|\nabla \bar{w}_n|^2 + |\bar{w}_n|^2\right) dx + \int_{\mathbb{R}^3} l(x)\phi_{\bar{w}_n}\bar{w}_n^2 dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(\bar{w}_n)\bar{w}_n dx - \mu\int_{\mathbb{R}^3} h(x)\bar{w}_n^2 dx + \beta'(\varepsilon), \end{aligned} \tag{3.38}$$

where $\beta'(\varepsilon)$ goes to zero as $\varepsilon$ goes to zero. Since $(y_n)_{n\in\mathbb{N}}$ is not bounded, for every $x \in \mathbb{R}^3$, there is $N_x' > 0$ such that, for all $n > N_x$, $|x-y_n| \ge 2\hat{R}$. Then, for $n$ large and every fixed $x$, one has $\bar{w}_n(x) = 0$. And then

$$\bar{w}_n \rightharpoonup 0 \ \text{ in } \ H^1(\mathbb{R}^3).$$

Thus, by Lemma 1.4.6 one obtains that $\int_{\mathbb{R}^3} h(x)\bar{w}_n^2 dx = o(1)$, which means that (3.38) becomes

$$\|\bar{w}_n\|^2 + F(\bar{w}_n) - \int_{\mathbb{R}^3} k(x)g(\bar{w}_n)\bar{w}_n dx = o(1) + \beta'(\varepsilon). \tag{3.39}$$

Furthermore, for $M_1 > 0$ as in (3.22), one obtains that

$$-\int_{\mathbb{R}^3} k(x)g(\bar{w}_n)\bar{w}_n dx = o(1) - \int_{|x|\ge M_1} k(x)g(\bar{w}_n)\bar{w}_n dx$$

with

$$\int_{|x|\geq M_1} k(x)g(\bar{w}_n)\bar{w}_n dx < 0,$$

which makes (3.39) become into

$$\|\bar{w}_n\|^2 = o(1) + \beta'(\varepsilon). \tag{3.40}$$

However, since $(y_n)_{n\in\mathbb{N}}$ is unbounded, one calculates by (3.24) that

$$\begin{aligned} \|\bar{w}_n\|^2 &= \int_{|x-y_n|\leq 2\hat{R}} \left(|\nabla \bar{w}_n|^2 + |\bar{w}_n|^2\right) dx \\ &\geq \int_{|x-y_n|\leq \frac{\hat{R}}{2}} \left(|\nabla \bar{w}_n|^2 + |\bar{w}_n|^2\right) dx \\ &= \int_{|x-y_n|\leq \frac{\hat{R}}{2}} \left(|\nabla u_n|^2 + u_n^2\right) \geq \alpha - \varepsilon, \end{aligned} \tag{3.41}$$

for large $n$. Clearly (3.40) and (3.41) contradict each other for the case that $(y_n)_{n\in\mathbb{N}}$ is bounded. Hence dichotomy does not happen.

Finally, by ruling out vanishing and dichotomy through above two steps, we conclude, by Lemma 3.1.2, that compactness necessarily takes place, i.e., there exists $(y_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^3$ such that, for any $\epsilon > 0$, there is $\bar{R} > 0$ satisfying $\int_{B_{\bar{R}}(y_n)} \rho_n(x)dx \geq \bar{\lambda} - \epsilon$, which yields that for each $n > N_{2,\epsilon}$ with $N_{2,\epsilon}$ depending on $\epsilon$

$$\int_{B^c_{\bar{R}}(y_n)} \rho_n(x)dx < \bar{\lambda} + \epsilon - \int_{B_{\bar{R}}(y_n)} \rho_n(x)dx = 2\epsilon \tag{3.42}$$

We claim that $(y_n)_{n\in\mathbb{N}}$ is bounded. Otherwise, if $(y_n)_{n\in\mathbb{N}}$ is not bounded, then for every $x_0 \in \mathbb{R}^3$ there exists $N_{x_0} > 0$ such that, for every $n > N_{x_0}$, $B_1(x_0) \cap B_{\bar{R}}(y_n) = \emptyset$ and then by (3.42)

$$\begin{aligned} \int_{B_1(x_0)} u_n^2 dx &= \int_{B_1(x_0)\cap B_{\bar{R}}(y_n)} u_n^2 dx + \int_{B_1(x_0)\cap B^c_{\bar{R}}(y_n)} u_n^2 dx \\ &= \int_{B_1(x_0)\cap B^c_{\bar{R}}(y_n)} u_n^2 dx \\ &\leq 2\epsilon, \end{aligned}$$

which implies $u_n \to 0$ a.e. in $\mathbb{R}^3$, and then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. It follows from Lemma 1.4.6 that $\int_{\mathbb{R}^3} h(x)u_n^2 dx = 0$, which inserts into the equality $o(1) = \langle I'_\mu(u_n), u_n\rangle$ yielding

$$o(1) = \|u_n\|^2 + F(u_n) - \int_{\mathbb{R}^3} k(x)g(u_n)u_n dx. \tag{3.43}$$

Since $|x| \geq \max\{R_0, M_1\}$ for all $x \in B_{\bar{R}}(y_n)$ with $n$ large, we have that

$$-\int_{B_{\bar{R}}(y_n)} k(x)g(u_n)u_n dx \geq 0,$$

and by Sobolev equality, (3.4) and (3.42) one deduces that

$$\left| \int_{B^c_{\bar{R}}(y_n)} k(x)g(u_n)u_n dx \right| \leq C \int_{B^c_{\bar{R}}(y_n)} (b_1|u_n|^p + b_2|u_n|^q)\, dx \leq C\left(\epsilon^{\frac{p}{2}} + \epsilon^{\frac{q}{2}}\right).$$

Hence one obtains that

$$-\int_{\mathbb{R}^3} k(x)g(u_n)u_n dx = o(1) + \gamma(\epsilon), \tag{3.44}$$

where $\gamma(\epsilon)$ goes to zero as $\epsilon$ goes to zero. Combining (3.43) with (3.44), one gets that $\|\rho_n\|_1 = \|u_n\|^2 = o(1) + \gamma(\epsilon)$, which is in contradiction with the assumption that $\|\rho_n\|_1 = \bar{\lambda} + o(1)$. Hence $(y_n)_{n\in\mathbb{N}}$ is bounded.

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, passing if necessary to a subsequence, one may assume that $u_n \rightharpoonup u$ and $u_n \to u$ in $L^t_{loc}(\mathbb{R}^3)$. According to the boundedness of $(y_n)_{n\in\mathbb{N}}$, (3.42) implies that there is $\bar{R}_2 > 0$ such that

$$\int_{B^c_{\bar{R}_2}(0)} \rho_n(x)dx < 2\epsilon.$$

Then it is clear that

$$u_n \to u \text{ in } L^t(\mathbb{R}^3) \text{ for } t \in [2,6). \tag{3.45}$$

From (3.4), (3.45) and the Sobolev equality, we obtain that

$$\begin{aligned}\left| \int_{\mathbb{R}^3} k(x)g(u_n)(u_n - u)dx \right| &\leq C \int_{\mathbb{R}^3} \left(b_1|u_n|^{p-1} + b_2|u_n|^{q-1}\right)|u_n - u|dx \\ &\leq C\left(\int_{\mathbb{R}^3} |u_n|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^3} |u_n - u|^p dx\right)^{\frac{1}{p}} \\ &\quad + C\left(\int_{\mathbb{R}^3} |u_n|^q dx\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^3} |u_n - u|^q dx\right)^{\frac{1}{q}} \\ &= o(1).\end{aligned}$$

Similarly, we also obtain that for $n$ large enough,

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n(u_n - u)dx = o(1)$$

and

$$\int_{\mathbb{R}^3} h(x)u_n(u_n - u)dx = o(1).$$

Hence using the fact that

$$\int_{\mathbb{R}^3} (\nabla(u_n - u)\nabla u + (u_n - u)u)\, dx = o(1),$$

one deduces that

$$\begin{aligned} o(1) &= \langle I'_\mu(u_n), u_n - u\rangle \\ &= \|u_n - u\|^2 + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n(u_n - u)dx \\ &\quad - \int_{\mathbb{R}^3} k(x)g(u_n)(u_n - u)dx - \int_{\mathbb{R}^3} h(x)u_n(u_n - u)dx \\ &= \|u_n - u\|^2. \end{aligned}$$

The proof of Lemma 3.1.4 is complete. □

## 3.2 Existence of two positive solutions

In this section, we will prove the existence and multiplicity of positive critical points of $I_\mu$ on $H^1(\mathbb{R}^3)$. Our main strategy is to study suitable minimization problem and minimax procedure. We emphasize that, with the help of Lemma 3.1.4, an important thing is to study the geometrical structure of $I_\mu$. We need Lemma 2.2.1 and eigenvalues of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h(x)$, which we state again in the following for the readers convenience.

**Lemma 3.2.1.** *Assume $h \in L^{3/2}(\mathbb{R}^3)$ and $h(x) \geq 0$. Then for every $u \in H^1(\mathbb{R}^3)$, there exists a unique $w \in H^1(\mathbb{R}^3)$ such that*

$$-\Delta w + w = h(x)u.$$

*Moreover, the operator $K_h : H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ defined by $K_h(u) = w$ is compact.*

Using the spectral theory of compact symmetric operators on Hilbert space, Lemma 3.2.1 implies the existence of a sequence of eigenvalues $(\mu_n)_{n\in\mathbb{N}}$ of

$$-\Delta u + u = \mu h(x)u, \text{ in } H^1(\mathbb{R}^3)$$

with $\mu_1 < \mu_2 \leq \cdots$ and each eigenvalue being of finite multiplicity. The associated normalized eigenfunctions are denoted by $e_1, e_2, \cdots$ with $\|e_i\| = 1$, $i = 1, 2, \cdots$. Moreover, since $K_h$ is a positive operator, one has $\mu_1 > 0$ with a positive eigenfunction $e_1 > 0$ in $\mathbb{R}^3$. In addition, we have the following variational characterization of $\mu_n$:

$$\mu_1 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx}, \quad \mu_n = \inf_{u\in S^\perp_{n-1}\setminus\{0\}} \frac{\|u\|^2}{\int_{\mathbb{R}^3} h(x)u^2dx}, \tag{3.46}$$

where $S^\perp_{n-1} = \{span\{e_1, e_2, \cdots, e_{n-1}\}\}^\perp$. Let $\bar{\mu}_1$ be the first eigenvalue of

$$-\Delta u + u = \mu h(x)u, \text{ in } H^1_0(\Omega^0).$$

Then clearly $\mu_1 < \bar{\mu}_1$ and we have that $\mu \notin \sigma(-\Delta + id, \Omega^0, h)$ for any $\mu < \bar{\mu}_1$.

In the following lemma, we prove the mountain pass geometry for the functional $I_\mu$ for $\mu$ less than $\mu_1$ and $\mu$ in the right neighborhood of $\mu_1$, respectively.

**Lemma 3.2.2.** *Assume that the hypotheses* $(G)$ *and* $(H)$ *hold.*

$(I_1)$ *If* $0 < \mu < \mu_1$*, then* $u = 0$ *is the local minimum of* $I_\mu$*;*

$(I_1')$ *There are positive constants* $\bar{\delta}, \rho$ *and* $\alpha$ *such that, for any* $\mu \in [\mu_1, \mu_1 + \bar{\delta})$*,* $I_\mu|_{\partial B_\rho} \geq \alpha$*; And*

$(I_2)$ *there is* $\bar{u} \in H^1(\mathbb{R}^3)$ *with* $\|\bar{u}\| > \rho$ *such that* $I_\mu(\bar{u}) < 0$ *for any* $\mu > 0$.

**Proof.** (i) *Proof of* $(I_1)$*:* From $(H_{l_1})$ it follows that $F(u) \geq 0$. $(H_{k_1})$ and $(H_{k_2})$ imply that $k$ is bounded in $\mathbb{R}^3$. Thus, by (3.46) and the continuity of the Sobolev embedding of $H^1(\mathbb{R}^3)$ in $L^p(\mathbb{R}^3)$, we deduce that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3} k(x)G(u)dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2dx \\ &\geq \frac{1}{2}\left(1 - \frac{\mu}{\mu_1}\right)\|u\|^2 - C\|u\|^p - C\|u\|^q. \end{aligned}$$

Therefore the statement $(I_1)$ follows.

(ii) *Proof of* $(I_1')$*:* For any $u \in H^1(\mathbb{R}^3)$, there exist $t \in \mathbb{R}$ and $v \in S_1^\perp$ such that

$$u = te_1 + v, \text{ where } \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + ve_1)\, dx = 0. \tag{3.47}$$

Hence we get by direct computation that

$$\|u\| = \left(\|\nabla(te_1 + v)\|_2^2 + \|te_1 + v\|_2^2\right)^{\frac{1}{2}} = \left(t^2 + \|v\|^2\right)^{\frac{1}{2}}, \tag{3.48}$$

$$\mu_2 \int_{\mathbb{R}^3} h(x)v^2dx \leq \|v\|^2, \ \ \mu_1 \int_{\mathbb{R}^3} h(x)e_1^2dx = \|e_1\|^2 = 1 \tag{3.49}$$

and

$$\mu_1 \int_{\mathbb{R}^3} h(x)e_1 v dx = \int_{\mathbb{R}^3} (\nabla v \nabla e_1 + ve_1)\, dx = 0. \tag{3.50}$$

Using the mean value theorem, we know that there exists $\vartheta$ with $0 < \vartheta < 1$ such that

$$\begin{aligned} |F(te_1 + v) - F(te_1)| &= 4\left|\int_{\mathbb{R}^3} l(x)\phi_{te_1+\vartheta v}(te_1 + \vartheta v)v dx\right| \\ &\leq 4\|l\|_\infty \|\phi_{te_1+\vartheta v}\|_6 \|te_1 + \vartheta v\|_2 \|v\|_3 \\ &\leq C\|l\|_\infty^2 \|te_1 + \vartheta v\|^3 \|v\| \\ &\leq C_0\left(|t|^3\|v\| + \|v\|^4\right). \end{aligned} \tag{3.51}$$

It follows from (3.4) that

$$|G(u)| \leq (b_1 2^p + b_2 2^q)\left(|t|^p|e_1|^p + |v|^p + |t|^q|e_1|^q + |v|^q\right). \tag{3.52}$$

We first consider the case that $\mu = \mu_1$ and estimate the value of $I_{\mu_1}$ for $u$ not too small. Denote $\theta := \frac{1}{2}\left(1 - \frac{\mu_1}{\mu_2}\right) > 0$. By (3.48)–(3.52) and the boundedness of the function $k$, one has that

$$\begin{aligned} I_{\mu_1}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)u^2dx - \int_{\mathbb{R}^3} k(x)G(u)dx \\ &= \frac{1}{2}\left(t^2 + \|v\|^2\right) + \frac{1}{4}F(te_1) + \frac{1}{4}F(te_1+v) - \frac{1}{4}F(te_1) \\ &\quad -\frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)(te_1+v)^2dx - \int_{\mathbb{R}^3} k(x)G(u)dx \\ &\geq \frac{1}{2}\|v\|^2 - \frac{\mu_1}{2}\int_{\mathbb{R}^3} h(x)v^2dx + \frac{1}{4}F(e_1)t^4 \\ &\quad -C_0\left(|t|^3\|v\| + \|v\|^4\right) - \int_{\mathbb{R}^3} k(x)G(u)dx \\ &\geq \theta\|v\|^2 + C_1t^4 - C_0|t|^3\|v\| - C_0\|v\|^4 \\ &\quad -C_5|t|^p - C_5\|v\|^p - C_6|t|^q - C_6\|v\|^q. \end{aligned} \tag{3.53}$$

Note that for some $2 < q_0 < 4$,

$$|t|^3\|v\| \leq \frac{1}{q_0}\|v\|^{q_0} + \frac{q_0-1}{q_0}|t|^{\frac{3q_0}{q_0-1}}.$$

We deduce from (3.53) that

$$\begin{aligned} I_{\mu_1}(u) &\geq \theta\|v\|^2 + C_1t^4 - \frac{C_0}{q_0}\|v\|^{q_0} - C_0\|v\|^4 - C_5\|v\|^p - C_6\|v\|^p \\ &\quad -\frac{C_0(q_0-1)}{q_0}|t|^{\frac{3q_0}{q_0-1}} - C_5|t|^p - C_6|t|^q. \end{aligned} \tag{3.54}$$

Therefore, from $2 < q_0 < 4$ and $\frac{3q_0}{q_0-1} > 4$, we know that there are positive constants $\theta_3, \theta_4$ and $\tilde{\theta}_3$, $\tilde{\theta}_4$ such that

$$I_{\mu_1}(u) \geq \theta_3\|v\|^2 + \theta_4|t|^4 \tag{3.55}$$

provided that $\|v\| \leq \tilde{\theta}_3$ and $|t| \leq \tilde{\theta}_4$. Hence there are positive constants $\theta_5$ and $\tilde{\theta}_5$ such that

$$I_{\mu_1}(u) \geq \theta_5\|u\|^4 \quad \text{for } \|u\|^2 \leq \left(\tilde{\theta}_5\right)^2.$$

Set

$$\bar{\delta} = \min\left\{\frac{\mu_1}{2}\theta_5\left(\tilde{\theta}_5\right)^2,\ \mu_2 - \mu_1,\ \bar{\mu}_1 - \mu_1\right\}.$$

Then for any $\mu \in [\mu_1, \mu_1 + \bar{\delta})$, we deduce from by (3.55) that

$$\begin{aligned} I_\mu(u) &= I_{\mu_1}(u) + \frac{1}{2}(\mu_1 - \mu)\int_{\mathbb{R}^3} h(x)u^2dx \\ &\geq \theta_5\|u\|^4 - \frac{\mu-\mu_1}{2\mu_1}\|u\|^2 \geq \|u\|^2\left(\theta_5\|u\|^2 - \frac{\mu-\mu_1}{2\mu_1}\right) \\ &\geq \|u\|^2\left(\frac{1}{2}\theta_5\left(\tilde{\theta}_5\right)^2 - \frac{1}{4}\theta_5\left(\tilde{\theta}_5\right)^2\right) \\ &= \frac{1}{4}\theta_5\left(\tilde{\theta}_5\right)^2\|u\|^2 \end{aligned}$$

for $\frac{1}{2}\left(\tilde{\theta}_5\right)^2 \leq \|u\|^2 \leq \left(\tilde{\theta}_5\right)^2$. Choosing

$$\rho^2 = \|u\|^2 \in \left[\frac{1}{2}\left(\tilde{\theta}_5\right)^2, \left(\tilde{\theta}_5\right)^2\right]$$

and $\alpha = \frac{1}{4}\theta_5\left(\tilde{\theta}_5\right)^2\rho^2$, we get the conclusion $(I_1')$.

(iii) *Proof of* $(I_2)$*:* Choose $\varphi \in H^1(\mathbb{R}^3)$ with $\mathrm{supp}\varphi \subset \Omega^+$ such that $\varphi(x) \geq 0$ for all $x \in \Omega^+$. Then for any $s > 0$ sufficiently large, by (3.5) one has that

$$I_\mu(s\varphi) \leq \frac{s^2}{2}\|\varphi\|^2 + \frac{s^4}{4}F(\varphi) - \frac{\mu s^2}{2}\int_{\mathbb{R}^3} h(x)\varphi^2 dx - b_4 s^p \int_{\Omega^+} k(x)|\varphi|^p dx.$$

From the choice of $\varphi$ we know that $I_\mu(s\varphi) < 0$ for $s$ sufficiently large. Thus the conclusion of $(I_2)$ follows by taking $\bar{u} = s\varphi$. □

**Remark 3.2.3.** *Set* $\bar{\epsilon} = \min\{\bar{\mu}_1 - \mu_1, \bar{\delta}\}$*. If* $0 < \mu < \mu_1 + \bar{\epsilon}$*, then* $\mu < \min\{\bar{\mu}_1, \mu_1 + \bar{\delta}\}$*. Hence, when* $0 < \mu < \mu_1 + \bar{\epsilon}$*, Lemma 3.2.2 implies that the functional* $I_\mu$ *has the mountain pass geometry and Lemma 3.1.4 means that the functional* $I_\mu$ *satisfies* $(PS)$*-condition.*

With the help of previous several lemmas, we are ready to prove Theorem 3.0.9.

**Proposition 3.2.4.** *Assume that the hypotheses* $(G)$ *and* $(H)$ *hold. If either there is* $\beta$ *with* $0 \leq \beta < 1$ *such that* $g(s) = |s|^{p-2}s + O\left(|s|^\beta\right)$ *as* $|s| \to \infty$*, or* $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$*, then problem (3.3) has a positive solution* $u_\mu$ *with* $I_\mu(u_\mu) > 0$ *for* $0 < \mu < \mu_1 + \bar{\epsilon}$*.*

**Proof.** We denote

$$c_{1,\mu} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\mu(\gamma(t)) \quad \text{with } \Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) \; : \; \gamma(0) = 0, \gamma(1) = \bar{u}\}.$$

By Remark 3.2.3, the Mountain Pass Theorem implies that $c_{1,\mu}$ is a critical value of $I_\mu$ and $c_{1,\mu} > 0$. The proof of positivity for at least one of the corresponding (nontrivial) critical point is inspired by the idea of Alama-Tarantello [1]. In fact, since $I_\mu(u) = I_\mu(|u|)$ in $H^1(\mathbb{R}^3)$, for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ with $\gamma_n(t) \geq 0$ (a.e. in $\mathbb{R}^3$) for all $t \in [0,1]$ such that

$$c_{1,\mu} \leq \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < c_{1,\mu} + \frac{1}{n}. \tag{3.56}$$

Consequently, by means of Ekeland's variational principle, there exists $\gamma_n^* \in \Gamma$ with the following properties:

$$\begin{cases} c_{1,\mu} \leq \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)) \leq \max_{t\in[0,1]} I_\mu(\gamma_n(t)) < c_{1,\mu} + \frac{1}{n}; \\ \max_{t\in[0,1]} \|\gamma_n(t) - \gamma_n^*(t)\| < \frac{1}{\sqrt{n}}; \\ \text{there exists } t_n \in [0,1] \text{ such that } z_n = \gamma_n^*(t_n) \text{ satisfies :} \\ I_\mu(z_n) = \max_{t\in[0,1]} I_\mu(\gamma_n^*(t)), \text{ and } \|I_\mu'(z_n)\| \leq \frac{1}{\sqrt{n}}. \end{cases} \tag{3.57}$$

In particular, we get a $(PS)_{c_{1,\mu}}$-sequence $(z_n)_{n\in\mathbb{N}}$. By Lemma 3.1.4 we get a convergent subsequence (still denoted by $(z_n)_{n\in\mathbb{N}}$). Let $z_n \to z$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$. On the other hand, by (3.57), we also arrive at $\gamma_n(t_n)\to z$ in $H^1(\mathbb{R}^3)$ as $n\to\infty$. Since $\gamma_n(t)\geq 0$, we conclude that $z\geq 0$ a.e. in $\mathbb{R}^3$ with $I_\mu(z)>0$ and $I'_\mu(z)=0$. The strong maximum principle implies that $z>0$ in $\mathbb{R}^3$. The conclusion of this proposition follows from choosing $u_\mu := z$. □

**Proposition 3.2.5.** *Assume that the hypotheses (G) and (H) hold. If either there is $\beta$ with $0\leq\beta<1$ such that $g(s)=|s|^{p-2}s+O\left(|s|^\beta\right)$ as $|s|\to\infty$, or $\overline{\Omega^+}\cap\overline{\Omega^-}=\emptyset$, then for each $\mu$ with $\mu_1<\mu<\mu_1+\bar{\epsilon}$, problem (3.3) has a positive solution $\omega_\mu$ with $I_\mu(\omega_\mu)<0$.*

**Proof.** Let $B_\rho$ denote the closed ball $B_\rho=\{u\in H^1(\mathbb{R}^3): \|u\|\leq\rho\}$ with $\rho$ as in Lemma 3.2.2. Set

$$c_{2,\mu} := \inf_{\|u\|\leq\rho} I_\mu(u). \tag{3.58}$$

It is clear that $c_{2,\mu}>-\infty$. We claim that $c_{2,\mu}<0$. In fact, given $R>0$, define $\eta_R\in C_0^\infty(\mathbb{R}^3)$ with $0\leq\eta_R(x)\leq 1$ and $|\nabla\eta_R(x)|\leq\frac{2}{R}$ for all $x\in\mathbb{R}^3$ and

$$\eta_R(x)=\begin{cases}1, & |x|\leq R,\\ 0, & |x|\geq 2R.\end{cases}$$

Then $\eta_R e_1\in H^1(\mathbb{R}^3)$. To complete the proof of the claim, it suffices to show that $I_\mu(t\eta_R e_1)<0$ for all $t>0$ small. First we have that

$$\begin{aligned}
I_\mu(t\eta_R e_1) &= \frac{t^2}{2}\|\eta_R e_1\|^2+\frac{t^4}{4}F(\eta_R e_1)\\
&\quad -\frac{\mu t^2}{2}\int_{\mathbb{R}^3}h(x)(\eta_R e_1)^2dx-\int_{\mathbb{R}^3}k(x)G(t\eta_R e_1)dx\\
&= \frac{t^2}{2}\int_{\mathbb{R}^3}\eta_R^2|\nabla e_1|^2dx+\frac{t^2}{2}\int_{\mathbb{R}^3}\eta_R^2 e_1^2dx\\
&\quad +t^2\int_{\mathbb{R}^3}\eta_R e_1\nabla\eta_R\nabla e_1dx+\frac{t^4}{4}F(\eta_R e_1)+\frac{t^2}{2}\int_{\mathbb{R}^3}e_1^2|\nabla\eta_R|^2dx\\
&\quad -\frac{\mu t^2}{2}\int_{\mathbb{R}^3}h(x)\eta_R^2e_1^2dx-\int_{\mathbb{R}^3}k(x)G(t\eta_R e_1)dx.
\end{aligned}\tag{3.59}$$

On the other hand, multiplying both sides of the equation

$$-\Delta e_1+e_1=\mu_1 h(x)e_1$$

by $\eta_R^2 e_1$ and integrating by parts, one obtains that

$$2\int_{\mathbb{R}^3}\eta_R e_1\nabla\eta_R\nabla e_1dx+\int_{\mathbb{R}^3}\eta_R^2e_1^2dx+\int_{\mathbb{R}^3}\eta_R^2|\nabla e_1|^2dx=\mu_1\int_{\mathbb{R}^3}h(x)\eta_R^2e_1^2dx. \tag{3.60}$$

Inserting (3.60) into (3.59), we get that

$$\begin{aligned} I_\mu(t\eta_R e_1) &= (\mu_1-\mu)\frac{t^2}{2}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx \\ &\quad + \frac{t^4}{4}F(\eta_R e_1) - \int_{\mathbb{R}^3} k(x)G(t\eta_R e_1)dx. \end{aligned} \tag{3.61}$$

By the definition of $\eta_R$, the Hölder inequality and the Sobolev inequality, we obtain that

$$\begin{aligned} \int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx &= \int_{R\le|x|\le 2R} e_1^2|\nabla\eta_R|^2 dx \\ &\le \left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \left(\int_{R\le|x|\le 2R} |\nabla\eta_R|^3 dx\right)^{\frac{2}{3}} \\ &\le \left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \left(\left(\frac{2}{R}\right)^3 \int_{R\le|x|\le 2R} dx\right)^{\frac{2}{3}} \\ &\le C\left(\int_{R\le|x|\le 2R} e_1^6 dx\right)^{\frac{1}{3}} \to 0, \text{ as } R\to\infty, \end{aligned} \tag{3.62}$$

since $\|e_1\| = 1$. Meanwhile, multiplying both sides of the equation

$$-\Delta e_1 + e_1 = \mu_1 h(x) e_1$$

by $e_1$ and integrating by parts, we get that

$$\mu_1 \int_{\mathbb{R}^3} h(x)e_1^2 dx = \|e_1\|^2 = 1. \tag{3.63}$$

Moreover, by choosing $R$ sufficiently large, we obtain that

$$\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx \ge \int_{|x|\le R} h(x)\eta_R^2 e_1^2 dx = \int_{|x|\le R} h(x)e_1^2 dx \ge \frac{1}{2\mu_1}, \tag{3.64}$$

and then choosing $R_2 > 0$ sufficiently large with $R \ge R_2$, we deduce from (3.62)–(3.64) that

$$\int_{\mathbb{R}^3} e_1^2|\nabla\eta_R|^2 dx \le \frac{\mu-\mu_1}{2}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx \tag{3.65}$$

for all $R > R_2$. From (3.4), (3.61) and (3.65), one deduces that

$$\begin{aligned} I_\mu(t\eta_R e_1) &\le (\mu_1-\mu)\frac{t^2}{4}\int_{\mathbb{R}^3} h(x)\eta_R^2 e_1^2 dx + \frac{t^4}{4}F(\eta_R e_1) \\ &\quad + \|k\|_\infty b_1 \int_{\mathbb{R}^3} |t\eta_R e_1|^q \, dx + \|k\|_\infty b_2 \int_{\mathbb{R}^3} |t\eta_R e_1|^p \, dx \\ &\le -C_7 t^2 + C_8 t^4 + C_9 t^p + C_{10} t^q, \end{aligned}$$

for all $R > R_2$, which means that $I_\mu(t\eta_R e_1) < 0$ for $t > 0$ small enough. Thus $c_{2,\mu} < 0$ and the proof of the claim is complete.

In addition, since $I_\mu(u) = I_\mu(|u|)$, given $n$, by (3.58), there exists $w_n^* \ge 0$ with $\|w_n^*\| \le \rho$

such that

$$c_{2,\mu} \le I_\mu(w_n^*) < c_{2,\mu} + \frac{1}{n}.$$

Then, according to the Ekeland's variational principle, there is $(w_n)_{n\in\mathbb{N}}$ with $\|w_n\| \le \rho$ satisfying

$$c_{2,\mu} \le I_\mu(w_n) \le I_\mu(w_n^*) < c_{2,\mu} + \frac{1}{n},$$

$$\|w_n - w_n^*\| \le \frac{1}{\sqrt{n}} \quad \text{and} \quad \|I'_\mu(w_n)\| \le \frac{1}{n}. \tag{3.66}$$

As $n \to \infty$, the sequence $(w_n)_{n\in\mathbb{N}}$ satisfies

$$I_\mu(w_n) \to c_{2,\mu} \quad \text{and} \quad I'_\mu(w_n) \to 0.$$

Then Lemma 3.1.4 implies the existence of a minimizer $w \in B_\rho$ for the functional $I_\mu$ and $w_n \to w$ in $H^1(\mathbb{R}^3)$. Hence, by (3.66), $w_n^* \to w$. Since $w_n^* \ge 0$, we get that $w \ge 0$ a.e. in $\mathbb{R}^3$ with $I_\mu(w) < 0$ and $I'_\mu(w) = 0$. The maximum principle implies that $w > 0$ in $\mathbb{R}^3$. The conclusion of this proposition follows from choosing $w_\mu := w$. The proof of Proposition 3.2.5 is complete. □

**Remark 3.2.6.** *In fact, for the case of $0 < \mu < \mu_1$, to get a positive solution, it is not necessary to involve the condition $(H_{l_2})$. Since this condition is used to get the boundedness of $(PS)$-sequence, for this case, one may use standard variational methods.*

**Proof of Theorem 3.0.9:** Since $\phi_u$ is always positive for every nonzero $u \in H^1(\mathbb{R}^3)$, we get that $(u_\mu, \phi_{u_\mu})$ and $(\omega_\mu, \phi_{\omega_\mu})$ are the positive solutions of problem (3.2) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ by Proposition 3.2.4 and Proposition 3.2.5, respectively. Hence we finish the proof of Theorem 3.0.9. □

# Chapter 4

# A positive solution of a Schrödinger-Poisson system with critical exponent

In this chapter, we study the existence of solutions of system $(\mathcal{SP})$ involving a critical growth with the following form

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^{2^*-2}u + \mu h(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.1}$$

where $2 \le q < 2^*$. Since we consider the problem in $\mathbb{R}^3$, $2^* = 6$. We use the standard Mountain Pass Theorem to show the existence of a positive solution. However, since the nonlinearity involves a critical exponent, the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 \le s \le 6)$ is not compact. This will create great difficulties in the proof of the Palais-Smale condition. We will transform the problem into a nonlocal elliptic equation in $\mathbb{R}^3$ and we also consider the limiting case $q = 2$.

We assume the following hypotheses $(H)$:

$(H_l)$ $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $l(x) \ge 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;

$(H_{k_1})$ $k(x) \ge 0$ for any $x \in \mathbb{R}^3$;

$(H_{k_2})$ There exists $x_0 \in \mathbb{R}^3$, $\delta_1 > 0$ and $\rho_1 > 0$ such that $k(x_0) = \max_{\mathbb{R}^3} k(x)$ and $|k(x) - k(x_0)| \le \delta_1 |x - x_0|^\alpha$ for $|x - x_0| < \rho_1$ with $1 \le \alpha < 3$;

$(H_{h_1})$ $h \in L^{6/(6-q)}(\mathbb{R}^3)$ and $h(x) \ge 0$ for any $x \in \mathbb{R}^3$ and $h \not\equiv 0$;

$(H_{h_2})$ There are $\delta_2 > 0$ and $\rho_2 > 0$ such that $h(x) \ge \delta_2 |x - x_0|^{-\beta}$ for $|x - x_0| < \rho_2$ and $2 - \frac{q}{2} < \beta < 3$, where $x_0$ is given by $(H_{k_2})$;

$(H_{h_\mu})$ $0 < \mu < \bar{\mu}$ when $2 \le q < 4$; $\mu > 0$ when $4 \le q < 6$, where $\bar{\mu}$ is defined by

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3)} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \ : \ \int_{\mathbb{R}^3} h(x)|u|^q dx = 1 \right\}.$$

**Remark 4.0.7.** *The hypotheses $(H_{k_1})$ and $(H_{k_2})$ mean that $k \in L^\infty(\mathbb{R}^3)$.*

**Remark 4.0.8.** *In Lemma 4.1.5, we show that $\bar{\mu}$ is achieved.*

The following theorem is the main result of this chapter.

**Theorem 4.0.9.** *Assume the hypotheses $(H)$ hold and $2 \le q < 6$. Then problem (4.1) has at least one positive solution $(u, \phi_u)$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

To prove the result above, we use a combination of techniques, e.g. techniques motivated by Willem [103], to overcome the lack of compactness of the Sobolev embedding, and methods used by Chen-Li-Li [28] and Zhao-Zhao [109], to estimate carefully the energy level.

The results presented here are published in [59].

## 4.1 Preliminaries

Let $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

We remind here that $F$ is already defined in (1.12) by

$$F(u) = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx.$$

Many works in the literature mention that $F \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, but we did not find any details of the proof. Let us start to prove this result with the case that $l \in L^\infty(\mathbb{R}^3)$.

**Lemma 4.1.1.** *(see Reed-Simon [86, p.31].) Let $0 < \beta < N$ and $f \in L^q(\mathbb{R}^N)$, $g \in L^r(\mathbb{R}^N)$ with $\frac{1}{q} + \frac{1}{r} + \frac{\beta}{N} = 2$ and $1 < q,\ r < \infty$. Then*

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f(x)||g(y)|}{|x-y|^\beta} dxdy \le C(q, r, \beta, N)\|f\|_q \|g\|_r, \quad x, y \in \mathbb{R}^N,$$

*where $C(q, r, \beta, N)$ is a positive constant depending on $q, r, \beta$ and $N$.*

**Lemma 4.1.2.** *If the hypothesis $(H_l)$ holds, then $F \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.*

**Proof.** From Lemma 4.1.1 and the hypothesis $(H_l)$ we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|l(x)u^2(x)||l(y)u(y)v(y)|}{|x-y|} dxdy \\ &\le C\|u\|_{12/5}^2 \|uv\|_{6/5} \\ &\le C\|u\|_{12/5}^2 \|u\|_{12/5} \|v\|_{12/5}\end{aligned}$$

for any $u, v \in H^1(\mathbb{R}^3)$. Then we may use the Lebesgue theorem and Fubini theorem and get

$$\begin{aligned}
&\lim_{t\to 0} \frac{F(u+tv)-F(u)}{t} \\
&= \lim_{t\to 0} \int_{\mathbb{R}^3} \frac{l(x)}{t} \left( (u+tv)^2 \left( \phi_u + 2t \int_{\mathbb{R}^3} \frac{l(y)u(y)v(y)}{|x-y|} dy + t^2 \phi_v \right) - \phi_u u^2 \right) dx \\
&= 2 \int_{\mathbb{R}^3} l(x) \left( u^2(x) \int_{\mathbb{R}^3} \frac{l(y)u(y)v(y)}{|x-y|} dy + u(x)v(x) \int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy \right) dx \\
&= 4 \int_{\mathbb{R}^3} l(x) \phi_u uv dx.
\end{aligned}$$

Hence the Gateaux derivative of $F$ on $H^1(\mathbb{R}^3)$ exists and $\langle \frac{1}{4} F'(u), v \rangle = \int_{\mathbb{R}^3} l(x)\phi_u uv dx$. Let $u_n \to u$ in $H^1(\mathbb{R}^3)$ and $v \in H^1(\mathbb{R}^3)$, then by $(H_l)$ we obtain

$$\begin{aligned}
&\|F'(u_n) - F'(u)\|_{H^{-1}} = \sup_{\|v\|=1} |\langle F'(u_n) - F'(u), v \rangle| \\
&= 4 \sup_{\|v\|=1} \left| \int_{\mathbb{R}^3} l(x)(\phi_{u_n} u_n - \phi_{u_n} u + \phi_{u_n} u - \phi_u u) v dx \right| \\
&\leq 4\|l\|_\infty \sup_{\|v\|=1} \left( \|\phi_{u_n}\|_6 \|u_n - u\|_{12/5} \|v\|_{12/5} + \int_{\mathbb{R}^3} |\phi_{u_n} - \phi_u||uv| dx \right).
\end{aligned} \tag{4.2}$$

It follows from Lemma 4.1.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\phi_{u_n} - \phi_u||uv| dx \\
&= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)v(x)||u_n^2(y) - u^2(y)|}{|x-y|} dx dy \\
&\leq C \|u_n^2 - u^2\|_{6/5} \|uv\|_{6/5} \\
&\leq C \|u_n^2 - u^2\|_{6/5} \|u\|_{12/5} \|v\|_{12/5}.
\end{aligned} \tag{4.3}$$

From (1.10), (4.2), (4.3) and the fact that $u_n \to u$ in $H^1(\mathbb{R}^3)$, we obtain

$$\|F'(u_n) - F'(u)\|_{H^{-1}} \to 0.$$

Thus $F$ has a continuous Gateaux derivative on $H^1(\mathbb{R}^3)$. Therefore $F \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. $\square$

In the setting of this chapter, for problem (4.1), the equation $(\mathcal{NSN})$ should become

$$-\Delta u + u + l(x)\phi_u u = k(x)|u|^4 u + \mu h(x)|u|^{q-2} u \quad \text{in } \mathbb{R}^3. \tag{4.4}$$

But since in this chapter we use a different method from Chapter 2 and Chapter 3 to get a positive solution, we consider the following corresponding modified equation

$$-\Delta u + u + l(x)\phi_u u = k(x)|u^+|^5 + \mu h(x)|u^+|^{q-1} \quad \text{in } \mathbb{R}^3. \tag{4.5}$$

Let us introduce the Euler functional associated to (4.5) by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \int_{\mathbb{R}^3} \left( \frac{1}{6}k(x)|u^+|^6 + \frac{\mu}{q}h(x)|u^+|^q \right) dx. \tag{4.6}$$

By Lemma 4.1.2 we know that the functional $I$ is of class $C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and its critical points are weak solutions of (4.5).

To prove Theorem 4.0.9, we still need some other preliminary lemmas.

**Lemma 4.1.3.** *Assume that the hypothesis* $(H_l)$ *holds. Then* $F$ *is a weakly continuous functional.*

**Proof.** Suppose $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Since $u_n \to u$ in $L^2_{loc}(\mathbb{R}^3)$, going if necessary to a subsequence, we can assume that

$$u_n \to u \;\text{ a.e. in }\; \mathbb{R}^3$$

and

$$\phi_{u_n} \to \phi_u \;\text{ a.e. in } \mathbb{R}^3.$$

In fact, the last statement is true since, by $(H_l)$ and the Hölder inequality, we have

$$\begin{aligned} |\phi_{u_n}(x) - \phi_u(x)| &\le \frac{1}{4\pi}\int_{\mathbb{R}^3} |l(y)||u_n^2(y) - u^2(y)| \frac{1}{|x-y|}dy \\ &\le C\|u_n^2 - u^2\|_{L^2(B_R(x))} \left( \int_{|x-y|\le R} \frac{1}{|x-y|^2}dy \right)^{1/2} \\ &+C\|u_n^2 - u^2\|_{L^{4/3}(B_R^c(x))} \left( \int_{|x-y|> R} \frac{1}{|x-y|^4}dy \right)^{1/4} \\ &\le C\|u_n^2 - u^2\|_{L^2(B_R(x))} + CR^{-\frac{1}{4}}\|u_n^2 - u^2\|_{L^{4/3}(B_R^c(x))} \\ &\to 0, \end{aligned} \tag{4.7}$$

as $n \to \infty$ and $R \to \infty$. Then $\phi_{u_n}u_n^2 \to \phi_u u^2$ a.e. in $\mathbb{R}^3$. Moreover, the sequence $(\phi_{u_n}u_n^2)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^3)$, since

$$\int_{\mathbb{R}^3} \left(\phi_{u_n}u_n^2\right)^2 dx \le \left( \int_{\mathbb{R}^3} \phi_{u_n}^6 dx \right)^{1/3} \left( \int_{\mathbb{R}^3} u_n^6 dx \right)^{2/3} = \|\phi_{u_n}\|_6^2 \|u_n\|_6^4 \le C\|u_n\|^6.$$

Hence $\phi_{u_n}u_n^2 \rightharpoonup \phi_u u^2$ in $L^2(\mathbb{R}^3)$. By $(H_l)$ we have

$$F(u_n) = \int_{\mathbb{R}^3} l(x)\phi_{u_n}u_n^2 dx \to \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx = F(u).$$

We have proven that $F$ is weakly continuous. □

**Lemma 4.1.4.** *Assume the hypothesis* $(H_l)$ *holds. Let* $u_n \rightharpoonup u$ *in* $H^1(\mathbb{R}^3)$*, then*

$$F(u_n - u) = F(u_n) - F(u) + o(1).$$

**Proof.** Since $(H_l)$ holds, from the proof of [109, Lemma 2.1], the result follows. $\square$

**Lemma 4.1.5.** *Suppose that the hypothesis $(H_{h_1})$ holds and $2 \le q < 4$. Then the following infimum*

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3)} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \ : \ \int_{\mathbb{R}^3} h(x)|u|^q dx = 1 \right\} \tag{4.8}$$

*is achieved.*

**Proof.** Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^3)$ be a minimizing sequence such that

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) dx \to \mu_h, \quad \text{as } n \to \infty.$$

So $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Then there exists a subsequence satisfying $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Since $h \in L^{6/(6-q)}(\mathbb{R}^3)$, by Lemma 1.4.6, we have

$$\int_{\mathbb{R}^3} h(x)|u_n|^q dx \to \int_{\mathbb{R}^3} h(x)|u|^q dx.$$

Hence

$$\int_{\mathbb{R}^3} h(x)|u|^q dx = 1.$$

Then, by the weakly lower semi-continuous property of the norm, we get

$$\mu_h = \liminf_{n\to\infty} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + u_n^2) dx \ge \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \ge \mu_h.$$

Thus the infimum $\mu_h$ is achieved. $\square$

**Lemma 4.1.6.** *Suppose that the hypotheses $(H_l)$, $(H_{k_1})$, $(H_{h_1})$ and $(H_{h_\mu})$ hold. Then $I(0) = 0$ and*

- $(I_1)$ *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \ge \alpha$; and*
- $(I_2)$ *there is $\bar{u} \in H^1(\mathbb{R}^3) \setminus \bar{B}_\rho$ such that $I(\bar{u}) < 0$.*

**Proof.** It is clear from the definition of $I$ that $I(0) = 0$. To prove $(I_1)$ and $(I_2)$, we consider $2 \le q < 4$ and $4 \le q < 6$, respectively. First, for $2 \le q < 4$, we have $0 < \mu < \bar{\mu}$ by $(H_{h_\mu})$. It follows from $(H_{k_1})$, Lemma 4.1.5 and the Sobolev inequality that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u^+|^6 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\|^6 - \frac{\mu}{q\bar{\mu}}\|u\|^2 \\ &= \|u\|^2 \left( \frac{1}{2} - \frac{\mu}{q\bar{\mu}} - C\|u\|^4 \right). \end{aligned}$$

Set $\rho = \|u\|$, small enough such that $C\rho^4 \le \frac{1}{2}(\frac{1}{2} - \frac{\mu}{q\bar{\mu}})$. Hence we have

$$I(u) \ge \frac{1}{2}\left(\frac{1}{2} - \frac{\mu}{q\bar{\mu}}\right)\rho^2. \tag{4.9}$$

Take $\alpha = \frac{1}{2}(\frac{1}{2} - \frac{\mu}{q\bar{\mu}})\rho^2$. Then we get the result $(I_1)$. By (1.10) and the fact that $\mu h(x) \ge 0$, for fixed $u_0$ with $\|u_0\| = 1$ and supp $u_0 \subset$ supp $k$, we have that

$$I(tu_0) \le t^6\left(\frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{C}{6}\int_{\mathbb{R}^3} k(x)|u_0^+|^6 dx\right).$$

Let $t$ be large enough such that $t > \rho$ and

$$\frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{C}{6}\int_{\mathbb{R}^3} k(x)|u_0^+|^6 dx < 0.$$

Take $\bar{u} = tu_0$. Then $(I_2)$ follows.

Next, we consider $4 \le q < 6$, so $\mu > 0$ by $(H_{h_\mu})$. Since $(H_{k_1})$ and $(H_{h_1})$ hold, the Hölder inequality and the Sobolev inequality imply that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u^+|^6 dx - \frac{\mu}{q}\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\|^6 - \frac{\mu}{q}\|h\|_{\frac{6}{6-q}}\|u\|_6^q \\ &\ge \|u\|^2\left(\frac{1}{2} - C\|u\|^4 - C\|u\|^{q-2}\right) \end{aligned}$$

for each $\mu > 0$ fixed. Let $\rho = \|u\|$ be small enough such that $C\|u\|^{q-2} \le \frac{1}{4}$ and then $I(u) \ge \frac{1}{4}\rho^2$. Take $\alpha = \frac{1}{4}\rho^2$. Thus one achieves the result $(I_1)$. The proof of $(I_2)$ is the same to the case that $2 \le q < 4$. The proof is complete. □

## 4.2 The proof of Palais-Smale condition

Since Lemma 4.1.6 shows that the functional $I$ has the mountain pass geometry, to apply the Mountain Pass Theorem to the functional $I$ on $H^1(\mathbb{R}^3)$, it is enough to prove that the $(PS)_c$-condition at some level $c$ that we are intended to solve.

**Lemma 4.2.1.** *Assume $(H_l)$, $(H_{k_1})$, $(H_{h_1})$ and $(H_{h_\mu})$ hold. Then the functional $I$ satisfies the $(PS)_c$-condition for $c \in \left(0, \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}\right)$, where $\mathcal{S}$ denotes the best Sobolev constant defined by*

$$\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3}|u|^6 dx\right)^{1/3}}. \tag{4.10}$$

**Proof.** Let $(u_n)_{n\in\mathbb{N}}$ be a $(PS)_c$-sequence of $I$ at the level $c \in \left(0, \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}\right)$, i.e.,

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^3). \tag{4.11}$$

**Step 1.** We consider $2 \leq q < 4$, so we get $0 < \mu < \bar{\mu}$ by $(H_{h_\mu})$. Then by the Sobolev inequality, Lemma 4.1.5 and $k(x) \geq 0$ for any $x \in \mathbb{R}^3$, for large $n$ we have that

$$\begin{aligned} c+1+\|u_n\| &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4}-\frac{1}{6}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^6 dx + \left(\frac{\mu}{4}-\frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \\ &\geq \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4}-\frac{1}{6}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^6 dx + \left(\frac{1}{4}-\frac{1}{q}\right)\frac{\mu}{\bar{\mu}}\|u_n\|^2 \\ &\geq \left(\frac{1}{4} + \left(\frac{1}{4}-\frac{1}{q}\right)\frac{\mu}{\bar{\mu}}\right)\|u_n\|^2, \end{aligned} \tag{4.12}$$

which implies $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, since $0 < \mu < \bar{\mu}$ and $2 \leq q < 4$. Passing if necessary to a subsequence, we can assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3),$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3,$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^3)$$

and

$$u_n \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^3).$$

Let us define $w_n = k(x)|u_n^+|^5$ and $w = k(x)|u^+|^5$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^6(\mathbb{R}^3)$ and $k \in L^\infty(\mathbb{R}^3)$, then $(w_n)_{n\in\mathbb{N}}$ is bounded in $L^{6/5}(\mathbb{R}^3)$ and so $w_n \rightharpoonup w$ in $L^{6/5}(\mathbb{R}^3)$. Note that for any $v \in H^1(\mathbb{R}^3)$, we have $v \in L^6(\mathbb{R}^3)$, $\nabla v \in L^2(\mathbb{R}^3)$ and $v \in L^2(\mathbb{R}^3)$. Hence

$$\int_{\mathbb{R}^3} w_n v dx \to \int_{\mathbb{R}^3} w v dx, \tag{4.13}$$

i.e.,

$$\int_{\mathbb{R}^3} k(x)|u_n^+|^5 v dx \to \int_{\mathbb{R}^3} k(x)|u^+|^5 v dx,$$

and

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v)\, dx \to \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx. \tag{4.14}$$

From the proof of Lemma 4.1.3 and Lemma 1.4.6 we also have

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \to \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx, \tag{4.15}$$

and

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \to \int_{\mathbb{R}^3} l(x)\phi_u u v dx. \tag{4.16}$$

Combining (4.13)–(4.16), for $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\langle I'(u_n), v\rangle &= \int_{\mathbb{R}^3}(\nabla u_n \nabla v + u_n v)dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \\ &\quad - \int_{\mathbb{R}^3} k(x)|u_n^+|^5 v dx - \mu \int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \\ &\to \int_{\mathbb{R}^3}(\nabla u \nabla v + uv)dx + \int_{\mathbb{R}^3} l(x)\phi_u uv dx - \int_{\mathbb{R}^3} k(x)|u^+|^5 v dx \\ &\quad -\mu \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx \\ &= \langle I'(u), v\rangle.\end{aligned} \tag{4.17}$$

On the other hand, by the fact $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$, we get that $\langle I'(u_n), v\rangle \to 0$ for any $v \in H^1(\mathbb{R}^3)$. So $\langle I'(u), v\rangle = 0$ for any $v \in H^1(\mathbb{R}^3)$, i.e.

$$-\Delta u + u + l(x)\phi_u u = k(x)|u^+|^5 + \mu h(x)|u^+|^{q-1}. \tag{4.18}$$

In particular, $\langle I'(u), u\rangle = 0$. And then from Lemma 4.1.5, $k(x) \geq 0$, and the assumptions that $2 \leq q < 4$ and $0 < \mu < \bar{\mu}$, we obtain that

$$\begin{aligned} I(u) &= \frac{1}{4}\langle I'(u), u\rangle + \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\int_{\mathbb{R}^3} k(x)|u^+|^6 dx \\ &\quad + \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\geq \left(\tfrac{1}{4} + \left(\tfrac{1}{4} - \tfrac{1}{q}\right)\tfrac{\mu}{\bar{\mu}}\right)\|u\|^2 \geq 0.\end{aligned} \tag{4.19}$$

Let $v_n = u_n - u$ and so $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Hence, by using the given hypotheses, the Brézis-Lieb lemma [22] implies that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} k(x)|u_n^+|^6 dx = \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx + \int_{\mathbb{R}^3} k(x)|u^+|^6 dx + o(1),$$

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx = \int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + \int_{\mathbb{R}^3} h(x)|u^+|^q dx + o(1),$$

and hence by Lemma 4.1.4 we have

$$I(u_n) = I(u) + \frac{1}{2}\|v_n\|^2 + \frac{1}{4}F(v_n) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx - \frac{1}{2}\int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1),$$

and

$$\begin{aligned}\langle I'(u_n), u_n\rangle &= \langle I'(u), u\rangle + \|v_n\|^2 + F(v_n) - \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \\ &\quad -\mu \int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1).\end{aligned}$$

Therefore it follows from Lemma 4.1.3, Lemma 1.4.6 and the hypotheses of $I(u_n) \to c$ and

$I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ that

$$c = \lim_{n\to\infty} I(u_n) = I(u) + \lim_{n\to\infty} \frac{1}{2}\|v_n\|^2 - \lim_{n\to\infty} \frac{1}{6}\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx, \tag{4.20}$$

and

$$\langle I'(u), u\rangle + \lim_{n\to\infty} \|v_n\|^2 - \lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx = 0. \tag{4.21}$$

Using (4.18) and (4.21) we obtain

$$\|v_n\|^2 - \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \to -\langle I'(u), u\rangle = 0.$$

Now we may assume that

$$\|v_n\|^2 \to b$$

and

$$\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \to b.$$

By the Sobolev's inequality we have

$$\|v_n\|^2 \geq \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \geq \mathcal{S}\left(\int_{\mathbb{R}^3} |v_n^+|^6 dx\right)^{1/3},$$

which means that

$$\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \leq \|k\|_\infty \int_{\mathbb{R}^3} |v_n^+|^6 dx \leq \|k\|_\infty \left(\mathcal{S}^{-1}\|v_n\|^2\right)^3,$$

i.e., $b \leq \|k\|_\infty \left(\mathcal{S}^{-1} b\right)^3$. So we get that $b = 0$ or $b \geq \mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Assume $b \geq \mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Then combining (4.19) and (4.20), we obtain that

$$c \geq \frac{1}{2}b - \frac{1}{6}b = \frac{1}{3}b \geq \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}},$$

which contradicts the fact that $c < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Hence $b = 0$.

**Step 2.** One computes by $4 \leq q < 6$ and $\mu > 0$ that

$$\begin{aligned} c + 1 + \|u_n\| &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\int_{\mathbb{R}^3} k(x)|u_n^+|^6 dx \\ &\quad + \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx \geq \frac{1}{4}\|u_n\|^2, \end{aligned}$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$. Passing if necessary to a subsequence, one can assume that

$$u_n \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^3),$$

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^3,$$

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^3)$$

and

$$u_n \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^3).$$

Let us define $w_n = k(x)|u_n^+|^5$ and $w = k(x)|u^+|^5$. Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^6(\mathbb{R}^3)$ and $k \in L^\infty(\mathbb{R}^3)$, then $(w_n)_{n\in\mathbb{N}}$ is bounded in $L^{6/5}(\mathbb{R}^3)$ and so $w_n \rightharpoonup w$ in $L^{6/5}(\mathbb{R}^3)$. Note that for any $v \in H^1(\mathbb{R}^3)$, one has $v \in L^6(\mathbb{R}^3)$, $\nabla v \in L^2(\mathbb{R}^3)$ and $v \in L^2(\mathbb{R}^3)$. Thus

$$\int_{\mathbb{R}^3} w_n v dx \to \int_{\mathbb{R}^3} w v dx, \tag{4.22}$$

i.e.,

$$\int_{\mathbb{R}^3} k(x)|u_n^+|^5 v dx \to \int_{\mathbb{R}^3} k(x)|u^+|^5 v dx,$$

and

$$\int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v)\, dx \to \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx. \tag{4.23}$$

From the proof of Lemma 4.1.3 and Lemma 1.4.6 we also have

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \to \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx, \tag{4.24}$$

and

$$\int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \to \int_{\mathbb{R}^3} l(x)\phi_u u v dx. \tag{4.25}$$

Combining (4.22)–(4.25), for $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, we obtain that

$$\begin{aligned} \langle I'(u_n), v\rangle &= \int_{\mathbb{R}^3} (\nabla u_n \nabla v + u_n v) dx + \int_{\mathbb{R}^3} l(x)\phi_{u_n} u_n v dx \\ &\quad - \int_{\mathbb{R}^3} k(x)|u_n^+|^5 v dx - \mu \int_{\mathbb{R}^3} h(x)|u_n^+|^{q-1} v dx \\ &\to \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx + \int_{\mathbb{R}^3} l(x)\phi_u u v dx - \int_{\mathbb{R}^3} k(x)|u^+|^5 v dx \\ &\quad -\mu \int_{\mathbb{R}^3} h(x)|u^+|^{q-1} v dx \\ &= \langle I'(u), v\rangle. \end{aligned} \tag{4.26}$$

On the other hand, by the fact $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$, we get that $\langle I'(u_n), v\rangle \to 0$ for any $v \in H^1(\mathbb{R}^3)$. So $\langle I'(u), v\rangle = 0$ for any $v \in H^1(\mathbb{R}^3)$, that is,

$$-\Delta u + u + l(x)\phi_u u = k(x)|u^+|^5 + \mu h(x)|u^+|^{q-1}.$$

In particular,

$$\langle I'(u), u\rangle = 0. \tag{4.27}$$

And then it follows from Lemma 4.1.5, $k(x) \geq 0$, $l(x) \geq 0$ and the assumptions that

$4 \leq q < 6$ and $\mu > 0$ that

$$\begin{aligned} I(u) &= \frac{1}{4}\langle I'(u), u\rangle + \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{6}\right)\int_{\mathbb{R}^3} k(x)|u^+|^6 dx \\ &\quad + \left(\frac{\mu}{4} - \frac{\mu}{q}\right)\int_{\mathbb{R}^3} h(x)|u^+|^q dx \\ &\geq \frac{1}{4}\|u\|^2 \geq 0. \end{aligned} \tag{4.28}$$

Let $v_n = u_n - u$ and so $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Hence, by using the given hypotheses, the Brézis-Lieb lemma [22] implies that

$$\|u_n\|^2 = \|v_n\|^2 + \|u\|^2 + o(1),$$

$$\int_{\mathbb{R}^3} k(x)|u_n^+|^6 dx = \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx + \int_{\mathbb{R}^3} k(x)|u^+|^6 dx + o(1),$$

$$\int_{\mathbb{R}^3} h(x)|u_n^+|^q dx = \int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + \int_{\mathbb{R}^3} h(x)|u^+|^q dx + o(1),$$

and hence by Lemma 4.1.4 we have

$$I(u_n) = I(u) + \frac{1}{2}\|v_n\|^2 + \frac{1}{4}F(v_n) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx - \frac{1}{2}\int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1),$$

and

$$\begin{aligned} \langle I'(u_n), u_n\rangle &= \langle I'(u), u\rangle + \|v_n\|^2 + F(v_n) \\ &\quad - \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx - \mu\int_{\mathbb{R}^3} h(x)|v_n^+|^q dx + o(1). \end{aligned}$$

Therefore it follows from Lemma 4.1.3, Lemma 1.4.6 and the hypotheses $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ that

$$c = \lim_{n\to\infty} I(u_n) = I(u) + \lim_{n\to\infty}\frac{1}{2}\|v_n\|^2 - \lim_{n\to\infty}\frac{1}{6}\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx, \tag{4.29}$$

and

$$\langle I'(u), u\rangle + \lim_{n\to\infty}\|v_n\|^2 - \lim_{n\to\infty}\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx = 0. \tag{4.30}$$

Using (4.27) and (4.30) we obtain

$$\|v_n\|^2 - \int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \to -\langle I'(u), u\rangle = 0.$$

Now we may assume that

$$\|v_n\|^2 \to b$$

and

$$\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \to b.$$

By the Sobolev's inequality we have

$$\|v_n\|^2 \geq \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \geq \mathcal{S} \left( \int_{\mathbb{R}^3} |v_n^+|^6 dx \right)^{1/3},$$

which means that

$$\int_{\mathbb{R}^3} k(x)|v_n^+|^6 dx \leq \|k\|_\infty \int_{\mathbb{R}^3} |v_n^+|^6 dx \leq \|k\|_\infty \left(\mathcal{S}^{-1}\|v_n\|^2\right)^3,$$

i.e., $b \leq \|k\|_\infty \left(\mathcal{S}^{-1} b\right)^3$. So we get that $b = 0$ or $b \geq \mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Assume $b \geq \mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Then combining (4.28) and (4.29), we obtain that

$$c \geq \frac{1}{2}b - \frac{1}{6}b = \frac{1}{3}b \geq \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}},$$

which contradicts the fact that $c < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Then $b = 0$. Hence we know that every $(PS)_c$-sequence $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence. This proves Lemma 4.2.1. □

**Lemma 4.2.2.** *Suppose the hypotheses* $(H)$ *hold. Then* $c < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$, *where* $c$ *is defined by*

$$c = \inf_{g\in\Gamma} \max_{u\in g[0,1]} I(u)$$

*with*

$$\Gamma = \{g \in C([0,1], H^1(\mathbb{R}^3)) \ : \ g(0) = 0, g(1) = \bar{u}\}$$

*and* $\bar{u}$ *is defined by Lemma* 4.1.6, *which belongs to* $H^1(\mathbb{R}^3) \setminus \bar{B}_\rho$ *and satisfies* $I(\bar{u}) < 0$.

**Proof.** The idea here is to find a path in $\Gamma$ such that the maximum of the functional $I$ at this path is strictly less than $\frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-1/2}$. To construct this path, we need the extremal function $u_{\varepsilon,x_0}$ for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, where

$$u_{\varepsilon,x_0} = C\frac{\varepsilon^{1/4}}{(\varepsilon + |x - x_0|^2)^{1/2}}.$$

Here $C$ is a normalizing constant and $x_0$ is given in $(H_{k_2})$. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $0 \leq \varphi \leq 1, \varphi|_{B_{R_2}} \equiv 1$ and supp $\varphi \subset B_{2R_2}$ for some $R_2 > 0$. Set

$$v_\varepsilon = \varphi u_{\varepsilon,x_0}$$

and then $v_\varepsilon \in H^1(\mathbb{R}^3)$ with $v_\varepsilon(x) \geq 0$ for each $x \in \mathbb{R}^3$. The following asymptotic estimates hold if $\varepsilon$ is small enough (see Brézis-Nirenberg [23]):

$$\|\nabla v_\varepsilon\|_2^2 = k_1 + O(\varepsilon^{\frac{1}{2}}), \quad \|v_\varepsilon\|_6^2 = k_2 + O(\varepsilon), \tag{4.31}$$

$$\|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{s}{4}}) & s \in [2,3), \\ O(\varepsilon^{\frac{s}{4}}|\ln \varepsilon|) & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}) & s \in (3,6), \end{cases} \tag{4.32}$$

with $k_1/k_2 = \mathcal{S}$, and $2 \le s < 6$. We know that the path $tv_\varepsilon \in \Gamma$. In the rest, we will prove that

$$\max_{t\ge 0} I(tv_\varepsilon) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-1/2} \tag{4.33}$$

for small $\varepsilon$. Since $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t\ge0} I(tv_\varepsilon)$. Also by Lemma 4.1.6, $\max_{t\ge0} I(tv_\varepsilon) \ge \alpha > 0$. Then we have that

$$I(t_\varepsilon v_\varepsilon) \ge \alpha > 0.$$

Thus from the continuity of $I$, we may assume that there exists some positive $t_0$ such that $t_\varepsilon \ge t_0 > 0$. Moreover from $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$ and $I(t_\varepsilon v_\varepsilon) \ge \alpha > 0$, we get that there exists $T_0$ such that $t_\varepsilon \le T_0$. Hence $t_0 \le t_\varepsilon \le T_0$. Let

$$I(t_\varepsilon v_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon),$$

where

$$A(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx,$$

$$B(\varepsilon) = \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x)|v_\varepsilon|^6 dx,$$

and

$$C(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \frac{t_\varepsilon^4}{4}F(v_\varepsilon) - \frac{t_\varepsilon^2\mu}{2}\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^q dx,$$

since $v_\varepsilon^+ = v_\varepsilon$. First, we claim that

$$A(\varepsilon) \le \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + C\varepsilon^{1/2}. \tag{4.34}$$

Indeed, let

$$g(t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx.$$

It is clear that $g(t)$ achieves its maximum value at some $T_\varepsilon$. So

$$0 = g'(T_\varepsilon) = T_\varepsilon\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - T_\varepsilon^5\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx.$$

That is,

$$T_\varepsilon = \left(\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx}\right)^{\frac{1}{4}}.$$

Therefore, from (4.31), we have that

$$g(T_\varepsilon) = \sup_{t\geq 0} g(t) = \frac{1}{3}\frac{\left(\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2dx\right)^{3/2}}{\left(\int_{\mathbb{R}^3}k(x_0)|v_\varepsilon|^6dx\right)^{1/2}} = \frac{1}{3}\mathcal{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + C\varepsilon^{1/2}.$$

Then (4.34) follows. Second, we claim that

$$B(\varepsilon) \leq C\varepsilon^{1/2}. \tag{4.35}$$

In fact, since $t_0 \leq t_\varepsilon \leq T_0$ and $k \in L^\infty(\mathbb{R}^3)$, by the definition of $v_\varepsilon$, $(H_{k_2})$ and using a change of variables with $1 \leq \alpha < 3$, we have

$$\begin{aligned}
B(\varepsilon) &= \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3}(k(x_0)-k(x))|v_\varepsilon|^6dx \\
&\leq C\delta_1\int_{|x-x_0|<\rho_1}\frac{|x-x_0|^\alpha\varepsilon^{3/2}}{(\varepsilon+|x-x_0|^2)^3}dx + C\int_{|x-x_0|\geq\rho_1}\frac{\varepsilon^{3/2}}{(\varepsilon+|x-x_0|^2)^3}dx \\
&\leq C\delta_1\varepsilon^{\frac{3}{2}}\int_0^{\rho_1}\frac{r^{2+\alpha}}{(\varepsilon+r^2)^3}dr + C\varepsilon^{\frac{3}{2}}\int_{\rho_1}^\infty r^{-4}dr \\
&= C\delta_1\varepsilon^{\frac{\alpha}{2}}\int_0^{\rho_1\varepsilon^{-\frac{1}{2}}}\frac{\rho^{2+\alpha}}{(1+\rho^2)^3}d\rho + C\rho_1^{-3}\varepsilon^{3/2} \\
&\leq C\delta_1\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{3/2} \\
&\leq C\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

So we have proved the claim (4.35). Therefore, to finish the proof, it is enough to show

$$\lim_{\varepsilon\to 0^+}\frac{C(\varepsilon)}{\varepsilon^{1/2}} = -\infty. \tag{4.36}$$

Actually, from the definition of $v_\varepsilon$, $(H_{h_2})$ and for any $\varepsilon$ such that $0 < \varepsilon \leq \rho_2^2$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3}h(x)|v_\varepsilon|^qdx &\geq C\delta_2\int_{|x-x_0|<\rho_2}\frac{|x-x_0|^{-\beta}\varepsilon^{q/4}}{(\varepsilon+|x-x_0|^2)^{q/2}}dx + \int_{|x-x_0|\geq\rho_2}h(x)|v_\varepsilon|^qdx \\
&\geq C\delta_2\varepsilon^{q/4}\int_0^{\rho_2}\frac{r^2}{r^\beta\,(\varepsilon+r^2)^{q/2}}dr \\
&= C\delta_2\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}\int_0^{\rho_2\varepsilon^{-\frac{1}{2}}}\frac{\rho^2}{\rho^\beta\,(1+\rho^2)^{q/2}}d\rho \\
&\geq C\delta_2\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}\int_0^1\frac{\rho^2}{2^q\rho^\beta}d\rho \\
&= C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}.
\end{aligned}$$

Therefore, by the fact that $t_0 \leq t_\varepsilon \leq T_0$ and hypothesis $(H_l)$, we have

$$\begin{aligned} C(\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{4}F(v_\varepsilon) - \frac{t_\varepsilon^2\mu}{2}\int_{\mathbb{R}^3}h(x)|v_\varepsilon|^q dx \\ &\leq C\|v_\varepsilon\|_2^2 + C\|v_\varepsilon\|_{12/5}^4 - \mu C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}} \\ &\leq C\varepsilon^{\frac{1}{2}} + C\varepsilon - \mu C\varepsilon^{\frac{3}{2}-\frac{q}{4}-\frac{\beta}{2}}. \end{aligned}$$

It follows from $2 - \frac{q}{2} < \beta < 3$ that for fixed $\mu$ we have

$$\frac{C(\varepsilon)}{\varepsilon^{1/2}} \leq C + C\varepsilon^{\frac{1}{2}} - \mu C\varepsilon^{1-\frac{q}{4}-\frac{\beta}{2}} \to -\infty, \text{ as } \varepsilon \to 0.$$

So we prove the claim (4.36). Therefore (4.33) follows. We finish the proof of this lemma. □

## 4.3 The proof of Theorem 4.0.9

To prove Theorem 4.0.9, we will apply the Mountain Pass Theorem to find a solution of problem (4.5) and then prove that it is a positive solution.

**Proof of Theorem 4.0.9.** By Lemma 4.1.6, the functional $I$ has the mountain pass geometry. Moreover, it follows from Lemma 4.2.1 and Lemma 4.2.2 that the functional $I$ satisfies the $(PS)_c$-condition at the level $c$ defined by

$$c = \inf_{g\in\Gamma}\max_{u\in g[0,1]} I(u)$$

with

$$\Gamma = \{g \in C([0,1], H^1(\mathbb{R}^3)) \ : \ g(0) = 0, g(1) = T_0 v_\varepsilon\},$$

for some $T_0 > 0$ such that $T_0 v_\varepsilon \in H^1(\mathbb{R}^3) \setminus \bar{B}_\rho$ and $I(T_0 v_\varepsilon) < 0$, where $\rho$ is defined by Lemma 4.1.6. Hence the functional $I$ has a critical value $c > 0$. That is, there exists a nontrivial $u \in H^1(\mathbb{R}^3)$ such that $I'(u) = 0$, which means that $u$ is the nontrivial solution of system (4.5).

Since

$$0 = \langle I'(u), u^- \rangle = \|u^-\|^2 + \int_{\mathrm{R}^3} l(x)\phi_u|u^-|^2dx \geq \|u^-\|^2,$$

then $u \geq 0$ in $\mathbb{R}^3$. So $u$ is the nontrivial solution of system (4.4). By standard arguments as in DiBenedetto [45] and Tolksdorf [97], we have that $u \in L^\infty(\mathbb{R}^3)$ and $u \in C^{1,\gamma}_{loc}(\mathbb{R}^3)$ with $0 < \gamma < 1$. Furthermore, by Harnack's inequality (see Trudinger [98]), $u(x) > 0$ for any $x \in \mathbb{R}^3$. Thus $u$ is a positive solution of system (4.4) and then $(u, \phi_u)$ is a positive solution of system (4.1). □

# Chapter 5

# Positive and sign changing solutions of a Schrödinger-Poisson system involving a critical nonlinearity

In this chapter, we continue to study the system with the critical Sobolev exponent in the nonlinear term, but the different thing from Chapter 4 is that here we use different methods to find a positive solution. More importantly, we obtain a pair of sign changing solutions. The results obtained here are published in [61].

We, in the present chapter, study the existence and multiplicity of fixed sign and sign changing solutions of the following nonlinear Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^4 u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5.1}$$

where $l, k$ and $h$ are nonnegative functions, $\mu$ is a positive constant, and the nonlinear growth of $|u|^4 u$ reaches the critical Sobolev exponent since the critical exponent $2^* = 6$ in three spatial dimensions, which is why we call critical nonlinearity in the title.

As we have seen in Chapter 1, system (5.1) can be easily reduced into a nonlinear Schrödinger equation with a nonlocal term as

$$-\Delta u + u + l(x)\phi_u u = k(x)|u|^4 u + \mu h(x)u \quad \text{in } \mathbb{R}^3. \tag{5.2}$$

And the corresponding functional is

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}F(u) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u|^6 dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)u^2 dx,$$

where $F(u) = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx$.

Our main purpose in this chapter is to study the existence of the sign changing solutions to equation (5.2). In general, finding a sign changing solution of an equation is much more difficult than finding a mere solution. Although there were several abstract theories or methods to study sign changing solutions, they are only applicable to some specific situations. For example, when a problem involves a small parameter, usually Lyapunov-Schmidt reduction procedure (see Ambrosetti-Malchiodi [4]) can be used to find sign changing solutions. In [13], Bartsch established an abstract critical point theory in partially order Hilbert spaces by virtue of critical groups and studied superlinear problems. In Li-Wang [73], one kind of Ljusternik-Schnirelman theory was established to study sign changing critical points of an even functional. Some linking type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of Bartsch [13], Bartsch-Weth [15] and Li-Wang [73] involved the dense Banach space $C(\Omega)$ ($\Omega$ is a smooth bounded domain) of continuous functions in the Hilbert space $H_0^1(\Omega)$, in which the cone has nonempty interior and this framework requires strong hypotheses such as boundedness of the domain. In [91], Schechter and Zou established relationships between sign changing critical point theorems and the linking type theorem of Schechter and the saddle point theorem of Rabinowitz, and applied them to study sign changing solutions for the nonlinear Schrödinger equation with jumping or oscillating nonlinearities and of double resonance.

It seems that all the methods mentioned above can not be applied directly to equation (5.2), which is considered in the whole space $\mathbb{R}^3$ with nonlocal term. Our methods used here involve neither the Palais-Smale sequence nor Ekeland variational principle. Our idea is inspired by Hirano-Shioji [56], but the procedure is a little simpler than that in Hirano-Shioji [56]. With the help of several lemmas (see Section 3), we get at least a pair of fixed sign solutions as well as a pair of sign changing solutions.

Many mathematicians have been devoted to the study of the similar system (5.1) with various nonlinearities $f(x,u)$ as we mentioned in the introduction. However, all these results are about existence, multiplicity and behavior of positive solutions. Only little information about sign changing solutions for the similar system is known. Recently, Ianni [63] used a dynamical approach together with a limit procedure to study the existence of infinitely many radially symmetric sign changing solutions in the case of $f(x,u) = |u|^{p-2}u$ ($4 \leq p < 6$) and function $l(x) \equiv 1$. To our best knowledge, we have not seen any results related to sign changing solutions to system (5.1).

In the present chapter, we assume the following hypotheses $(H)$:

$(H_l)$ $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;

$(H_{k_1})$ $k(x) \geq 0$ for any $x \in \mathbb{R}^3$;

$(H_{k_2})$ There exist $x_0 \in \mathbb{R}^3$, $\delta_1 > 0$ and $\rho_1 > 0$ such that $k(x_0) = \max_{\mathbb{R}^3} k(x)$ and $|k(x) - k(x_0)| \leq \delta_1 |x - x_0|^\alpha$ for $|x - x_0| < \rho_1$ with $1 \leq \alpha < 3$;

$(H_{h_1})$ $h \in L^{3/2}(\mathbb{R}^3)$ and $h(x) \geq 0$ for any $x \in \mathbb{R}^3$ ;

$(H_{h_2})$ There are $\delta_2 > 0$ and $\rho_2 > 0$ such that $h(x) \geq \delta_2|x - x_0|^{-\beta}$ for $|x - x_0| < \rho_2$, where $x_0$ is given by $(H_{k_2})$;

$(H_{h_\mu})$ $0 < \mu < \bar{\mu}$, where $\bar{\mu}$ is defined by

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx \ : \ \int_{\mathbb{R}^3} h(x)|u|^2 dx = 1 \right\}.$$

*Remark 1.* From the above assumptions, we have the following two remarks.

**(1)** The hypotheses $(H_{k_1})$ and $(H_{k_2})$ mean that $k \in L^\infty(\mathbb{R}^3)$.

**(2)** Lemma 5.1.2 (iii) shows that $\bar{\mu}$ is achieved.

The main results in this chapter read as follows.

**Theorem 5.0.1.** *Assume that the hypotheses $(H)$ hold with $1 < \beta < 3$. Then the system (5.1) has at least one positive solution $(\psi_1, \phi_{\psi_1})$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

**Theorem 5.0.2.** *Assume that the hypotheses $(H)$ hold with $\frac{3}{2} < \beta < 3$. Then the system (5.1) has at least one sign changing solution $(\psi_2, \phi_{\psi_2})$ in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.*

The remainder of this chapter is organized as follows. In Section 1, we give some useful preliminaries. In Section 2, we study the existence of a positive solution of (5.1), where we not only prove Theorem 5.0.1 but also prove several lemmas which pave the way for getting sign changing solutions. Then Section 3 is devoted to proving Theorem 5.0.2.

## 5.1 Preliminaries

In this section, our aim is to give some useful preliminary lemmas. Let us start with the following easy lemma–Calculus Lemma.

**Lemma 5.1.1.** *(see Ghoussoub-Yuan [53].) For every $1 \leq q \leq 3$, there exists a constant $C$ (depending on $q$) such that for $a, b \in \mathbb{R}$ we have*

$$\left||a+b|^q - |a|^q - |b|^q - qab\left(|a|^{q-2} + |b|^{q-2}\right)\right| \leq \begin{cases} C|a||b|^{q-1} & \text{if } |a| \geq |b|, \\ C|a|^{q-1}|b| & \text{if } |a| \leq |b|. \end{cases}$$

*For $q \geq 3$, there exists a constant $C$ (depending on $q$) such that for $a, b \in \mathbb{R}$ we have*

$$\left||a+b|^q - |a|^q - |b|^q - qab\left(|a|^{q-2} + |b|^{q-2}\right)\right| \leq C\left(|a|^{q-2}b^2 + a^2|b|^{q-2}\right).$$

From this inequality, we can actually deduce the following more convenient result for any $q \geq 1$:

$$\left||a+b|^q - |a|^q - |b|^q - qab\left(|a|^{q-2} + |b|^{q-2}\right)\right| \leq 2C\left(|a|^{q-1}|b| + |a||b|^{q-1}\right).$$

**Lemma 5.1.2.** *Assume that the hypotheses* $(H_l)$ *and* $(H_{h_1})$ *hold. Then the following statements are valid:*

**(i)** *$F$ is a weakly continuous functional.*

**(ii)** *If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $F(u_n - u) = F(u_n) - F(u) + o(1)$.*

**(iii)** *The following infimum $\bar{\mu}$ is achieved*

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3)} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \ : \ \int_{\mathbb{R}^3} h(x) u^2 dx = 1 \right\}. \tag{5.3}$$

**Proof.** The proofs of (i) and (iii) are the same as Lemma 4.1.3 and Lemma 4.1.5 in Chapter 4, respectively. And it follows from (i) that (ii) holds. □

**Lemma 5.1.3.** *If the hypotheses* $(H_l)$, $(H_{k_1})$, $(H_{h_1})$ *and* $(H_\mu)$ *hold, then* $I(0) = 0$ *and*

$(I_1)$ *there are constants $\rho$, $\alpha_0 > 0$ such that $I|_{\partial B_\rho} \geq \alpha_0$;*

$(I_2)$ *for every $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\| = 1$ and meas(supp$(ku_0)$) $> 0$, there exists $\rho_* > 0$ such that $I(tu_0) < 0$ for any $t > \rho_*$.*

**Proof.** It is clear that $I(0) = 0$. It follows from Lemma 5.1.2 (iii) and the Sobolev inequality that

$$I(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^6 - \frac{\mu}{2\bar{\mu}}\|u\|^2 = \|u\|^2 \left( \frac{1}{2} - \frac{\mu}{2\bar{\mu}} - C\|u\|^4 \right).$$

Set $\rho = \|u\|$ small enough such that $C\rho^4 \leq \frac{1}{4}(1 - \frac{\mu}{\bar{\mu}})$. Hence we have

$$I(u) \geq \frac{1}{4}(1 - \frac{\mu}{\bar{\mu}})\rho^2. \tag{5.4}$$

Choosing $\alpha_0 = \frac{1}{4}(1 - \frac{\mu}{\bar{\mu}})\rho^2$, we get the statement $(I_1)$.

Let $u = tu_0$, with $\|u_0\| = 1$. By (1.11) and the assumptions $(H_{h_1})$ and $(H_\mu)$, we have that

$$I(u) = I(tu_0) \leq t^6 \left( \frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u_0|^6 dx \right).$$

Let $\rho_* > 0$ be fixed such that for all $t > \rho_*$ we have

$$\frac{1}{2t^4}\|u_0\|^2 + \frac{C}{4t^2}\|u_0\|^4 - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u_0|^6 dx < 0.$$

Then $(I_2)$ follows. □

Next, we prove an important lemma, by which we analyze the behavior of the Nehari set $\mathcal{N}$ defined by

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} \ : \ G(u) = 0\}, \text{ where } G(u) = \langle I'(u), u \rangle.$$

**Lemma 5.1.4.** *Suppose that the hypotheses* $(H_l)$ *and* $(H_\mu)$ *hold. Then we have the following conclusions:*

**(1)** *For every* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$*, there exists a unique* $t_u \equiv t(u) > 0$ *such that* $t_u u \in \mathcal{N}$*.*

**(2)** *If* $\langle I'(u), u\rangle < 0$*, then* $0 < t_u < 1$*; if* $\langle I'(u), u\rangle > 0$*, then* $t_u > 1$*.*

**(3)** $t_u$ *is a continuous functional in* $H^1(\mathbb{R}^3)$ *with respect to* $u$*.*

**(4)** *If* $\|u\| \to 0$*, then* $t_u \to +\infty$*.*

**Proof. (1)** For every $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, define $g(t) = I(tu)$ and

$$f(t) = \|u\|^2 + t^2 F(u) - t^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx.$$

Then we have $g'(t) = tf(t)$. By the definition of $\mathcal{N}$, for $t > 0$, we obtain that

$$g'(t) = \langle I'(tu), u\rangle = 0 \;\Leftrightarrow\; tu \in \mathcal{N}. \tag{5.5}$$

From the structure of the functional $I$, we know that $\sup_{t>0} g(t)$ is achieved at some $t_u = t(u) > 0$, and then $g'(t_u) = 0$. Hence, by (5.5), $t_u u \in \mathcal{N}$. It remains to prove that such $t_u$ with $g'(t_u) = 0$ is unique, i.e.,

it is sufficient to prove that the solution of $f(t) = 0$ in $(0, +\infty)$ is unique.

In fact, from

$$f'(t) = 2tF(u) - 4t^3 \int_{\mathbb{R}^3} k(x)|u|^6 dx = 0,$$

we obtain a unique

$$t_u^* = \sqrt{\frac{F(u)}{2\int_{\mathbb{R}^3} k(x)|u|^6 dx}} > 0 \tag{5.6}$$

such that $f'(t_u^*) = 0$ and $f'(t) > 0$ for any $t \in (0, t_u^*)$; $f'(t) < 0$ for any $t \in (t_u^*, +\infty)$. Moreover, since $0 < \mu < \bar{\mu}$, by Lemma 5.1.2 (iii),

$$f(0) = \|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx > 0.$$

Therefore, from $f(t_u) = 0$, $t_u \in (t_u^*, +\infty)$ and so $t_u$ must be unique. That is, for any $u \in H^1(\mathbb{R}^3)$, there exists a unique $t_u$ satisfying

$$\|u\|^2 + t_u^2 F(u) - t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx = 0. \tag{5.7}$$

This proves (1).

**(2)** If $\langle I'(u), u\rangle < 0$, using Lemma 5.1.2 (iii) and the assumption of $0 < \mu < \bar{\mu}$, we get that

$$\|u\|^2 + F(u) < \int_{\mathbb{R}^3} k(x)|u|^6 dx + \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \leq \int_{\mathbb{R}^3} k(x)|u|^6 dx + \frac{\mu}{\bar{\mu}}\|u\|^2,$$

which means that

$$F(u) < \int_{\mathbb{R}^3} k(x)|u|^6 dx. \tag{5.8}$$

It is deduced from (5.6) and (5.8) that $t_u^* < 1$. Moreover, we have that

$$f(1) = \langle I'(u), u\rangle < 0.$$

Then $t_u^* < t_u < 1$, because $f(t)$ decreases in $(t_u^*, +\infty)$ and $f(t_u) = 0$. Thus, in the case of $\langle I'(u), u\rangle < 0$, we have proven that $0 < t_u < 1$.

In the case of $\langle I'(u), u\rangle > 0$, we have $f(1) = \langle I'(u), u\rangle > 0$. If $t_u^* \geq 1$, we deduce that $t_u > t_u^* \geq 1$. Now we consider $t_u^* < 1$. Since $f(t)$ decreases in $(t_u^*, +\infty)$ and $f(1) > 0$, to be sure that $f(t_u) = 0$ for $t_u \in (t_u^*, +\infty)$, it must have $t_u > 1$. Therefore, in this case we have that $t_u > 1$. This finishes the proof of (2).

**(3)** Let $(u_n)_{n\in\mathbb{N}}$ be such that $u_n \to u$ in $H^1(\mathbb{R}^3)$. By (5.7) there exists a unique positive real sequence $(t_{u_n})_{n\in\mathbb{N}}$ satisfying

$$\|u_n\|^2 + t_{u_n}^2 F(u_n) - t_{u_n}^4 \int_{\mathbb{R}^3} k(x)|u_n|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u_n|^2 dx = 0, \tag{5.9}$$

which implies $(t_{u_n})_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Going if necessary to a subsequence, still denoted by $(t_{u_n})_{n\in\mathbb{N}}$, we may assume that there is $t_0 > 0$ such that $\lim_{n\to\infty} t_{u_n} = t_0$ and then as $n \to \infty$ passing to the limit in (5.9) we get that

$$g'(t_0) = t_0 \left( \|u\|^2 + t_0^2 F(u) - t_0^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \right) = 0.$$

Hence it follows from (5.5) that $t_0 u \in \mathcal{N}$. According to the uniqueness of $t_u$, we arrive at $t_0 = t_u$, i.e., $\lim_{n\to\infty} t_{u_n} = t_u$. We have proved that $t_u$ is continuous with respect to $u \in H^1(\mathbb{R}^3)$.

**(4)** When $\|u\| \to 0$, we claim that $t_u \to +\infty$. Otherwise, if $\|u\| \to 0$ and there exists $M > 0$ such that $|t_u| \leq M$, then by the Sobolev inequality and $k \in L^\infty(\mathbb{R}^3)$, we obtain that

$$t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx = o(\|u\|^2). \tag{5.10}$$

From (5.10) and Lemma 5.1.2 (iii) we deduce that

$$\|u\|^2 + t_u^2 F(u) - t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \geq \left(1 - \frac{\mu}{\bar{\mu}}\right) \|u\|^2 - o(\|u\|^2) > 0,$$

which contradicts (5.7). Hence $t_u \to +\infty$. This proves (4) of Lemma 5.1.4 and so we finish the proof of Lemma 5.1.4. □

For any $u \in \mathcal{N}$, by (1) of Lemma 5.1.4, we have that $t_u = 1$. Moreover, by the proof (1) of Lemma 5.1.4, we have the following corollary.

**Corollary 5.1.5.** *If $u \in \mathcal{N}$, then $\max_{t>0} I(tu) = I(u)$.*

## 5.2 Existence of a positive solution

**Lemma 5.2.1.** *Assume that the hypotheses $(H_l)$, $(H_{k_1})$, $(H_{h_1})$ and $(H_\mu)$ hold. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ be such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $I(u_n) \to d$, but any subsequence of $(u_n)_{n\in\mathbb{N}}$ does not converge strongly to $u$ in $H^1(\mathbb{R}^3)$. Then one of the following conclusions holds:*

**(1)** $d \geq \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ *if $u = 0$;*

**(2)** $d > I(t_u u)$ *if $u \neq 0$ and $\langle I'(u), u\rangle < 0$;*

**(3)** $d > \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ *if $u \neq 0$ and $\langle I'(u), u\rangle \geq 0$,*

*where $\bar{S}$ denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ defined by*

$$\bar{S} = \inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{1/3}}$$

*and $t_u$ is defined as in Lemma 5.1.4.*

**Proof.** We borrow an idea from Hirano-Shioji [56] to prove this lemma. From $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ we have that $u_n - u \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then by Lemma 1.4.6 and Lemma 5.1.2 (i) we arrive at

$$\int_{\mathbb{R}^3} h(x)|u_n - u|^2 dx \to 0 \text{ and } F(u_n - u) \to 0. \tag{5.11}$$

Going if necessary to a subsequence, we may assume that for some $a \geq 0$ and $b \geq 0$

$$\|u_n - u\|^2 \to a \text{ and } \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx \to b. \tag{5.12}$$

Since any subsequence of $(u_n)_{n\in\mathbb{N}}$ does not converge strongly to $u$ in $H^1(\mathbb{R}^3)$, one has $a \neq 0$. By the Brézis-Lieb lemma [22], (5.11) and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, we get that

$$d + o(1) = I(u_n) = I(u) + \frac{1}{2}\|u_n - u\|^2 - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx + o(1) \tag{5.13}$$

and

$$0 = \langle I'(u_n), u_n\rangle = \langle I'(u), u\rangle + \|u_n - u\|^2 - \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx + o(1). \tag{5.14}$$

Let

$$g(t) = I(tu),$$

$$\beta(t) = \frac{a}{2}t^2 - \frac{b}{6}t^6$$

and

$$\gamma(t) = g(t) + \beta(t).$$

It follows from (5.12) and (5.14) that $\gamma'(1) = g'(1) + \beta'(1) = 0$ and $t = 1$ is the only critical point of $\gamma(t)$ in $(0, +\infty)$, which implies that

$$\gamma \text{ achieves its maximum at } t = 1. \tag{5.15}$$

In the following, we will prove the three possibilities of the conclusions, respectively.

**(1)** If $u = 0$, then from (5.12) and (5.14) one deduces that

$$0 = \langle I'(u_n), u_n\rangle = \|u_n\|^2 - \int_{\mathbb{R}^3} k(x)|u_n|^6 dx + o(1) \to a - b,$$

which implies that $a = b$ and $b \neq 0$, since $a \neq 0$. Using the Sobolev inequality, we have that

$$\|u_n\|^2 \geq \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \bar{S}\left(\int_{\mathbb{R}^3} |u_n|^6 dx\right)^{\frac{1}{3}}$$

and then

$$\int_{\mathbb{R}^3} k(x)|u_n|^6 dx \leq \|k\|_\infty \int_{\mathbb{R}^3} |u_n|^6 dx \leq \|k\|_\infty \left(\bar{S}^{-1}\|u_n\|^2\right)^3,$$

i.e., $b \leq \|k\|_\infty \left(\bar{S}^{-1} b\right)^3$. Therefore, by the fact that $b \neq 0$, we obtain that

$$b \geq \bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \tag{5.16}$$

Thus, combining (5.12), (5.13), (5.16) with the assumption of $u = 0$, we get that

$$d = \frac{1}{2}\lim_{n\to\infty} \|u_n - u\|^2 - \frac{1}{6}\lim_{n\to\infty} \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx \geq \frac{1}{3}\bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}.$$

This proves the case (1).

**(2)** Now we consider the case that $u \neq 0$ and $\langle I'(u), u\rangle < 0$. In this case, by (5.14), we get that $a > b \geq 0$. Then we arrive at

$$\beta'(t) = at - bt^5 > bt(1 - t^4) \geq 0$$

for any $t \in (0, 1)$, which implies that $\beta$ strictly increases in $(0, 1)$ and then

$$\beta(t) > \beta(0) = 0 \text{ for any } t \in (0, 1). \tag{5.17}$$

By applying Lemma 5.1.4 to the $u$ with $\langle I'(u), u\rangle < 0$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $0 < t_u < 1$. Then by (5.17) we arrive at $\beta(t_u) > 0$. Therefore, combining (5.13) with (5.15), we get that

$$d = \gamma(1) > \gamma(t_u) = g(t_u) + \beta(t_u) > I(t_u u).$$

This proves the second statement.

**(3)** For the third case, we separate it into two steps. First, we consider that $u \neq 0$ and $\langle I'(u), u\rangle = 0$. Then from Corollary 5.1.5 and $(I_1)$ of Lemma 5.1.3 we obtain that

$$I(u) = \max_{t>0} I(tu) > 0. \tag{5.18}$$

By (5.14) and the same process as in the proof of (5.16), we can deduce that

$$a = b \geq \bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \tag{5.19}$$

Thus from (5.13), (5.18) and (5.19) we obtain that

$$d = \gamma(1) = I(u) + \frac{a}{2} - \frac{b}{6} > \frac{1}{3}\bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}.$$

Next, we prove the case that $u \neq 0$ and $\langle I'(u), u\rangle > 0$. In this case (5.14) implies that $b > a$. So $b > a > 0$. Since $\beta'(t) = at - bt^5$, we get $t_u^{**} = \left(\frac{a}{b}\right)^{\frac{1}{4}} < 1$ such that $\beta'(t_u^{**}) = 0$. Note that

$$\beta'(t) \geq 0 \text{ for any } \ t \in (0, t_u^{**}) \text{ and } \ \beta'(t) \leq 0 \text{ for any } \ t \in (t_u^{**}, \infty). \tag{5.20}$$

We get the maximum of $\beta$ as follows

$$\max_{t>0} \beta(t) = \beta(t_u^{**}) = \frac{a^{\frac{3}{2}}}{3b^{\frac{1}{2}}}. \tag{5.21}$$

It is now deduced from

$$\int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx \leq \|k\|_\infty \int_{\mathbb{R}^3} |u_n - u|^6 dx \leq \|k\|_\infty \left(\bar{S}^{-1}\|u_n - u\|^2\right)^3$$

that

$$\frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} \geq \bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \tag{5.22}$$

Inserting (5.22) into (5.21), we get that

$$\beta(t_u^{**}) = \max_{t>0} \beta(t) \geq \frac{1}{3}\bar{S}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \tag{5.23}$$

By Lemma 5.1.4 we know that $t_u > 1$. Hence

$$0 < t_u^{**} < 1 < t_u.$$

By the definition of $t_u$, we know that $I(t_u^{**}u) \geq 0$. Hence from (5.15) and (5.23) we obtain that

$$d = \gamma(1) > \gamma(t_u^{**}) = I(t_u^{**}u) + \beta(t_u^{**}) \geq \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}.$$

This proves the third statement. In sum, we finish the proof of this lemma. □

Next, we define the following minimization problem

$$c_1 = \inf_{u\in\mathcal{N}} I(u).$$

The following estimate to the minimum $c_1$ will be useful in what follows.

**Lemma 5.2.2.** *Suppose the hypotheses* $(H)$ *hold with* $1 < \beta < 3$. *Then*

$$c_1 < \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}.$$

**Proof.** The idea here is to find an element in $\mathcal{N}$ such that the value of the functional $I$ at this element is strictly less than $\frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. To construct this element, we need the extremal function $u_{\varepsilon,x_0}$ of the embedding $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, where

$$u_{\varepsilon,x_0} = C\frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x - x_0|^2)^{\frac{1}{2}}}$$

and $C$ is a normalizing constant and $x_0$ is given in $(H_{k_2})$. Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $0 \leq \varphi \leq 1, \varphi|_{B_{R_2}} \equiv 1$ and supp$\varphi \subset B_{2R_2}$ for some $R_2 > 0$. Set

$$v_\varepsilon = \varphi u_{\varepsilon,x_0}$$

and then $v_\varepsilon \in H^1(\mathbb{R}^3)$ with $v_\varepsilon(x) \geq 0$ for any $x \in \mathbb{R}^3$. The following asymptotic estimates hold for $\varepsilon$ small enough (see Brézis-Nirenberg [23]):

$$\|\nabla v_\varepsilon\|_2^2 = K_1 + O(\varepsilon^{\frac{1}{2}}), \quad \|v_\varepsilon\|_6^2 = K_2 + O(\varepsilon), \tag{5.24}$$

$$\|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{s}{4}}) & s \in [2,3), \\ O(\varepsilon^{\frac{s}{4}}|\ln \varepsilon|) & s = 3, \\ O(\varepsilon^{\frac{6-s}{4}}) & s \in (3,6), \end{cases} \tag{5.25}$$

with $\frac{K_1}{K_2} = \bar{S}$. For this $v_\varepsilon$, by Lemma 5.1.4, we know that there exists a unique $t_{v_\varepsilon} > 0$ such that $t_{v_\varepsilon}v_\varepsilon \in \mathcal{N}$. Thus $c_1 \leq I(t_{v_\varepsilon}v_\varepsilon)$. To prove $c_1 < \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$, it is enough to prove that

$$\max_{t>0} I(tv_\varepsilon) < \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}. \tag{5.26}$$

Since $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t>0} I(tv_\varepsilon)$. And by Lemma 5.1.3, $\max_{t>0} I(tv_\varepsilon) \geq \alpha_0 > 0$. Then we have that

$$I(t_\varepsilon v_\varepsilon) \geq \alpha_0 > 0.$$

Thus from the continuity of $I$, we may assume that there exists some positive $t_0$ such that $t_\varepsilon \geq t_0 > 0$. Moreover, from $I(tv_\varepsilon) \to -\infty$ as $t \to \infty$ and $I(t_\varepsilon v_\varepsilon) \geq \alpha_0 > 0$, we get that there exists $T_0$ such that $t_\varepsilon \leq T_0$. Hence $t_0 \leq t_\varepsilon \leq T_0$. Let

$$I(t_\varepsilon v_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon),$$

where

$$A(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx,$$

$$B(\varepsilon) = \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx - \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3} k(x)|v_\varepsilon|^6 dx,$$

and

$$C(\varepsilon) = \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \frac{t_\varepsilon^4}{4}F(v_\varepsilon) - \frac{t_\varepsilon^2\mu}{2}\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^2 dx.$$

First, we claim

$$A(\varepsilon) \leq \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + C\varepsilon^{\frac{1}{2}}. \tag{5.27}$$

Indeed, let

$$z(t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t^6}{6}\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx.$$

It is easy to see that $z(t)$ achieves its maximum at $T_\varepsilon$ with

$$T_\varepsilon = \left(\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx}\right)^{\frac{1}{4}}.$$

Therefore, from (5.24), we have that

$$z(T_\varepsilon) = \sup_{t\geq 0} z(t) = \frac{1}{3}\frac{\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx\right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^3} k(x_0)|v_\varepsilon|^6 dx\right)^{\frac{1}{2}}} = \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}). \tag{5.28}$$

This proves (5.27). Second, we claim that

$$B(\varepsilon) \leq C\varepsilon^{1/2}.$$

In fact, since $t_0 \leq t_\varepsilon \leq T_0$ and $k \in L^\infty(\mathbb{R}^3)$, by the definition of $v_\varepsilon$, $(H_{k_2})$ and using a change of variables, we obtain that for $\varepsilon$ small enough

$$\begin{aligned}
B(\varepsilon) &= \frac{t_\varepsilon^6}{6}\int_{\mathbb{R}^3}(k(x_0)-k(x))|v_\varepsilon|^6dx\\
&\le C\delta_1\int_{|x-x_0|<\rho_1}\frac{|x-x_0|^\alpha\varepsilon^{\frac{3}{2}}}{(\varepsilon+|x-x_0|^2)^3}dx + C\int_{|x-x_0|\ge\rho_1}\frac{\varepsilon^{\frac{3}{2}}}{(\varepsilon+|x-x_0|^2)^3}dx\\
&\le C\delta_1\varepsilon^{\frac{3}{2}}\int_0^{\rho_1}\frac{r^{2+\alpha}}{(\varepsilon+r^2)^3}dr + C\varepsilon^{\frac{3}{2}}\int_{\rho_1}^{\infty}r^{-4}dr\\
&= C\delta_1\varepsilon^{\frac{\alpha}{2}}\int_0^{\rho_1\varepsilon^{-\frac{1}{2}}}\frac{\rho^{2+\alpha}}{(1+\rho^2)^3}d\rho + C\rho_1^{-3}\varepsilon^{\frac{3}{2}}\\
&\le C\delta_1\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{\frac{3}{2}}\\
&\le C\varepsilon^{\frac{1}{2}}.
\end{aligned}\tag{5.29}$$

So we obtain $B(\varepsilon)\le C\varepsilon^{1/2}$. Therefore, to finish the proof, it suffices to show that

$$\lim_{\varepsilon\to 0^+}\frac{C(\varepsilon)}{\varepsilon^{1/2}} = -\infty. \tag{5.30}$$

Actually, from the definition of $v_\varepsilon$, $(H_{h_2})$ and for any $\varepsilon$ such that $0<\varepsilon\le\rho_2^2$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3}h(x)|v_\varepsilon|^2dx\\
&\ge C\delta_2\int_{|x-x_0|<\rho_2}\frac{|x-x_0|^{-\beta}\varepsilon^{\frac{1}{2}}}{\varepsilon+|x-x_0|^2}dx + \int_{|x-x_0|\ge\rho_2}h(x)|v_\varepsilon|^2dx\\
&\ge C\delta_2\varepsilon^{\frac{1}{2}}\int_0^{\rho_2}\frac{r^2}{r^\beta\,(\varepsilon+r^2)}dr\\
&= C\delta_2\varepsilon^{1-\frac{\beta}{2}}\int_0^{\rho_2\varepsilon^{-\frac{1}{2}}}\frac{\rho^2}{\rho^\beta\,(1+\rho^2)}d\rho\\
&\ge C\delta_2\varepsilon^{1-\frac{\beta}{2}}\int_0^1\frac{\rho^2}{2\rho^\beta}d\rho\\
&= C\varepsilon^{1-\frac{\beta}{2}}.
\end{aligned}\tag{5.31}$$

Thus, by the fact that $t_0\le t_\varepsilon\le T_0$ and hypothesis $(H_l)$, we have that

$$\begin{aligned}
C(\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2dx + \frac{t_\varepsilon^4}{4}F(v_\varepsilon) - \frac{t_\varepsilon^2\mu}{2}\int_{\mathbb{R}^3}h(x)|v_\varepsilon|^2dx\\
&\le C\|v_\varepsilon\|_2^2 + C\|v_\varepsilon\|_{12/5}^4 - \mu C\varepsilon^{1-\frac{\beta}{2}}\\
&\le C\varepsilon^{\frac{1}{2}} + C\varepsilon - \mu C\varepsilon^{1-\frac{\beta}{2}}.
\end{aligned}$$

It is deduced from $1<\beta<3$ that for fixed $\mu$ we have that

$$\frac{C(\varepsilon)}{\varepsilon^{1/2}} \le C + C\varepsilon^{\frac{1}{2}} - \mu C\varepsilon^{\frac{1}{2}-\frac{\beta}{2}} \to -\infty,\ \text{as } \varepsilon\to 0.$$

Hence (5.30) holds. Then (5.26) follows and the proof of Lemma 5.2.2 is complete. □

**Theorem 5.2.3.** *Suppose that the hypotheses* $(H)$ *hold with* $1<\beta<3$. *Then there exists* *a positive $\psi_1 \in \mathcal{N}$ such that $c_1 = I(\psi_1)$ and then $\psi_1$ is a positive critical point of the functional $I$ in $H^1(\mathbb{R}^3)$.*

**Proof.** By the definition of $c_1$, we may assume that there exists $(v_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ such that $I(v_n) \to c_1$ as $n \to \infty$. It is also known from Lemma 5.2.2 that

$$c_1 < \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}. \tag{5.32}$$

Since $(v_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, we have that

$$\|v_n\|^2 + F(v_n) - \mu\int_{\mathbb{R}^3} h(x)v_n^2 dx = \int_{\mathbb{R}^3} k(x)|v_n|^6 dx. \tag{5.33}$$

Using (5.33) and Lemma 5.1.2 (iii), we get that

$$\begin{aligned} c_1 + o(1) &= \frac{1}{2}\left(\|v_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)v_n^2 dx\right) + \frac{1}{4}F(v_n) - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|v_n|^6 dx \\ &= \frac{1}{N}\left(\|v_n\|^2 - \mu\int_{\mathbb{R}^3} h(x)v_n^2 dx\right) + \left(\frac{1}{4} - \frac{1}{6}\right)F(v_n) \\ &\geq \frac{1}{N}\left(1 - \frac{\mu}{\bar{\mu}}\right)\|v_n\|^2, \end{aligned}$$

which implies $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, since $0 < \mu < \bar{\mu}$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Suppose that any subsequence of $(v_n)_{n\in\mathbb{N}}$ does not converge strongly to $v$ in $H^1(\mathbb{R}^3)$ and then by Lemma 5.2.1 we obtain one of the following three cases:

(1) $c_1 \geq \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ if $v = 0$;

(2) $c_1 > I(t_v v)$ if $v \neq 0$ and $\langle I'(v), v\rangle < 0$;

(3) $c_1 > \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ if $v \neq 0$ and $\langle I'(v), v\rangle \geq 0$.

However, from (5.32) we know that the both cases (1) and (3) do not occur. Moreover, from the definition of $t_v$, we know that $t_v v \in \mathcal{N}$. So $I(t_v v) \geq c_1$, and then $c_1 > I(t_v v) \geq c_1$ by (2), which is a contradiction. Hence the second case (2) is also impossible to happen. Therefore there must exist some subsequence of $(v_n)_{n\in\mathbb{N}}$ converging strongly to $v$ in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $v_n \to v$ in $H^1(\mathbb{R}^3)$. By the Sobolev inequality, Lemma 5.1.2 (iii) and (5.33), we arrive at

$$\left(1 - \frac{\mu}{\bar{\mu}}\right)\|v_n\|^2 \leq \int_{\mathbb{R}^3} k(x)|v_n|^6 dx \leq \|k\|_\infty \left(\bar{S}^{-1}\|v_n\|^2\right)^3. \tag{5.34}$$

Since $(v_n)_{n\in\mathbb{N}} \subset \mathcal{N}$, $\|v_n\| \neq 0$ for any $n \in \mathbb{N}$ and hence by (5.34) we have $\|v_n\| \geq C$ for some positive $C$, which depends on the constants $\mu, \bar{\mu}, \|k\|_\infty$ and $\bar{S}$. Therefore $v \neq 0$ and then $v \in \mathcal{N}$ and $I(v) = c_1$. By Lagrange multiplier rule, there exists $\theta \in \mathbb{R}$ such that

$$I'(v) = \theta G'(v).$$

Then

$$0 = \langle I'(v), v\rangle = \theta \left( 2\|v\|^2 + 4F(v) - 6\int_{\mathbb{R}^3} k(x)|v|^6 dx - 2\mu \int_{\mathbb{R}^3} h(x)|v|^2 dx \right),$$

which implies that

$$\theta \left( -4 \left( \|v\|^2 - \mu \int_{\mathbb{R}^3} h(x)|v|^2 dx \right) - 2F(v) \right) = 0$$

since $v \in \mathcal{N}$. Then $\theta = 0$ and hence $v$ is a nontrivial critical point of the functional $I$ in $H^1(\mathbb{R}^3)$.

Note that if $(v_n)_{n\in\mathbb{N}} \subset \mathcal{N}$ and $I(v_n) \to c_1$ as $n \to \infty$, then $(|v_n|)_{n\in\mathbb{N}} \subset \mathcal{N}$ and $I(|v_n|) \to c_1$ as $n \to \infty$. Hence we may assume that $v \geq 0$ in $\mathbb{R}^3$. By standard arguments as in DiBenedetto [45] and Tolksdorf [97], we have that $v \in L^\infty(\mathbb{R}^3)$ and $v \in C^{1,\omega}_{loc}(\mathbb{R}^3)$ with $0 < \omega < 1$. Furthermore, by Harnack's inequality (see Trudinger [98]), $v(x) > 0$ for any $x \in \mathbb{R}^3$. Thus $v$ is a positive critical point of $I$ in $H^1(\mathbb{R}^3)$. We finish the proof of Theorem 5.2.3 by choosing $\psi_1 = v$. □

## 5.3 Existence of sign changing solutions

In this section, we will prove the existence of sign changing solutions of (5.1). A function $w$ is called sign changing if $w^+ \neq 0$ and $w^- \neq 0$, where $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Denote

$$\mathcal{N}_* = \{w = w^+ - w^- \in H^1(\mathbb{R}^3) \ : \ w^+ \in \mathcal{N}, w^- \in \mathcal{N}\}$$

and define

$$c_2 = \inf_{w\in\mathcal{N}_*} I(w).$$

We will prove that $c_2$ is achieved at some point $\psi_2$ and $\psi_2$ is a sign changing critical point of the functional $I$ in $H^1(\mathbb{R}^3)$.

**Lemma 5.3.1.** *Suppose that the hypotheses* $(H)$ *hold with* $\frac{3}{2} < \beta < 3$. *Then* $c_2 < c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$.

**Proof.** We are going to find an element in $\mathcal{N}_*$ such that the value of $I$ at this element is strictly less than $c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$. Let $\psi_1$ be the positive critical point of $I$ obtained in Theorem 5.2.3 and $v_\varepsilon$ be constructed in Lemma 5.2.2.

First, we claim that there exist $a_0 > 0$ and $b_0 \in \mathbb{R}$ such that

$$a_0\psi_1 + b_0 v_\varepsilon \in \mathcal{N}_*. \tag{5.35}$$

In fact, denote $\varphi(s) = \psi_1 + s v_\varepsilon$ with $s \in \mathbb{R}$, and define $s_1 \in [-\infty, +\infty)$ and $s_2 \in (-\infty, +\infty]$ by $s_1 = \inf\{s \in \mathbb{R} : \varphi^+(s) \neq 0\}$ and $s_2 = \sup\{s \in \mathbb{R} : \varphi^-(s) \neq 0\}$. We know $s_1 < s_2$, because $\varphi(s)$ is strictly increasing. Since $t(\varphi^+(s)) - t(\varphi^-(s)) \to +\infty$ as $s \to s_1 + 0$ and $t(\varphi^+(s)) - t(\varphi^-(s)) \to -\infty$ as $s \to s_2 - 0$ by (3) and (4) of Lemma 5.1.4, there exists $s_0 \in (s_1, s_2)$ such that $t(\varphi^+(s_0)) = t(\varphi^-(s_0))$ by (3) of Lemma 5.1.4. Thus

$$\begin{aligned} t(\varphi^+(s_0))\varphi(s_0) &= t(\varphi^+(s_0))(\varphi^+(s_0) - \varphi^-(s_0)) \\ &= t(\varphi^+(s_0))\varphi^+(s_0) - t(\varphi^-(s_0))\varphi^-(s_0) \in \mathcal{N}_*. \end{aligned}$$

By the definition of $t_u$, we have $t(\varphi^+(s)) > 0$, which implies that (5.35) holds.

Second, we claim that there is $\varepsilon > 0$ such that

$$\sup_{a>0, b\in\mathbb{R}} I(a\psi_1 + bv_\varepsilon) < c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}. \tag{5.36}$$

In fact, it follows from $(I_2)$ of Lemma 5.1.3 that for any $a > 0, b \in \mathbb{R}$ such that $\|a\psi_1 + bv_\varepsilon\| > \rho_*$ we have that

$$I(a\psi_1 + bv_\varepsilon) < 0.$$

Thus it suffices to consider the case that $\|a\psi_1 + bv_\varepsilon\| \leq \rho_*$, which means that it is sufficient to consider that $a$ and $b$ are contained in a bounded interval. Since $\psi_1$ is a solution of (5.2), it holds

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla(a\psi_1)\nabla(bv_\varepsilon) + (a\psi_1)(bv_\varepsilon) - \mu h(x)(a\psi_1)(bv_\varepsilon))\,dx \\ &= ab\left(\int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx - \int_{\mathbb{R}^3} l(x)\phi_{\psi_1}\psi_1 v_\varepsilon dx\right) \end{aligned} \tag{5.37}$$

Let

$$\begin{aligned} g_{1,\varepsilon}(b) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla(bv_\varepsilon)|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx, \\ g_{2,\varepsilon}(b) &= \frac{1}{6}\int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|bv_\varepsilon|^6 dx, \\ g_{3,\varepsilon}(a,b) &= \frac{1}{4}F(a\psi_1 + bv_\varepsilon) - \frac{1}{4}F(a\psi_1) \end{aligned}$$

and

$$g_{4,\varepsilon}(a,b) = \frac{1}{6}\int_{\mathbb{R}^3} k(x)\left(|a\psi_1|^6 + |bv_\varepsilon|^6 - |a\psi_1 + bv_\varepsilon|^6\right) dx.$$

It follows from Lemma 5.1.1 that

$$g_{4,\varepsilon}(a,b) \leq C\int_{\mathbb{R}^3} k(x)\left(|\psi_1|^5 v_\varepsilon + \psi_1|v_\varepsilon|^5\right) dx. \tag{5.38}$$

Using (5.37), (5.38) and Corollary 5.1.5, we deduce that

$$\begin{aligned} I(a\psi_1 + bv_\varepsilon) &= I(a\psi_1) + g_{1,\varepsilon}(b) + g_{2,\varepsilon}(b) + g_{3,\varepsilon}(a,b) + g_{4,\varepsilon}(a,b) \\ &+\frac{1}{2}\int_{\mathbb{R}^3} |bv_\varepsilon|^2 dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)|bv_\varepsilon|^2 dx \\ &+\int_{\mathbb{R}^3} \left(\nabla(a\psi_1)\nabla(bv_\varepsilon) + (a\psi_1)(bv_\varepsilon) - \mu h(x)(a\psi_1)(bv_\varepsilon)\right) dx \\ &\le I(\psi_1) + g_{1,\varepsilon}(b) + g_{2,\varepsilon}(b) + g_{3,\varepsilon}(a,b) + \frac{1}{2}\int_{\mathbb{R}^3} |bv_\varepsilon|^2 dx \\ &+C\int_{\mathbb{R}^3} k(x)|v_\varepsilon|^5\psi_1 dx + C\int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx \\ &+C\int_{\mathbb{R}^3} l(x)\phi_{\psi_1}\psi_1 v_\varepsilon dx - \frac{\mu}{2}\int_{\mathbb{R}^3} h(x)|bv_\varepsilon|^2 dx. \end{aligned} \tag{5.39}$$

By (5.28) we obtain that

$$\sup_{b\in\mathbb{R}} \; g_{1,\varepsilon}(b) \quad = \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + O(\varepsilon^{\frac{1}{2}}). \tag{5.40}$$

Since $b$ is bounded and $1 \le \alpha < 3$, we get from (5.29) that

$$g_{2,\varepsilon}(b) = \frac{1}{6}\int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx - \frac{1}{6}\int_{\mathbb{R}^3} k(x)|bv_\varepsilon|^6 dx \le C\varepsilon^{\frac{1}{2}}. \tag{5.41}$$

By (5.25) and the fact of $\psi_1 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we obtain that

$$\int_{\mathbb{R}^3} k(x)|v_\varepsilon|^5\psi_1 dx \le \|k\|_\infty\|\psi_1\|_\infty \int_{\mathbb{R}^3} |v_\varepsilon|^5 dx \le C\varepsilon^{\frac{1}{4}} \tag{5.42}$$

and

$$\int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx \le \|k\|_\infty\|\psi_1^5\|_\infty \int_{\mathbb{R}^3} v_\varepsilon dx \le C\varepsilon^{\frac{1}{4}}. \tag{5.43}$$

We claim that

$$g_{3,\varepsilon}(a,b) \le C\varepsilon^{\frac{1}{4}}. \tag{5.44}$$

Actually by calculation we arrive at

$$\begin{aligned} g_{3,\varepsilon}(a,b) \quad &= \frac{1}{4}\int_{\mathbb{R}^3} l(x)\left(\phi_{a\psi_1+bv_\varepsilon}(a\psi_1+bv_\varepsilon)^2 - \phi_{a\psi_1}(a\psi_1)^2\right) dx \\ &= ab\int_{\mathbb{R}^3} l(x)\phi_{a\psi_1}\psi_1 v_\varepsilon dx + ab\int_{\mathbb{R}^3} l(x)\phi_{bv_\varepsilon}\psi_1 v_\varepsilon dx \\ &+\frac{1}{2}b^2\int_{\mathbb{R}^3} l(x)\phi_{a\psi_1}(v_\varepsilon)^2 + \frac{1}{4}b^2\int_{\mathbb{R}^3} l(x)\phi_{bv_\varepsilon}(v_\varepsilon)^2 dx \\ &+a^2b^2\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{|x-y|} l(y)\psi_1(y)v_\varepsilon(y)l(x)\psi_1(x)v_\varepsilon(x)dxdy. \end{aligned} \tag{5.45}$$

Using the Hölder inequality, (1.10), (5.24), (5.25) and the fact that $a, b$ are bounded, we obtain that

$$\int_{\mathbb{R}^3} l(x)\phi_{a\psi_1}\psi_1 v_\varepsilon dx \le \|l\|_\infty\|\phi_{a\psi_1}\|_6\|\psi_1\|_{12/5}\|v_\varepsilon\|_{12/5} \le C\|v_\varepsilon\|_{12/5} \le C\varepsilon^{\frac{1}{4}}, \tag{5.46}$$

$$\int_{\mathbb{R}^3} l(x)\phi_{bv_\varepsilon}\psi_1 v_\varepsilon dx \le \|l\|_\infty \|\phi_{bv_\varepsilon}\|_6 \|\psi_1\|_{12/5}\|v_\varepsilon\|_{12/5} \le C\|v_\varepsilon\|_{12/5}^3 \le C\varepsilon^{\frac{3}{4}}, \tag{5.47}$$

$$\int_{\mathbb{R}^3} l(x)\phi_{bv_\varepsilon}(v_\varepsilon)^2 dx \le \|l\|_\infty \|\phi_{bv_\varepsilon}\|_6 \|v_\varepsilon\|_{12/5}^2 \le C\|v_\varepsilon\|_{12/5}^4 \le C\varepsilon \tag{5.48}$$

and

$$\int_{\mathbb{R}^3} l(x)\phi_{a\psi_1}(v_\varepsilon)^2 dx \le \|l\|_\infty \|\phi_{a\psi_1}\|_6 \|v_\varepsilon\|_{12/5}^2 \le C\varepsilon^{\frac{1}{2}}. \tag{5.49}$$

Moreover, by Lemma 4.1.1, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{l(y)\psi_1(y)v_\varepsilon(y)l(x)\psi_1(x)v_\varepsilon(x)}{|x-y|}dxdy \\
&\le \left(\int_{\mathbb{R}^3} |l(x)\psi_1(x)v_\varepsilon(x)|^{\frac{6}{5}}dx\right)^{\frac{5}{3}} \\
&\le C\|\psi_1\|_{\frac{12}{5}}^2 \|v_\varepsilon\|_{\frac{12}{5}}^2 \le C\varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{5.50}$$

It follows from (5.45)–(5.50) that the claim (5.44) holds. Hence combining (5.31) with (5.39)–(5.44), for $\frac{3}{2} < \beta < 3$, we obtain that

$$\begin{aligned}
I(a\psi_1 + bv_\varepsilon) &\le I(\psi_1) + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} - C\varepsilon^{1-\frac{\beta}{2}} + C\varepsilon^{\frac{1}{4}} \\
&< I(\psi_1) + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} \\
&= c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}},
\end{aligned} \tag{5.51}$$

as $\varepsilon \to 0$. Hence the claim (5.36) follows. Thus by (5.35) and (5.36) we deduce that

$$c_2 < c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}.$$

This proves Lemma 5.3.1. □

**Lemma 5.3.2.** *If the hypotheses (H) hold with* $\frac{3}{2} < \beta < 3$*, then there exists* $\psi_2$ *in* $\mathcal{N}_*$ *such that* $I(\psi_2) = c_2$.

**Proof.** From the definition of $c_2$ we may assume that there exists $(w_n)_{n\in\mathbb{N}} \subset \mathcal{N}_*$ such that $I(w_n) \to c_2$. And we may assume that there exist constants $d_1$ and $d_2$ such that $I(w_n^+) \to d_1$ and $I(w_n^-) \to d_2$ and $c_2 = d_1 + d_2$. By the definition of $c_1$, $w_n^+$ and $w_n^-$, it holds that

$$d_1 \ge c_1 \text{ and } d_2 \ge c_1. \tag{5.52}$$

Just as the proof of Theorem 5.2.3, there are positive constants $C_1, C_2, C_3$ and $C_4$ such that

$$C_1 \le \|w_n^+\| \le C_2 \text{ and } C_3 \le \|w_n^-\| \le C_4. \tag{5.53}$$

Going if necessary to a subsequence, we may assume that $w_n^+ \rightharpoonup w^+$ and $w_n^- \rightharpoonup w^-$. If

$w^+ = 0$ or $w^- = 0$, by (1) of Lemma 5.2.1 and (5.52), we obtain that

$$c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} \le d_1 + d_2 = c_2,$$

which contradicts Lemma 5.3.1. Hence we may assume that $w^+ \neq 0$ and $w^- \neq 0$. Using Lemma 5.2.1, we get one of the following:

$(I_1)$ there is a subsequence of $(w_n^+)_{n\in\mathbb{N}}$ converging strongly to $w^+$ in $H^1(\mathbb{R}^3)$;

$(I_2)$ $d_1 > I(t_{w^+}w^+)$ if $w^+ \neq 0$ and $\langle I'(w^+), w^+\rangle < 0$;

$(I_3)$ $d_1 > \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ if $u^+ \neq 0$ and $\langle I'(w^+), w^+\rangle \ge 0$;

and we also have one of the following:

$(II_1)$ there is a subsequence of $(w_n^-)_{n\in\mathbb{N}}$ converging strongly to $w^-$ in $H^1(\mathbb{R}^3)$;

$(II_2)$ $d_2 > I(t_{w^-}w^-)$ if $w^- \neq 0$ and $\langle I'(w^-), w^-\rangle < 0$;

$(II_3)$ $d_2 > \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}}$ if $w^- \neq 0$ and $\langle I'(w^-), w^-\rangle \ge 0$.

We claim that only $(I_1)$ and $(II_1)$ happen. In fact, if the pair $(I_1)$ and $(II_2)$ or the pair $(I_2)$ and $(II_2)$ holds, then from $w^+ - t_{w^-}w^- \in \mathcal{N}_*$ or $t_{w^+}w^+ - t_{w^-}w^- \in \mathcal{N}_*$ respectively, we arrive at respectively

$$c_2 \le I(w^+ - t_{w^-}w^-) = I(w^+) + I(t_{w^-}w^-) < d_1 + d_2 = c_2$$

or

$$c_2 \le I(t_{w^+}w^+ - t_{w^-}w^-) = I(t_{w^+}w^+) + I(t_{w^-}w^-) < d_1 + d_2 = c_2.$$

Any one of the above two inequalities is not true. If the pair $(I_1)$ and $(II_3)$ or the pair $(I_2)$ and $(II_3)$, or the pair $(I_3)$ and $(II_3)$ occurs, then by Lemma 5.2.2 we obtain the following three possibilities:

$$c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} \le I(w^+) + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} < d_1 + d_2 = c_2;$$

$$c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} \le I(t_{w^+}w^+) + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} < d_1 + d_2 = c_2;$$

$$c_1 + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} \le \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} + \frac{1}{3}\bar{S}^{\frac{3}{2}}\|k\|_\infty^{-\frac{1}{2}} < d_1 + d_2 = c_2,$$

while any one of the above three possibilities contradicts the conclusions of Lemma 5.3.1. Hence all the pairs $(I_1)$ and $(II_2)$, $(I_2)$ and $(II_2)$, $(I_1)$ and $(II_3)$, $(I_2)$ and $(II_3)$, $(I_3)$ and $(II_3)$ do not occur. The pairs $(I_2)$ and $(II_1)$, $(I_3)$ and $(II_1)$, $(I_3)$ and $(II_2)$ also do not happen by a similar proof. Since we have considered all the cases, we know that only the pair $(I_1)$ and $(II_1)$ holds. We may assume that $w_n^+ \to w^+$ strongly in $H^1(\mathbb{R}^3)$ and

$w_n^- \to w^-$ strongly in $H^1(\mathbb{R}^3)$. From these and (5.53) we have that $w^+ \neq 0$ and $w^- \neq 0$. Set $\psi_2 = w^+ - w^-$. Then $\psi_2 \in \mathcal{N}_*$ and $I(\psi_2) = d_1 + d_2 = c_2$. This proves Lemma 5.3.2. □

According to Lemma 5.3.2 we know that $\psi_2 \in \mathcal{N}_*$ is sign changing and $I(\psi_2) = c_2$. Since $\mathcal{N}_*$ usually is not a manifold, Lagrange multiplier rule may not be applied. In order to show that $\psi_2$ is a critical point of the functional $I$ in $H^1(\mathbb{R}^3)$, i.e., $I'(\psi_2) = 0$, we need an idea from Castro-Cossio-Neuberger [24] and Hirano-Shioji [56].

**Theorem 5.3.3.** *If the hypotheses* $(H)$ *hold with* $\frac{3}{2} < \beta < 3$*, then* $\psi_2$ *is a sign changing critical point of the functional* $I$ *in* $H^1(\mathbb{R}^3)$*.*

**Proof.** Suppose that $\psi_2$ is not a critical point of $I$, i.e., $I'(\psi_2) \neq 0$. For any $u \in \mathcal{N}$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^3} k(x)|u|^6 dx &= \|u\|^2 + F(u) - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \\
&\geq \left(1 - \frac{\mu}{\bar{\mu}}\right) \|u\|^2 + F(u) \\
&\geq F(u)
\end{aligned}$$

and then

$$\begin{aligned}
\langle G'(u), u\rangle &= 2\left(\|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx\right) + 4F(u) - 6\int_{\mathbb{R}^3} k(x)|u|^6 dx \\
&= -4\int_{\mathbb{R}^3} k(x)|u|^6 dx + 2F(u) \\
&\leq -2F(u) < 0.
\end{aligned} \tag{5.54}$$

Therefore, for any $u \in \mathcal{N}$, $\|G'(u)\|_{H^{-1}} = \sup_{\|v\|=1} |\langle G'(u), v\rangle| \neq 0$. Set

$$\Phi(u) = I'(u) - \left\langle I'(u), \frac{G'(u)}{\|G'(u)\|}\right\rangle \frac{G'(u)}{\|G'(u)\|}, \quad u \in \mathcal{N} \tag{5.55}$$

Then we get that $\Phi(\psi_2) \neq 0$. In fact, if $\Phi(\psi_2) = 0$, then, by (5.54) and (5.55), it holds that

$$0 = \langle I'(\psi_2), \psi_2\rangle = \left\langle I'(\psi_2), \frac{G'(\psi_2)}{\|G'(\psi_2)\|}\right\rangle \left\langle \frac{G'(\psi_2)}{\|G'(\psi_2)\|}, \psi_2\right\rangle \neq 0,$$

which is a contradiction. Let $\delta \in (0, \min\{\|\psi_2^+\|, \|\psi_2^-\|\}/3)$ such that

$$\|\Phi(v) - \Phi(\psi_2)\| \leq \frac{1}{2}\|\Phi(\psi_2)\| \text{ for each } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \leq 2\delta.$$

Let $\chi : \mathcal{N} \to [0, 1]$ be a Lipschitz mapping such that

$$\chi(v) = \begin{cases} 1, & \text{for } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \leq \delta, \\ 0, & \text{for } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \geq 2\delta. \end{cases}$$

Let $\eta : [0, s_0] \times \mathcal{N} \to \mathcal{N}$ be the solution of the differential equation

$$\eta(0,v)=v, \quad \frac{d\eta(s,v)}{ds}=-\chi(\eta(s,v))\Phi(\eta(s,v)), \quad \text{for } (s,v)\in[0,s_0]\times\mathcal{N},$$

where $s_0$ is a positive number. We set

$$r(\tau)=t((1-\tau)\psi_2^+ +\tau\psi_2^-)((1-\tau)\psi_2^+ +\tau\psi_2^-)$$

and $\sigma(\tau)=\eta(s_0,r(\tau))$, for $0\le\tau\le1$. If $\tau\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$, we have

$$I(\sigma(\tau))\le I(r(\tau))=I(r^+(\tau))+I(r^-(\tau))<I(\psi_2^+)+I(\psi_2^-)=I(\psi_2),$$

and $I(\sigma(\frac{1}{2}))=I(r(\frac{1}{2}))<I(\psi_2)$, i.e., $I(\sigma(\tau))<I(\psi_2)$ for all $\tau\in(0,1)$. Since $t(\sigma^+(\tau))-t(\sigma^-(\tau))\to-\infty$ as $\tau\to0+0$ and $t(\sigma^+(\tau))-t(\sigma^-(\tau))\to+\infty$ as $\tau\to1-0$, there exists $\tau_1\in(0,1)$ such that $t(\sigma^+(\tau))=t(\sigma^-(\tau))$. Then we have $\sigma(\tau_1)\in\mathcal{N}_*$ and $I(\sigma(\tau_1))<I(\psi_2)$, which is a contradiction. This proves Theorem 4.3. □

*Remark 2.* Theorem 5.2.3 means that $(\psi_1,\phi_{\psi_1})$ is a positive solution of system (5.1) and Theorem 5.3.3 means that $(\psi_2,\phi_{\psi_2})$ is a sign changing solution of system (5.1). Furthermore, if $(\psi,\phi_\psi)$ is the solution of system (5.1), then $(-\psi,\phi_{-\psi})$ is also its solution. Hence, by Theorem 5.2.3 and Theorem 5.3.3, we know that system (5.1) has at least one pair of fixed sign solutions and at least one pair of sign changing solutions under the hypotheses $(H)$ with $\frac{3}{2}<\beta<3$, respectively. □

# Chapter 6

# Some considerations and future research

In this last chapter, we will present some final comments about problems under study and we will give some directions of future research.

## 6.1 Some considerations

Once one reads this thesis, maybe the most common question he or she raises is that it is possible to extend the operator Laplace $\Delta u$ in system $(\mathcal{SP})$ to the more general $p$-Laplace $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$. To our best knowledge, for now we have not seen any results extending the operator Laplace in all kinds of the Schrödinger-Poisson systems to more general $p$-Laplace, although there are large amount of works related to these systems in recent years. There are three possible reasons to explain this situation according to our understanding.

We think the main reason is based on the physical meaning that the Schrödinger-Poisson systems arise in an interesting physical context as we mention in the introduction of this thesis unless extending the operator is theoretical or mathematical needs. The second reason, in our opinion, is that for the Schrödinger-Poisson systems there are still a lot of work to be done for the time being. And from the point of theoretical view the final reason, why we have seen nothing about the extending, is that one needs to find some new methods, at least some different ways from the methods used until now. So it is not only a big challenge in some sense, but also an interesting thing to think and try.

## 6.2 Some directions of future research

The problems studied in this work involve interesting research points. Moreover, we introduce some methods originally, which can be applied to many other problems. We believe there are many remains to be done. Here we will give some possible research directions that can use our methods, or which turn to be some kind of generalization of already obtained results, situations without being considered.

**To apply our methods to other cases**

- In Chapter 2 and Chapter 3, we introduce a method to improve the previous results in the literature with indefinite nonlinearities. We find an interesting phenomenon in Chapter 2 or Chapter 3 that we do not need the condition $\int_{\mathbb{R}^3} k(x)e_1^p dx < 0$ with an indefinite non-corercive case, which has been shown to be a sufficient condition to the existence of positive solutions for semilinear elliptic equations with indefinite nonlinearities (see e.g. Alama and Tarantello [1], Costa and Tehrani [36]), where $e_1$ is the first eigenfunction of $-\Delta + id$ in $H^1(\mathbb{R}^3)$ with weight function $h$. Moreover, the process used in this case can be applied to study the other aspects of the Schrödinger-Poisson systems and it also gives a way to study the Kirchhoff systems and quasilinear Schrödinger systems.

- In Chapter 5, to get sign changing solutions, we follow the spirit of Hirano and Shioji [56], but the procedure is simpler than that in that paper [56]. One can also apply this simpler method to find sign changing solutions of other problems.

**To extend the left side of system** $(\mathcal{SP})$

- To extend the system $(\mathcal{SP})$ to semiclassical case. Let us give the following example.

$$\begin{cases} -\epsilon^2 \Delta u + u + l(x)\phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x) u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{6.1}$$

  where $\epsilon \to 0$. There are already many results about the similar system to (6.1) with $g(x,u) = |u|^{p-2}u$, such as [3, 37, 90, 105] and their references therein. But so far we have not seen any information on our cases.

- To extend system $(\mathcal{SP})$ to a case with steep potential well

$$\begin{cases} -\Delta u + a_\lambda(x) u + l(x)\phi u = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x) u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{6.2}$$

  where $a_\lambda(x) = 1 + \lambda f(x)$, $\lambda$ is a positive parameter, and $f$ satisfies

  $(f_1)$ $f(x) \geq 0$, $f \in L^\infty(\mathbb{R}^3)$;

  $(f_2)$ $\{x \in \mathbb{R}^3 : f(x) = 0\}$ is bounded and has nonempty interior;

  $(f_3)$ $\liminf_{|x|\to\infty} f(x) = c$, where $c$ is a positive constant.

  The above conditions imply that $a_\lambda$ represents a potential well whose depth is controlled by $\lambda$. $a_\lambda$ is called a steep potential well if $\lambda$ large, see Jiang-Zhou [66], where the authors study the case that $g(x,u) = |u|^{p-2}u$.

- To extend the usual Laplace operator $\Delta$ to a second order operator with a weight function, for example, may we replace the $\Delta u$ by $div(A(x)\nabla u)$ with suitable conditions on $A(x)$? We believe that this case will be very interesting because for the usual Schrödinger equation, the study of $-div(A(x)\nabla u) + V(x)u = f(x,u)$ is an interesting problem and has been studied by many mathematicians. For system $(\mathcal{SP})$ with $\Delta u$ replaced by $div(A(x)\nabla u)$, we do not know if such kind of system arises from what kind of physical phenomena. However, from the pure mathematical point of view, this problem will be quite interesting since in this case it is not easy to establish the relation between the Poisson term $\int_{\mathbb{R}^3} \phi_u u^2 dx$ and $\int_{\mathbb{R}^3} A(x)|\nabla u|^2 dx$. We will think these problems.

- Note that for system $(\mathcal{SP})$, a new phenomenon is the appear of Poisson term

$$\int_{\mathbb{R}^3} \phi_u u^2 dx = \int_{\mathbb{R}^3} u^2(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy dx = \int_{\mathbb{R}^3} u^2(x) \left(u^2 * |x|^{-1}\right) dx.$$

  A question is that: can we replace the $|x|^{-1}$ by a general symmetric function? Can we study the existence and multiplicity of positive solutions for the system in this case?

- For system $(\mathcal{SP})$, the second equation can be solved explicitly, and the system can be reduced to a single equation with a nonlocal Poisson term. A natural question is that if the second equation can not be solved, what will happen? For example, if the second equation is $-\Delta\phi + l_1(x)\phi = u^2$, what can we do by using the variational method? We believe that these will be also good questions for further studies.

**To extend the right side of system** $(\mathcal{SP})$

- To extend system $(\mathcal{SP})$ to a positive potential. In this case, one may not get Palais-Smale condition at any level. An interesting question is to study, for which level, the Palais-Smale conditions will hold. Another interesting question is to study the role played by the Poisson term to the existence and multiplicity of solutions, as well as under what conditions the system may not have positive solutions.

- To loose the conditions of system $(\mathcal{SP})$. In Chapter 3, we give an improvement to the homogenous nonlinearity in Chapter 2, namely, we assume that the nonlinear function $g(x,u)$ has the form $a(x)g(u)$ and $g(u)$ satisfies the conditions

$$\lim_{s\to 0} \frac{g(s)}{|s|^p} = 1$$

  and

$$\lim_{s\to\infty} \frac{g(s)}{|s|^q} = 1.$$

  An interesting question is that: can we get similar existence and multiplicity results if one of the following conditions holds?

1. $\lim_{s\to 0} \frac{g(s)}{|s|^p} = 0$ and $\lim_{s\to\infty} \frac{g(s)}{|s|^q} = 0$.
2. $\lim_{s\to 0} \frac{g(s)}{|s|^p} = \infty$ and $\lim_{s\to\infty} \frac{g(s)}{|s|^q} = 0$.
3. $\lim_{s\to 0} \frac{g(s)}{|s|^p} = 0$ and $\lim_{s\to\infty} \frac{g(s)}{|s|^q} = \infty$.
4. $\lim_{s\to 0} \frac{g(s)}{|s|^p} = \infty$ and $\lim_{s\to\infty} \frac{g(s)}{|s|^q} = \infty$.
5. $\lim_{s\to 0} \frac{g(s)}{|s|^p} = 0$ and $\lim_{s\to\infty} \frac{g(s)}{|s|^{2^*}} = 0$.

Another interesting question is that: can we get similar results for very general $g(x, u)$?

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