Universidade de Aveiro Departamento de Matemática,
2013

Mahdi Doostmohammadi

Estudo poliédrico de modelos de programação inteira mista que ocorrem em problemas de lot-sizing

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Tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações (PDMA 2009–2013), da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação cientifica do Professor Agostinho Miguel Mendes Agra, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro.

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This thesis is dedicated to my parents and my wife, who have given so much to make this possible, and who were willing to give so much more.

o júri

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palavras-chave Programação inteira mista, Problemas de gestão de stocks, Teoria poliédrica, Desigualdaes Válidas, Desigualdades que são facetes, Invólucro convexo, Problema de separação, Formulações estendidas, Experiências computacionais.

resumo

O algoritmo "branch-and-cut" é um dos métodos exatos mais eficientes para resolver problemas de programação inteira mista. Este algoritmo combina as vantagens do algoritmo branch-and-bound com o método de planos de corte. O algoritmo branch-and-cut recorre ao cálculo da relaxação linear em cada nó da árvore de pesquisa, a qual é melhorada com a utilização de cortes, isto é, com a inclusão de desigualdades válidas. Deve-se ter em conta que a escolha dos cortes mais fortes é crucial para a sua utilização efetiva no algoritmo branch-and-cut.

Esta tese centra-se na obtenção de desigualdades válidas e sua utilização como planos de corte para resolver problemas gerais de programação inteira mista, em particular, problemas que combinam a gestão de stocks com outros problemas, tais como: a distribuição, selecção de fornecedores, e determinação de rotas de veículos, etc. Para alcançar este objetivo, são consideradas, em primeiro lugar, subestruturas, isto é, modelos de programação inteira mista que definem conjuntos de soluções admissíveis resultantes de relaxações desses problemas gerais. A estrutura poliédrica desses modelos é estudada de modo a serem obtidas novas famílias de desigualdades válidas. Finalmente, essas desigualdades são incluídas em algoritmos de planos de corte para resolver os problemas gerais de programação inteira mista.

Nesta dissertação estudamos três modelos de programação inteira mista. Os dois primeiros modelos surgem como relaxações de problemas gerais tais como: dimensionamento de lotes com seleção de fornecedores, desenho de redes, e problemas que combinam a produção com a distribuição. Esses conjuntos constituem variantes do conhecido single node fixed-charge network set, onde uma variável binária ou inteira está associada a cada nó. O terceiro modelo ocorre como relaxação de problemas de programação inteira mista onde são consideradas incompatibilidades entre pares de variáveis binárias. Para os três modelos são geradas famílias de desigualdades válidas, são identificadas classes de desigualdades que definem facetas, e são discutidos os problemas de separação associados a essas desigualdades. Em seguida, essas desigualdades são utilizadas em algoritmos de planos de corte. É apresentada uma experiência computacional preliminar.

keywords

Mixed integer programming, Inventory problems, Polyhedral theory, Valid inequality, Facet-defining inequality, Convex hull, Separation problem, Extended formulation, Computational experiment.

abstract

"Branch-and-cut" algorithm is one of the most efficient exact approaches to solve mixed integer programs. This algorithm combines the advantages of a pure branch-and-bound approach and cutting planes scheme. Branch-and-cut algorithm computes the linear programming relaxation of the problem at each node of the search tree which is improved by the use of cuts, i.e. by the inclusion of valid inequalities. It should be taken into account that selection of strongest cuts is crucial for their effective use in branch-and-cut algorithm.

In this thesis, we focus on the derivation and use of cutting planes to solve general mixed integer problems, and in particular inventory problems combined with other problems such as distribution, supplier selection, vehicle routing, etc. In order to achieve this goal, we first consider substructures (relaxations) of such problems which are obtained by the coherent loss of information. The polyhedral structure of those simpler mixed integer sets is studied to derive strong valid inequalities. Finally those strong inequalities are included in the cutting plane algorithms to solve the general mixed integer problems.

We study three mixed integer sets in this dissertation. The first two mixed integer sets arise as a subproblem of the lot-sizing with supplier selection, the network design and the vendor-managed inventory routing problems. These sets are variants of the well-known single node fixed-charge network set where a binary or integer variable is associated with the node. The third set occurs as a subproblem of mixed integer sets where incompatibility between binary variables is considered. We generate families of valid inequalities for those sets, identify classes of facet-defining inequalities, and discuss the separation problems associated with the inequalities. Then cutting plane frameworks are implemented to solve some mixed integer programs. Preliminary computational experiments are presented in this direction.

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Chapter 1

Introduction

Many problems in science, technology, and business can be formulated as linear mixed integer programming problems. Although there is no unique method to solve all integer programming problems, the construction of strong formulations, based on cutting planes, has received a significant attention in solving many problems in this field over the last decades. The branch-and-cut method is based on the inclusion of cutting planes (cuts) in the branch-and-bound algorithm, and it is now considered as one of the main and successful tools to solve mixed integer programs.

A class of mixed integer programs which has interested researchers is inventory problems [24,42,48]. An inventory is a stock of goods which is held or stored for the purpose of future sale or production. An inventory problem is faced by a firm that must decide how much to produce in each time period in order to satisfy demand for its products. For instance, the problem of deciding how many spare parts to keep on hand for a given machine is of this type.

During the early studies of mixed integer programs, those inventory problems which dealt only with inventories had been investigated. Recently, more challenging inventory problems are considered and studied by researchers and practitioners. Complex inventory problems are those problems where inventory decisions are integrated with distribution, supplier selection, vehicle routing, etc. This class covers a very broad family of real problems with a wide range of applications (see [1, 7, 12, 15]).

This thesis primarily concerns the derivation of cutting planes and generating stronger formulations for complex inventory problems. To achieve this objective, we study the polyhedral structure of new mixed integer sets resulting from relaxation of the complex inventory problem. Then we derive valid inequalities for those substructures which generates valid inequalities for the main problem. Next, the inclusion of these cuts in the branch-and-bound framework is implemented to solve the given problem.

In this chapter, we give a brief overview of mixed integer programming and polyhedral theory. Our discussion provides a sufficient background for the reader less familiar with mixed integer programming. The reader more familiar with mixed integer programming and cutting plane theory may wish to skip ahead to the final section where we describe our contributions and outline the remainder of the thesis.

1.1 Mixed Integer Programming

Combinatorial optimization is to find the *optimal solutions* out of the finite set of *feasible solutions*. This can be formulated as

$$\begin{aligned} \min \quad & f(y) \\ s.t. \quad & y \in X, \end{aligned}$$

where X is the set of feasible solutions and f is a function associating a cost $f(y)$ (or quality measure when maximizing) to each feasible solutions y. The set X is called the feasible set and f is called the *objective function*. In the foregoing problem we want to find the solution(s) y for which the cost $f(y)$ is minimum, among all the feasible solutions $y \in X$. This optimization problem is called combinatorial if X is finite.

Notice that in interesting combinatorial optimization problems, f and X are given in a structured or implicit way. All problems which will be considered in this dissertation are *Mixed Integer Programs*, or MIPs for short, which can be defined as follows. Suppose that we have a Linear Program (LP)

$$\min \left\{cx : Ax \leq b, x \geq 0\right\},$$

where A is an m by n matrix, c is an n-dimensional row vector, b is an m-dimensional column vector, and x is an n-dimensional column vector of variables. Now we add the restriction that some variables must take integer values. Thus, we have the following cases.

If all variables are integer, then we have (linear) *Integer Program*, IP for short, written as

$$\begin{aligned} z = \min\ & cx \\ s.t.\ & Ax \leq b, \\ & x \geq 0 \ and\ integer. \end{aligned}$$

If some but not all variables are integer, we have a (linear) MIP, written as

$$\begin{aligned} \min\ & cx + hy \\ s.t.\ & Ax + By \leq b, \\ & x \geq 0 \ and\ integer,\ y \geq 0, \end{aligned}$$

where B is m by p matrix, h is a p row-vector, x ia a n-dimensional column vector of integer variables, and y is a p-dimensional column vector.

If all variables are restricted to be $0-1$ values, we have a *0-1 or Binary Integer Program* which is denoted by BIP and written as

$$\begin{aligned} \min\ & cx \\ s.t.\ & Ax \leq b, \\ & x \in \{0,1\}^n. \end{aligned}$$

Similarly, we define *Mixed Binary Integer Program*, denoted by MBIP, where some decision variables are binary, and other decision variables are either integer or continuous valued.

Definition 1.1.1. *Given the MIP, set*

$$X = \Big\{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + By \leq b\Big\},$$

is the set of feasible solutions.

Some mixed integer programs are listed here.

- The network design problem in which we have to decide on arcs to open in a network to allow a certain flow to pass through the network while minimizing the cost of opening the arcs at the same time (see [26]).
- The production planning problem that deals with decisions about the size of the production lots of the different products to manufacture or to process, about the time at which those lots have to be produced, and sometimes about the machine or production facility where the production must take place. In such problem, the financial objectives are usually represented by production costs, set-up costs, and inventory costs (see [40]).
- The inventory routing problem which is concerned with the coordination of the inventory management of the stock levels of a set of products with the distribution of those products by a fleet of vehicles (see [15]).
- Mixed integer programming also appears in airline crew scheduling problem (see [25]), train scheduling problem (see [14]), and telecommunications (see [13]).

1.2 Basics on Polyhedral Theory

In this section we provide a quick and oriented introduction to the field of polyhedral theory. Schrijver [41], Nemhauser and Wolsey [36], Wolsey [45], and Pochet and Wolsey [40] are comprehensive references which have been used in this chapter. The interested reader will find in those books more general and complete treatment of this topic.

In the first part we express the concept of *polyhedron.*

Definition 1.2.1. *A subset of* $\mathbb{R}^n$ *described by a finite set of linear inequalities* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *is a polyhedron.*

Definition 1.2.2. *A polyhedron* $P \subseteq \mathbb{R}^n$ *is* bounded *if there exists a scalar* $\omega \in \mathbb{R}_+$ *such that*

$$P \subseteq \Big\{x \in \mathbb{R}^n : -\omega \leq x_i \leq \omega \ for \ i \in 1, \ldots, n\Big\}.$$

A bounded polyhedron is called a polytope.

Now we state the concept of *convex hull* as follows.

Definition 1.2.3. *A point $x \in \mathbb{R}^n$ is said to be a* convex combination *of the points $x^1, \ldots, x^T$ if there exists vector $\lambda \in \mathbb{R}^T_+$, with property $\sum_{i=1}^T \lambda_i = 1$ and $x = \sum_{i=1}^T \lambda_i x^i$. If $X \subseteq \mathbb{R}^n$, the convex hull of X, denoted by $\mathrm{conv}(X)$, is the set of all points $x \in \mathbb{R}^n$ that are convex combination of points in X. In other words*

$$conv(X) = \Big\{ x \in \mathbb{R}^n : \textit{there exist } T \in \mathbb{Z}_+, x^1, \ldots, x^T \in X \textit{ and } \lambda_1, \ldots, \lambda_T \geq 0,$$
$$\textit{such that } x = \sum_{i=1}^T \lambda_i x^i, \ \sum_{i=1}^T \lambda_i = 1 \Big\}.$$

In fact, $\mathrm{conv}(X)$ is the smallest polyhedron (inclusionwise) containing X.

Example 1.2.4. *Let*

$$X = \Big\{ (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (4,2), (4,3) \Big\}.$$

The convex of hull of set X is represented by the shaded area in Figure 1.1.

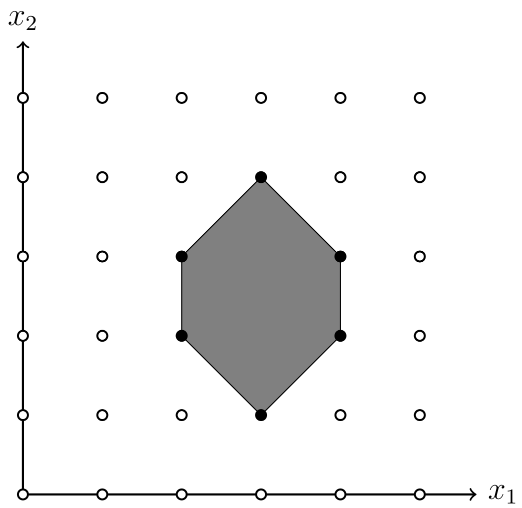

Figure 1.1: The convex hull of X.

The following proposition expresses the importance of the convex hull.

Proposition 1.2.5. *[40] Let $X \subseteq \mathbb{R}^n$ and $c \in \mathbb{R}^n$ and assume the problem min$\{cx : x \in X\}$ has an optimal solution. Then*

$$\min \Big\{ cx : x \in X \Big\} = \min \Big\{ cx : x \in conv(X) \Big\}.$$

The foregoing proposition states that in order to optimize a linear function over the set X it suffices to optimize it over $\mathrm{conv}(X)$. Therefore, the objective of many studies in polyhedral theory, as in this work, is to find classes of linear inequalities which describe partially or completely the convex hull of mixed integer sets.

1.2.1 Describing Polyhedra by Facets

We define the concept of *valid inequality* as follows.

Definition 1.2.6. *Let $X \subseteq \mathbb{R}^n$. A linear inequality $\pi x \leq \pi_0$ with $(\pi, \pi_0) \in \mathbb{R}^n \times \mathbb{R}$ is said a valid inequality for X if it is satisfied by all points in X, that is, if $\pi x \leq \pi_0$ for all $x \in X$.*

Observe that $\pi x \leq \pi_0$ is valid for X if and only if it is valid for $\mathrm{conv}(X)$.

Definition 1.2.7. *An inequality $\pi x \leq \pi_0$ is violated by the point x^* if $\pi x^* > \pi_0$.*

The dominance of inequalities is defined as follows.

Definition 1.2.8. *If $\pi x \leq \pi_0$ and $\mu x \leq \mu_0$ are two valid inequalities for polyhedron $P \subseteq \mathbb{R}^n_+$, then $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ if there exists $u > 0$ such that $\pi \geq u\mu$ and $\pi_0 \leq u\mu_0$, and $(\pi, \pi_0) \neq (u\mu, u\mu_0)$.*

According to the above definition, in Figure 1.2 inequality $\pi x \leq \pi_0$ dominates $\mu x \leq \mu_0$ (or $\mu x \leq \mu_0$ is dominated by $\pi x \leq \pi_0$).

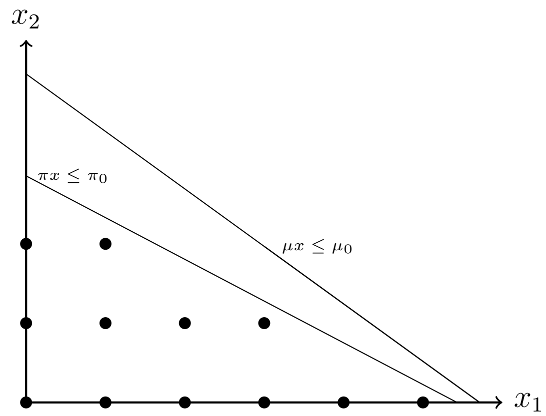

Figure 1.2: Dominance of inequalities.

Definition 1.2.9. *A valid inequality $\pi x \leq \pi_0$ is redundant in the description of polyhedron P if there exists a linear combination of the inequalities in the description that dominates inequality $\pi x \leq \pi_0$.*

Note that we can drop redundant inequalities from description of a polyhedron.

Definition 1.2.10. *A family of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is* linearly independent *if the system of linear equations $\sum_{i=1}^k \lambda_i x^i = 0$ has the unique solution $\lambda_i = 0$ for all $i = 1, \ldots, k$.*

Definition 1.2.11. *A family $x^0, \ldots, x^k$ of $k+1$ points in $\mathbb{R}^n$ is* affinely independent *if the system of linear equations $\sum_{i=0}^k \lambda_i x^i = 0, \sum_{i=0}^k \lambda_i = 0$ has the unique solution $\lambda_i = 0$ for all $i = 0, \ldots, k$, or equivalently if the family of directions $x^1 - x^0, \ldots, x^k - x^0$ in $\mathbb{R}^n$ is linearly independent.*

Dimension of a polyhedron can be defined in the following way.

Definition 1.2.12. *A polyhedron $P \subseteq \mathbb{R}^n$ is of dimension k, denoted by $dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$.*

Definition 1.2.13. *A polyhedron $P \subseteq \mathbb{R}^n$ is full-dimensional if $dim(p) = n$.*

Example 1.2.14. (continued) *$dim(conv(X)) = 2$ because $(2, 2), (2, 3)$ and $(3, 2)$ are affinely independent. So the polyhedron conv(X) is full-dimensional.*

Definition 1.2.15. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron and $\pi x \leq \pi_0$ be a valid inequality for P. The* face *of P induced by $\pi x \leq \pi_0$ is the set of points $F = \{x \in P : \pi x = \pi_0\}$. Notice that F is a polyhedron as well. A face F of P is said to be* proper *if $F \neq \emptyset$ and $F \neq P$. If $dim(F) = 0$, then F contains only one point. If $dim(F) = dim(P) - 1$, then F is called a* facet *of P. In this case we say that the valid inequality $\pi x \leq \pi_0$ is facet-defining for P.*

Example 1.2.16. (continued) *Inequality $2x_1 + x_2 \leq 11$ is valid for X and inequality $x_1 + x_2 \leq 7$ is facet-defining for the polyhedron conv(X).*

To show that a valid inequality $\pi x \leq \pi_0$ for a polyhedron $P \subseteq \mathbb{R}^n$ defines a facet, it suffices to exhibit $dim(P)$ affinely independent points belonging to the set $\{x \in P : \pi x = \pi_0\}$. In this dissertation, we use this idea in many proofs to establish that certain valid inequalities define facets.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. We denote by $(A^=, b^=)$ the submatrix of (A, b) corresponding to these inequalities that are satisfied at equality by all points $x \in P$. The following theorem establishes that any polyhedron has a minimal facet representation.

Theorem 1.2.17. *[36]*

(i) *A full-dimensional polyhedron P has a unique minimal representation by a finite set of linear inequalities.*

(ii) *If $dim(P) = n - k$ with $k > 0$, then $P = \{x \in \mathbb{R}^n : a^i x = b_i \ for \ i = 1, \ldots, k, a^i x < b_i \ for \ i = k + 1, \ldots, k + t\}$, where (a^i, b_i) for $i = 1, \ldots, k$ are linearly independent rows of $(A^=, b^=)$, and $a^i x \leq b_i$ for $i = k + 1, \ldots, k + t$ is any inequality from the equivalence class of inequalities defining each facet of P.*

Corollary 1.2.18. *[36] If F is a facet of polyhedron P, then in any description of P, there exists some inequality representing F.*

Corollary 1.2.19. *[36] Every inequality that represents a face of polyhedron P that is not a facet is unnecessary in the description of P.*

It follows that we are interested in facet-defining inequalities because they are the strongest valid inequalities.

1.2.2 Describing Polyhedra by Extreme Point and Extreme Ray

A *vertex* of a polyhedron P is an *extreme point* which can be defined as follows.

Definition 1.2.20. *$x \in P$ is an extreme point of polyhedron P if there do not exist two points $x^1, x^2 \in P, x^1 \neq x^2$ with $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.*

In other words, an extreme point of P is a point of P that cannot be written as the convex combination of two other points in P.

Definition 1.2.21. *$r \neq 0$ is a ray of a polyhedron $P \neq \emptyset$ if $x \in P$ implies $x + \lambda r \in P$ for all $\lambda \geq 0$.*
A ray r of P is an extreme ray *if there do not exist two rays r^1, r^2 of P, $r^1 \neq \lambda r^2$ for some $\lambda > 0$, with $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$.*

Any polyhedron P has a finite number of extreme points and extreme rays. In addition, any polyhedron can be described in terms of extreme points and extreme rays as follows.

Theorem 1.2.22. (Minkowski's Theorem) *Every polyhedron $P \neq \emptyset$ can be represented as a convex combination of extreme points $\{x^t\}_{t=1}^T$ and a non-negative combination of extreme rays $\{r^s\}_{s=1}^S$:*

$$P = \Big\{x : x = \sum_{t=1}^{T} \lambda_t x^t + \sum_{s=1}^{S} \mu_s r^s, \sum_{t=1}^{T} \lambda_t = 1, \lambda \in \mathbb{R}_+^T, \mu \in \mathbb{R}_+^S \Big\}.$$

Corollary 1.2.23. *A polytope is the convex hull of its extreme points.*

A characteristic cone of a polyhedron is defined as follows.

Definition 1.2.24. *Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. Then*

$$char.cone(P) = \Big\{r \in \mathbb{R}^n : Ar \leq 0\Big\}. \tag{1.1}$$

Observe that an extreme ray of char.cone(P) is also called an extreme ray of P.

1.2.3 Formulation and Integral Polyhedra

Definition 1.2.25. *A polyhedron $P \subseteq \mathbb{R}^n$ is a* formulation *for a set $X \subseteq \mathbb{Z}^n$ if and only if $X = P \cap \mathbb{Z}^n$, that is X is precisely the set of integer points in P.*

Definition 1.2.26. *Given two formulations P_1 and P_2 for X, we say P_1 is stronger (better) than P_2 if $P_1 \subset P_2$.*

Observe that for any objective function $c \in \mathbb{R}^n$ we get

$$z \geq \min \Big\{cx : x \in P_1\Big\} \geq \min \Big\{cx : x \in P_2\Big\}.$$

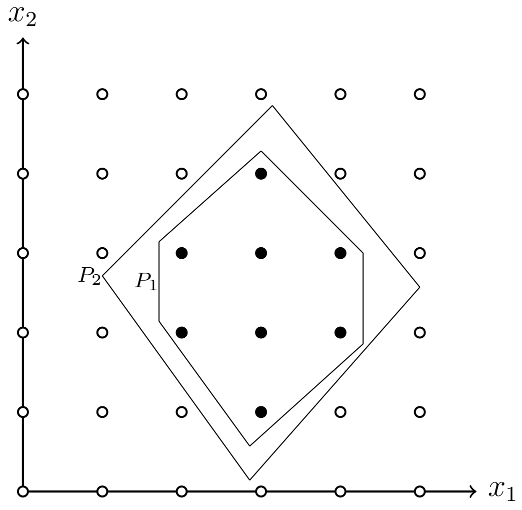

Figure 1.3: Two formulations P_1 and P_2 for X.

Example 1.2.27. (continued) *In Figure 1.3 we present two different formulations for the following set X. It is easy to see that formulation P_1 is better than formulation P_2.*

Definition 1.2.28. *A formulation with a polynomial number of variables and constraints is said to be* compact.

Observe that adding a facet-defining inequality (that is not already presented) to a formulation necessarily provides a stronger formulation because they are the strongest valid inequalities.

A *relaxation* of a problem is defined by the following definition.

Definition 1.2.29. *A program $z^R = \min\{f(x) : x \in T \subseteq \mathbb{R}^n\}$ is a relaxation of program $z = \min\{c(x) : x \in X \subseteq \mathbb{R}^n\}$ if:*

(i) $X \subseteq T$*, and*

(ii) $f(x) \leq c(x)$ *for all* $x \in X$.

Solving a relaxation of a problem provides a bound on the optimal value of the original problem.

Proposition 1.2.30. *[45] If program $z^R = \min\{f(x) : x \in T \subseteq \mathbb{R}^n\}$ is a relaxation of program $z = \min\{c(x) : x \in X \subseteq \mathbb{R}^n\}$, then $z^R \leq z$.*

Definition 1.2.31. *Let $X = \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$. The* Linear Programming (LP) relaxation *of X is*

$$LP(X) = \Big\{x \in \mathbb{R}^n_+ : Ax \leq b\Big\}.$$

This definition states that LP relaxation is obtained by dropping the integrality constraints to obtain a linear program.

Using valid inequalities for a LP relaxation can be stated as follows.

Proposition 1.2.32. *[36] Any inequality valid for a relaxation of an IP is valid for the IP itself.*

Now we define an *integral polyhedron* and a *totally unimodular* matrix.

Definition 1.2.33. *A nonempty polyhedron $P \subseteq \mathbb{R}^n$ is said to be integral if each of its nonempty faces contains an integral point.*

Definition 1.2.34. *The maximum number of linearly independent rows (columns) of matrix A is called* rank *of A and denoted by rank(A).*

The following proposition states that for a polyhedron to be integral it suffices to check its extreme points.

Proposition 1.2.35. *[36] A nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ with rank$(A) = n$ is integral if and only if all of its extreme points are integral.*

Also, if $P = \{x \in \mathbb{R}^n : Ax \leq b\} \subseteq \mathbb{R}^n_+$ and is not empty, then rank$(A) = n$. Thus, we have the following corollary.

Corollary 1.2.36. *[36] A nonempty polyhedron $p \subseteq \mathbb{R}^n_+$ is integral if and only if all of its extreme points are integral.*

The concept of totally unimodularity can be defined as follows.

Definition 1.2.37. *An $m \times n$ integral matrix A is totally unimodular (TU) if the determinant of each square submatrix of A is equal to $0, 1,$ or -1.*

Obviously, only matrices with entries $0, 1,$ and -1 can be TU.

The following theorem proposes a way to recognize totally unimodular matrices.

Theorem 1.2.38. *[36] The following statements are equivalent.*
(i) Matrix A is TU.
(ii) For every $J \subseteq N = \{1, \ldots, n\}$, there exists a partition J_1, J_2 of J such that

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1, \ for \ i = 1, \ldots, m.$$

Proposition 1.2.39. *[36] If matrix A is totally unimodular, then*

$$P(b) = \Big\{ x \in \mathbb{R}^n_+ : Ax \leq b \Big\},$$

is integral for all $b \in \mathbb{Z}^m$ for which it is not empty.

1.2.4 Separation Problem

Polyhedral structure of different mixed integer sets has been studied and large classes of valid inequalities and facet-defining inequalities have been derived to improve the formulations of those sets by adding them to the formulation. However since in many cases, there is an infinity number of valid inequalities and even the number of facet-defining inequalities can be large, it is not always desirable to add all these inequalities to the formulation a priori.

One possibility is to add valid inequalities as *cuts* or *cutting planes* such that cut off a point x^* that is not integral. Such points are typically obtained as the optimal solution of the linear programming relaxation of the problem. See the following example.

Example 1.2.40. *In Figure 1.4 the direction in which the objective function decreases is shown. We mentioned that inequality* $2x_1 + x_2 \leq 11$ *is valid for* X*. As shown in the figure, this inequality cuts off point* x^**, considering formulation* P_2*.*

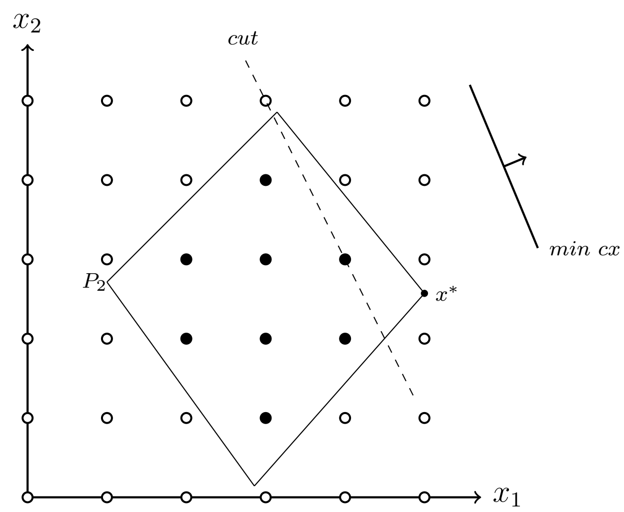

Figure 1.4: A cut removing point x^*.

The problem of finding whether there is a valid inequality for X cutting off x^* is of interest.

Definition 1.2.41. *Let mixed integer set* X *and point* $x^* \in \mathbb{R}^n$ *with* $x^* \notin conv(X)$ *are given. The* separation *problem, denoted by* $SEP(X, x^*)$*, is the problem of finding a valid inequality* $\pi x \leq \pi_0$ *cutting off point* x^* *(*$\pi x^* > \pi_0$*), or deciding that there is no such inequality.*

If we do not have the complete description of the conv(X) (which is almost always the case), we may have a family of valid inequalities $\mathcal{F}$. These give us implicitly the polyhedron

$$P_{\mathcal{F}} = \left\{x \in \mathbb{R}^n : \pi x \leq \pi_0 \ for\ all\ (\pi, \pi_0) \in \mathcal{F}\right\},$$

for which we wish to solve the separation problem $SEP(P_{\mathcal{F}}, x^*)$.

Theorem 1.2.42. *[40] Finding an optimal solution to the problem min$\{cx : x \in X\}$ is polynomially solvable if and only if SEP problem is polynomially solvable.*

The consequence of this result is that there is only hope of finding a complete description of conv(X) if the problems min$\{cx : x \in X\}$ and $SEP(X, x^*)$ are polynomially solvable. On the other hand, for problems which are difficult (NP-hard), we can hope to find the partial description of conv(X).

1.3 Optimization Algorithms

In this section we explain three successful algorithms for finding optimal solutions of various optimization problems, especially in the field of mixed integer programming.

Branch-and-Bound Algorithm

Branch-and-bound (B&B) is one of the exact solution techniques used in practice for solving mixed integer programming problems. This algorithm is basically a tree where each node of the tree is an LP problem. We describe it for a minimization problem as follows.

There is a value called the incumbent, that is the value of the best feasible solution found so far, and therefore, is an upper bound of the value of the optimal solution. In the beginning, if no feasible solution is known, the incumbent is set to $+\infty$. At the root node, B&B solves the LP relaxation, and in case a fractional solution k for an integer variable x is obtained, a constraint $x \leq \lfloor k \rfloor$ or $x \geq \lceil k \rceil$ is added to the LP relaxation to obtain two child nodes which are called subproblems. At each tree node, the LP relaxation is solved. If the solution is integral the incumbent is updated and the tree node is pruned. If the LP relaxation problem is infeasible the node is also pruned since the corresponding subproblem is infeasible as well. In addition, if the value of the incumbent is less than the value of the LP solution, the node can be pruned since the optimal solution of the subproblem is worse than a known feasible solution. When the node is not pruned, a variable with fractional value in the LP solution is chosen and branching is implemented. If the set of subproblems is empty, the B&B algorithm stops, and the optimal solution is found. Otherwise we need to branch and solve the resulting subproblems, recursively. The B&B scheme is summarized in Figure 1.5.

Branch-and-Cut Algorithm

The idea of a branch-and-cut algorithm is to use some cutting planes within the branch-and-bound algorithm. This produces tighter bounds and LP solutions closer to actual feasible integer solutions. The cutting phase can be carried out either at the root node by generating globally valid inequalities or during the branching phase. In the latter case, the cutting planes generated are only valid locally. Adding valid inequalities can strengthen the

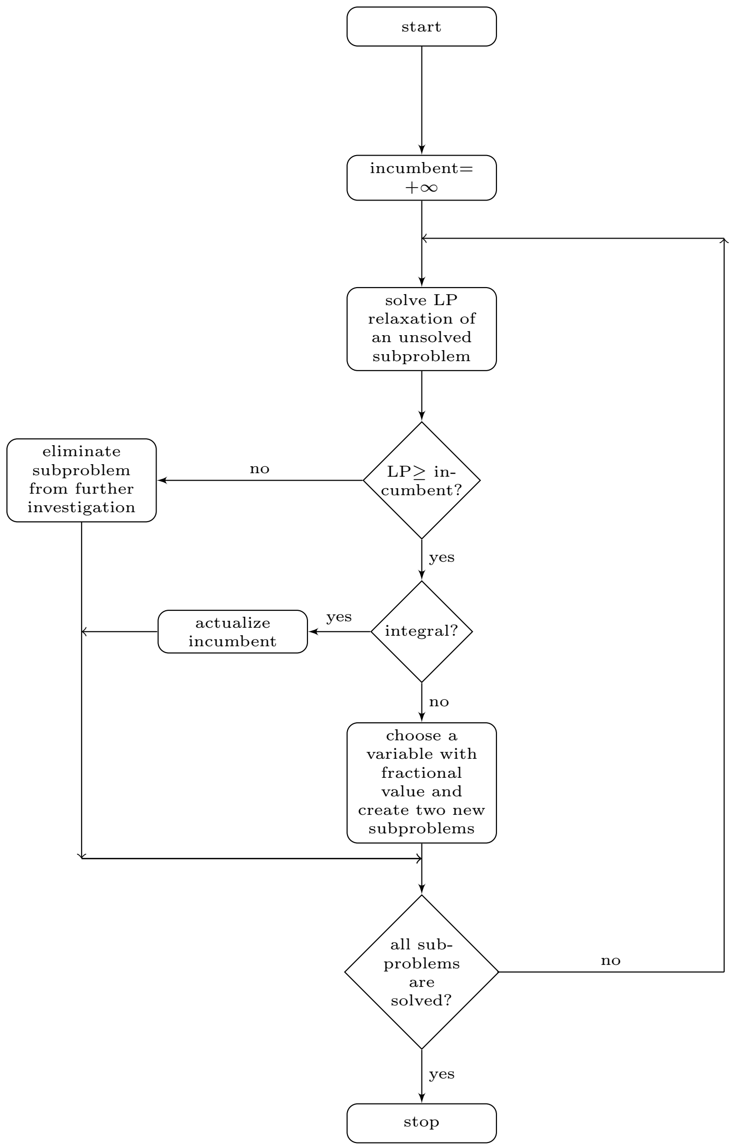

Figure 1.5: Branch and bound algorithm

formulation which tends to reduce the number of enumerated nodes. The branch-and-cut algorithm is shown in Figure 1.6.

Cutting Planes

Suppose that we have different families of valid inequalities for X. In general, we do not add these inequalities directly to the formulation but add them when they are needed: they are appended to the formulation through the cutting plane procedure which can be described as follows.

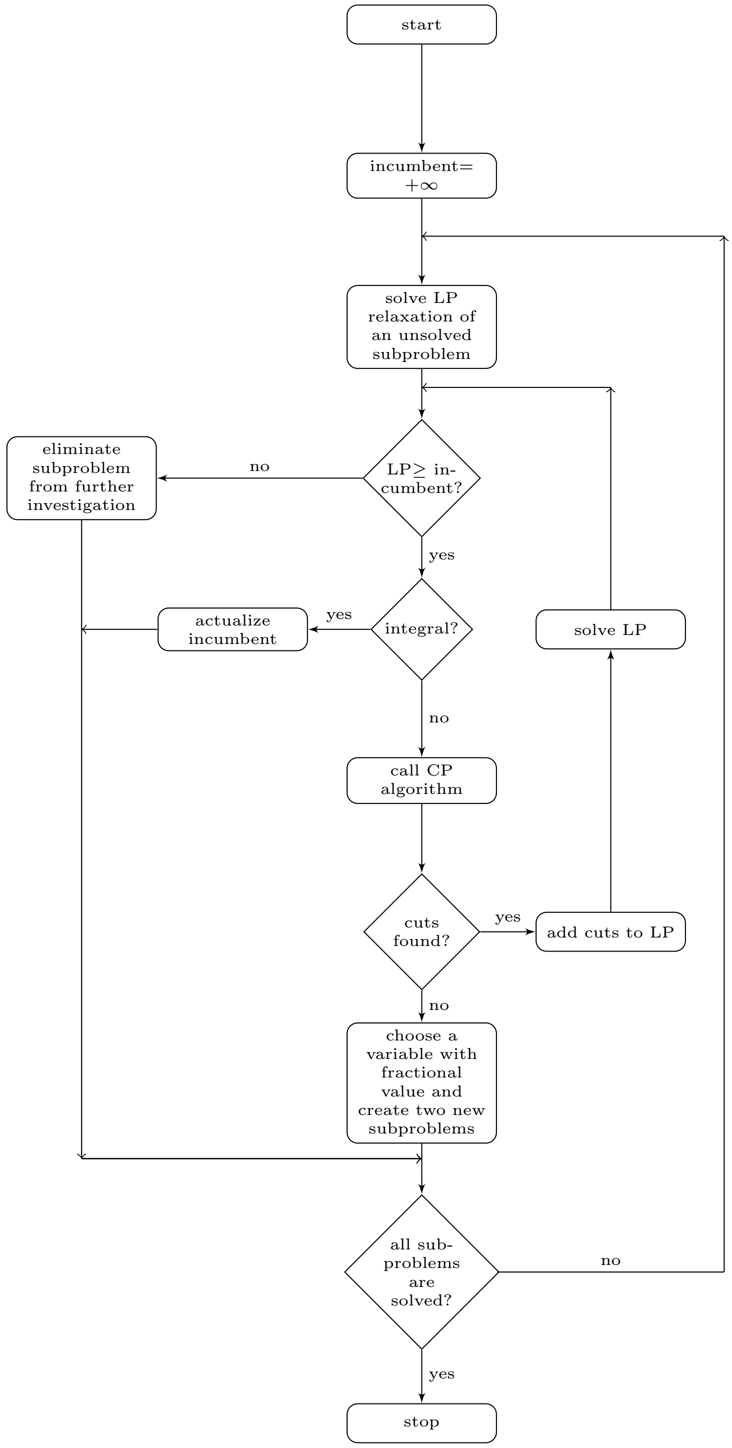

Figure 1.6: Branch and cut algorithm

A cutting plane procedure is essentially a two-step procedure.

Step 1. Find x^* which is the solution to the linear relaxation of the integer program $min\{cx : x \in X\}$. If x^* is integer, then STOP; otherwise go to Step 2.

Step 2. Find valid inequalities for X violated by point x^* and then add them to the formulation of the problem. If no violated inequality is found, STOP, otherwise go back to Step 1.

1.3.1 Extended Formulations

Another way to strengthen a formulation is to look for an *extended formulation* involving *additional variables.* For $X = \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$, suppose that it can be shown that

$$X = \Big\{x \in \mathbb{Z}^n_+ : Bx + Gz \leq d,\ for\ some\ z \in \mathbb{R}^q\Big\}.$$

Definition 1.3.1. *Let $Q = \{(x, z) \in \mathbb{R}^n_+ \times \mathbb{R}^q : Bx + Gz \leq d\}$. The* projection *of Q into the x-space, denoted by $proj_x Q$, is the polyhedron given by*

$$proj_x Q = \Big\{x \in \mathbb{R}^n :\ there\ exists\ z\ for\ which\ (x, z) \in Q\Big\}.$$

Now $\tilde{P} = proj_x Q$ is a formulation for X as $X = \tilde{P} \cap \mathbb{Z}^n$. An extended formulation is defined as follows.

Definition 1.3.2. *The polyhedron $Q = \{(x, z) \in \mathbb{R}^n_+ \times \mathbb{R}^q : Bx + Gz \leq d\}$ is an extended formulation for $X = \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$ if $proj_x Q$ is a formulation for X.*

Notice that there are extended formulations, that we call *tight (exact)*, whose projection gives conv(X).

One way to derive an extended formulation for a given polyhedron is to follow the idea of union of polyhedra. We review a result of Balas [10, 11] about the union of k polyhedra as follows.

Consider k polyhedra $P_i = \{x \in \mathbb{R}^n : A^i x \leq b^i\}, i = 1, \ldots, k$ and their union $\bigcup_{i=1}^k P_i$. Then $\overline{conv}(\bigcup_{i=1}^k P_i)$, the smallest closed convex set that contains $\bigcup_{i=1}^k P_i$, is a polyhedron.

Disjunctive programming is optimization over unions of polyhedra. While polyhedra are convex sets, their unions are not. It is clear that, for instance, a linear program over a feasible set X is amended with the condition that variable x_j has to be an integer between 0 and k, which can be written as $(x_j = 0) \vee (x_j = 1) \vee \cdots \vee (x_j = k)$ (where "$\vee$" is the logical "or" symbol), then it becomes an optimization problem over a union of polyhedra $P_0 \cup P_1 \cup \cdots \cup P_k$, where $P_i = \{x \in X : x_j = i\}$, for $i = 0, \ldots, k$.

A compact representation of the convex hull of a union of polyhedra in a higher dimensional space is given by the following theorem which can be projected back to the original space of variables.

Theorem 1.3.3. *Given polyhedra* $P_i = \{x \in \mathbb{R}^n : A^i x \leq b^i\} \neq \emptyset, i = 1, \ldots, k,$ *the closed convex hull of* $\bigcup_{i=1}^{k} P_i$ *is the set of those* $x \in \mathbb{R}^n$ *for which there exist vectors* $(y^i, y_0^i) \in \mathbb{R}^{n+1}, i = 1, \ldots, k,$ *satisfying*

$$\begin{aligned}
&x - \sum_{i=1}^{k} y^i = 0,\\
&A^i y^i - b^i y_0^i \leq 0,\\
&y_0^i \geq 0, i = 1, \ldots, k,\\
&\sum_{i=1}^{k} y_0^i = 1.
\end{aligned}$$

Example 1.3.4. *Let* $k = 2, x_j \in \{0, 1\}$*, and*

$$\begin{aligned}
P_{j0} &= \Big\{x \in \mathbb{R}_+^n : Ax \leq b, x_j = 0\Big\},\\
P_{j1} &= \Big\{x \in \mathbb{R}_+^n : Ax \leq b, x_j = 1\Big\}.
\end{aligned}$$

Then $\overline{conv}(P_{j0} \cup P_{j1})$ *is the set of those* $x \in \mathbb{R}_+^n$ *for which there exist vectors* $(y, y_0), (z, z_0) \in \mathbb{R}_+^{n+1}$ *such that*

$$\begin{aligned}
&x - y - z = 0,\\
&Ay - by_0 \leq 0,\\
&y_j = 0,\\
&Az - bz_0 \leq 0,\\
&z_j - z_0 = 0,\\
&y_0 + z_0 = 1.
\end{aligned}$$

1.3.2 Lifting and Superadditivity

In this dissertation we use the notations of lifting and superadditivity several times to derive families of valid and facet-defining inequalities for the sets which have been studied. We review the results investigated by Gu, Nemhauser, and Savelsbergh [22] on lifting process.

Consider the following set

$$X = \Big\{x \in \mathbb{R}_+^{|N|} : \sum_{j \in N} a_j x_j \leq d, \sum_{j \in C_k} w_j x_j \leq r_k, k = 0, \ldots, t, x_j \in \{0, 1\}, j \in I \subseteq N\Big\},$$

where $\{C_k, k = 0, \ldots, t\}$ is a partition of N, $a_j, j \in N$ and d are $m \times 1$ and $w_j, j \in N$, and r_k are $m_k \times 1$. Moreover, we assume that $a_j, d,$ and r_k, but not necessarily w_j, are nonnegative. Initially, we consider the subset of X with $x_j = 0$ for $j \in N \setminus C_0$ given by

$$X^0 = \Big\{x \in \mathbb{R}_+^{|C_0|} : \sum_{j \in C_0} a_j x_j \leq d, \sum_{j \in C_0} w_j x_j \leq r_0, x_j \in \{0, 1\}, j \in I \cap C_0\Big\}.$$

Let inequality

$$0 \leq \alpha_0 - \sum_{j \in C_0} \alpha_j x_j, \tag{1.2}$$

be an arbitrary valid inequality for X^0. We aim to construct a valid inequality for X of the from

$$0 \leq \alpha_0 - \sum_{0 \leq k \leq t} \sum_{j \in C_k} \alpha_j x_j. \tag{1.3}$$

To construct such an inequality, we start with inequality (1.2) and lift the variables in $N \setminus C_0$. Without loss of generality, we assume that the sets of variables $C_1, \ldots, C_t$ are lifted sequentially in that order and that the variables within the sets $C_1, \ldots, C_t$ are lifted simultaneously. The intermediate sets of feasible points X^i for $i = 1, \ldots, t$ are defined by

$$X^i = \Big\{ x \in \mathbb{R}_+^{\sum_{0 \leq k \leq i} |C_k|} : \sum_{0 \leq k \leq i} \sum_{j \in C_k} a_j x_j \leq d, \sum_{j \in C_k} w_j x_j \leq r_k, k = 0, \ldots, i, \\ x_j \in \{0,1\}, j \in \cap \Big(\bigcup_{k=0}^{i} C_k \Big) \Big\}.$$

The lifting problem associated with C_i, given a valid inequality

$$0 \leq \alpha_0 - \sum_{0 \leq k < i} \sum_{j \in C_k} \alpha_j x_j,$$

for X^{i-1}, is to find coefficients α_j for $j \in C_i$ such that

$$\sum_{j \in C_i} \alpha_j x_j \leq \alpha_0 - \sum_{0 \leq k < i} \sum_{j \in C_k} \alpha_j x_j, \tag{1.4}$$

is a valid inequality for X^i.

Now let $Z = [0, d]$. Furthermore, for $z \in Z$ let

$$\begin{aligned} h_i(z) = \max \ & \sum_{j \in C_i} \alpha_j x_j \\ s.t. \ & \sum_{j \in C_i} a_j x_j = z, \\ & \sum_{j \in C_i} w_j x_j \leq r_i, \\ & x_j \in \{0,1\}, j \in C_i \cap I, x \in \mathbb{R}_+^{|C_i|}, \end{aligned}$$

and let

$$
\begin{aligned}
f_i(z) = \min\ & \alpha_0 - \sum_{0\le k<i} \sum_{j\in C_k} \alpha_j x_j \\
s.t.\ & \sum_{0\le k<i} \sum_{j\in C_k} a_j x_j \le d - z, \\
& \sum_{j\in C_k} w_j x_j \le r_k, k = 0, \ldots, i-1, \\
& x_j \in \{0,1\}, j \in C_i \cap \Big(\bigcup_{k=0}^{i-1} C_k\Big), x \in \mathbb{R}_+^{\sum_{k=0}^{i-1}|C_k|}.
\end{aligned}
$$

Proposition 1.3.5. *Inequality* (1.4) *is valid for X^i for any choice of $\alpha_j, j \in C_i$ such that $h_i(z) \le f_i(z)$ for all $z \in Z$.*

When α_j for $j \in C_i$ are such that $h_i(z) = f_i(z)$ has $|C_i|$ solutions $x^1, x^2, \ldots, x^{|C_i|}$ such that the components in C_i of $x^1, x^2, \ldots, x^{|C_i|}$ are linearly independent, we say that the lifting is *maximal* which leads to a strongest lifted inequality.

Theorem 1.3.6. *If inequality* (1.2) *is facet-defining for conv*(X^0), *conv*(X^i) *for* $i = 0, \ldots, t-1$, *is full-dimensional, and at each step i the lifting is maximal, then inequality* (1.3) *defines a facet of conv*(X).

It is clear that lifting coefficients are, in general, dependent on the lifting sequence $C_1, C_2, \ldots, C_t$.

Sequence Independent Lifting

We now present the concept of sequence independent lifting and its relation to superadditive functions.

Definition 1.3.7. *The lifting function f with respect to valid inequality* (1.2) *for X^0 is defined to be $f(z) = f_1(z)$ for all $z \in Z$.*

Definition 1.3.8. *If $f(z) = f_i(z)$ for $z \in Z, i = 2, \ldots, t$, and all lifting sequences, then the lifting is said to be sequence independent.*

We define the concept of *superadditive* function as follows.

Definition 1.3.9. *A function f is superadditive on Z if f is bounded for all $z \in Z$ and*

$$f(z_1) + f(z_2) \le f(z_1 + z_2),\ \textit{for all}\ z_1, z_2\ \textit{and}\ z_1 + z_2 \in Z.$$

Now we give a sufficient condition for sequence independent lifting.

Theorem 1.3.10. *If f is superadditive on Z, then lifting is sequence independent.*

Obviously, a superadditive lifting function greatly reduces the computational burden of the lifting process. In this approach, instead of computing lifting functions f_i for all i, we only have to compute f. But unfortunately f is often not superadditive. In order to benefit from the property of a superadditive function to reduce the computational cost, we consider the class of superadditive lifting functions as follows.

Definition 1.3.11. *A superadditive function g is called a superadditive valid lifting function for f, if $g(z) \leq f(z)$ for all $z \in Z$.*

Theorem 1.3.12. *If g is a superadditive valid lifting function and if α_j for $j \in C_i$ are such that $h_i(z) \leq g(z)$ for $z \in Z$ and for $i = 1, \ldots, t$, then the lifted inequality* (1.3) *is valid for X.*

1.4 Basic Mixed Integer Programming Models

An inequality which is valid for a set X is also valid for a set Y if Y is a subset of X. This simple observation propose a general method to derive valid inequalities for a mixed integer set Y which has been used as a fundamental step in this dissertation. The first step of this method is to identify a superset (also called in this context a relaxation) X of Y. The second step is to derive valid inequalities for the mixed integer set X. Clearly, this is only meaningful and fruitful if it is easier to find valid inequalities for X than for Y. One way to ensure this is to restrict ourselves in the definition of X to those mixed integer sets whose polyhedral structure is simpler to study.

In this section, we review polyhedral results for four basic mixed integer sets which have been used in this thesis. The first set is the two-variable mixed integer set for which a famous class of valid inequalities, which are called *mixed integer rounding inequalities*, is introduced; the second set is the single node fixed-charge network set; the next one is the Mixed 0-1 Knapsack set, and the last one is called the vertex packing set.

Two-Variable Mixed Integer Set

We present the Mixed Integer Rounding (MIR) inequality by reviewing the polyhedral results for the two-variable mixed integer set. We consider a mixed integer set with only two variables as

$$X^{MI} = \Big\{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s + y \geq b\Big\}.$$

Let $f = b - \lfloor b \rfloor \geq 0$ be the fractional part of b. Then the following proposition gives the complete description of $\text{conv}(X^{MI})$.

Proposition 1.4.1. *[40] (i) The mixed inter rounding inequality*

$$s \geq f(\lceil b \rceil - y),$$

is valid for X^{MI}.
(*ii*) *The polyhedron*

$$\begin{aligned} &s + y \geq b, \\ &s + fy \geq f\lceil b \rceil, \\ &s \geq 0, \end{aligned}$$

describes the convex hull of X^{MI}.

Example 1.4.2. *Consider the set* $X^{MI} = \{(s, y) \in \mathbb{R}^1_+ \times \mathbb{Z}^1 : s + y \geq 2.25)\}$ *shown in Figure 1.7. Proposition 1.4.1 implies that inequality*

$$s + 0.25y \geq 0.75,$$

is a MIR inequality which states that $s \geq 0$ *when* $y = 3$ *and* $s \geq 0.25$ *when* $y = 2$. *As it is shown in Figure 1.7, the two points* $(0, 3)$ *and* $(0.25, 2)$ *are the extreme points of conv(*X^{MI}*) limiting the shaded region cut off by the MIR inequality. Observe that these two points prove that inequality* $s + 0.25y \geq 0.75$ *is a facet-defining valid inequality of conv(*X^{MI}*).*

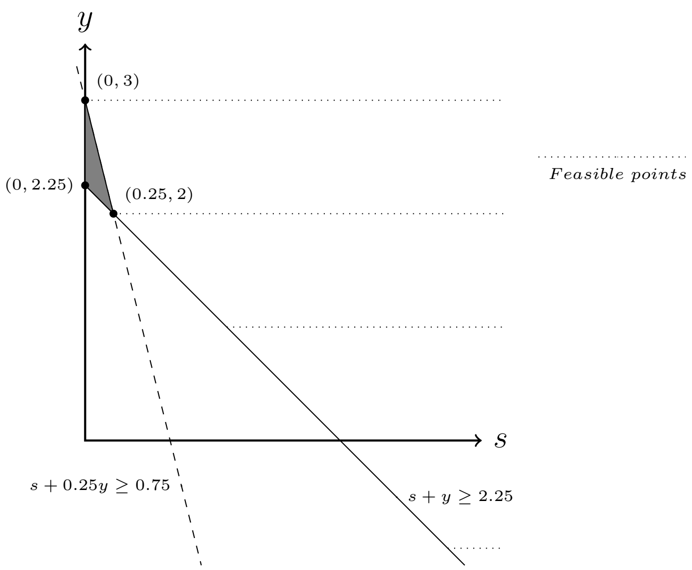

Figure 1.7: Mixed Integer Rounding (MIR) inequality for X^{MI}.

There has been considerable research on the generation of MIR inequalities and their use as cuts in solving mixed integer programs. Marchand et al. [31] discussed on the generation of MIR inequalities from constraints or simple aggregations of constraints of the original problem. This idea is motivated by the observation that several strong valid inequalities based on specific problem structure can be derived as MIR inequalities. Agra et al. [3, 4] investigated the use of MIR inequalities in solving maritime inventory routing problems.

Single Node Fixed-Charge Network Set

Consider a single node fixed-charge network set, denoted by X^{SNFCN}, containing only variable upper bounds which is defined as

$$X^{SNFCN} = \Big\{(x,z) \in \mathbb{R}^n_+ \times \mathbb{B}^n : \sum_{j \in N} x_j \le d, x_j \le c_j z_j, j \in N\Big\}, \tag{1.5}$$

where $0 < c_j \le d$ and $|N| = n$.

Definition 1.4.3. *$S \subseteq N$ is a* cover *if $\sum_{j \in S} c_j > d$, and then we associate the value $\lambda = \sum_{j \in S} c_j - d > 0$ with each cover.*

Padberg, Van Roy and Wolsey [38] studied the foregoing set X^{SNFCN} by considering constraints $\sum_{j \in N} x_j$ $(\le, =, \ge)$ d instead of $\sum_{j \in N} x_j \le d$ and derived the so-called *flow cover inequalities* for these sets. Moreover, they considered the constant capacitated case, where all upper bounds $c_j, j \in N$ are equal to the constant value c, and they characterized the complete description of the convex hull of the set.

Goemans [20] considered a variant of X^{SNFCN} and introduced a family of facet-defining inequalities for the set. Furthermore, Van Roy and Wolsey [44] studied the case where variable lower bounds are also taken into account. They derived the *generalized flow cover inequalities* for this set.

Mixed 0-1 Knapsack Set

Consider the knapsack problem with a single continuous variable, called the mixed $0-1$ knapsack problem which is

$$X^{MK} = \Big\{(s,x) \in \mathbb{R}^1_+ \times \mathbb{B}^n : \sum_{j \in N} a_j x_j \le b + s\Big\},$$

where $a_j > 0, j \in N$, $b \ge 0$, and $\sum_{j \in N} a_j > b$.

The set X^{MK} was studied by Marchand and Wolsey (see [32]). They derived two classes of facet-defining inequalities which are called knapsack and complemented knapsack facets. Then they introduced continuous cover inequalities for such a set.

Vertex Packing Set

Consider a finite, and undirected graph $G = (V, E)$ where V and E are the vertex and edge sets of G, respectively.

Definition 1.4.4. *The vertices i, j are adjacent in graph G if $(i, j) \in E$.*

Definition 1.4.5. *A* vertex packing (independent set) *in graph G is a subset $P \subseteq V$ for which all $i, j \in P$ satisfy $(i, j) \notin E$. In fact, P is a subset of vertices such that no two of which are adjacent.*

The vertex packing set can be defined as

$$X^{VP} = \Big\{ x \in \mathbb{B}^n : x_i + x_j \leq 1, (i,j) \in E \Big\}.$$

where $|V| = n$.

The vertex packing problem was studied by Padberg [37]. He derived two families of fact-defining valid inequalities for such a set. Nemhauser and Trotter in [35] discussed on properties of the vertex packing polyhedron and introduced a class of facets for this polyhedron which subsumes the class investigated by Padberg. In addition, Alper Atamtürk et al. [8] studied a generalization of the vertex packing problem having both binary and bounded continuous variables, called the mixed vertex packing problem.

1.5 Purpose and Outline of the Thesis

In this dissertation, we focus on the role of cutting planes in mixed integer programming and, in particular, in complex inventory problems. We aim to generate and improve the quality of cuts used in a cutting plane framework. Within this context, we wish to obtain cuts that produce better bounds earlier in the solution process.

Two possible approaches arise in the generation of cutting planes for a specific problem. The first one is to study the generic structure of the problem to derive cuts. The second one is to study elementary substructures of this model, or structures which can be obtained by the inherent loss of information from a relaxation of the original problem. The second approach is followed in this dissertation.

The main goal of this thesis is to provide valuable theoretical contributions for solving general inventory problems and more specifically lot-sizing with supplier selection, network design and vendor-managed inventory routing problems. These theoretical contributions are essentially the derivation of stronger formulations (formulations whose corresponding linear relaxation bound provides a better bound to the value of the optimal solution than the bound obtained from the linear relaxation of the initial formulation) for such problems either by the inclusion of strong valid inequalities and/or extended formulations for mixed integer subsets of the original problem. In order to achieve the main goal, the first stage is to find those simpler mixed integer models that retain the main characteristics of the general problem. These mixed integer sets are obtained by aggregation, relaxation or decomposition of the general problem. In the next stage, polyhedral structure of those simpler mixed integer sets is studied to derive efficient valid inequalities and extended formulations.

In Chapter 2 we consider a variant of the well-known single node fixed-charge flow set that arises from the lot-sizing with supplier selection problem, where we introduce a new set-up binary variable which is associated with the node. Both variable and constant capacitated cases are considered in this research. A major point to study this variant is that the structure of this set is richer than the structure of the single node fixed-charge flow set, namely, new facet-defining inequalities appear in the description of the convex hull of the set we have considered. We investigate the polyhedral structure of such a set and as a result, the well-known flow cover inequalities are generalized into the set-up flow cover inequalities. Furthermore, a class of lifted set-up flow cover inequalities are presented. The full polyhedral description of the convex hull of this set is provided where constant capacitated case is considered.

Another simpler mixed integer set which arises as a relaxation of complex inventory problems such as lot-sizing combined with supplier selection decisions and vendor-managed inventory routing problems is studied in depth in Chapter 3. This mixed integer set can be represented as a variant of the single arc design set where a binary variable is imposed on each arc. On the other hand, the set which has been introduced in this chapter is a variant of the mixed integer set defined in Chapter 2 where the binary variable associated with the node is imposed to be integer and a new set of constraints is added to the set. In Chapter 3 we generalize the well-know flow cover inequalities and the arc residual capacity inequalities. Moreover, we derive families of strong valid inequalities for that mixed integer set where the variable and constant capacitated case are taken into account. For the constant capacitated case we provide a compact extended formulation and give a partial description of the convex hull in the original space of variables which is exact under a certain condition. Finally, all these cuts are added to the branch-and-cut algorithm to check their effectiveness in improving the integrality gap and solving the randomly generated instances of the lot-sizing with supplier selection problem.

In Chapter 4 we study the polyhedral structure of a mixed integer set which results from an intersection of a simple mixed integer set and a vertex packing set. In fact, the concept of conflict graph is combined with a mixed integer set in this study. The set we consider in this chapter arises as a subproblem of mixed integer sets and more particularly inventory routing problems. We focus on deriving conflict mixed integer rounding inequalities which are variant of the MIR inequalities where the incompatibility between binary variables is considered. Moreover, families of strong valid inequalities which maintain the structure of simple mixed integer set and the vertex packing set simultaneously, are generated. Lastly, computational experiment in improving the integrality gap of the randomly generated instances of the single node fixed-charge set with conflicts on arcs is reported.

In Chapter 5 we review the main results of the dissertation and give some directions for future research.

Chapter 2

Facets for the Single Node Fixed-Charge Network Set with a Node Set-Up Variable

2.1 Introduction

In this chapter we consider the first simpler mixed integer set that can be obtained as a relaxation of inventory problems. This set is a variant of the well-known Single Node Fixed-Charge Network (SNFCN) set (1.5) where a set-up variable is associated with the node, indicating whether the node is open or not. This mixed integer set is of the form

$$X_{binary} = \Big\{(x, z, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{B} \mid \sum_{j \in N} x_j \leq dy, x_j \leq c_j z_j,\ j \in N\Big\},$$

where $N = \{1, \ldots, n\}, d > c_j > 0, \forall j \in N$, and integer.

Set X_{binary} is much related with the well-known SNFCN set which is the restriction of X_{binary} to the subspace defined by $y = 1$. We also consider set

$$X^1 = \Big\{(x, z, y) \in X_{binary} \mid y = 1\Big\}.$$

The SNFCN set is the projection of X^1 into the (x, z) space. Variable y can be regarded as a set-up variable associated to the node itself. Thus, y indicates whether the capacity of the node is installed or not. So, in the classical SNFCN set the capacity of the node is assumed to be installed. As usual, the binary variables z_j are associated with the arcs entering the node and indicating whether the arc is open or not (see Figure 2.1).

The convex hull of X_{binary} will be denoted by P_{binary} and the convex hull of the restricted set X^1 by P^1.

Set X_{binary} arises as a relaxation of several mixed integer problems. Next we provide a few examples. In the single-item Lot-sizing with Supplier Selection Problem (LSSP) we are given a set N of suppliers. In each time period one needs to decide lot-sizes and a

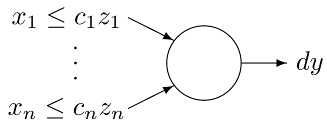

Figure 2.1: Single node fixed-charge network set with a node set-up variable.

subset of suppliers to use in order to satisfy the demands while minimizing the costs. For each time period, set X_{binary} arises as follows: y represents the binary variable indicating whether there is a set-up for production or not, z_j indicates whether the supplier $j \in N$ is selected or not, x_j is the amount supplied by supplier j, d is the production capacity and c_j is the supplying capacity of supplier j, see [47]. Other examples occur in inventory-routing problems such as the Vendor-Managed Inventory-Routing Problem (see [7]), where, for each time period t, y is a binary variable indicating whether the supplier is visited at time t or not, z_j is a binary variable equal to 1 if the retailer j is served at time t, and 0 otherwise, d is the capacity of the vehicle, and c_j is the maximum inventory level in retailer j. In some related problems, d may also represent the inventory capacity and y indicates whether the warehouse is set-up to receive goods or not. Variables x_j represent the supplied quantities and variables z_j indicate the suppliers selected (as in the LSSP). Other examples can be found where such relaxations occur under particular cases. See, for instance, the Capacitated Location Problem presented in [17] where the binary variable y indicates whether a facility is installed at a given node, z_j indicates whether client j is served or not from that node, x_j indicates the quantity that the facility sends to client $j \in N$. d represents the facility capacity and c_j represents the capacity of the facility-client link.

Although, as we will show later, valid inequalities derived for X^1 are, under general conditions, valid for X_{binary} (and, therefore, can be used to tighten the general mixed integer problems with set-ups on the nodes), a deep study of this particular set is of practical interest, in particular, when the set-up variable may play an important role. Our goal is to provide a better understanding of such sets and explain what can be gained with the explicit inclusion of the set-up variable in the model.

The single node fixed-charge set has been intensively studied in the past, and different variants of the model have been considered. Padberg et al. [38] considered sets of the type

$$X_{[\triangle]} = \left\{(x,z) \in \mathbb{R}^n_+ \times \mathbb{B}^n \mid \sum_{j \in N} x_j \triangle d, \; x_j \leq c_j z_j, \; j \in N\right\},$$

where $\triangle \in \{\leq, =, \geq\}$. They introduced the well-known "flow cover" inequalities. For the case $\leq$ these inequalities can be stated as follows.

Proposition 2.1.1. *Let S be a cover such that $\sum_{j \in S} c_j = d + \lambda, \lambda > 0$ and $\bar{c} = \max_{j \in S} c_j > \lambda$. Then the simple flow cover inequality*

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - \lambda)^+ z_j \leq d - \sum_{j \in S} (c_j - \lambda)^+, \tag{2.1}$$

defines a facet of P^1*, and for* $L \subseteq N \setminus S$ *with* $0 < \bar{c} - \lambda < c_k \leq \bar{c}$ *for all* $k \in L$*, the extended flow cover*

$$\sum_{j \in S \cup L} x_j - \sum_{j \in S} (c_j - \lambda)^+ z_j \leq d - \sum_{j \in S} (c_j - \lambda)^+ + \sum_{j \in L} (\bar{c} - \lambda) z_j, \tag{2.2}$$

defines a facet of P^1.

They showed that inequalities (2.2) together with the defining inequalities are enough to describe P^1 when $c_j = c, \forall j \in N$. Gu et al. [23] provided a strategy for sequence independent lifting of the flow cover inequalities using valid superadditive lifting functions. In particular, the lifted inequalities generalize inequalities (2.2).

Our main contribution is to extend the well-known polyhedral results for the SNFCN set to set X_{binary} and establish relations between the results for both. This chapter is organized as follows. In Section 2.2 we establish basic properties of P_{binary}, introduce a simple family of facet-defining inequalities and relate set X_{binary} with set X^1. In Section 2.3 we introduce the set-up flow cover inequalities and relate this class of inequalities with the well-known flow cover inequalities. We show that the new class of inequalities together with the inequalities defining X_{binary}, and the simple family introduced in Section 2.2, give the complete characterization of P_{binary} when the capacities are constant. In Section 2.4 we discuss the lifting of the set-up flow cover inequalities. Preliminary computational experiments are reported in Section 2.5. Finally, a summary of this chapter is addressed in Section 2.6.

2.2 Properties of P_{binary}

In this section we establish basic properties for P_{binary} and relate polyhedron P_{binary} with the SNFCN polyhedron.

We assume that for each $k \in N$, $0 < c_k < d$ and $\sum_{i=1}^{n} c_i > d + c_k$. Under these assumptions we trivially have the following result.

Proposition 2.2.1. P_{binary} *and* P^1 *are full-dimensional polyhedra.*

Proof. First, we prove that P_{binary} is a full-dimensional polyhedron. The following points belong to P_{binary}.

- $v_0 : y = 0; x_j = 0, j \in N, z_j = 0, j \in N;$
- $v_1 : y = 1; x_j = 0, j \in N, z_j = 0, j \in N;$
- $v_2, \ldots, v_{n+1}$: for all $k \in N$, set $y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\};$
- $v_{n+2}, \ldots, v_{2n+1}$: for all $k \in N$, set $y = 0; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\}.$

We demonstrate that the listed points are affinely independent. Since $(\mathbf{0}, \mathbf{0}, 0)$ is among them so it suffices to prove that points $v_1, \ldots, v_{2n+1}$ are linearly independent. So we consider the system $\sum_{j=1}^{2n+1} \lambda_j v_j = \mathbf{0}$, for scalars $\lambda_j, j = 1, \ldots, 2n+1$ which are not all zero. Thus, we get

$$\begin{cases} c_{i-1}\lambda_i = 0, i = 2, \ldots, n+1 \\ \lambda_i + \lambda_{n+i} = 0, i = 2, \ldots, n+1 \\ \sum_{i=1}^{n+1} \lambda_i = 0. \end{cases} \tag{2.3}$$

The first equation of system (3.2) provides $\lambda_2 = \cdots = \lambda_{n+1} = 0$. The second equation implies $\lambda_{n+2} = \cdots = \lambda_{2n+1} = 0$ and finally, the last equation of system (3.2) gives $\lambda_1 = 0$.

Next, we show that P^1 is full-dimensional. We introduce $2n$ points belonging to P^1 as follows by considering the fact that y is not a variable.

- $v_0 : y = 1; x_j = 0, j \in N, z_j = 0, j \in N;$
- $v_1, \ldots, v_n$: for all $k \in N, y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\};$
- $v_{n+1}, \ldots, v_{2n}$: for all $k \in N, y = 1; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\}.$

Similar to the first part of the proof, it can be concluded that these points are affinely independent. □

Trivial facets of P_{binary} are given by the following proposition.

Proposition 2.2.2. *1. for every $i \in N, x_i \geq 0$ defines a facet of P_{binary}.*

2. for every $i \in N, z_i \leq 1$ defines a facet of P_{binary}.

3. for every $i \in N, x_i \leq c_i z_i$ defines a facet of P_{binary}.

4. $y \leq 1$ defines a facet of P_{binary}.

5. If $\sum_{j \in N} c_j > d + c_k, \forall k \in N$, then $\sum_{j \in N} x_j \leq dy$ defines a facet of P_{binary}.

Proof. Proof of 1. For a fixed i, let $K = P_{binary} \cap \{(x, z, y) \mid (x, z, y) \text{ satisfies } x_i = 0\}$. Then we prove that inequality $x_i \geq 0$ is facet-defining by showing that whenever the inequality $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y \leq \gamma_0$ is valid for X_{binary} and satisfies the condition that

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0, \forall (x, z, y) \in K, \tag{2.4}$$

then equality (3.3) is a multiple of $x_i = 0$. We provide the following feasible points belonging to K.

(*i*) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N;$

(*ii*) $y = 1; x_j = 0, j \in N; z_j = 0, j \in N;$

(*iii*) for all $k \in N, y = 1; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\};$

(*iv*) for all $k \in N, k \neq i, y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\}.$

Substituting point (*i*) and (*ii*) in equation (3.3) gives $\gamma_0 = 0$ and $\gamma = 0$ respectively. Then it follows by replacing solution (*iii*) in (3.3) that $\beta_j = 0, j \in N$. Finally, substituting solution (*iv*) in equation (3.3) implies $\alpha_j = 0, j \in N \setminus \{i\}$. Thus, equation (3.3) is equivalent to $\alpha x_i = 0$ which is a multiple of $x_i = 0$.

Proof of 2. Following the technique used in part 1, we give the following points belong to K.

(*i*) $y = 0; x_j = 0, j \in N; z_i = 1; z_j = 0, j \in N \setminus \{i\};$

(*ii*) $y = 1; x_j = 0, j \in N; z_i = 1; z_j = 0, j \in N \setminus \{i\};$

(*iii*) $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = 1; z_j = 0, j \in N \setminus \{i\};$

(*iv*) for all $k \in N \setminus \{i\}$, set $y = 1; x_j = 0, j \in N; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\};$

(*v*) for all $k \in N \setminus \{i\}$, set $y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}.$

Then substituting points (*i*) and (*ii*) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ and subtracting imply $\gamma = 0$. Replacing (*i*) and (*iv*) in the foregoing equation and subtracting give $\beta_j = 0, j \in N \setminus \{i\}$. Then it follows from substituting solutions (*i*) and (*iii*) in the equation that $\alpha_i = 0$ and substituting points (*iv*) and (*v*) provides $\alpha_j = 0, j \in N \setminus \{i\}$. Finally, replacing point (*i*) in the equation gives $\gamma_0 = \beta_i = \beta$. Thus, we get $\beta z_i = \beta$ which is a multiple of $z_i = 1$.

Proof of 3. Similarly, the following points are in K.

(*i*) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N;$

(*ii*) $y = 1; x_j = 0, j \in N; z_j = 0, j \in N;$

(*iii*) $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = 1; z_j = 0, j \in N \setminus \{i\};$

(*iv*) for all $k \in N \setminus \{i\}, y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\};$

(*v*) for all $k \in N \setminus \{i\}, y = 1; x_i = c_i; x_k = \varepsilon_k$ such that $c_i + \varepsilon_k \leq d; x_j = 0, j \in N \setminus \{i, k\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}.$

Now substituting points (i) and (ii) in equation $\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y = \gamma_0$ gives $\gamma_0 = 0$ and $\gamma = 0$ respectively. Then replacing points (iii) and (iv) in the foregoing equation implies $\beta_j = 0, j \in N \setminus \{i\}$. It follows from substituting points (iv) and (v) and subtracting that $\alpha_j = 0, j \in N \setminus \{i\}$. Therefore, the equation is $\alpha x_i + \beta z_i = 0$. Lastly, replacing point (iii)in this equation gives $\beta = -\alpha c_i$ which completes the proof.

Proof of 4. The following points belong to K.

(i) $y = 1; x_j = 0, j \in N; z_j = 0, j \in N;$

(ii) for all $k \in N, y = 1; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\};$

(iii) for all $k \in N, y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\}.$

Then substituting points (i) and (ii) in equation $\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y = \gamma_0$ and subtracting give $\beta_j = 0, j \in N$. It can be concluded from replacing points (i) and (iii) in the equation that $\alpha_j = 0, j \in N$. So we obtain an equation $\gamma y = \gamma_0$. Substituting point (i) in this equation implies $\gamma_0 = \gamma U$ which proves that $\gamma y = \gamma U$ is a multiple of $y = U$.

Proof of 5. The points belonging to K are listed as follows.

(i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N;$

(ii) for all $k \in N$, set $y = 1; x_j = c_j, j \in S \subset N \setminus \{k\}; x_t = d - \sum_{j\in S} c_j < c_t$, where $t \neq k, t \notin S; x_i = 0, i \in N \setminus (S \cup \{t\}); z_j = 1, j \in S \cup \{t\}, z_i = 0, i \in N \setminus (S \cup \{t\});$

(iii) for all $k \in N$, set $y = 1; x_j = c_j, j \in S \subset N \setminus \{k\}; x_t = d - \sum_{j\in S} c_j < c_t$, where $t \neq k, t \notin S; x_i = 0, i \in N \setminus (S \cup \{t\}); z_j = 1, j \in S \cup \{t, k\}; z_i = 0, i \in N \setminus (S \cup \{t, k\}).$

First, note that the condition $\sum_{j\in N} c_j > d + c_k, \forall k \in N$ guarantees that we can create points of type (ii) and (iii). Then replacing point (i) in equation $\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y = \gamma_0$ implies $\gamma_0 = 0$. Next, substituting points (ii) and (iii) in the equation and subtracting provide $\beta_j = 0, j \in N$.

Now let $i_1, i_2 \in N$. We consider a point of type (ii) where $x_{i_1} = c_{i_1}$ and $x_{i_2} = d - \sum_{j\in S} c_j$. Then we create a new solution by decreasing the value of x_{i_1} by 1 and increasing the value of x_{i_2} by the same value which belongs to K. Substituting these two solutions in the equation and subtracting implies $\alpha_{i_1} = \alpha_{i_2}$. Thus, $\alpha_j = \alpha, j \in N$. So the initial equation becomes $\alpha \sum_{j\in N} x_j + \gamma y = 0$ and finally, replacing point (ii) in this equation gives $\gamma = -\alpha d$ which completes the proof. □

Proposition 2.2.3. *Consider a non-trivial facet-defining inequality*

$$\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j \leq \delta y + \gamma. \tag{2.5}$$

Then (i) $\beta_j \leq 0, \forall j \in N$, (ii) $\gamma = 0$, (iii) $\delta \geq 0$, (iv) $\alpha_j \geq 0, \forall j \in N$, (v) *if* $\beta_j < 0$ *then* $\alpha_j > 0, \forall j \in N$.

Proof. Let $\mathcal{F}$ be the facet defined by (2.5). Proof of (i). Suppose $\beta_j > 0$. We show that $z_j = 1, \forall (x,z,y) \in \mathcal{F}$. So assume to the contrary that there exists a point $(x^*, z^*, y^*) \in \mathcal{F} \cap X_{binary}$ satisfying $z_j^* = 0$, then the point $(x^*, z', y^*) \in X_{binary}$ where $z_j' = 1, z_k' = z_k^*, k \neq j$ violates (2.5). Hence, $\mathcal{F} \subseteq \{(x,z,y) \mid z_j = 1\}$, which is a contradiction.

Proof of (ii). Since $(\mathbf{0}, \mathbf{0}, 0) \in X_{binary}$ and inequality (2.5) is valid for X_{binary}, then $\gamma \geq 0$. Suppose $\gamma > 0$. Since there can be no point in $\mathcal{F}$ with $y = 0$ (because $x_j = 0, j \in N$ and $\beta_j \leq 0, j \in N$) then $y = 1, \forall (x,z,y) \in \mathcal{F}$. Thus, $\mathcal{F} \subseteq \{(x,z,y) \mid y = 1\}$, which is a contradiction.

Proof of (iii). Since $(\mathbf{0}, \mathbf{0}, 1) \in X_{binary}$, $\gamma = 0$, and (2.5) is valid for X_{binary}, then $\delta \geq 0$.

Proof of (iv). Suppose to the contrary that $\alpha_j < 0$ for some $j \in N$. There must exist a point $(x^*, z^*, y^*) \in \mathcal{F} \cap X_{binary}$ satisfying $x_j^* > 0$, since otherwise $\mathcal{F} \subseteq \{(x,z,y) \mid x_j = 0\}$. As $(x^*, z^*, y^*) \in \mathcal{F}$, then $\sum_{i \in N} \alpha_i x_i^* + \sum_{i \in N} \beta_i z_i^* = \delta y^*$. Then we generate a new point $(x', z^*, y^*) \in X_{binary}$ such that $x_i' = x_i^*, \forall i \neq j, x_j' = 0$. Clearly, $(x', z^*, y^*) \in X_{binary}$ violates inequality (2.5). Therefore $\alpha_j \geq 0, \forall j \in N$.

Proof of (v). Suppose that for some $j \in N$, $\beta_j < 0$ and $\alpha_j = 0$. Then as in the proof of (iv) we can show that all the points in $\mathcal{F}$ satisfy $z_j = 0$ and so $\mathcal{F} \subseteq \{(x,z,y) \mid z_j = 0\}$, which is a contradiction. □

Set X_{binary} is very closely related to X^1. The following property relates valid inequalities for the two sets: X_{binary} and X^1.

Proposition 2.2.4. *Consider the following inequality*

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j \leq \delta, \tag{2.6}$$

(i) *If* (2.6) *is valid for* X^1, *then* (2.6) *is valid for* X_{binary}.
(ii) *If* $\beta_j \leq 0, \forall j \in N$, *then inequality* (2.6) *is valid for* X^1, *if and only if*

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j \leq \delta y, \tag{2.7}$$

is valid for X_{binary}.

Proof. (i) Suppose $(x,z,y) \in X_{binary}$ violates (2.6). Hence $y = 0$, which implies $x_j = 0, \forall j \in N$. Thus, the point $(\mathbf{0}, z, 1) \in X^1$ and violates inequality (2.6), which is a contradiction.
(ii) Consider $(x,z,y) \in X_{binary}$ and suppose that (2.6) is valid for X^1. If $y = 1$, then validity of (2.6) implies that (x,z,y) satisfies (2.7). If $y = 0$, then $\sum_{j \in N} x_j \leq dy = 0$ and $x_j \geq 0$ imply that $x_j = 0, \forall j \in N$. Since $\beta_j \leq 0$ and $z_j \geq 0$, then $\sum_{j \in N} \beta_j z_j \leq 0$ which shows that (x,z,y) satisfies (2.7). Now suppose that (2.7) is valid for X_{binary}. Since X^1 is a restriction of X_{binary} with $y = 1$ so (2.6) is valid for X^1. □

From Proposition 2.2.4, part (i), it follows that the flow covers are valid for X_{binary}. Moreover, the inequalities

$$\sum_{j \in S} x_j - \sum_{j \in S^+} (c_j - \lambda) z_j \leq \Big(d - \sum_{j \in S^+} (c_j - \lambda)\Big) y, \forall S \subseteq N, \tag{2.8}$$

where S is a cover and $S^+ = \{j \in S : c_j > \lambda\}$, are valid for X_{binary}. Inequalities (2.8) can be regarded as strengthened simple flow cover inequalities.

Observing that the point $(x, z, y) = (\mathbf{0}, \mathbf{0}, 0)$ satisfies (2.7) as equation, it is straightforward to check the following result.

Proposition 2.2.5. *If* (2.6) *defines a facet of* P^1, *then* (2.7) *defines a facet of* P_{binary}.

However, the structure of P_{binary} is richer than the structure of P^1 since it includes many new facet-defining inequalities. Next we introduce a new family of facet-defining inequalities.

Proposition 2.2.6. *The inequality*

$$x_j \leq c_j y, \; j \in N, \tag{2.9}$$

is valid for X_{binary} *and defines a non-trivial facet of* P_{binary}.

Proof. In order to show the validity, let $(x, z, y) \in X_{binary}$. If $y = 0$, then $\sum_{i \in N} x_i \leq dy$ and $x_i \geq 0, i \in N$ imply $x_j = 0$. Now let $y = 1$. Then the validity of (2.9) follows from $x_j \leq c_j z_j$ and $z_j \leq 1$.

To prove that (2.9) defines a facet it suffices to generate $2n + 1$ affinely independent points as follows.

- $v_0 : y = 0; x_i = 0, i \in N; z_i = 0, i \in N;$
- $v_1, \ldots, v_n$: for some $k \in N$, we set $y = 0; x_i = 0, i \in N; z_k = 1; z_i = 0, i \in N \setminus \{k\};$
- $v_{n+1} : y = 1; x_j = c_j; x_i = 0, i \in N \setminus \{j\}; z_j = 1; z_i = 0, i \in N \setminus \{j\};$
- $v_{n+2}, \ldots, v_{2n}$: for some $k \in N \setminus \{j\}$, we set $y = 1; x_j = c_j; x_k = b_k; x_i = 0, i \in N \setminus \{j, k\}; z_j = z_k = 1; z_i = 0, i \in N \setminus \{j, k\}$; where $b_k = \min\{c_k, d - c_j\}$.

Now we justify that these points are affinely independent. Since $v_0 = (\mathbf{0}, \mathbf{0}, 0)$ is one of the points so it suffices to show that $v_1, \ldots, v_{2n}$ are linearly independent. Let $\sum_{i=1}^{2n} \alpha_i v_i = \mathbf{0}$, where $\alpha_i, i = 1, \ldots, 2n$ are scalars which are not all zero. This equation gives the following system.

$$\begin{cases} b_1\alpha_{n+2} = 0, \\ \vdots \\ b_{j-1}\alpha_{n+j} = 0, \\ c_j\alpha_{n+1} + \cdots + c_j\alpha_{2n} = 0, \\ b_{j+1}\alpha_{n+j+1} = 0, \\ \vdots \\ b_n\alpha_{2n} = 0, \\ \alpha_1 + \alpha_{n+2} = 0, \\ \vdots \\ \alpha_{j-1} + \alpha_{n+j} = 0, \\ \alpha_j + \alpha_{n+1} + \cdots + \alpha_{2n} = 0, \\ \alpha_{j+1} + \alpha_{n+j+1} = 0, \\ \vdots \\ \alpha_n + \alpha_{2n} = 0, \\ \alpha_{n+1} + \cdots + \alpha_{2n} = 0. \end{cases}$$

Then the first n equations of the above-mentioned system imply $\alpha_{n+1} = \cdots = \alpha_{2n} = 0$ and the next n equations give $\alpha_1 = \cdots = \alpha_n = 0$. □

2.3 Set-Up Flow Cover Inequalities

In this section we introduce the set-up flow cover inequalities which can be seen as an extension of the flow cover inequalities to set X_{binary}.

Proposition 2.3.1. *Let S be a cover with $\max_{j\in S} c_j > \lambda$. For each $\emptyset \neq \overline{S}^+ \subseteq S^+ = \{j \in S : c_j > \lambda\}$, the simple set-up flow cover inequality*

$$\sum_{j\in S} x_j - \sum_{j\in\overline{S}^+} (c_j - \lambda) z_j \leq \Big(d - \sum_{j\in\overline{S}^+} (c_j - \lambda)\Big) y, \tag{2.10}$$

is a facet of P_{binary}.

Proof. First, we justify the validity. If $y = 0$, then $x_j = 0, \forall j \in N$. Since, for $j \in \overline{S}^+$, $c_j - \lambda > 0$ and $z_j \geq 0$, then $-\sum_{j\in\overline{S}^+}(c_j - \lambda)z_j \leq 0$, which implies (2.10).

Now assume $y = 1$. Let (x, z, y) be a point of X_{binary} with $z_i = 1$ for $i \in T$, and $z_i = 0$ otherwise. We consider the following cases.

Case 1. $|\overline{S}^+ \setminus T| = 0$. It implies $z_j = 1, \forall j \in \overline{S}^+$ and so the validity of (2.10) follows from $\sum_{j\in S} x_j \leq d$ clearly.

Case 2. $|\overline{S}^+ \setminus T| \geq 1$. Then

$$\sum_{j\in S} x_j - \sum_{j\in\overline{S}^+}(c_j-\lambda)z_j = \sum_{j\in S\cap T} x_j - \sum_{j\in\overline{S}^+\cap T}(c_j-\lambda) \leq \sum_{j\in S\cap T} c_j - \sum_{j\in\overline{S}^+\cap T} c_j + |\overline{S}^+\cap T|\lambda$$
$$= \sum_{j\in S\cap T} c_j + \sum_{j\in\overline{S}^+\setminus T} c_j - \sum_{j\in\overline{S}^+} c_j - \lambda + (|\overline{S}^+\cap T|+1)\lambda \leq \sum_{j\in S} c_j - \lambda - \sum_{j\in\overline{S}^+} c_j$$
$$+ (|\overline{S}^+\cap T|+1)\lambda = d - \sum_{j\in\overline{S}^+} c_j + (|\overline{S}^+\cap T|+1)\lambda \leq d - \sum_{j\in\overline{S}^+}(c_j-\lambda),$$

where the last inequality follows from $|\overline{S}^+ \setminus T| \geq 1$ which implies $|\overline{S}^+ \cap T| \leq |\overline{S}^+| - 1$.

To prove (2.10) defines a facet we construct $2n+1$ affinely independent points of the form $(X_{S\setminus\overline{S}^+}, X_{\overline{S}^+}, X_{N\setminus S}, Z_{S\setminus\overline{S}^+}, Z_{\overline{S}^+}, Z_{N\setminus S}, y)$, satisfying (2.10) as equation, where X_J is the vector of x_j's for $j \in J \subseteq N$. Since S is a cover, there exist $s = | S |$ affinely independent points $(X^k_{S\setminus\overline{S}^+}, X^k_{\overline{S}^+})$, $k \in S$ satisfying $0 \leq x_j \leq c_j$ for $j \in S$ and $\sum_{j\in S} x_j = d$. We assume $S = \{1, \ldots s\}$. Now, for $k \in \overline{S}^+$, let

$$l_k = \max\Big\{\sum_{j\in S} x_j - \sum_{j\in\overline{S}^+}(c_j-\lambda)z_j + \sum_{j\in\overline{S}^+\setminus\{k\}}(c_j-\lambda) \mid \sum_{j\in S} x_j \leq d,$$
$$x_j \leq c_j z_j, j \in S, z_j \in \{0,1\}, j \in S, z_k = 0\Big\},$$

and let $\overline{X}^k = (\overline{X}^k_{S\setminus\overline{S}^+}, \overline{X}^k_{\overline{S}^+})$ be an optimal solution of this maximization problem. From validity of (2.10), and $z_k = 0$, we have $l_k + (c_k - \lambda) \leq d$. On the other hand, as $c_k > \lambda$, then $\sum_{j\in S\setminus\{k\}} c_j \leq d$. Hence, considering the solution $z_j = 1$, and $x_j = c_j$ for all $j \in S \setminus \{k\}$, $y = 1$, and $z_k = x_k = 0$, we have $l_k \geq d - (c_k - \lambda)$. Thus, $l_k = d - (c_k - \lambda)$.

Combining the assumptions $\sum_{j\in N} c_j > d + c_k$ and $\max_{j\in S} c_j > \lambda$ gives $S \subsetneq N$. Without loss of generality, assume that $1 \in \overline{S}^+$. For each vector $\overline{X}^k$ with the property $\sum_{j\in S} \overline{X}^k_j = d - (c_k - \lambda)$, we define $\varepsilon_k > 0$ such that $\sum_{j\in S} \overline{X}^k_j + \varepsilon_k \sum_{j\in N\setminus S} c_j = d$. In fact, $\varepsilon_k = (c_k - \lambda)/(\sum_{j\in N\setminus S} c_j)$.

Let e_j denote the jth unit vector, $\mathbf{1}$ denote the vector whose components are all one, and $\mathbf{0}$ denote the vector whose components are all zero. Then consider the following points:

(i) $(X^k_{S\setminus\overline{S}^+}, X^k_{\overline{S}^+}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, 1)$,	$k \in S$,
(ii) $(\overline{X}^k_{S\setminus\overline{S}^+}, \overline{X}^k_{\overline{S}^+}, \mathbf{0}, \mathbf{1}, \mathbf{1} - e_k, \mathbf{0}, 1)$,	$k \in \overline{S}^+$,
(iii) $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, e_j, 0)$,	$j \in N \setminus S$,
(iv) $(\overline{X}^1_{S\setminus\overline{S}^+}, \overline{X}^1_{\overline{S}^+}, \varepsilon_1 c_j e_j, \mathbf{1}, \mathbf{1} - e_1, e_j, 1)$,	$j \in N \setminus S$,
(v) $(\mathbf{0}, \mathbf{0}, \mathbf{0}, e_k, \mathbf{0}, \mathbf{0}, 0)$,	$k \in S \setminus \overline{S}^+$,
(vi) $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, 0)$.	

The set of given points belong to X_{binary} and satisfies inequality (2.10) at equality. Suppose that these points lie on the following hyperplane.

$$\sum_{j\in N}\alpha_j x_j + \sum_{j\in N}\beta_j z_j = \gamma y + \gamma_0. \tag{2.11}$$

Substituting point (vi) in hyperplane (2.11) gives $\gamma_0 = 0$. Using points of type (iii) and (v) we have $\beta_j = 0, j \in N\backslash\overline{S}^+$. Since points (i) lie in the hyperplane $\sum_{j\in S}\alpha_j x_j + \sum_{j\in\overline{S}^+}\beta_j = \gamma$, and uniquely define $\sum_{j\in S} x_j = d$, then $\alpha_j = \alpha, j \in S$ and $\alpha d + \sum_{j\in\overline{S}^+}\beta_j = \gamma$. Considering the point of type (ii) with $k = 1$ and points in (iv) we obtain $\alpha_j = 0, j \in N \setminus S$. By substituting the points (ii) in (2.11) it follows that $\alpha\sum_{j\in S}\overline{X}_j^k + \sum_{j\in\overline{S}^+}\beta_j - \beta_k = \gamma$, for $k \in \overline{S}^+$. Combining this equation with $\alpha d + \sum_{j\in\overline{S}^+}\beta_j = \gamma$ and $\sum_{j\in S}\overline{X}_j^k = d - (c_k - \lambda)$ implies $\beta_k = -\alpha(c_k - \lambda), k \in \overline{S}^+$. Finally, using any point of type (i) it follows that $\gamma = \alpha\big(d - \sum_{j\in\overline{S}^+}(c_j - \lambda)\big)$. Hence, (2.11) is a positive multiple of the hyperplane defined by (2.10). □

Notice that the simple flow covers can be obtained from (2.10), setting $y = 1$ and considering $\overline{S}^+ = S^+$.

Next we give the extended set-up flow cover inequalities. The result is given without proof since justification can be derived from the lifting of inequalities (2.10) discussed in Section 2.4.

Proposition 2.3.2. *Let S be a cover with $\max_{j\in S} c_j > \lambda$. For each $\emptyset \neq \overline{S}^+ \subseteq S^+ = \{j \in S : c_j > \lambda\}$, and for each $L \subseteq N \setminus S$ where for $k \in L$, $\bar{c} - \lambda < c_k \leq \bar{c}$, and $\bar{c} = \max_{j\in\overline{S}^+} c_j$, the extended set-up flow cover inequality*

$$\sum_{j\in S\cup L} x_j - \sum_{j\in\overline{S}^+}(c_j - \lambda)z_j - \sum_{j\in L}(\bar{c} - \lambda)z_j \leq \Big(d - \sum_{j\in\overline{S}^+}(c_j - \lambda)\Big)y, \tag{2.12}$$

is a facet of P_{binary}.

Observe that proof of this proposition is given in Section 2.4.

When the capacities are constant ($c_j = c, \forall j \in N$) we obtain the following class of inequalities.

Corollary 2.3.3. *Assume $c_j = c, \forall j \in N$, $d > c > 0$, $nc > d$, and assume d is not a multiple of c. Define $r = d - \lfloor \frac{d}{c} \rfloor c$. Let $S_1, S_2 \subseteq N$ such that $S_1 \cap S_2 = \emptyset$ and $|S_1| \leq \lfloor \frac{d}{c} \rfloor$, $\lceil \frac{d}{c} \rceil \leq |S_1| + |S_2|$. Then the following inequality is non-trivial facet of P_{binary}.*

$$\sum_{j\in S_1} x_j + \sum_{j\in S_2}(x_j - rz_j) \leq (d - kr)y, \tag{2.13}$$

where $k = \lceil \frac{d}{c} \rceil - |S_1|$.

Proof. Here we show how to obtain inequality (2.13) from inequality (2.12). Consider inequality (2.12) and let $S \subseteq N$ such that $|S| = \lceil \frac{d}{c} \rceil$. It implies that $\lambda = c - r$ and so $c - \lambda = r$. Then we define $S_1 = S \setminus \overline{S}^+$, $S_2 = \overline{S}^+ \cup L$ and $k = |\overline{S}^+|$. It follows from this definition that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S \cup L$. Therefore, $|S_1| + |S_2| = |S_1 \cup S_2| = |S \cup L| \geq \lceil \frac{d}{c} \rceil$. Since $|\overline{S}^+| \geq 1$, we get $|S_1| = |S| - |\overline{S}^+| \leq \lceil \frac{d}{c} \rceil - 1 = \lfloor \frac{d}{c} \rfloor$. Furthermore, $|S_1| = |S| - |\overline{S}^+| = \lceil \frac{d}{c} \rceil - k$ which is equivalent with $k = \lceil \frac{d}{c} \rceil - |S_1|$. □

Example 2.3.4. *Consider an instance with $n = 4$, $d = 14$, and $c = 5$. So $r = 4$. Using the software PORTA [16], we obtain 18 facet-defining inequalities for P^1 and 57 facet-defining inequalities for P_{binary}. The non-trivial facet-defining inequalities for P^1 are the following.*

$$\begin{aligned}
&x_2 + x_3 + x_4 - 4z_2 - 4z_3 - 4z_4 \leq 2,\\
&x_1 + x_3 + x_4 - 4z_1 - 4z_3 - 4z_4 \leq 2,\\
&x_1 + x_2 + x_4 - 4z_1 - 4z_2 - 4z_4 \leq 2,\\
&x_1 + x_2 + x_3 - 4z_1 - 4z_2 - 4z_3 \leq 2,\\
&x_1 + x_2 + x_3 + x_4 - 4z_1 - 4z_2 - 4z_3 - 4z_4 \leq 2.
\end{aligned}$$

For P_{binary} we have 43 non-trivial inequalities. For instance, considering $S = \{1, 2, 3\}$, we have the following facet-defining inequalities of type (2.13) for $k = 1, 2,$ and 3 :

$$\begin{aligned}
&x_1 + x_2 + x_3 - 4z_2 \leq 10y, \quad k = 1,\\
&x_1 + x_2 + x_3 - 4z_2 - 4z_3 \leq 6y, \quad k = 2,\\
&x_1 + x_2 + x_3 - 4z_1 - 4z_2 - 4z_3 \leq 2y, \quad k = 3.
\end{aligned}$$

Note that for $k = 3$, the inequality appears in P^1 as a facet-defining inequalities by setting $y = 1$. However for $k = 1$ and $k = 2$ the corresponding inequalities for P^1, obtained by setting $y = 1$, are not facet-defining.

Next we give the full polyhedral description of P when the $c_j = c, \forall j \in N$, which is the constant capacitated case.

Theorem 2.3.5. *If $c_j = c, j \in N$, the defining inequalities of X_{binary} with inequalities (2.9) and (2.13) suffice to describe P_{binary}.*

Proof. Set X_{binary} can be decomposed into two mixed-integer sets whose polyhedral characterization is known: set X^1, obtained by restricting $y = 1$, and set X^0 obtained by restricting $y = 0$. The convex hull of X^1, denoted by P^1, was derived in [38], and is given by the trivial facet-defining inequalities and the simple flow cover inequalities. The convex hull of X^0, P^0, is given by

$$P^0 = \Big\{(x, z, y) \in \mathbb{R}^{2n+1} : x_j = 0, j \in N, 0 \leq z_j \leq 1, j \in N, y = 0\Big\}.$$

Using Theorem 1.3.3 implies that polyhedron P_{binary} is the closed convex hull of $P^0 \bigcup P^1$ and can be represented as a linear program in a higher dimensional space as follows:

$$\Big\{(x, z, y, x^0, z^0, y^0, x^1, z^1, y^1, \delta_0, \delta_1) \in \mathbb{R}^{6n+5} :$$
$$x_j^0 = 0, j \in N, 0 \leq z_j^0 \leq \delta^0, j \in N, y^0 = 0,$$
$$x_j^1 \geq 0, j \in N, x_j^1 \leq cz_j^1, j \in N, z_j^1 \leq \delta^1, j \in N, \sum_{j \in N} x_j^1 \leq d\delta^1, y^1 = \delta^1,$$
$$\sum_{j \in S}(x_j^1 - rz_j^1) \leq \Big(d - r\Big\lceil\frac{d}{c}\Big\rceil\Big)\delta^1, \forall S \subseteq N : | S | \geq \Big\lceil\frac{d}{c}\Big\rceil,$$
$$x_j = x_j^0 + x_j^1, z_j = z_j^0 + z_j^1, y = y^0 + y^1, \delta^0 + \delta^1 = 1\Big\}.$$

Projecting out variables $x_j^0, x_j^1, z_j^1, j \in N, \delta^0, \delta^1, y^0, y^1$ (using the equations $x_j^0 = 0$, $x_j^1 = x_j - x_j^0$, $z_j^1 = z_j - z_j^0$, $\delta^0 = 1 - \delta^1$, $\delta^1 = y^1$, $y^0 = 0$, $y^1 = y - y^0$) we obtain:

$$\Big\{(x, z, y, z^0) \in \mathbb{R}^{3n+1} :$$
$$x_j \geq 0, j \in N, \tag{2.14}$$
$$\sum_{j \in N} x_j \leq dy, \tag{2.15}$$
$$\sum_{j \in S}(x_j - rz_j + rz_j^0) \leq \Big(d - r\Big\lceil\frac{d}{c}\Big\rceil\Big)y, \forall S \subseteq N : | S | \geq \Big\lceil\frac{d}{c}\Big\rceil, \tag{2.16}$$
$$z_j - z_j^0 \leq y, j \in N, \tag{2.17}$$
$$z_j^0 \leq 1 - y, j \in N, \tag{2.18}$$
$$x_j \leq c(z_j - z_j^0), j \in N, \tag{2.19}$$
$$z_j^0 \geq 0, j \in N\Big\}. \tag{2.20}$$

Now we use the Fourier-Motzkin elimination (see [36]) to project out variables $z_j^0, j \in N$. Inequalities (2.17) and (2.18) imply $z_j \leq 1$, $\forall j \in N$; inequalities (2.17) and (2.19) imply $x_j \leq cy$ $\forall j \in N$; (2.18) and (2.20) imply $y \leq 1$; (2.19) and (2.20) imply $x_j \leq cz_j, \forall j \in N$. Finally, combining (2.16), with (2.17) for $j \in S_1 \subseteq S$ and (2.20) for $j \in S_2 = S \setminus S_1$ we have (2.13). Notice that when $|S_1| \geq \lceil\frac{d}{c}\rceil$ the projected inequality does not define a facet. Hence, the projected polyhedron is P. □

Next we explain the relation between the polyhedra defined by the simple flow covers (2.1), P_{SFC}, the strengthened simple flow covers (2.8), P_{SFC}^Y, and the polyhedron defined by the simple set-up flow covers (2.10), P_{SSFC}^Y.

Proposition 2.3.6. *The inclusions $P_{SSFC}^Y \subseteq P_{SFC}^Y \subseteq P_{SFC}$ hold. Moreover, we have*

$$P_{SFC}^Y \bigcap \Big\{(x, z, y) : z_j \leq y, \forall j \in N\Big\} = P_{SSFC}^Y \bigcap \Big\{(x, z, y) : z_j \leq y, \forall j \in N\Big\}.$$

Proof. The first two inclusions are trivial. Suppose $z_j \leq y, \forall j \in N$. Since $P^Y_{SSFC} \subseteq P^Y_{SFC}$ we have

$$P^Y_{SSFC} \bigcap \Big\{(x,z,y) : z_j \leq y, \forall j \in N\Big\} \subseteq P^Y_{SFC} \bigcap \Big\{(x,z,y) : z_j \leq y, \forall j \in N\Big\}.$$

To prove the inclusion $\subseteq$ we show that inequalities (2.10) with $\overline{S}^+ \subsetneq S^+$ do not define facets. For each $\emptyset \subsetneq \overline{S}^+ \subsetneq S^+$ we have $\sum_{j \in S^+ \setminus \overline{S}^+} z_j \leq \sum_{j \in S^+ \setminus \overline{S}^+} y$. Thus,

$$\sum_{j \in S^+ \setminus \overline{S}^+} (c_j - \lambda) z_j \leq \sum_{j \in S^+ \setminus \overline{S}^+} (c_j - \lambda) y.$$

Adding this inequality to (2.8) (which is (2.10) with $\overline{S}^+ = S^+$) we obtain the set-up flow cover (2.10) defined by S^+ and $\overline{S}^+$. □

Restrictions $z_j \leq y$ occur in some practical problems where a set-up of an arc can occur only if the node is open, see for example [7]. Proposition 2.3.6 states that in such cases inequalities with $\overline{S}^+ \subsetneq S^+$ are dominated and it suffices to strengthen the flow covers (2.8) to get the non-dominated inequalities.

Example 2.3.7. *Consider the data in Example 2.3.4 and the fractional solution* $y^* = 0.7, z_1^* = 1, z_2^* = 1, z_3^* = 0.5, z_4^* = 0, x_1^* = 5, x_2^* = 2.3, x_3^* = 2.5, x_4^* = 0$. *There is no flow cover inequality* (2.2) *and no strengthened flow cover inequality* (2.8) *cutting off the extreme point. However the inequality* $x_1 + x_2 + x_3 - 4z_3 \leq 10y$ *is violated.*

Now we consider the separation problem associated with the set-up flow cover inequalities. Consider a fractional solution (x^*, z^*, y^*). If there is an inequality (2.10) cutting off (x^*, z^*, y^*) for a given set S and S^+, the most violated inequality is obtained by considering $\overline{S}^+ = \{j \in S^+ | z_j^* \leq y^*\}$. Thus, any separation heuristic for flow covers directly leads to a separation heuristic for inequalities (2.10). Following [38], for each λ one can find the most violated inequality (2.10) by solving the knapsack problem:

$$\eta_\lambda = \max \Big\{ \sum_{j \in N} \tau_j(\lambda) w_j \mid \sum_{j \in N} c_j w_j = d + \lambda, \quad w_j \in \{0,1\}, j \in N \Big\},$$

where

$$\tau_j(\lambda) = x_j^* + (c_j - \lambda)^+ \times (y^* - z_j^*)^+.$$

Let $U = \{j \in N | w_j = 1\}$ and $\overline{U}^+ = \{j \in U | c_j > \lambda \wedge y^* > z_j^*\}$. If $\eta_\lambda > dy^*$ then a violated inequality (2.10) with $S = U$ and $\overline{S}^+ = \overline{U}^+$ has been found. Otherwise, no such violated inequality exists.

In the constant capacitated case, the separation of (2.13) amounts to checking whether

$$\max_{S_1, S_2 \subseteq N, S_1 \cap S_2 = \emptyset, k = \lceil \frac{d}{c} \rceil - |S_1|} \Big\{ \sum_{j \in S_1} x_j^* + \sum_{j \in S_2} (x_j^* - r z_j^*) + k r y^* \Big\},$$

is strictly greater than dy^* (the inequality induced by S_1, S_2, and k is violated) or not. In the last case there are no violated inequalities in this family.

The foregoing maximization problem is equivalent to the following integer program.

$$
\begin{aligned}
(IP) \quad \max \; & \sum_{j \in N} \big(u_j x_j^* + v_j(x_j^* - r z_j^*)\big) + kry^* \\
s.t. \; & \sum_{j \in N} u_j + k = \left\lceil \frac{d}{c} \right\rceil, \\
& u_j + v_j \leq 1, j \in N, \\
& k \leq \left\lceil \frac{d}{c} \right\rceil, \\
& u_j, v_j \in \{0,1\}, j \in N, k \in \mathbb{Z}^+,
\end{aligned}
$$

where $u_j = 1$ if and only if $j \in S_1$ and $v_j = 1$ if and only if $j \in S_2$. Observe that Theorem 1.2.38 implies the coefficient matrix of the foregoing program is totally unimodular and so it is enough to solve the linear relaxation of this program to get the optimal integer solution. In fact, integer program (IP) is a special case of the *Transportation Problem* for which there are very efficient combinatorial algorithms.

2.4 Lifting the Set-Up Flow Cover Inequalities

In this section we discuss the lifting of inequalities (2.10), following the approach presented in [23].

For $T \subset N$, let

$$\overline{X} = \Big\{(x, z, y) \in X | (x_j, z_j) = (0, 0), j \in T\Big\},$$

and consider a valid inequality (2.10) for $\overline{X}$. For a given variable pair $(x_k, z_k), k \in T$, we want to determine the coefficients α_k, β_k such that the inequality

$$\sum_{j \in S} x_j + \alpha_k x_k - \sum_{j \in \overline{S}^+} (c_j - \lambda) z_j + \beta_k z_k \leq \Big(d - \sum_{j \in \overline{S}^+} (c_j - \lambda)\Big) y, \tag{2.21}$$

is valid for $\overline{X}^k = \big\{(x, z, y) \in X | (x_j, z_j) = (0, 0), j \in T \setminus \{k\}\big\}$. Let

$$h_k(u) = \max \Big\{\alpha_k x_k + \beta_k z_k | x_k = u, 0 \leq x_k \leq c_k z_k, z_k \in \{0, 1\}\Big\},$$

and consider the lifting function:

$$
\begin{aligned}
f(u) \quad = \min \quad & \Big(d - \sum_{j\in\overline{S}^+}(c_j-\lambda)\Big)y - \sum_{j\in S} x_j + \sum_{j\in\overline{S}^+}(c_j-\lambda)z_j \\
s.t. \quad & \sum_{j\in N\setminus T} x_j \le dy - u, \\
& 0 \le x_j \le c_j z_j,\ j\in N\setminus T, \\
& z_j \in \{0,1\},\ j \in N\setminus T, \\
& y \in \{0,1\},
\end{aligned}
$$

where $S \subseteq N\setminus T$ is a cover with $\max_{j\in S} c_j > \lambda$. Then inequality (2.21) is valid for $\overline{X}^k$ if and only if $h_k(u) \le f(u), 0 \le u \le c_k$. Moreover, in order to obtain a strongest lifted inequality (known as *maximal* lifting), α_k and β_k should be such that the equation $h_k(u) = f(u)$ has two linearly independent solutions. If (2.10) defines a facet for $conv(\overline{X})$ and the lifting is maximal, then the resulting inequality defines a facet for $conv(\overline{X}^k)$.

First we characterize function f. Feasibility of the lifting problem associated with the lifting function $f(u)$, for $u > 0$, implies $y = 1$ because $x_j \ge 0, j \in N\setminus T$ and $\sum_{j\in N\setminus T} x_j \le dy - u$. Hence the lifting function is similar to the one given in [23], p. 450, for the flow covers on $[0, d]$.

Let $\overline{S}^+ = \{\ell_1, \ldots, \ell_r\}$ with $c_{\ell_i} \ge c_{\ell_{i+1}}$ for $i = 1,\ldots,r-1$. Function f, for $u \ge 0$, can be written as

$$
f(u) = \begin{cases}
i\lambda, & M_i \le u \le M_{i+1} - \lambda, i = 0,\ldots,r-1, \\
u - M_i + i\lambda, & M_i - \lambda \le u \le M_i, i = 1,\ldots,r-1, \\
u - M_r + r\lambda, & M_r - \lambda \le u \le d,
\end{cases}
$$

where $M_0 = 0$ and $M_i = \sum_{k=1}^{i} c_{\ell_k}$ for $i = 1,\ldots,r$.

From Theorem 6 in [23], function f is superadditive on $[0, d]$. Hence, the lifting of all variable pairs $(x_j, z_j), j \in T$ can be done simultaneously. Different functions $h_j(u)$ can be defined for each $j \in T$, leading to maximal lifted inequalities. For each $j \in T$ we define $h_j(u)$ as a line passing through the points $(u, h_j(u))$ for $u = c_j$ and $u = M_i - \lambda$ where $i = argmax\{t \in \{1,\ldots,r\} | M_t - \lambda \le c_j\}$. It can be easily checked that $h_j(u)$ underestimates f in $[0, c_j]$. From this discussion, and computing the values of α_j, β_j such that $h_j(u) = f(u)$ for the two points given above, it follows that the following inequalities are valid for X_{binary}.

$$
\sum_{j\in S} x_j + \sum_{j\in T}\alpha_j x_j - \sum_{j\in\overline{S}^+}(c_j-\lambda)z_j + \sum_{j\in T}\beta_j z_j \le \Big(d - \sum_{j\in\overline{S}^+}(c_j-\lambda)\Big)y, \qquad (2.22)
$$

where,

$$(\alpha_j, \beta_j) = \begin{cases} \left(\frac{\lambda}{c_j - M_i + \lambda}, (i-1)\lambda - \frac{\lambda(M_i - \lambda)}{c_j - M_i + \lambda}\right), \; if \; M_i \le c_j \le M_{i+1} - \lambda, \\ \\ (1, i\lambda - M_i), \; otherwise. \end{cases}$$

Here we explain how to obtain inequality (2.12) by lifted inequality (2.22). First, note that $\bar{c} = max\{c_j | j \in \overline{S}^+\} = c_{l_1} = M_1$. So condition $\bar{c} - \lambda < c_j \le \bar{c}, \forall j \in L$ is equivalent to $M_1 - \lambda < c_j \le M_1, \forall j \in L$. It follows from this condition that $\alpha_j = 1$ and $\beta_j = \lambda - M_1 = \lambda - \bar{c}$. Substituting these values in (2.22) implies inequality (2.12). Furthermore, since $\overline{X}$ is full-dimensional, inequality (2.10) defines a facet of $conv(\overline{X})$, and equation $h_j(u) = f(u)$ has two linearly independent solutions, so inequality (2.12) define a facet of P_{binary} (see Theorem 1.3.6).

An interesting question arises when we consider more general sets X' obtained from X_{binary} replacing the inequality $\sum_{j \in N} x_j \le dy$ by $\sum_{j \in N} x_j - \sum_{j \in N^-} x_j \le dy$ or by $\sum_{j \in N} x_j \le dy + s$ with $s \ge 0$. In both cases (2.10) is valid for the restriction of X' to the subspace defined by $(x_j, z_j) = (0, 0), j \in T \cup N^-$, or $s = 0$, respectively. For these cases, in order to lift (2.10), we need to consider $f(u)$ for $u < 0$. For negative u, the minimum of the lifting function is obtained by setting $y = 0$. For example, as long as $u \ge -\sum_{j \in S \setminus \overline{S}^+} c_j$, the value of $f(u)$ is obtained by setting $y = 0$, $x_j = z_j = 0, j \in \overline{S}^+$ and $\sum_{j \in S \setminus \overline{S}^+} x_j = u$. Hence, for $u < 0$, we have

$$f(u) = \begin{cases} -\gamma - r\lambda, & u \le -\sum_{j \in S} c_j, \\ u + N_j - j\lambda, & -\gamma - N_j \le u \le -\gamma - N_j + \lambda, j = 1, \dots, r, \\ -\gamma - j\lambda, & -\gamma - N_{j+1} + \lambda \le u \le -\gamma - N_j, j = 0, \dots, r-1, \\ u, & -\gamma \le u \le 0, \end{cases}$$

where $\gamma = \sum_{j \in S \setminus \overline{S}^+} c_j$, $N_0 = 0$ and $N_j = \sum_{k=r-j+1}^{r} c_{\ell_k}$ for $j = 1, \dots, r$.

Function $f(u)$ is not superadditive in all its domain. In order to perform a simultaneous lifting we derive superadditive functions that underestimate f (called valid lifting functions). Since $f(u)$ is superadditive for $u \ge 0$, one such function can be obtained by underestimating f on the negative side:

$$g_1(u) = \begin{cases} u & u < 0, \\ f(u) & u \ge 0. \end{cases}$$

Proposition 2.4.1. *Function g_1 is a valid superadditive lifting function for f.*

Proof. Obviously we have $g_1(u) \le f(u)$, for all $u \in [-\infty, d]$. Then we justify superadditivity. Since g_1 is superadditive on $[0, d]$ and on $[-\infty, 0]$, separately, we only need to prove that $g_1(u_1) + g_1(u_2) \le g_1(u_1 + u_2)$ when u_1 and u_2 have opposite signs. So we consider two following cases.

Case I. $u_1 < 0$ and $M_i \le u_2 \le M_{i+1} - \lambda$. Then $g_1(u_1) = u_1$ and $g_1(u_2) = i\lambda$.

If $u_1 + u_2 < 0$, then

$$g_1(u_1 + u_2) = u_1 + u_2 \geq u_1 + M_i = u_1 + \sum_{t=1}^{i} c_{\ell_t} \geq u_1 + i\lambda = g_1(u_1) + g_1(u_2).$$

Now assume $u_1 + u_2 \geq 0$. We consider the following subcases.

Subcase 1. $M_j \leq u_1 + u_2 \leq M_{j+1} - \lambda$, for some $j \leq i$. As $u_1 + u_2 \leq M_{j+1} - \lambda$ and $-u_2 \leq -M_i$, then

$$u_1 \leq M_{j+1} - \lambda - M_i = M_{j+1} - \lambda - M_{j+1} - \sum_{t=j+2}^{i} c_{\ell_t} \leq -\lambda - (i - j - 1)\lambda = (j - i)\lambda.$$

Hence

$$g_1(u_1 + u_2) = j\lambda = (j - i)\lambda + i\lambda \geq u_1 + i\lambda = g_1(u_1) + g_1(u_2).$$

Subcase 2. $M_j - \lambda \leq u_1 + u_2 \leq M_j$, for some $j \leq i$. As $u_2 \geq M_i = M_j + \sum_{t=j+1}^{i} c_{\ell_t} \geq M_j + (i - j)\lambda$, then

$$g_1(u_1 + u_2) = u_1 + u_2 - M_j + j\lambda \geq u_1 + (i - j)\lambda + j\lambda = u_1 + i\lambda = g_1(u_1) + g_1(u_2).$$

Case II. $u_1 < 0$ and $M_i - \lambda \leq u_2 \leq M_i$. So $g_1(u_1) = u_1$ and $g_1(u_2) = u_2 - M_i + i\lambda$.

If $u_1 + u_2 < 0$, then

$$\begin{aligned} g_1(u_1 + u_2) &= u_1 + u_2 = u_1 + u_2 - M_i + M_i = u_1 + u_2 - M_i + \sum_{t=1}^{i} c_{\ell_t} \\ &\geq u_1 + u_2 - M_i + i\lambda = g_1(u_1) + g_1(u_2). \end{aligned}$$

Now let $u_1 + u_2 \geq 0$. We have two subcases as follows.

Subcase 1. $M_j \leq u_1 + u_2 \leq M_{j+1} - \lambda$, for some $j \leq i$. Then $g_1(u_1 + u_2) = j\lambda$. So

$$\begin{aligned} g_1(u_1) + g_1(u_2) &= u_1 + u_2 - M_i + i\lambda \leq M_{j+1} - \lambda - M_i + i\lambda \\ &= M_{j+1} - \lambda - M_{j+1} - \sum_{t=j+2}^{i} c_{\ell_t} + i\lambda = -\lambda - \sum_{t=j+2}^{i} c_{\ell_t} + i\lambda \\ &\leq -\lambda - (i - j - 1)\lambda + i\lambda = j\lambda = g_1(u_1 + u_2). \end{aligned}$$

Subcase 2. $M_j - \lambda \leq u_1 + u_2 \leq M_j$, for some $j \leq i$. Then $g_1(u_1 + u_2) = u_1 + u_2 - M_j + j\lambda$. Therefore

$$\begin{aligned} g_1(u_1) + g_1(u_2) &= u_1 + u_2 - M_i + i\lambda = u_1 + u_2 - M_i + i\lambda - M_j + M_j \\ &= u_1 + u_2 - M_j - \sum_{t=j+1}^{i} c_{\ell_t} + i\lambda - M_j + M_j = u_1 + u_2 - \sum_{t=j+1}^{i} c_{\ell_t} + i\lambda - M_j \\ &\leq u_1 + u_2 - (i - j)\lambda + i\lambda - M_j = u_1 + u_2 + j\lambda - M_j = g_1(u_1 + u_2). \end{aligned}$$

□

This function may differ from f largely when $u < -\gamma$. In such cases, and when $\gamma > \lambda$ we can use the following function, g_2, that provides a better approximation of f for $u < 0$ but differs from f on the positive side.

$$g_2(u) = \begin{cases} u + N_r + kc_r - (r+k)\lambda, & -\gamma - N_r - kc_r \le u \le -\gamma - N_r - kc_r + \lambda, k \ge 1, \\ -\gamma - (r+k)\lambda, & -\gamma - N_r - (k+1)c_r + \lambda \le u \le -\gamma - N_r - kc_r, k \ge 0, \\ u + N_j - j\lambda, & -\gamma - N_j \le u \le -\gamma - N_j + \lambda, j = 1, \dots, r, \\ -\gamma - j\lambda, & -\gamma - N_{j+1} + \lambda \le u \le -\gamma - N_j, j = 0, \dots, r-1, \\ u, & -\gamma \le u \le 0, \\ i\lambda, & ic_1 \le u \le (i+1)c_1 - \lambda, i \ge 0, \\ u - ic_1 + i\lambda, & ic_1 - \lambda \le u \le ic_1, i \ge 1, \end{cases}$$

where $c_1 = max\{c_j | j \in \overline{S}^+\}$ and $c_r = min\{c_j | j \in \overline{S}^+\}$.

Proposition 2.4.2. *Function g_2 is a valid superadditive lifting function for f if $\gamma > \lambda$.*

Proof. It can be checked readily that $g_2(u) \le f(u)$, for all $u \in [-\infty, +\infty]$. Then since function g_2 is superadditive on $[0, +\infty]$ (see [23]) so we only prove that $g_2(u_1) + g_2(u_2) \le g_2(u_1 + u_2)$ when u_1 and u_2 have opposite signs or negative signs by considering the following cases.

Case *I*. $ic_1 \le u_1 \le (i+1)c_1 - \lambda$ and $-\gamma \le u_2 \le 0$. Then let $u_1 = ic_1 + \delta_1$ where $0 \le \delta_1 \le c_1 - \lambda$, and $u_2 = -\delta_2$ where $0 \le \delta_2 \le \gamma$. So $g_2(u_1) = i\lambda$ and $g_2(u_2) = -\delta_2$. Then it can be seen that $g_2(u_1) + g_2(u_2) = i\lambda - \delta_2 \ge -\gamma$. So we consider two subcases as follows.

Subcase 1. If $-\gamma \le g_2(u_1) + g_2(u_2) \le 0$. Then

$$u_1 + u_2 = ic_1 + \delta_1 - \delta_2 \ge ic_1 + \delta_1.$$

Since g_2 is non-decreasing then the foregoing inequality implies

$$g_2(u_1 + u_2) \ge g_2(ic_1 + \delta_1) = ic_1 + \delta_1 = g_2(u_1) + g_2(u_2).$$

Subcase 2. If $g_2(u_1) + g_2(u_2) > 0$. So assume $i\lambda - \delta_2 = t\lambda + \delta$ for some $t \ge 0$ and $0 \le \delta < \lambda$. Observe that case $i < t$ cannot occur. Moreover, case $i = t$ implies $\delta = \delta_2 = 0$ where the superadditivity is trivial. So let $i > t$. Then

$$\begin{aligned} u_1 + u_2 = ic_1 + \delta_1 - \delta_2 \ge ic_1 - \delta_2 = ic_1 - (i-t)\lambda + \delta = (t+1)c_1 + (i-t-1)c_1 \\ - (i-t)\lambda + \delta > (t+1)c_1 + (i-t-1)\lambda - (i-t)\lambda + \delta = (t+1)c_1 - \lambda + \delta. \end{aligned}$$

g_2 is non-decreasing so

$$g_2(u_1 + u_2) \ge g_2((t+1)c_1 - \lambda + \delta) = t\lambda + \delta = g_2(u_1) + g_2(u_2).$$

Case II. $ic_1 \leq u_1 \leq (i+1)c_1 - \lambda$ and $-\gamma - N_{j+1} + \lambda \leq u_2 \leq -\gamma - N_j$ where $0 \leq j \leq r-1$. Then let $u_1 = ic_1 + \delta_1$ where $0 \leq \delta_1 \leq c_1 - \lambda$ and $u_2 = -\gamma - N_j - \delta_2$ where $0 \leq \delta_2 \leq c_{l_{r-j}} - \lambda$. So $g_2(u_1) = i\lambda, g_2(u_2) = -\gamma - j\lambda$ and $g_2(u_1) + g_2(u_2) = -\gamma - (j-i)\lambda$. We consider two subcases.

Subcase 1. If $i \leq j$. Then let $k = j - i \geq 0$. So

$$u_1 + u_2 = -\gamma + ic_1 - N_j + \delta_1 - \delta_2 \geq -\gamma + ic_1 - N_{j+1} + \lambda = -\gamma + ic_1 - N_{k+i+1} + \lambda$$
$$= -\gamma + ic_1 - c_{l_{r-k-i}} - \cdots - c_{l_{r-k-1}} - c_{l_{r-k}} - \cdots - c_{l_r} + \lambda \geq -\gamma - N_{k+1} + \lambda,$$

where the last inequality holds because $c_1 \geq c_{l_t}, r-k-i \leq t \leq r-k-1$. Thus

$$g_2(u_1+u_2) \geq g_2(-\gamma - N_{k+1} + \lambda) = -\gamma - k\lambda = -\gamma - (j-i)\lambda = g_2(u_1) + g_2(u_2).$$

Subcase 2. If $i > j$. Let $k = i - j > 0$ which implies $g_2(u_1) + g_2(u_2) = -\gamma + k\lambda \geq -\gamma + \lambda$. Regarding $\gamma > \lambda$, we consider two cases: (a) $-\gamma + \lambda \leq g_2(u_1) + g_2(u_2) \leq 0$, and (b) $g_2(u_1) + g_2(u_2) > 0$.

Let case (a) occurs. Then

$$u_1 + u_2 = -\gamma + ic_1 - N_j + \delta_1 - \delta_2 \geq -\gamma + ic_1 - N_{j+1} + \lambda = -\gamma + (j+1)c_1 + (k-1)c_1$$
$$- N_{j+1} + \lambda > -\gamma + (j+1)c_1 + (k-1)\lambda - N_{j+1} + \lambda \geq -\gamma + (k-1)\lambda + \lambda = -\gamma + k\lambda,$$

where the last inequality follows from $(j+1)c_1 \geq N_{j+1}$. Therefore

$$g_2(u_1+u_2) \geq g_2(-\gamma + k\lambda) = -\gamma + k\lambda = g_2(u_1) + g_2(u_2).$$

Now assume that case (b) happens. Let $\gamma = t\lambda + \delta$ where $0 \leq \delta < \lambda$. So $g_2(u_1) + g_2(u_2) = -\gamma + k\lambda > 0$ implies $0 \leq t < k$. Thus, $g_2(u_1) + g_2(u_2) = (k-t)\lambda - \delta$. Then

$$u_1 + u_2 \geq -\gamma + ic_1 - N_{j+1} + \lambda = -\gamma + (i-j-1)c_1 + (j+1)c_1 - N_{j+1} + \lambda \geq -\gamma$$
$$+ (k-1)c_1 + \lambda = -t\lambda - \delta + (k-1)c_1 + \lambda = -t\lambda - \delta + (k-t)c_1 + (t-1)c_1 + \lambda$$
$$\geq -\delta - (t-1)\lambda + (k-t)c_1 + (t-1)\lambda = (k-t)c_1 - \delta.$$

Hence

$$g_2(u_1+u_2) \geq g_2((k-t)c_1 - \delta) = (k-t)\lambda - \delta = g_2(u_1) + g_2(u_2).$$

Case III. $ic_1 \leq u_1 \leq (i+1)c_1 - \lambda$ and $-\gamma - N_j \leq u_2 \leq -\gamma - N_j + \lambda$ where $1 \leq j \leq r$. Then suppose $u_1 = ic_1 + \delta_1$ where $0 \leq \delta_1 \leq c_1 - \lambda$ and $u_2 = -\gamma - N_j + \delta_2$ where $0 \leq \delta_2 \leq \lambda$. So $g_2(u_1) = i\lambda, g_2(u_2) = -\gamma - j\lambda + \delta_2$ and $g_2(u_1) + g_2(u_2) = -\gamma - (j-i)\lambda + \delta_2$. We consider two subcases as follows.

Subcase 1. If $i < j$. We set $k = j - i$ where $0 < k \leq r$. Then

$$u_1 + u_2 = -\gamma + ic_1 - N_j + \delta_1 + \delta_2 \geq -\gamma + ic_1 - N_{i+k} + \delta_2 = -\gamma + ic_1 - c_{l_{r-k-i-1}}$$
$$- \cdots - c_{l_{r-k}} - c_{l_{r-k+1}} - \cdots - c_{l_r} + \delta_2 \geq -\gamma - N_k + \delta_2,$$

where the last inequality holds since $c_1 \geq c_{l_t}, r-k-i+1 \leq t \leq r-k$. Thus

$$g_2(u_1+u_2) \geq g_2(-\gamma - N_k + \delta_2) = -\gamma - k\lambda + \delta_2 = -\gamma - (j-i)\lambda + \delta_2 = g_2(u_1) + g_2(u_2).$$

Subcase 2. If $i \geq j$. We set $k = i - j \geq 0$. Since $g_2(u_1) + g_2(u_2) = -\gamma + k\lambda + \delta_2 \geq -\gamma$, so we consider two cases: (a) $-\gamma \leq g_2(u_1) + g_2(u_2) \leq 0$, (b) $g_2(u_1) + g_2(u_2) > 0$

Let case (a) happens. Then

$$u_1 + u_2 = -\gamma + ic_1 - N_j + \delta_1 + \delta_2 \geq -\gamma + ic_1 - N_j + \delta_2 = -\gamma + jc_1 + (i-j)c_1 - N_j + \delta_2$$
$$\geq -\gamma + k\lambda + \delta_2,$$

where the last inequality follows from $jc_1 \geq N_j$. So

$$g_2(u_1+u_2) \geq g_2(-\gamma + k\lambda + \delta_2) = -\gamma + k\lambda + \delta_2 = g_2(u_1) + g_2(u_2).$$

Consider case (b) happens. Then let $g_2(u_1) + g_2(u_2) = -\gamma + k\lambda + \delta_2 = t\lambda + \delta$ where $0 \leq \delta < \lambda$. Then one can check that condition $\gamma > \lambda$ implies that case $k \leq t$ cannot occur. So assume $k > t$. So

$$u_1 + u_2 \geq -\gamma + ic_1 - N_j + \delta_2 = ic_1 - N_j - (k-t)\lambda + \delta = kc_1 + (i-k)c_1 - N_j$$
$$-(k-t)\lambda + \delta \geq kc_1 - (k-t)\lambda + \delta = (t+1)c_1 + (k-t-1)c_1 - (k-t)\lambda + \delta$$
$$> (t+1)c_1 + (k-t-1)\lambda - (k-t)\lambda + \delta = (t+1)c_1 - \lambda + \delta.$$

Then

$$g_2(u_1+u_2) \geq g_2((t+1)c_1 - \lambda + \delta) = t\lambda + \delta = g_2(u_1) + g_2(u_2).$$

Case IV. $ic_1 \leq u_1 \leq (i+1)c_1 - \lambda$ and $-\gamma - N_r - (k+1)c_r + \lambda \leq u_2 \leq -\gamma - N_r - kc_r$ where $k \geq 0$. Then let $u_1 = ic_1 + \delta_1$ where $0 \leq \delta_1 \leq c_1 - \lambda$ and $u_2 = -\gamma - N_r - kc_r - \delta_2$ where $0 \leq \delta_2 \leq c_r - \lambda$. Therefore, $g_2(u_1) + g_2(u_2) = -\gamma - (r+k-i)\lambda$. We consider the following subcases.

Subcase 1. If $i - k \leq r$. Then we set $j = r - i + k \geq 0$. So $g_2(u_1) + g_2(u_2) = -\gamma - j\lambda \leq -\gamma$. So two cases (a) $j \leq r-1$, and (b) $j \geq r$ must be considered.

Let case (a) happens. Then it implies $i > k$. Then

$$u_1 + u_2 = -\gamma + ic_1 - N_r - kc_r + \delta_1 - \delta_2 \geq -\gamma + ic_1 - N_r - (k+1)c_r + \lambda = -\gamma$$
$$+ (i-k-1)c_1 + (k+1)c_1 - c_{l_1} - \cdots - c_{l_{r-j-1}} - c_{l_{r-j}} - \cdots - c_{l_r} - (k+1)c_r + \lambda$$
$$\geq -\gamma + (k+1)c_r - N_{j+1} - (k+1)c_r + \lambda = -\gamma - N_{j+1} + \lambda,$$

where the last inequality holds since $c_1 \geq c_r$ and $c_1 \geq c_{l_t}, 1 \leq t \leq r-j-1$. Hence

$$g_2(u_1+u_2) \geq g_2(-\gamma - N_{j+1} + \lambda) = -\gamma - j\lambda = g_2(u_1) + g_2(u_2).$$

Now assume case (b) takes place. It follows from (b) that $k \geq i$. Then

$$u_1 + u_2 \geq -\gamma + ic_1 - N_r - (k+1)c_r + \lambda \geq -\gamma + ic_r - N_r - (k+1-i)c_r - ic_r + \lambda$$
$$= -\gamma - N_r - (k+1-i)c_r + \lambda.$$

Therefore

$$g_2(u_1+u_2) \geq g_2(-\gamma - N_r - (k+1-i)c_r + \lambda) = -\gamma - j\lambda = g_2(u_1) + g_2(u_2).$$

Subcase 2. If $i-k \geq r+1$. Let $j = i-k-r-1 \geq 0$. Then $g_2(u_1)+g_2(u_2) = -\gamma+(j+1)\lambda \geq -\gamma+\lambda$. So regarding $\gamma > \lambda$, two cases (a) $-\gamma+\lambda \leq g_2(u_1)+g_2(u_2) \leq 0$, and (b) $g_2(u_1)+g_2(u_2) > 0$ are considered.

Let case (a) happens. Then

$$\begin{aligned} u_1+u_2 &\geq -\gamma + ic_1 - N_r - (k+1)c_r + \lambda = -\gamma + (j+k+r+1)c_1 - N_r - (k+1)c_r + \lambda \\ &= -\gamma + jc_1 + (k+1)c_1 + rc_1 - N_r - (k+1)c_r + \lambda \geq -\gamma + j\lambda + (k+1)c_r - (k+1)c_r \\ &+ \lambda = -\gamma + (j+1)\lambda, \end{aligned}$$

where the last inequality holds because $rc_1 \geq N_r$. Thus

$$g_2(u_1+u_2) \geq g_2(-\gamma+(j+1)\lambda) = -\gamma+(j+1)\lambda = g_2(u_1)+g_2(u_2).$$

Assume that case (b) occurs. Let $\gamma = t\lambda + \delta$ where $0 \leq \delta < \lambda$ and since $\gamma > \lambda, t \geq 1$. So we get $g_2(u_1)+g_2(u_2) = (j-t+1)\lambda - \delta$ and then (b) implies $j+1 > t$. Then

$$\begin{aligned} u_1+u_2 &\geq -\gamma + ic_1 - N_r - (k+1)c_r + \lambda = -t\lambda - \delta + (j+k+r+1)c_1 - N_r - (k+1)c_r \\ &+ \lambda = (-t+1)\lambda - \delta + jc_1 + (k+1)c_1 + rc_1 - N_r - (k+1)c_r \geq (-t+1)c_1 - \delta + jc_1 \\ &+ (k+1)c_r - (k+1)c_r = (j-t+1)c_1 - \delta. \end{aligned}$$

Therefore

$$g_2(u_1+u_2) \geq g_2((j-t+1)c_1 - \delta) = (j-t+1)\lambda - \delta = g_2(u_1)+g_2(u_2).$$

Case V. $ic_1 \leq u_1 \leq (i+1)c_1 - \lambda$ and $-\gamma - N_r - kc_r \leq u_2 \leq -\gamma - N_r - kc_r + \lambda$ where $k \geq 1$. Then let $u_1 = ic_1 + \delta_1$ where $0 \leq \delta_1 \leq c_1 - \lambda$ and $u_2 = -\gamma - N_r - kc_r + \delta_2$ where $0 \leq \delta_2 \leq \lambda$. Thus, $g_2(u_1)+g_2(u_2) = -\gamma - (r+k-i)\lambda + \delta_2$. We consider the following subcases.

Subcase 1. If $i-k \leq r-1$. Let $j = r+k-i-1 \geq 0$ which gives $g_2(u_1)+g_2(u_2) = -\gamma-(j+1)\lambda+\delta_2 \leq -\gamma$. Thus, two cases (a) $j \leq r-1$, and (b) $j \geq r$ must be considered.

Let case (a) happens which implies $i \geq k$. So

$$\begin{aligned} u_1+u_2 &= -\gamma + ic_1 - N_r - kc_r + \delta_1 + \delta_2 \geq -\gamma + ic_1 - N_r - kc_r + \delta_2 = -\gamma + (i-k)c_1 \\ &+ kc_1 - c_{l_1} - \cdots - c_{l_{r-j-1}} - c_{l_{r-j}} - \cdots - c_{l_r} - kc_r + \delta_2 \geq -\gamma + kc_r - N_{l+1} - kc_r + \delta_2 \\ &= -\gamma - N_{l+1} + \delta_2, \end{aligned}$$

where the last inequality follows from $c_1 \geq c_{l_t}, 1 \leq t \leq r-j-1$. Thus

$$g_2(u_1+u_2) \geq g_2(-\gamma - N_{l+1} + \delta_2) = -\gamma - (j+1)\lambda + \delta_2 = g_2(u_1)+g_2(u_2).$$

Now assume that case (b) takes place. Then it follows that $k > i$. So

$$u_1 + u_2 \geq -\gamma + ic_1 - N_r - kc_r + \delta_2 \geq -\gamma + ic_r - N_r - ic_r - (k-i)c_r + \delta_2$$
$$= -\gamma - N_r - (k-i)c_r + \delta_2.$$

Hence we get

$$g_2(u_1 + u_2) \geq g_2(-\gamma - N_r - (k-i)c_r + \delta_2) = -\gamma - (r + k - i)\lambda + \delta_2$$
$$= -\gamma - (j+1)\lambda + \delta_2 = g_2(u_1) + g_2(u_2).$$

Subcase 2. If $i - k \geq r$. We set $j = i - k - r \geq 0$ and so $g_2(u_1) + g_2(u_2) = -\gamma + j\lambda + \delta_2 \geq -\gamma$. Therefore, we consider two cases: (a) $-\gamma \leq g_2(u_1) + g_2(u_2) \leq 0$, and (b) $g_2(u_1) + g_2(u_2) > 0$.

Suppose that case (a) happens. Then

$$u_1 + u_2 \geq -\gamma + ic_1 - N_r - kc_r + \delta_2 = -\gamma + (j + k + r)c_1 - N_r - kc_r + \delta_2$$
$$= -\gamma + jc_1 + kc_1 + rc_1 - N_r - kc_r + \delta_2 \geq -\gamma + j\lambda + kc_r - kc_r + \delta_2 = -\gamma + j\lambda + \delta_2,$$

where the last inequality holds because $rc_1 \geq N_r$. Thus

$$g_2(u_1 + u_2) \geq g_2(-\gamma + j\lambda + \delta_2) = -\gamma + j\lambda + \delta_2 = g_2(u_1) + g_2(u_2).$$

Now let case (b) occurs and assume $-\gamma + j\lambda + \delta_2 = t\lambda + \delta$ where $0 \leq \delta < \lambda$. Then it can be checked easily that condition $\gamma > \lambda$ implies that $j \leq t$ cannot happen. So let $t < j$. Then

$$u_1 + u_2 \geq -\gamma + ic_1 - N_r - kc_r + \delta_2 = (j + k + r)c_1 - N_r - kc_r - (j-t)\lambda + \delta = jc_1 + kc_1$$
$$+ rc_1 - N_r - kc_r - (j-t)\lambda + \delta \geq jc_1 + kc_r - kc_r - (j-t)\lambda + \delta = jc_1 - (j-t)\lambda + \delta$$
$$= (t+1)c_1 + (j-t-1)c_1 - (j-t)\lambda + \delta \geq (t+1)c_1 + (j-t-1)\lambda - (j-t)\lambda + \delta$$
$$= (t+1)c_1 - \lambda + \delta.$$

Therefore

$$g_2(u_1 + u_2) \geq g_2((t+1)c_1 - \lambda + \delta) = t\lambda + \delta = g_2(u_1) + g_2(u_2).$$

Observe that other cases can be done similar to the cases presented here and so we omit them. □

2.5 Computational Experiments

In this section we illustrate the use of the proposed inequalities to improve the integrality gap on a set of randomly generated instances. The conducted experiments are preliminary, since it is outside of the scope of this dissertation to provide a deep study of the effectiveness of these inequalities in benchmark instances.

Table 2.1: Average dual gaps and average closed dual gaps using inequalities.

d=100	I_1			I_2			I_3		
	IG	CGS	CGL	IG	CGS	CGL	IG	CGS	CGL
$co_y = -10$	0.47	16.06	69.54	0.84	17.64	87.38	0.24	28.96	70.18
$co_y = -1000$	68.79	87.5	93.16	50.95	55.17	93.57	53.81	90.66	94.45
d=500	I_4			I_5			I_6		
	IG	CGS	CGL	IG	CGS	CGL	IG	CGS	CGL
$co_y = -10$	0.36	3.31	39.04	0.62	14.92	56.55	0.35	21.70	56.25
$co_y = -1000$	0.47	3.31	42.66	0.79	14.92	72.61	0.45	21.70	56.25

In order to test the impact of the inequalities developed for X_{binary}, with different capacities, in the reduction of the integrality gap, we generate different sets of instances considering a maximization problem and compute, for each set, the average initial gap (IG), the average closed gap using inequalities (2.10) (CGS), and the average closed gap using the lifted inequalities (2.22) (CGL). Initial gaps are computed as $\frac{UB-OPT}{UB} \times 100$ where OPT indicates the optimal value and UB denotes the upper bound obtained by the linear relaxation of the problem. Moreover, closed gaps are calculated as $\frac{UB-IUB}{UB-OPT} \times 100$ where IUB denotes the linear relaxation with inequalities (2.10) for CGS and, linear relaxation with inequalities (2.22) for CGL. For CGS, inequalities are added using the separation algorithm of Section 2.3 to the linear relaxation solution, and then the linear relaxation is solved again. The process is repeated until no new cuts are found. For inequalities (2.22) we use the same procedure while we only lift inequalities (2.10). All computations are performed using the optimization software Xpress-Optimizer version 23.01.03 [46].

The test instances are generated randomly on the basis of the following data. We consider $n = 50$, two possible values for d (100 and 500), two possible values for the objective coefficient of y, denoted by co_y, and for each possible combination of d and co_y we randomly generate the values of c_j from three sets. For $d = 100$ we consider two uniform distributions for intervals $I_1 = [4, 5]$, $I_2 = [10, 20]$, and another set $I_3 = [4, 6] \cup [9, 11] \cup [14, 16]$, where c_j is assigned to each interval with probability 1/3 and then it is generated using the uniform distribution for the corresponding interval. Similarly, for $d = 500$ we consider $I_4 = [15, 17]$, $I_5 = [40, 60]$, $I_6 = [10, 30] \cup [40, 60] \cup [70, 90]$. These intervals allow us to test the cases where the coefficients are almost constant and the cases where coefficients belong to different magnitudes. Coefficients of z_j are randomly generated in the interval $[\theta_j - 20, \theta_j + 20]$ where $\theta_j = -5\mu_j$ and μ_j denotes the average value of the interval for c_j. Coefficient of x_j is randomly generated in the interval $[10, 15]$. For each possible combination of d, co_y and interval for c_j, we generate 5 instances. In Table 2.1 we report the average results of those 5 instances.

It can be concluded from Table 2.1 that the improvement from use of the simple set-up flow covers (2.10), and the lifted inequalities (2.22) decreases as d increases. Also lifting has a clear impact on the reduction of the initial gap in all tested cases. The impact of inequalities (2.10) depends on the coefficients c_j considered. Besides, in most cases this impact is greater when the values of c_j increase.

2.6 Summary

This chapter can be summarized as follows. We derived a family of valid inequalities, the set-up flow cover inequalities, for a feasible set X_{binary}, which can be regarded as a variant of the SNFCN set where a new binary variable y is associated with the capacity of the node. We related the polyhedral structure of this variant with the polyhedral structure of the SNFCN set. We showed that in the presence of the node set-up variable new facet-defining inequalities appear and established the relation between the new family of inequalities with the flow cover inequalities. Based on these inequalities we provided a complete polyhedral characterization of the convex hull of X_{binary} when capacities on the arcs are constant. For the case of varying capacities, we lifted the set-up flow cover inequalities. The preliminary computational results were encouraging, suggesting further tests to study the effectiveness of these inequalities should be carried out in benchmark instances sets.

Chapter 3

Valid Inequalities for the Single Arc Design Problem with Set-Ups

3.1 Introduction

In this chapter we study the polyhedral structure of the second mixed integer set which generalizes two well-known sets: the single node fixed-charge network set and the single arc design set. This mixed integer set is of the form

$$X_{integer} = \Big\{(x, z, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{Z}_+ \mid \sum_{j \in N} x_j \leq dy,\ x_j \leq c_j z_j,\\ z_j \leq y, j \in N, y \in \{0, \ldots, U\}\Big\},$$

where $N = \{1, \ldots, n\}$, $\sum_{j \in N} c_j > d$, $0 < c_j < d, j \in N$, d, U and $c_j, j \in N$, are integer, and $U \leq \left\lceil \frac{\sum_{j \in N} c_j}{d} \right\rceil$.

The set $X_{integer}$ is related to two well-known sets: the single node fixed-charge network set (1.5) which can be represented as

$$X_{y=a} = \Big\{(x, z) \in \mathbb{R}^n_+ \times \mathbb{B}^n \mid \sum_{j \in N} x_j \leq d',\ x_j \leq c_j z_j\Big\},$$

obtained from $X_{integer}$ by setting y to a constant, and the Single Arc Design (SAD) set [30]

$$X_{z=1} = \Big\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{Z}_+ \mid \sum_{j \in N} x_j \leq dy,\ x_j \leq c_j, y \in \{0, \ldots, U\}\Big\},$$

obtained from $X_{integer}$ by setting $z_j = 1, j \in N$. Therefore the set $X_{integer}$ can be regarded as an extension of the SNFCN and the SAD sets. Moreover, observe that the set $X_{integer}$ can be obtained by imposing the variable y to take integer values and adding constraints $z_j \leq y, j \in N$ to the set X_{binary} studied in Chapter 2.

Notice that optimizing an arbitrary objective function over the set $X_{y=a}, a \in \{1, \ldots, U\}$ is a NP-hard problem (see [38]) which implies that optimizing an objective function over the set $X_{integer}$ is NP-hard as well.

The set $X_{integer}$ arises as a relaxation of several mixed integer problems such as lot-sizing and network design problems. Next we provide a few examples. In the single-item Lot-sizing with Supplier Selection Problem (LSSP) we are given a set N of suppliers. In each time period one needs to decide lot-sizes and a subset of suppliers to use in order to satisfy the demands while minimizing the costs. For each time period, the set $X_{integer}$ arises as follows: y represents the integer variable indicating the number of batches to produce, z_j indicates whether the supplier $j \in N$ is selected or not, x_j is the amount supplied by supplier j, d is the size of each batch and c_j is the supplying capacity of supplier j, see [47]. Other examples occur in inventory-routing problems such as the Vendor-Managed Inventory-Routing Problem (see [7]), where, for each time period t, y is an integer variable indicating the number of vehicles used at time t, z_j is a binary variable equal to 1 if the retailer j is served at time t, and 0 otherwise, d is the capacity of each vehicle (assuming a homogenous fleet), and c_j is the maximum inventory level in retailer j. In [7] the model considers only a single vehicle.

Next we introduce some notations used throughout this chapter: for any $S \subseteq N$, $\mu(S) = \lceil \frac{\sum_{j \in S} c_j}{d} \rceil$, and $r(S) = \sum_{j \in S} c_j - (\mu(S) - 1)d$. We denote by $P_{integer}, P_{y=a}, P_{z=1}$ the convex hull of $X_{integer}, X_{y=a}, X_{z=1}$, respectively. We use the notation $(a)^+ = \max\{a, 0\}$.

As we stated in Chapter 2, for the SNFCN set, Padberg et al. [38] introduced the flow cover inequalities (2.1) and the extended flow cover inequalities (2.2) which are obtained by lifting of the flow cover inequalities.

For the SAD set, Magnanti et al. [30] introduce the arc residual capacity inequalities.

Proposition 3.1.1. *For each $S \subseteq N$ the inequality*

$$\sum_{j \in S} x_j - r(S)y \leq (\mu(S) - 1)(d - r(S)),$$

is valid for $X_{z=1}$ and defines a facet of $P_{z=1}$ if S satisfies the following conditions: (i) if $\mu(S) = 1$, then $|S| = 1$; (ii) if $r(S) = d$, then $S = N$.

They show that the inequalities defining $X_{z=1}$ with the arc residual capacities inequalities suffice to describe $P_{z=1}$.

In a companion paper, Agra and Doostmohammadi [6], discuss the polyhedral structure of the set $X_{integer}$ when $U = 1$, and its relaxation obtained by removing constraints $z_j \leq y, j \in N$. They introduce the set-up flow cover inequalities and provide a full polyhedral description for the constant capacitated case. For the set $X_{integer}$ with $U = 1$, the set-up flow cover inequalities are obtained from the flow-cover inequalities (2.1) multiplying the RHS by y :

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - \lambda)^+ z_j \leq \Big(d - \sum_{j \in S} (c_j - \lambda)^+\Big) y. \tag{3.1}$$

We now describe the contents of this chapter. In Section 3.2 we establish basic properties of $P_{integer}$, derive families of facet-defining inequalities which generalize the residual capacity inequalities and flow cover inequalities. In Section 3.3 we consider the constant capacitated case, provide a compact extended formulation for $P_{integer}$, and introduce several valid inequalities in the original space of variables. In addition, we provide the complete characterization of $P_{integer}$ when the capacities are constant and a particular condition is considered. In Section 3.4 we discuss the lifting of a class of valid inequalities derived in Section 3.3. In section 3.5 we study the separation problem associated to those valid inequalities derived for the constant capacitated case. Preliminary computational experiments are reported in Section 3.6. Lastly, a summary of this chapter is presented in Section 3.7.

3.2 Valid Inequalities for $P_{integer}$

In this section we investigate the polyhedral structure of $P_{integer}$. The following propositions establish basic properties of $P_{integer}$.

Proposition 3.2.1. *$P_{integer}$ is a full-dimensional polyhedron.*

Proof. Consider the following $2n+2$ points belonging to $P_{integer}$.

- $v_0 : y = 0; x_j = 0, j \in N, z_j = 0, j \in N;$
- $v_1 : y = 1; x_j = 0, j \in N, z_j = 0, j \in N;$
- $v_2, \dots, v_{n+1}$: for all $k \in N$, $y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\};$
- $v_{n+2}, \dots, v_{2n+1}$: for all $k \in N$, $y = 1; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\}.$

We show that the foregoing points are affinely independent. Since $(\mathbf{0}, \mathbf{0}, 0)$ is listed here so it suffices to show that points $v_1, \dots, v_{2n+1}$ are linearly independent. So we consider the system $\sum_{j=1}^{2n+1} \lambda_j v_j = \mathbf{0}$, for scalars $\lambda_j, j = 1, \dots, 2n+1$ which are not all zero. Thus, we get

$$\begin{cases} c_{i-1}\lambda_i = 0, i = 2, \dots, n+1 \\ \lambda_i + \lambda_{n+i} = 0, i = 2, \dots, n+1 \\ \sum_{i=1}^{2n+1} \lambda_i = 0. \end{cases} \tag{3.2}$$

The first equation of system (3.2) provides $\lambda_2 = \dots = \lambda_{n+1} = 0$. The second equation implies $\lambda_{n+2} = \dots = \lambda_{2n+1} = 0$ and finally, the last equation of system (3.2) gives $\lambda_1 = 0$ which justify that $P_{integer}$ is a full-dimensional polyhedron. □

Proposition 3.2.2. *The extreme points of $P_{integer}$ are of one of the following forms:*

(i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N;$

(ii) $y = 1; x_j = 0, j \in N; z_j = 1, j \in T \subseteq N, z_j = 0, j \in N \setminus T$, *where* $T \neq \emptyset$;

(iii) $y = a; x_j = c_j, j \in S, x_j = 0, j \in N \setminus S; z_j = 1, j \in T, S \subseteq T \subseteq N, z_j = 0, j \in N \setminus T$; *where* $a \in \{\mu(S), U\}$;

(iv) $y = a \in \{1, \ldots, U\}; x_j = c_j, j \in S \subseteq N, x_t = ad - \sum_{j \in S} c_j, x_j = 0, j \in N \setminus S \cup \{t\}; z_j = 1, j \in T, S \cup \{t\} \subseteq T, z_j = 0, j \in N \setminus T$; *where* $ad - \sum_{j \in S} c_j < c_t$.

The following proposition states the trivial facets of $P_{integer}$.

Proposition 3.2.3. *1. For every* $i \in N$, $x_i \geq 0$ *defines a facet of* $P_{integer}$.

2. If $U \geq 2$, *then for every* $i \in N$, $z_i \leq 1$ *defines a facet of* $P_{integer}$.

3. For every $i \in N$, $x_i \leq c_i z_i$ *defines a facet of* $P_{integer}$.

4. For every $i \in N$, $z_i \leq y$ *defines a facet of* $P_{integer}$.

5. $y \leq U$ *defines a facet of* $P_{integer}$.

6. If $\sum_{j \in N} c_j > d + c_k, \forall k \in N$, *then* $\sum_{j \in N} x_j \leq dy$ *defines a facet of* $P_{integer}$.

Proof. Proof of 1. For a fixed i, let $K = P_{integer} \cap \{(x, z, y) \mid (x, z, y)$ satisfies $x_i = 0\}$. Then we prove that inequality $x_i \geq 0$ is facet-defining by showing that whenever the inequality $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y \leq \gamma_0$ is valid for $X_{integer}$ and satisfies the condition that

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0, \forall (x, z, y) \in K, \tag{3.3}$$

then equality (3.3) is a multiple of $x_i = 0$. We provide the following feasible points belonging to K.

(i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N$;

(ii) $y = 1; x_j = 0, j \in N; z_j = 0, j \in N$;

(iii) for all $k \in N$, $y = 1; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\}$;

(iv) for all $k \in N \setminus \{i\}$, $y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\}$.

Substituting point (i) and (ii) in equation (3.3) gives $\gamma_0 = 0$ and $\gamma = 0$ respectively. Then it follows by replacing solution (iii) in (3.3) that $\beta_j = 0, j \in N$. Finally, substituting solution (iv) in equation (3.3) implies $\alpha_j = 0, j \in N \setminus \{i\}$. Thus, equation (3.3) is equivalent to $\alpha x_i = 0$ which is a multiple of $x_i = 0$.

Proof of 2. Following the technique used in part 1, we give the following points belong to K.

- (i) $y = 1; x_j = 0, j \in N; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;
- (ii) $y = 2; x_j = 0, j \in N; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;
- (iii) $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;
- (iv) for all $k \in N \setminus \{i\}$, $y = 1; x_j = 0, j \in N; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$;
- (v) for all $k \in N \setminus \{i\}$, $y = 1; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$.

Then substituting points (i) and (ii) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ and subtracting the resultant equalities imply $\gamma = 0$. Replacing (i) and (iv) in the foregoing equation and subtracting them give $\beta_j = 0, j \in N \setminus \{i\}$. Then it follows from substituting solutions (i) and (iii) in the equation that $\alpha_i = 0$ and substituting points (iv) and (v) provides $\alpha_j = 0, j \in N \setminus \{i\}$. Finally, replacing point (i) in the equation gives $\gamma_0 = \beta_i = \beta$. Thus, we get $\beta z_i = \beta$ which is a multiple of $z_i = 1$.

Proof of 3. Similarly, the following points are in K.

- (i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N$;
- (ii) $y = 1; x_j = 0, j \in N; z_j = 0, j \in N$;
- (iii) $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;
- (iv) for all $k \in N \setminus \{i\}$, set $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$;
- (v) for all $k \in N \setminus \{i\}$, set $y = 1; x_i = c_i; x_k = \varepsilon_k$ such that $c_i + \varepsilon_k \leq d; x_j = 0, j \in N \setminus \{i, k\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$.

Now substituting points (i) and (ii) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ gives $\gamma_0 = 0$ and $\gamma = 0$ respectively. Then replacing points (iii) and (iv) in the foregoing equation implies $\beta_j = 0, j \in N \setminus \{i\}$. It follows from substituting points (iv) and (v) and subtracting the resultant equalities that $\alpha_j = 0, j \in N \setminus \{i\}$. Therefore, the equation is $\alpha x_i + \beta z_i = 0$. Lastly, replacing point (iii) in this equation gives $\beta = -\alpha c_i$ which completes the proof.

Proof of 4. We introduce the points belonging to K as follows.

- (i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N$;

(*ii*) $y = 1; x_j = 0, j \in N; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;

(*iii*) $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = 1; z_j = 0, j \in N \setminus \{i\}$;

(*iv*) for all $k \in N \setminus \{i\}$, $y = 1; x_i = c_i; x_j = 0, j \in N \setminus \{i\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$;

(*v*) for all $k \in N \setminus \{i\}$, $y = 1; x_i = c_i; x_k = \varepsilon_k$ such that $c_i + \varepsilon_k \leq d; x_j = 0, j \in N \setminus \{i, k\}; z_i = z_k = 1; z_j = 0, j \in N \setminus \{i, k\}$.

Then substituting point (*i*) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ implies $\gamma_0 = 0$. Replacing solutions (*iii*) and (*iv*) in the equation and subtracting them give $\beta_j = 0, j \in N \setminus \{i\}$. Next, replacing points (*ii*), (*iii*), (*iv*), and (*v*) in the foregoing equation provides $\alpha_i = 0, i \in N$. Finally, it follows from substituting point (*ii*) in equation $\beta z_i + \gamma y = 0$ that $\gamma = -\beta$ which completes the justification.

Proof of 5. The following points belong to K.

(*i*) $y = U; x_j = 0, j \in N; z_j = 0, j \in N$;

(*ii*) for all $k \in N$, set $y = U; x_j = 0, j \in N; z_k = 1; z_j = 0, j \in N \setminus \{k\}$;

(*iii*) for all $k \in N$, set $y = U; x_k = c_k; x_j = 0, j \in N \setminus \{k\}; z_k = 1; z_j = 0, j \in N \setminus \{k\}$.

Then substituting points (*i*) and (*ii*) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ and subtracting the resultant equalities give $\beta_j = 0, j \in N$. It can be concluded from replacing points (*i*) and (*iii*) in the equation that $\alpha_j = 0, j \in N$. So we obtain an equation $\gamma y = \gamma_0$. Substituting point (*i*) in this equation implies $\gamma_0 = \gamma U$ which proves that $\gamma y = \gamma U$ is a multiple of $y = U$.

Proof of 6. The points belonging to K are listed as follows.

(*i*) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N$;

(*ii*) for all $k \in N, y = 1; x_j = c_j, j \in S \subset N \setminus \{k\}; x_t = d - \sum_{j \in S} c_j < c_t$, where $t \neq k, t \notin S; x_i = 0, i \in N \setminus (S \cup \{t\}); z_j = 1, j \in S \cup \{t\}, z_i = 0, i \in N \setminus (S \cup \{t\})$;

(*iii*) for all $k \in N, y = 1; x_j = c_j, j \in S \subset N \setminus \{k\}; x_t = d - \sum_{j \in S} c_j < c_t$, where $t \neq k, t \notin S; x_i = 0, i \in N \setminus (S \cup \{t\}); z_j = 1, j \in S \cup \{t, k\}; z_i = 0, i \in N \setminus (S \cup \{t, k\})$.

First, note that the condition $\sum_{j \in N} c_j > d + c_k, \forall k \in N$ guarantees that we can create points of type (*ii*) and (*iii*). Then replacing point (*i*) in equation $\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0$ implies $\gamma_0 = 0$. Next, substituting points (*ii*) and (*iii*) in the equation and subtracting them provide $\beta_j = 0, j \in N$.

Now let $i_1, i_2 \in N$. We consider a point of type (*ii*) where $x_{i_1} = c_{i_1}$ and $x_{i_2} = d - \sum_{j \in S} c_j$. Then we create a new solution by decreasing the value of x_{i_1} by 1 and increasing the value of x_{i_2} by the same value which belongs to K. Substituting these two solutions in the equation and subtracting the resultant equalities imply $\alpha_{i_1} = \alpha_{i_2}$. Thus, $\alpha_j = \alpha, j \in N$. So the initial equation becomes $\alpha \sum_{j \in N} x_j + \gamma y = 0$ and finally, replacing point (*ii*) in this equation gives $\gamma = -\alpha d$ which completes the proof. □

Next we introduce a family of inequalities that generalizes the arc residual capacity inequalities and the flow cover inequalities.

Proposition 3.2.4. *Let $S \subseteq N$ such that $\sum_{j\in S} c_j > d$ and $c_j \leq d, j \in S$. Then*

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S}(c_j - r(S))^+, \qquad (3.4)$$

is valid for $X_{integer}$, and defines a facet of $P_{integer}$ if $\bar{c} = \max\{c_j | j \in S\} > r(S)$ and $\mu(S) \leq U$.

Proof. First we prove validity. Consider a point $(x,z,y) \in X_{integer}$. We consider two cases. Case 1: $y \geq \mu(S)$. Since $x_j - (c_j - r(S))^+ z_j \leq c_j - (c_j - r(S))^+, j \in S$, then

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq \sum_{j\in S} c_j - \sum_{j\in S}(c_j - r(S))^+ = r(S)\mu(S) + (\mu(S)-1)(d-r(S))$$
$$- \sum_{j\in S}(c_j - r(S))^+ \leq r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S}(c_j - r(S))^+.$$

Case 2: $y \leq \mu(S) - 1$. Let $T = \{j \in S | z_j = 1\}$ and $k = |\{j \in S \setminus T | c_j > r(S)\}|$. If $k \geq \mu(S) - y$, then

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq \sum_{j\in T} c_j - \sum_{j\in T}(c_j - r(S))^+ = \sum_{j\in S} c_j - \sum_{j\in S}(c_j - r(S))^+$$
$$- \sum_{j\in S\setminus T} c_j + \sum_{j\in S\setminus T}(c_j - r(S))^+ \leq (\mu(S)-1)d + r(S) - r(S)k - \sum_{j\in S}(c_j - r(S))^+$$
$$\leq r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S}(c_j - r(S))^+.$$

If $k < \mu(S) - y$, then

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq dy - \sum_{j\in T}(c_j - r(S))^+ = r(S)y + (\mu(S)-1)(d-r(S))$$
$$- (\mu(S)-1-y)(d-r(S)) - \sum_{j\in T}(c_j - r(S))^+ \leq r(S)y + (\mu(S)-1)(d-r(S))$$
$$- k(d-r(S)) - \sum_{j\in T}(c_j - r(S))^+ = r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S\setminus T | c_j > r(S)} (d-r(S))$$
$$- \sum_{j\in T}(c_j - r(S))^+ \leq r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S}(c_j - r(S))^+.$$

To prove that (3.4) defines a facet of $P_{integer}$ it suffices to notice that restricting the face defined by (3.4) to the hyperplane defined by $y = \mu(S) - 1$, we obtain a facet of $P_{y=\mu(S)-1}$, see [38], hence it includes $2n$ affinely independent points $(x^t, z^t), t \in \{1, \ldots, 2n\}$. Therefore, the points $(x^t, z^t, \mu(S)-1), t \in \{1, \ldots, 2n\}$ are affinely independent. We can easily construct a new affinely independent point in $X_{integer}$ satisfying (3.4) as equation, setting $y = \mu(S)$, $x_j = c_j, j \in S$, and $z_j = 1, j \in S$. □

Setting $y = \mu(S) - 1$ in (3.4) we obtain the flow cover inequality presented in [38]. Setting $z_j = 1, \forall j \in S$ in (3.4) we obtain the arc residual capacity inequality. Hence, (3.4) generalizes the flow cover inequalities and the residual inequalities for the set $X_{z=1}$.

Following the idea of extended flow cover inequalities, the following proposition extends inequalities (3.4).

Proposition 3.2.5. *Let $S \subseteq N$ such that $\sum_{j\in S} c_j > d$ and $c_j \leq d, j \in S$. If $U \leq \mu(S) - 1$, then the following inequality is valid for $X_{integer}$:*

$$\sum_{j\in S\cup L} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq r(S)y + (\mu(S)-1)(d-r(S)) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j, \tag{3.5}$$

where $\bar{c}_j = \max\{c_j, \bar{c}\}, \bar{c} = \max\{c_j | j \in S\}$ and $L \subseteq N \setminus S$.

Proof. Let $T = \{j \in S \cup L | z_j = 1\}$ and $k = |\{j \in S \setminus T | c_j > r(S)\}|$ and $p = |\{j \in L | z_j = 1\}|$. We consider two cases as follows.

Case 1: $k - p \geq \mu(S) - y$. Then

$$\begin{aligned}
&\sum_{j\in S\cup L} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq \sum_{j\in S\cap T} c_j + \sum_{j\in L\cap T} c_j - \sum_{j\in S\cap T}(c_j - r(S))^+ \\
&+ \sum_{j\in S}(c_j - r(S))^+ - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j - \sum_{j\in L\cap T}(\bar{c}_j - r(S)) \\
&= \sum_{j\in S\cap T} c_j + \sum_{j\in L\cap T} c_j + \sum_{j\in S\setminus T}(c_j - r(S))^+ - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j \\
&- \sum_{j\in L\cap T}(\bar{c}_j - r(S)) \leq \sum_{j\in S} c_j - r(S)k + \sum_{j\in L\cap T}(c_j - \bar{c}_j) + r(S)p - \sum_{j\in S}(c_j - r(S))^+ \\
&+ \sum_{j\in L}(\bar{c}_j - r(S))z_j \leq \sum_{j\in S} c_j - r(S)(\mu(S) - y) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j \\
&= r(S)y + (\mu(S) - 1)(d - r(S)) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j.
\end{aligned}$$

Case 2: $k - p \leq \mu(S) - y - 1$. So

$$\begin{aligned}
&\sum_{j\in S\cup L} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq dy - \sum_{j\in S\cap T}(c_j - r(S))^+ = dy - \sum_{j\in S\cap T}(c_j - r(S))^+ \\
&+ \sum_{j\in S}(c_j - r(S))^+ - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j - \sum_{j\in L\cap T}(\bar{c}_j - r(S))z_j \\
&= dy + \sum_{j\in S\setminus T}(c_j - r(S))^+ - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j - \sum_{j\in L\cap T}(\bar{c}_j - r(S))
\end{aligned}$$

$$\begin{aligned}
&\leq dy + k\bar{c} - r(S)k - \sum_{j\in S}(c_j - r(S))^+ - p\bar{c} + r(S)p + \sum_{j\in L}(\bar{c}_j - r(S))z_j \\
&= dy + (\bar{c} - r(S))(k-p) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j \\
&\leq dy + (d - r(S))(\mu(S) - y - 1) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j \\
&= r(S)y + (\mu(S) - 1)(d - r(S)) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\bar{c}_j - r(S))z_j.
\end{aligned}$$

□

The following example shows that inequality (3.5) may not be valid for $X_{integer}$ if $U \geq \mu(S)$.

Example 3.2.6. *Let* $N = \{1,2,3,4\}, c = (8,8,8,8), d = 10, S = \{1,2,3\}, \mu(S) = 3, r(S) = 4$. *Inequality* (3.5) *with* $L = \{4\}$ *is*

$$x_1 + x_2 + x_3 + x_4 - (8-4)(z_1 + z_2 + z_3) \leq 4y + 2(10-4) - 12 + (8-4)z_4.$$

The point $(x, z, y) \in X_{integer}$ *with* $y = 3$, $x_1 = x_2 = x_3 = 8, x_4 = 6, z_1 = z_2 = z_3 = z_4 = 1$ *violates the inequality.*

Flow cover inequalities can be generalized in a different way leading to a different class of facet-defining inequalities.

Proposition 3.2.7. *Let* $S \subseteq N$ *such that* $\sum_{j\in S} c_j > d$ *and* $c_j \leq d, j \in S$. *The inequality*

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \leq \left(d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S) - 1}\right) y, \tag{3.6}$$

is valid for $X_{integer}$ *if*

$$L(k) \leq kd - \frac{k\sum_{j\in S}(c_j - r(S))^+}{\mu(S) - 1}, k = 1, \ldots, \mu(S) - 2, \tag{3.7}$$

where

$$L(k) = \max\Big\{\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j | \sum_{j\in S} x_j \leq dk,$$
$$0 \leq x_j \leq c_j z_j, j \in S, z_j \in \{0,1\}, j \in S\Big\},$$

and defines a facet of $P_{integer}$ *if* $\bar{c} = \max\{c_j | j \in S\} > r(S)$ *and* $\mu(S) - 1 \leq U$.

Proof. Condition (3.7) ensures validity of (3.6) for $y = 1, \dots, \mu(S) - 2$. For $y = \mu(S) - 1$, (3.6) is a flow cover, so validity follows from validity of flow covers for $X_{y=\mu(S)-1}$. Inequality (3.6) is trivially valid for $y = 0$. Now assume $y > \mu(S) - 1$. Let $S^+ = \{j \in S | c_j > r(S)\}$. If $|S^+| \leq \mu(S) - 1$, as $c_j \leq d$ and $r(S) < d$, then $(\mu(S)-1)d \geq \sum_{j \in S^+} c_j + (\mu(S) - 1 - |S^+|)r(S)$ and so $(\mu(S)-1)d - \sum_{j \in S^+} c_j + |S^+|\, r(S) \geq (\mu(S)-1)r(S)$ which implies $d - \frac{\sum_{j \in S}(c_j - r(S))^+}{\mu(S)-1} \geq r(S)$. If $|S^+| \geq \mu(S)$, then

$$\sum_{j \in S}(c_j - r(S))^+ \leq \sum_{j \in S} c_j - \left|S^+\right| r(S) \leq (\mu(S) - 1)d + r(S) - \mu(S)r(S)$$
$$= (\mu(S) - 1)(d - r(S)),$$

which implies $d - \frac{\sum_{j \in S}(c_j - r(S))^+}{\mu(S)-1} \geq r(S)$. Hence,

$$\Big(d - \frac{\sum_{j \in S}(c_j - r(S))^+}{\mu(S) - 1}\Big)y = d(\mu(S) - 1) - \sum_{j \in S}(c_j - r(S))^+$$
$$+ \Big(d - \frac{\sum_{j \in S}(c_j - r(S))^+}{\mu(S) - 1}\Big)(y - \mu(S) + 1) \geq d(\mu(S) - 1) - \sum_{j \in S}(c_j - r(S))^+$$
$$+ r(S)(y - \mu(S) + 1) \geq \sum_{j \in S} x_j - \sum_{j \in S}(c_j - r(S))^+ z_j,$$

where the last inequality is a flow cover inequality (3.4).

To prove that (3.6) defines a facet it suffices to notice that since (3.6) is a flow cover for the restricted set obtained by setting $y = \mu(S) - 1$. Hence, there are $2n$ affinely independent points satisfying $y = \mu(S) - 1$. Another affinely independent point can be given by the null vector $y = 0, z_j = x_j = 0, j \in N$. □

When $\mu(S) = 2$, Proposition 3.2.7 states that the set-up flow cover inequalities (3.1) are valid for $X_{integer}$.

3.3 The Constant Case $c_j = c, j \in N$

In this section we consider the constant capacitated case, that is, we assume $c_j = c, j \in N$. In Section 3.3.1 we provide a compact linear extended formulation for $P_{integer}$. From the theoretical point of view this formulation proves that optimizing a linear function over $X_{integer}$ can be done in polynomial time. In Section 3.3.2 we introduce several facet-defining inequalities in the original space of variables.

We assume $nc > d > c > 0$; d, c are integer; d is not a multiple of c, and $U \leq \lceil \frac{nc}{d} \rceil$. For $u \in \{1, \dots, U\}$, we define $r_u = ud \ mod \ c$.

3.3.1 A Compact Formulation

In this section we provide a compact linear formulation for $P_{integer}$. First we provide an extended formulation for the set

$$X_{y=u} = \Big\{(x,z) \in \mathbb{R}^n_+ \times \mathbb{B}^n \mid \sum_{j\in N} x_j \leq du,\ x_j \leq cz_j\Big\}$$

obtained by restricting y to u, for $u = 1,\dots,U$. Set $X_{y=u}$ is the single node flow set with constant bounds. Padberg et al. [38] showed that adding to the defining inequalities of $X_{y=u}$, the flow cover inequalities

$$\sum_{j\in S}(x_j - r_u z_j) \leq \Big\lfloor \frac{du}{c} \Big\rfloor (c - r_u), \quad \forall S \subseteq N, \quad |S| \geq \Big\lfloor \frac{du}{c} \Big\rfloor + 1, \tag{3.8}$$

completely describes $P_{y=u}$.

Since the family of flow cover inequalities has an exponential number of inequalities, in order to derive a compact formulation, we follow Martin [33] to derive an compact extended formulation for $P_{y=u}$. Consider the following linear formulation with the additional nonnegative variables $\delta_j = (x_j - r_u z_j)^+, j \in N$.

$$\sum_{j\in N} x_j \leq du, \tag{3.9}$$

$$\delta_j \geq x_j - r_u z_j, j \in N, \tag{3.10}$$

$$\sum_{j\in N} \delta_j \leq \Big\lfloor \frac{du}{c} \Big\rfloor (c - r_u), \tag{3.11}$$

$$x_j \leq cz_j, j \in N, \tag{3.12}$$

$$z_j \leq 1, j \in N, \tag{3.13}$$

$$x_j \geq 0, j \in N, \tag{3.14}$$

$$\delta_j \geq 0, j \in N. \tag{3.15}$$

This formulation has $\mathcal{O}(n)$ variables and $\mathcal{O}(n)$ constraints. Let Q_u be the set of those points (x, z, δ) that satisfy (3.9)–(3.15). Next we show that the projection of Q_u onto the space of variables (x, z) is $P_{y=u}$.

Theorem 3.3.1. *$Proj_{(x,z)} Q_u = P_{y=u}$.*

Proof. Consider the representation of $P_{y=u}$ given by (3.8) and the defining inequalities (3.9), (3.12)–(3.14). Since each inequality defining $P_{y=u}$ is valid for Q_u (inequalities (3.8) are obtained from (3.10), (3.11) and (3.15) by Fourier-Motzkin elimination) it follows that $proj_{(x,z)} Q_u \subseteq P_{y=u}$. Conversely, let $(x, z) \in P_{y=u}$ and define $\delta_j = \max\{0, x_j - r_u z_j\}$. We need to show that $(x, z, \delta) \in Q_u$. From the definition of δ, constraints (3.10) and (3.15) are trivially satisfied. Constraints (3.11) are implied by (3.8) taking $S = \{j \in N | \delta_j = x_j - r_u z_j\}$. □

We can now write $P_{integer}$ as the union of polyhedra $P_{y=u}$ for each $u \in \{0, \dots, U\}$, where $P_{y=0} = \{0\}$.

Theorem 3.3.2. $P_{integer} = conv(\bigcup_{u=0,\dots,U} P_{y=u})$.

Proof. In order to obtain the first inclusion, since $P_{y=u} \subseteq P_{integer}$ and $P_{y=u}$ is bounded for all $u \in \{0, \dots, U\}$, then we get $conv(\bigcup_{u=0,\dots,U} P_{y=u}) \subseteq P_{integer}$. Conversely, since each extreme point (x^*, z^*, y^*) of $P_{integer}$ belongs to $X_{integer}$ and satisfies $y^* = u$ for some $u \in \{0, \dots, U\}$, then $(x^*, z^*, y^*) \in P_{y=u}$. Therefore $P_{integer} \subseteq conv(\bigcup_{u=0,\dots,U} P_{y=u})$. □

As a compact formulation for P_u is known for each $u \in \{0, \dots, U\}$, and since U is bounded by n, using a result from Balas [10] on the union of polyhedra we can now easily derive a compact formulation for $P_{integer} = conv(\bigcup_{u=0,\dots,U} P_{y=u})$.

Theorem 3.3.3. *The following formulation is a compact extended formulation for $P_{integer}$.*

$$
\begin{aligned}
&\delta_j = \sum_{u=0}^{U} \delta_j^u, j \in N,\\
&x_j = \sum_{u=0}^{U} x_j^u, j \in N,\\
&z_j = \sum_{u=0}^{U} z_j^u, j \in N,\\
&\delta_j^u \geq x_j^u - r_u z_j^u, j \in N, u \in \{1, \dots, U\},\\
&\sum_{j \in N} \delta_j^u \leq \left\lfloor \frac{du}{c} \right\rfloor (c - r_u) y_0^u, u \in \{1, \dots, U\},\\
&\sum_{j \in N} x_j^u \leq du y_0^u, u \in \{1, \dots, U\},\\
&x_j^u \leq c z_j^u, j \in N, u \in \{1, \dots, U\},\\
&z_j^u \leq y_0^u, j \in N, u \in \{1, \dots, U\},\\
&x_j^u \geq 0, j \in N, u \in \{1, \dots, U\},\\
&\delta_j^u \geq 0, j \in N, u \in \{1, \dots, U\},\\
&\sum_{u=0}^{U} y_0^u = 1,\\
&\delta_j^0 = z_j^0 = x_j^0 = 0, j \in N.
\end{aligned}
$$

The formulation has $\mathcal{O}(nU)$ variables and $\mathcal{O}(nU)$ constraints.

In theory, by projecting out the additional variables $\delta_j^u, x_j^u, z_j^u, y_0^u$ we obtain an exact description of $P_{integer}$ on the original space of variables (x, z, y). This task seems not to be easy. In the next section we provide valid inequalities in the original space and explain why such a full polyhedral description is not trivial.

3.3.2 Valid Inequalities for the Constant Capacitated Case

Here we establish several valid inequalities for $P_{integer}$. The first class of valid inequalities is given by the following proposition.

Proposition 3.3.4. *Let $S \subseteq N$ such that $|S| \geq \lceil \frac{d}{c} \rceil$. The inequality*

$$\sum_{j \in S} x_j \leq r(S)y + (\mu(S) - 1)(d - r(S)), \tag{3.16}$$

is valid for $X_{integer}$, and defines a non-trivial facet of $P_{integer}$ if $c \leq r(S) < d$.

Proof. To prove validity we show that (3.16) is an MIR (Mixed Integer Rounding) inequality. Let $W = \sum_{j \in S} x_j$, $Z = \sum_{j \in S} z_j$. Then

$$\Big\{(W, Z, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \mid W \leq dy, W \leq cZ, Z \leq |S|, y \leq U\Big\},$$

is a relaxation of $X_{integer}$. Now consider the restriction of this set defined by setting $Z = |S|$ which is

$$\Big\{(W, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid W \leq dy, W \leq |S| c\Big\}.$$

Setting $s =\mid S \mid c - W$, we obtain the MIP set

$$\Big\{(s, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid s + dy \geq |S| c\Big\}.$$

Proposition 1.4.1 implies that the MIR inequality for this MIP set is

$$s \geq r(S)(\mu(S) - y).$$

In the original space of variables this inequality gives inequality (3.16).

Then we show that inequality (3.16) is facet-defining. Assume that $c \leq r(S) < d$. Consider an equation

$$\sum_{j \in S} x_j = r(S)y + (\mu(S) - 1)(d - r(S)), \tag{3.17}$$

and let $K = P \cap \big\{(x, z, y) \bigm| (x, z, y) \text{ satisfies (3.17)}\big\}$. Now we show that inequality (3.16) defines a facet of $P_{integer}$ by showing that whenever any inequality

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y \leq \gamma_0,$$

is valid for $X_{integer}$ and satisfies the condition that

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0, \forall (x, z, y) \in K, \tag{3.18}$$

then the equation (3.18) is a multiple of (3.17). Let $S = \{1, 2, \ldots, s\}$. We generate the following points belonging to K.

(1) Set

$$y = \mu(S),\ x_j = \begin{cases} c\ ,\ j \in S, \\ 0\ ,\ otherwise, \end{cases},z_j = \begin{cases} 1\ ,\ j \in S, \\ 0\ ,\ otherwise. \end{cases}$$

Considering solution (1), since $r(S) < d$ we can create a new point belonging to K by increasing the flow x_k from 0 to 1, for some $k \geq s+1$. So the following points are in K.

(2) $\forall\ k \in N \setminus S$,

$$y = \mu(S),\ x_j = \begin{cases} c\ ,\ j \in S, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},z_j = \begin{cases} 1\ ,\ j \in S, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise. \end{cases}$$

Let $q = argmax_{\ i<s} \left\{ i : ic \leq d(\mu(S)-1) \right\} = \lfloor \frac{d(\mu(S)-1)}{c} \rfloor$ and $Q = \{1, 2, \ldots, q\} \subset S$. Notice that $q < s$. Then we define the following points.

(3) $\forall\ k \in S \setminus Q$,

$$y = \mu(S) - 1,\ x_j = \begin{cases} c\ ,\ j \in Q, \\ d(\mu(S)-1) - qc\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},z_j = \begin{cases} 1\ ,\ j \in Q, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

where $d(\mu(S)-1) - qc = d(\mu(S)-1) - \lfloor \frac{d(\mu(S)-1)}{c} \rfloor c = d(\mu(S)-1)\ mod\ c < c$. Observe that if $r(S) = c$, then $(|S|-1)c = d(\mu(S)-1)$ which implies $q = |S|-1$. So points (3) can be rewritten as follows.

(4) $\forall\ k \in S$,

$$y = \mu(S) - 1,\ x_j = \begin{cases} c\ ,\ j \in S \setminus \{k\}, \\ 0\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},z_j = \begin{cases} 1\ ,\ j \in S \setminus \{k\}, \\ \{0,1\}\ ,\ for\ k, \\ 0\ ,\ otherwise. \end{cases}$$

The following points belong to K as well.

(5) $\forall\ k \in N \setminus S$,

$$y = \mu(S),\ x_j = \begin{cases} c\ ,\ j \in S, \\ 0\ ,\ otherwise, \end{cases},z_j = \begin{cases} 1\ ,\ j \in S, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise. \end{cases}$$

Since $r(S) > c$, so $|S| \geq q+2$. Now let $k_1 \in S$. Then we generate other points belonging to K by considering the following subcases: (a) $k_1 \in Q$, and (b) $k_1 \in S \setminus Q$. Assume that subcase (a) occurs. Then the following points belong to K.

(6) $\forall\ k_1 \in Q, \forall\ k_2 \in S \setminus Q, \forall\ k_3 \in S \setminus (Q \cup \{k_2\})$,

$$y = \mu(S) - 1,\ x_j = \begin{cases} c\ ,\ j \in Q \setminus \{k_1\}, \\ 0\ ,\ for\ k_1, \\ d(\mu(S) - 1) - qc\ ,\ for\ k_2, \\ c\ ,\ for\ k_3, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in Q \setminus \{k_1\}, \\ \{0,1\}\ ,\ for\ k_1, \\ 1\ ,\ for\ k_2, \\ 1\ ,\ for\ k_3, \\ 0\ ,\ otherwise. \end{cases}$$

Now let subcase (b) happens. So the following points are in K.

(7) $\forall\ k_1 \in S \setminus Q, \forall k_2 \in S \setminus (Q \cup \{k_1\})$,

$$y = \mu(S) - 1,\ x_j = \begin{cases} c\ ,\ j \in Q, \\ 0\ ,\ for\ k_1, \\ d(\mu(S) - 1) - qc\ ,\ for\ k_2, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in Q, \\ \{0,1\}\ ,\ for\ k_1, \\ 1\ ,\ for\ k_2, \\ 0\ ,\ otherwise. \end{cases}$$

Now let $k \in N \setminus S$. Then substituting points (1) and (2) in equation (3.18) and subtracting the resultant equations imply $\alpha_k + \beta_k = 0$. Moreover, substituting points (1) and (5) in equality (3.18) and subtracting them give $\beta_k = 0$. Thus, $\alpha_k = \beta_k = 0, \forall k \in N \setminus S$.

Next let $k \in Q$. Then points (6) with $x_k = 0$ and $z_k \in \{0, 1\}$ imply $\beta_k = 0, \forall k \in Q$. Then suppose $k \in S \setminus Q$. So points (7) with $x_k = 0$ and $z_k \in \{0, 1\}$ imply $\beta_k = 0, \forall k \in S \setminus Q$. Therefore, $\beta_k = 0, \forall k \in S$.

Let $k_1 \in Q$ and $k_2 \in S \setminus Q$. Considering point (3), we create a new point by decreasing the flow x_{k_1} by 1 and increasing the flow x_{k_2} by the same quantity. Since this point belongs to K, so substituting these points in equation (3.18) and subtracting them imply $\alpha_{k_1} = \alpha_{k_2}, \forall k_1 \in Q, \forall k_2 \in S \setminus Q$. On the other hand, assume $k_1, k_2 \in S \setminus Q$. Applying the similar argument on the flows x_{k_1} and x_{k_2} implies $\alpha_{k_1} = \alpha_{k_2}$. Thus, $\alpha_j = \alpha, j \in S$.

Substituting points (1) and (3) in equality (3.18) and subtracting them imply $\gamma = -\alpha r(S)$ and finally we get $\gamma_0 = \alpha[(\mu(S) - 1)(d - r(S))]$ by substituting point (1) in equation (3.18). □

Now we rewrite inequalities (3.4) and (3.6) for the constant case. First we consider inequalities (3.4).

Proposition 3.3.5. *Let $S \subseteq N$ such that $|S| \geq \lceil \frac{d}{c} \rceil$. The inequality*

$$\sum_{j \in S} x_j - \bar{r}(S) \sum_{j \in S} z_j \leq r(S)y + (\mu(S) - 1)(d - r(S)) - \bar{r}(S)\,|S|\,, \tag{3.19}$$

where $\bar{r}(S) = (\mu(S) - 1)d$ mod c, is valid for $X_{integer}$, and defines a non-trivial facet of $P_{integer}$ if $r(S) < c$ and $\mu(S) \leq U$.

As stated above, inequalities (3.16) and (3.19) generalize the facet-defining inequalities proposed and studied by Magnanti et al. [30]. When $\mu(S) = 2$, then $\bar{r}(S) = r_1$, inequalities (3.16) and (3.19) can be written, respectively, as follows:

$$\sum_{j \in S} x_j \leq d + r(S)(y-1),$$

$$\sum_{j \in S} (x_j - r_1 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + (c - r_1)(y-1), \tag{3.20}$$

where the latter inequality is obtained by using the fact that property $r(S) < c$ implies $r(S) + \bar{r}(S) = c$.

Example 3.3.6. *Assume that $n = 5$, $d = 11$, and $c = 5$ and $U = \lceil \frac{5 \times 5}{11} \rceil = 3$. So $y \in \{0, 1, 2, 3\}$. Using the software PORTA, we obtain 89 facet-defining inequalities for $X_{integer}$ which includes the following inequalities of type* (3.16) *and* (3.19).

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 \leq 9y + 2,\\
&x_1 + x_2 + x_3 + x_5 \leq 9y + 2,\\
&x_1 + x_2 + x_4 + x_5 \leq 9y + 2,\\
&x_1 + x_3 + x_4 + x_5 \leq 9y + 2,\\
&x_2 + x_3 + x_4 + x_5 \leq 9y + 2,\\
&x_1 + x_2 + x_3 \leq 4y + 4, \;\; x_1 + x_2 + x_4 \leq 4y + 4,\\
&x_1 + x_2 + x_5 \leq 4y + 4, \;\; x_1 + x_3 + x_4 \leq 4y + 4,\\
&x_1 + x_3 + x_5 \leq 4y + 4, \;\; x_1 + x_4 + x_5 \leq 4y + 4,\\
&x_2 + x_3 + x_4 \leq 4y + 4, \;\; x_2 + x_3 + x_5 \leq 4y + 4,\\
&x_2 + x_4 + x_5 \leq 4y + 4, \;\; x_3 + x_4 + x_5 \leq 4y + 4,\\
&x_1 + x_2 + x_3 + x_4 + x_5 - 2z_1 - 2z_2 - 2z_3 - 2z_4 - 2z_5 \leq 3y + 6.
\end{aligned}$$

Now we consider the particular case of inequalities (3.6) when $c_j = c$. First observe that condition $\bar{c} = \max\{c_j | j \in S\} > r(S)$ implies $r(S) < c$. By restricting inequality (3.6) to this case ($r(S) < c$) it follows that $r_{\mu(S)-1} = c - r(S)$. In this case (3.6) can be written as follows.

Proposition 3.3.7. *Let $S \subseteq N$ such that $r(S) < c$ and $\mu(S) - 1 \leq U$. The inequality*

$$\sum_{j \in S} x_j - \sum_{j \in S} r_{\mu(S)-1} z_j \leq \frac{|\, S \,| - 1}{\mu(S) - 1} r(S) y, \tag{3.21}$$

is a valid facet-defining inequality of $P_{integer}$, if

$$r_k - c + r(S) \left\lceil \frac{kd}{c} \right\rceil \leq \frac{k(|\, S \,| - 1)}{\mu(S) - 1} r(S), k = 1, \ldots, \mu(S) - 2.$$

When $r_{\mu(S)-1} = (\mu(S)-1)r_1 < c$, inequality (3.21) can be written as:

$$\sum_{j\in S} x_j - \sum_{j\in S} r_{\mu(S)-1} z_j \le \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_{\mu(S)-1}) y,$$

which in case of $\mu(S) = 2$ leads to the inequality

$$\sum_{j\in S} x_j - \sum_{j\in S} r_1 z_j \le \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) y. \tag{3.22}$$

The following proposition extends inequalities (3.20) and (3.22).

Proposition 3.3.8. *Let $S \subseteq N$, then for $k \in \{1, \dots, \lfloor \frac{d}{c} \rfloor\}$, the inequality*

$$\sum_{j\in S} (x_j - r_1 z_j) \le k(c - r_1)y + \Big(\Big\lfloor \frac{d}{c} \Big\rfloor - k\Big)(c - r_1), \tag{3.23}$$

is valid facet-defining inequality of $P_{integer}$, when
(i) $|S| \in \{\lfloor \frac{d}{c} \rfloor + 1, \dots, min\{2\lfloor \frac{d}{c} \rfloor, n\}$ if $k = \lfloor \frac{d}{c} \rfloor$,
(ii) $|S| = \lfloor \frac{d}{c} \rfloor + k$, if $k \in \Big\{1, 2, \dots, min\{\lfloor \frac{d}{c} \rfloor - 1, n - \lfloor \frac{d}{c} \rfloor\}\Big\}$.

We omit the proof here since we provide a proof for a more general result below. Notice that by setting $k = 1$ in (ii), the inequality (3.23) becomes (3.20).

The following theorem establishes that the described inequalities are enough to characterize $P_{integer}$ when $n \le 2\lfloor \frac{d}{c} \rfloor$.

Theorem 3.3.9. *Assume $d > c > 0$, d is not a multiple of c, and $n \le 2\lfloor \frac{d}{c} \rfloor$. Then the trivial facet-defining inequalities of Proposition 3.2.3 in addition to the inequalities* (3.16) *and* (3.23)*, give the complete description of $P_{integer}$.*

Proof. We prove this theorem using a technique introduced by Lovasz [29]. Assume $(x, z, y) \in X_{integer}$ and $(\alpha, \beta, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that $(\alpha, \beta, \gamma) \neq (\mathbf{0}, \mathbf{0}, 0)$. Let $M(\alpha, \beta, \gamma)$ be the set of optimal solutions to the problem $max\{\ h(x, z, y) \mid (x, z, y) \in X_{integer}\ \}$, where $h(x, z, y) = \sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y$. Let R be a polyhedron defined by inequalities of Proposition 3.2.3, inequalities (3.16), and (3.23). So we show that if $M(\alpha, \beta, \gamma) \neq \emptyset$ and $M(\alpha, \beta, \gamma) \neq X_{integer}$, then $M(\alpha, \beta, \gamma)$ is contained in one of the hyperplanes defining R. Alternatively, one can consider the subset of points in $M(\alpha, \beta, \gamma)$ that are extreme in $P_{integer}$ instead of the set $M(\alpha, \beta, \gamma)$.

If $\alpha_j < 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid x_j = 0\}$. If $c\alpha_j + \beta_j < 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid z_j = 0\}$. If $\gamma > 0$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid y = 2\}$. If $\beta_j + \gamma > 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid z_j = 1\}$. Thus, we assume $\alpha_j \ge 0, c\alpha_j + \beta_j \ge 0, \beta_j + \gamma \le 0, j \in N$, and $\gamma \le 0$.

We define the following value function defined for $\lambda \in \{0,1,2\}$:

$$f(\lambda) = max\Big\{ \sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j \mid \sum_{j \in N} x_j \leq d\lambda, x_j \leq cz_j, j \in N$$
$$z_j \leq \lambda, j \in N, z_j \in \{0,1\}, x_j \geq 0, j \in N \Big\}.$$

If $f(1)+\gamma < 0$ and $f(2)+2\gamma < 0$, then $M(\alpha,\beta,\gamma) \subseteq \{(x,z,y) \mid y = 0\}$. Thus, we assume $\max\{f(1)+\gamma, f(2)+2\gamma\} \geq 0$, and consider the following cases.

Case 1: $f(2)+2\gamma > f(1)+\gamma$. Then if $f(1)+\gamma \geq 0$, so $f(2)+2\gamma > f(1)+\gamma \geq 0$ implies $M(\alpha,\beta,\gamma) \subseteq \{(x,z,y) \mid y = 2\}$. Now consider $f(1)+\gamma < 0$. As $f(2)+2\gamma \geq 0$, we show it cannot happen $f(2)+2\gamma = 0$. Assume $f(2)+2\gamma = 0$. We claim that $f(2) \leq 2f(1)$. In order to prove the claim, assume without loss of generality that $c\alpha_1 + \beta_1 \geq \cdots \geq c\alpha_n + \beta_n$. Then $f(1) \geq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j+\beta_j)^+$ and it can be concluded from $n \leq 2\lfloor \frac{d}{c} \rfloor$ that $\sum_{j=\lfloor \frac{d}{c} \rfloor+1}^{n}(c\alpha_j+\beta_j)^+ \leq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j+\beta_j)^+$. Thus, using these inequalities gives

$$f(2) = \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j+\beta_j)^+ + \sum_{j=\lfloor \frac{d}{c} \rfloor+1}^{n}(c\alpha_j+\beta_j)^+ \leq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j+\beta_j)^+ + \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j+\beta_j)^+ \leq 2f(1), \tag{3.24}$$

which proves the claim. Now the following contradiction $-\gamma < f(2) - f(1) \leq f(1) < -\gamma$ holds, where the first inequality follows from $f(2)+2\gamma > f(1)+\gamma$, the second inequality comes from $f(2) \leq 2f(1)$, and the last one follows from $f(1)+\gamma < 0$. Hence, from $f(2)+2\gamma > 0$ follows $M(\alpha,\beta,\gamma) \subseteq \{(x,z,y) \mid y = 2\}$.

Case 2: $f(2)+2\gamma < f(1)+\gamma$. This implies $y \leq 1$ for every $(x,z,y) \in M(\alpha,\beta,\gamma)$. The case $y \leq 1$ was studied in [6] where it was shown that in addition to the defining inequalities the facet defining inequalities are of type (3.23) with $k = \lfloor \frac{d}{c} \rfloor$.

Case 3: $f(2)+2\gamma = f(1)+\gamma \geq 0$. Hence, there are extreme points maximizing function h with $y = 1$, $y = 2$, and the null vector (with $y = 0$) if $f(2)+2\gamma = f(1)+\gamma = 0$. Let $S = \{j \in N | c\alpha_j + \beta_j > 0\}$. Since $n \leq 2\lfloor \frac{d}{c} \rfloor$, then $f(2)$ is obtained by setting $x_j = c, z_j = 1$ for all $j \in S$. Thus, all extreme points with $y = 2$ maximizing function h satisfy (a) $x_j = c, z_j = 1, j \in S$ and $\sum_{j\in S} x_j = c|S| = d + r(S)$. The extreme points with $y = 1$ maximizing function h belong to one of the following two types: (b.1) $y = 1, \sum_{j\in S} x_j = d$; (b.2) $y = 1, \sum_{j\in S} x_j = c\lfloor \frac{d}{c} \rfloor, \sum_{j\in S} z_j = \lfloor \frac{d}{c} \rfloor$. We consider three subcases accordingly to the extreme points maximizing function h, where extreme points of type (a) are considered in all subcases.

Subcase 3.a: If all extreme points maximizing function h with $y = 1$ are of type (b.2), then $M(\alpha,\beta,\gamma) \subseteq \{(x,z,y) \mid x_j = cz_j\}, j \in S$ whether the null vector belongs to $M(\alpha,\beta,\gamma)$ or not.

Subcase 3.b: If all the extreme points maximizing h with $y = 1$ are of type (b.1), then $M(\alpha,\beta,\gamma) \subseteq \{(x,z,y) \mid \sum_{j\in S} x_j = d + r(S)(y-1)\}$. In this case we must show the null vector cannot be optimal. Assume to the contrary that $f(2)+2\gamma = f(1)+\gamma =$

0. Then $f(1) = -\gamma$, and $f(2) = 2f(1)$. So considering inequality (3.24), the condition $f(2) = 2f(1)$ implies $c\alpha_j + \beta_j = \sigma$, where σ is constant, $\forall j \in S, |S| = n = 2\lfloor \frac{d}{c} \rfloor$, and $f(1) = \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j + \beta_j)$. The last equality ensures that there is at least one extreme point with $y = 1$ of type (b.2) maximizing h, which is a contradiction.

Subcase 3.c: Assume there are extreme points maximizing function h with $y = 1$ of both types (b.1) and (b.2). Then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid \sum_{j\in S}(x_j - r_1 z_j) = k(c - r_1)y + (\lfloor \frac{d}{c} \rfloor - k)(c - r_1)\}$, where $k = |S| - \lfloor \frac{d}{c} \rfloor$. Notice that, as in the proof of the subcase 3.b, if null vector is optimal, then $|S| = n = 2\lfloor \frac{d}{c} \rfloor$. Hence, the null vector belongs to $M(\alpha, \beta, \gamma)$ because $k = \lfloor \frac{d}{c} \rfloor$. □

It is easy to verify that for the general case $n > 2\lfloor \frac{d}{c} \rfloor$ the inequalities presented above only provide a partial description of $P_{integer}$. Next we generalize inequalities (3.23).

In the following we will use the remark presented next.

Remark 3.3.10. *One can check that for $j = 2, \ldots, U$, if $jr_1 < c$ then $r_j = jr_1$ and $\lfloor \frac{jd}{c} \rfloor = j\lfloor \frac{d}{c} \rfloor$, and if $jr_1 \geq c$, we have $r_j = jr_1 - \lfloor \frac{jr_1}{c} \rfloor c$ and $\lfloor \frac{jd}{c} \rfloor = j\lfloor \frac{d}{c} \rfloor + \lfloor \frac{jr_1}{c} \rfloor$.*

Proposition 3.3.11. *Assume $d > c > 0$, d is not a multiple of c, and $2\lfloor \frac{d}{c} \rfloor < n$. If $r_a = ar_1 < c$, for some $a \in \{2, \ldots, U-1\}$, and $S \subseteq N$, where $|S| \leq (a+1)\lfloor \frac{d}{c} \rfloor$, then*

$$\sum_{j\in S}(x_j - r_a z_j) \leq k(c - r_a)y + a\Big(\Big\lfloor \frac{d}{c} \Big\rfloor - k\Big)(c - r_a)\ ,\ k = 1, \ldots, \Big\lfloor \frac{d}{c} \Big\rfloor, \tag{3.25}$$

is valid facet-defining inequality of $P_{integer}$, when
(i) $|S| \geq a\lfloor \frac{d}{c} \rfloor + 1$, if $k = \lfloor \frac{d}{c} \rfloor$;
(ii) $|S| = a\lfloor \frac{d}{c} \rfloor + k$, if $k \in \Big\{1, 2, \ldots, min\{\lfloor \frac{d}{c} \rfloor - 1, n - \lfloor \frac{ad}{c} \rfloor\}\Big\}$.

Proof. First, assume that (i) happens. Then we prove validity by considering the following cases.

1. Case $y \geq a + 1$: If $\sum_{j\in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\begin{aligned}\sum_{j\in S}(x_j - r_a z_j) &\leq \sum_{j\in S} c z_j - \sum_{j\in S} r_a z_j = (c - r_a)\sum_{j\in S} z_j \leq \Big\lfloor \frac{ad}{c} \Big\rfloor (c - r_a) = a\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a)\\ &\leq (a+1)\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a) \leq \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a)y.\end{aligned}$$

If $\sum_{j\in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq (c - r_a)\sum_{j\in S} z_j \leq |S|\,(c - r_a) \leq (a+1)\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a) \leq \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a)y.$$

2. Case $y = a$: If $\sum_{j\in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq \Big\lfloor \frac{ad}{c} \Big\rfloor (c - r_a) = a\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a) = \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a)y.$$

If $\sum_{j\in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq ad - \left(\left\lfloor \frac{ad}{c} \right\rfloor + 1\right) r_a = a\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

3. Case $y = b < a$: If $\sum_{j\in S} z_j \leq \lfloor \frac{bd}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq \left\lfloor \frac{bd}{c} \right\rfloor (c - r_a) = b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

If $\sum_{j\in S} z_j \geq \lceil \frac{bd}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq bd - \left(\left\lfloor \frac{bd}{c} \right\rfloor + 1\right) r_a = b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) + r_b - r_a < b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

where the last inequality follows from $r_b < r_a$.

Next, we prove that inequality (3.25) defines a facet of $P_{integer}$. Consider the following points satisfying (3.25) as equation:

(1) $y = 0,\ x_j = 0, z_j = 0, j \in N,$

(2) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}$$

(3) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in S \setminus S_1,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(4) $\forall S_1 \subset S, |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus S,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(5) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus S,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(6) $\forall S_1 \subset S,\ |S_1| = \lfloor \frac{d}{c} \rfloor$,

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases} ,z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 0\ ,\ otherwise. \end{cases}$$

Now consider the following inequality which defines a face of $P_{integer}$.

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y \leq \gamma_0.$$

Then we show that the following equality

$$\sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y = \gamma_0, \tag{3.26}$$

is a multiple of (3.25) as equality where points (1)–(6) satisfy equation (3.26).

It follows by replacing solution (1) in equation (3.26) that $\gamma_0 = 0$. Then substituting solutions (2) and (4) in equation (3.26) and subtracting the resultant equalities imply $r_a\alpha_k + \beta_k = 0, k \in N \setminus S$. In addition, substituting points (2) and (5) in (3.26) and subtracting them give $\beta_k = 0, k \in N \setminus S$. Combining these equations giving $\alpha_k = \beta_k = 0, k \in N \setminus S$.

Now let $i_1, i_2 \in S$. We consider solution (3) with $x_{i_1} = c$ and $x_{i_2} = r_a$. Considering this point, we construct a new point by decreasing the flow of x_{i_1} by 1 and increasing the flow of x_{i_2} by the same value. This new point satisfies (3.25) as equation. Substituting these two solutions in equation (3.26) and subtracting the equalities imply $\alpha_j = \alpha, j \in S$.

Next, for $i_1, i_2 \in S$, we consider solution (2) where $x_{i_1} = c$, $z_{i_1} = 1$ and $x_{i_2} = z_{i_2} = 0$. Then we create a new solution by setting $x_{i_1} = z_{i_1} = 0$ and $x_{i_2} = c$, $z_{i_2} = 1$ which is of type (2) as well. Substituting these points in equation (3.26) and subtracting the resultant equalities give $\beta_j = \beta, j \in S$. Substituting solutions (2) and (3) in equality (3.26) and subtracting them imply $\beta = -\alpha r_a$. Finally, substituting points (6) in (3.26) gives $\gamma = -\alpha \lfloor \frac{d}{c} \rfloor (c - r_a)$ which completes the proof of part (i).

Now let case (ii) occurs. Validity can be proved as follows.

1. Case $y \geq a + 1$. If $\sum_{j \in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\begin{aligned}\sum_{j \in S}(x_j - r_a z_j) &\leq \sum_{j \in S} c z_j - \sum_{j \in S} r_a z_j = (c - r_a) \sum_{j \in S} z_j \leq \Big\lfloor \frac{ad}{c} \Big\rfloor (c - r_a) \\ &\leq a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a) + k(c - r_a)(y - a),\end{aligned}$$

where the last inequality results from $k(c - r_a)(y - a) \geq 1$.

If $\sum_{j \in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\begin{aligned}\sum_{j \in S}(x_j - r_a z_j) &\leq (c - r_a) \sum_{j \in S} z_j \leq |S|\,(c - r_a) = \Big(a \Big\lfloor \frac{d}{c} \Big\rfloor + k\Big)(c - r_a) \\ &\leq a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a) + k(c - r_a)(y - a),\end{aligned}$$

where that last inequality follows from $y - a \geq 1$.

2. Case $y = a$. If $\sum_{j\in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq (c - r_a)\sum_{j\in S} z_j \leq \Big\lfloor \frac{ad}{c}\Big\rfloor (c - r_a) = a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a) + k(c - r_a)(y - a).$$

If $\sum_{j\in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq ad - \Big(\Big\lfloor \frac{ad}{c}\Big\rfloor + 1\Big) r_a = a\Big\lfloor \frac{d}{c}\Big\rfloor c + r_a - a\Big\lfloor \frac{d}{c}\Big\rfloor r_a - r_a = a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a)$$
$$= a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a) + k(c - r_a)(y - a).$$

3. Case $y = b < a$. If $\sum_{j\in S} z_j \leq \lfloor \frac{bd}{c} \rfloor$. Notice that $b < a$ implies $br_1 < ar_1 < c$ and so $\lfloor \frac{bd}{c} \rfloor = b\lfloor \frac{d}{c} \rfloor$ holds. Then

$$\sum_{j\in S}(x_j - r_a z_j) \leq \Big\lfloor \frac{bd}{c}\Big\rfloor (c - r_a) = \Big\lfloor \frac{ad}{c}\Big\rfloor (c - r_a) + \Big(\Big\lfloor \frac{bd}{c}\Big\rfloor - \Big\lfloor \frac{ad}{c}\Big\rfloor\Big)(c - r_a)$$
$$= a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a) + \Big\lfloor \frac{d}{c}\Big\rfloor (b - a)(c - r_a) < a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a) + k(y - a)(c - r_a),$$

where last inequality follows from $b - a < 0$ and $k < \lfloor \frac{d}{c} \rfloor$.

If $\sum_{j\in S} z_j \geq \lceil \frac{bd}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq bd - \Big(\Big\lfloor \frac{bd}{c}\Big\rfloor + 1\Big) r_a = b\Big\lfloor \frac{d}{c}\Big\rfloor c + r_b - r_a - b\Big\lfloor \frac{d}{c}\Big\rfloor r_a = \Big\lfloor \frac{bd}{c}\Big\rfloor (c - r_a)$$
$$+ r_b - r_a < \Big\lfloor \frac{bd}{c}\Big\rfloor (c - r_a) = \Big\lfloor \frac{ad}{c}\Big\rfloor (c - r_a) + \Big(\Big\lfloor \frac{bd}{c}\Big\rfloor - \Big\lfloor \frac{ad}{c}\Big\rfloor\Big)(c - r_a) = a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a)$$
$$+ \Big\lfloor \frac{d}{c}\Big\rfloor (b - a)(c - r_a) < a\Big\lfloor \frac{d}{c}\Big\rfloor (c - r_a) + k(y - a)(c - r_a).$$

In order to prove that inequality (3.25) defines a facet in this case, we follow the approach applied in part (i) and we present the following points belonging to K.

(1) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor$,

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}$$

(2) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in S \setminus S_1$,

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(3) $\forall S_1 \subset S, |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus S,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases} ,z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(4) $\forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus S,$

$$y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases} ,z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(5) Set

$$y = a+1,\ x_j = \begin{cases} c\ ,\ j \in S, \\ 0\ ,\ otherwise, \end{cases} ,z_j = \begin{cases} 1\ ,\ j \in S, \\ 0\ ,\ otherwise. \end{cases}$$

□

At the end of this section, we derive other classes of valid inequalities.

Proposition 3.3.12. *Assume $d > c > 0$, d is not a multiple of c, and $2\lfloor \frac{d}{c} \rfloor < n$. Then*

(i) If $r_2 = 2r_1$, then for $S_1 \subset N$ such that $|S_1| = 2\lfloor \frac{d}{c} \rfloor$ and $S_2 \subseteq N \setminus S_1$, the inequality

$$\sum_{j \in S_1} (x_j - r_1 z_j) + \sum_{j \in S_2} (x_j - r_2 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y, \tag{3.27}$$

is valid for $X_{integer}$ and defines a facet of $P_{integer}$.

(ii) If $r_2 = 2r_1 - c$, for $S \subseteq N$ and for some $i \in S$, the inequality

$$\sum_{j \in S \setminus \{i\}} (x_j - r_1 z_j) + (x_i - r_2 z_i) \leq \left\lceil \frac{d}{c} \right\rceil (c - r_1) y, \tag{3.28}$$

is valid for $X_{integer}$. Moreover, it defines a facet of $P_{integer}$ if $|S| \geq 2\lfloor \frac{d}{c} \rfloor + 1$.

Proof. (i) First, we show that inequality (3.27) is valid. Let $(x, z, y) \in X_{integer}$. For $y = 0$, the validity is straightforward. Let $y = 1$. Then we consider the following cases.

Case I: If $\sum_{j \in S_1} z_j + \sum_{j \in S_2} z_j \leq \lfloor \frac{d}{c} \rfloor$. Since $c - r_2 < c - r_1$, then

$$\sum_{j \in S_1} (x_j - r_1 z_j) + \sum_{j \in S_2} (x_j - r_2 z_j) \leq \sum_{j \in S_1} (c - r_1) z_j + \sum_{j \in S_2} (c - r_2) z_j$$
$$\leq (c - r_1) \sum_{j \in S_1 \cup S_2} z_j \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1).$$

Case II: If $\sum_{j\in S_1} z_j + \sum_{j\in S_2} z_j \geq \lceil \frac{d}{c} \rceil$ or equivalently $\sum_{j\in S_1} z_j \geq \lceil \frac{d}{c} \rceil - \sum_{j\in S_2} z_j$. So

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) = \sum_{j\in S_1\cup S_2} x_j - r_1\sum_{j\in S_1} z_j - r_2\sum_{j\in S_2} z_j$$
$$\leq d - r_1\Big\lceil\frac{d}{c}\Big\rceil + r_1\sum_{j\in S_2} z_j - 2r_1\sum_{j\in S_2} z_j = c\Big\lfloor\frac{d}{c}\Big\rfloor - r_1\Big\lfloor\frac{d}{c}\Big\rfloor - r_1\sum_{j\in S_2} z_j \leq \Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1),$$

where the last inequality holds because $-r_1\sum_{j\in S_2} z_j \leq 0$.

Now let $y = a$ where $2 \leq a \leq U$. Then we have the following cases. Case 1: $ar_1 < c$; Case 2: $ar_1 \geq c$.

Let Case 1 occurs. Then we have $r_a = ar_1$ and $\lfloor \frac{ad}{c} \rfloor = a\lfloor \frac{d}{c} \rfloor$. We consider two subcases as follows.

Subcase I: If $\sum_{j\in S_1} z_j + \sum_{j\in S_2} z_j \leq \lfloor \frac{ad}{c} \rfloor$. Then since $c - r_2 < c - r_1$, so

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) \leq \sum_{j\in S_1}(c - r_1)z_j + \sum_{j\in S_2}(c - r_2)z_j$$
$$\leq (c - r_1)\sum_{j\in S_1\cup S_2} z_j \leq a\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1).$$

Subcase II: If $\sum_{j\in S_1} z_j + \sum_{j\in S_2} z_j \geq \lceil \frac{ad}{c} \rceil$ or equivalently $\sum_{j\in S_2} z_j \geq \lceil \frac{ad}{c} \rceil - \sum_{j\in S_1} z_j$. Thus

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) = \sum_{j\in S_1\cup S_2} x_j - r_1\sum_{j\in S_1} z_j - r_2\sum_{j\in S_2} z_j$$
$$\leq ad - r_1\sum_{j\in S_1} z_j - r_2\Big\lceil\frac{ad}{c}\Big\rceil + r_2\sum_{j\in S_1} z_j = ad - r_1\sum_{j\in S_1} z_j - 2r_1\Big(\Big\lfloor\frac{ad}{c}\Big\rfloor + 1\Big) + 2r_1\sum_{j\in S_1} z_j$$
$$\leq ad - 2ar_1\Big\lfloor\frac{d}{c}\Big\rfloor - 2r_1 + r_1\sum_{j\in S_1} z_j \leq ad - 2ar_1\Big\lfloor\frac{d}{c}\Big\rfloor - 2r_1 + 2r_1\Big\lfloor\frac{d}{c}\Big\rfloor$$
$$= ac\Big\lfloor\frac{d}{c}\Big\rfloor + ar_1 - 2ar_1\Big\lfloor\frac{d}{c}\Big\rfloor - 2r_1 + 2r_1\Big\lfloor\frac{d}{c}\Big\rfloor = a\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1) - (2 - a)r_1 + (2 - a)r_1\Big\lfloor\frac{d}{c}\Big\rfloor$$
$$\leq a\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1) + \Big(\Big\lfloor\frac{d}{c}\Big\rfloor - 1\Big)(2 - a)r_1 \leq a\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1),$$

where the last inequality is obtained by using $(\lfloor \frac{d}{c} \rfloor - 1)(2 - a)r_1 \leq 0$.

Next, let Case 2 happens. So $r_a = ar_1 - \lfloor \frac{ar_1}{c} \rfloor c$ and $\lfloor \frac{ad}{c} \rfloor = a\lfloor \frac{d}{c} \rfloor + \lfloor \frac{ar_1}{c} \rfloor$. Then we consider the following subcases similarly.

Subcase I: If $\sum_{j\in S_1} z_j + \sum_{j\in S_2} z_j \leq \lfloor \frac{ad}{c} \rfloor$. Then applying $-\lfloor \frac{d}{c} \rfloor \leq -1$ and $-\lfloor \frac{ar_1}{c} \rfloor \leq -1$ give

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) \le \Big\lfloor \frac{2d}{c}\Big\rfloor(c-r_1) + \Big(\Big\lfloor \frac{ad}{c}\Big\rfloor - \Big\lfloor \frac{2d}{c}\Big\rfloor\Big)(c-r_2)$$
$$= 2\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) + \Big(a\Big\lfloor \frac{d}{c}\Big\rfloor + \Big\lfloor \frac{ar_1}{c}\Big\rfloor\Big)(c-r_1-r_1) - 2\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1-r_1)$$
$$= 2\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) + a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) - ar_1\Big\lfloor \frac{d}{c}\Big\rfloor + c\Big\lfloor \frac{ar_1}{c}\Big\rfloor - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor - 2\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) + 2r_1\Big\lfloor \frac{d}{c}\Big\rfloor$$
$$= a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) - (a-2)r_1\Big\lfloor \frac{d}{c}\Big\rfloor + ar_1 - r_a - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor \le a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) - (a-2)r_1 + ar_1$$
$$- r_a - 2r_1 = a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) - r_a \le a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1)$$

Subcase II: If $\sum_{j\in S_2} z_j \ge \lceil \frac{ad}{c}\rceil - \sum_{j\in S_1} z_j$. Thus, considering $-\lfloor \frac{d}{c}\rfloor \le -1$ implies

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) = \sum_{j\in S_1\cup S_2} x_j - r_1\sum_{j\in S_1} z_j - r_2\sum_{j\in S_2} z_j$$
$$\le ad - r_1\sum_{j\in S_1} z_j - r_2\Big\lceil \frac{ad}{c}\Big\rceil + r_2\sum_{j\in S_1} z_j = ad - 2r_1\Big(\Big\lfloor \frac{ad}{c}\Big\rfloor + 1\Big) + r_1\sum_{j\in S_1} z_j$$
$$\le ac\Big\lfloor \frac{d}{c}\Big\rfloor + ar_1 - 2ar_1\Big\lfloor \frac{d}{c}\Big\rfloor - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor - 2r_1 + 2r_1\Big\lfloor \frac{d}{c}\Big\rfloor = a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) + ar_1$$
$$-(a-2)r_1\Big\lfloor \frac{d}{c}\Big\rfloor - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor - 2r_1 \le a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) + ar_1 - (a-2)r_1 - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor$$
$$-2r_1 = a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1) - 2r_1\Big\lfloor \frac{ar_1}{c}\Big\rfloor < a\Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1)$$

where the last inequality holds since $-2r_1\lfloor \frac{ar_1}{c}\rfloor < 0$.

Next we show that inequality (3.27) defines a facet. Under those conditions, we consider an equation

$$\sum_{j\in S_1}(x_j - r_1 z_j) + \sum_{j\in S_2}(x_j - r_2 z_j) = \Big\lfloor \frac{d}{c}\Big\rfloor(c-r_1)y, \tag{3.29}$$

and let $K = P_{integer} \cap \{(x,z,y) \mid (x,z,y) \text{ satisfies } (3.29)\}$. Now we show that inequality (3.27) is facet-defining by showing that whenever the inequality $\sum_{j\in N}\alpha_j x_j + \sum_{j\in N}\beta_j z_j + \gamma y \le \gamma_0$ is valid for $X_{integer}$ and satisfies the condition that

$$\sum_{j\in N}\alpha_j x_j + \sum_{j\in N}\beta_j z_j + \gamma y = \gamma_0, \forall (x,z,y) \in K, \tag{3.30}$$

then equality (3.30) is a multiple of (3.29). Now we create the following feasible points which belong to K.

(1) $y = 0,\ x_j = 0, z_j = 0, j \in N,$

(2) $\forall S' \subset S_1, |S'| = \lfloor \frac{d}{c} \rfloor,$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S', \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S', \\ 0\ ,\ otherwise, \end{cases}$$

(3) $\forall S' \subset S_1, |S'| = \lfloor \frac{d}{c} \rfloor, \forall k \in S_1 \setminus S',$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S', \\ r_1,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S', \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(4) $\forall S' \subset S_1, |S'| = \lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus (S_1 \cup S_2),$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S', \\ r_1,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S', \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(5) $\forall S' \subset S_1, |S'| = \lfloor \frac{d}{c} \rfloor, \forall k \in N \setminus (S_1 \cup S_2),$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S', \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S', \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(6) Set

$$y = 2,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}$$

(7) $\forall k \in S_2,$

$$y = 2,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_2,\ for\ k, \\ 0\ ,\ otherwise, \end{cases},\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise. \end{cases}$$

We conclude $\gamma_0 = 0$ by replacing solution (1) in equation (3.30). Substituting solutions (2) and (4) in equation (3.30) and subtracting the resultant equalities imply $r_1\alpha_k + \beta_k = 0, k \in N \setminus (S_1 \cup S_2)$. In addition, substituting points (2) and (5) in (3.30) and subtracting them give $\beta_k = 0, k \in N \setminus (S_1 \cup S_2)$. Combining these equations giving $\alpha_k = \beta_k = 0, k \in N \setminus (S_1 \cup S_2)$. Therefore, equation (3.30) is

$$\sum_{j \in S_1 \cup S_2} \alpha_j x_j + \sum_{j \in S_1 \cup S_2} \beta_j z_j + \gamma y = 0. \tag{3.31}$$

Let $i_1, i_2 \in S_1$. Then we know that solution (3) such that $x_{i_1} = c$ and $x_{i_2} = r_1$ belongs to K. Considering this feasible point, we construct a new solution by decreasing the value of x_{i_1} by 1 and increasing the value of x_{i_2} by the same value. This new solution is in K. Substituting these two solutions in equation (3.31) and subtracting the resultant equalities imply $\alpha_j = \alpha, j \in S_1$. Next, let $i_1 \in S_1$ and $i_2 \in S_2$. We consider solution (7) where $x_{i_1} = c$ and $x_{i_2} = r_2$. Similarly, we conclude $\alpha_j = \alpha, j \in S_2$.

Now for $i_1, i_2 \in S_1$, solution (2) where $x_{i_1} = c$, $z_{i_1} = 1$ and $x_{i_2} = z_{i_2} = 0$ belongs to K. Considering this solution, we create a new solution such that $x_{i_1} = z_{i_1} = 0$ and $x_{i_2} = c$, $z_{i_2} = 1$ which is in K. Substituting these feasible points in equation (3.31) and subtracting them give $\beta_j = \beta_1, j \in S_1$. Applying the same technique with solutions (7) implies $\beta_j = \beta_2, j \in S_2$. Substituting solutions (2) and (3) in equation (3.31) and subtracting the equalities imply $\beta_1 = -\alpha r_1$. In a similar way, solutions (6) and (7) give $\beta_2 = -\alpha r_2$. Finally, substituting points (2) and (6) in (3.31) and subtracting them imply $\gamma = -\alpha\lfloor \frac{d}{c} \rfloor(c - r_1)$ which completes the proof of part (i).

(ii) We justify the validity of inequality (3.28) as follows. Let $(x, z, y) \in X_{integer}$. For $y = 0$, the validity is trivial. Assume $y = 1$. Then we follow two cases.

Case I: If $\sum_{j\in S\setminus\{i\}} z_j + z_i \leq \lfloor \frac{d}{c} \rfloor$. Then since $c - r_2 = 2(c - r_1)$

$$\sum_{j\in S\setminus\{i\}} (x_j - r_1 z_j) + (x_i - r_2 z_i) \leq \sum_{j\in S\setminus\{i\}} (c - r_1)z_j + (c - r_2)z_i = (c - r_1)\sum_{j\in S} z_j + (c - r_1)z_i$$
$$\leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + (c - r_1) = \left\lceil \frac{d}{c} \right\rceil (c - r_1).$$

Case II: If $\sum_{j\in S\setminus\{i\}} z_j + z_i \geq \lceil \frac{d}{c} \rceil$ or equivalently $\sum_{j\in S\setminus\{i\}} z_j \geq \lceil \frac{d}{c} \rceil - z_i$. Then applying $r_1 - r_2 = c - r_1$ gives

$$\sum_{j\in S} x_j - r_1 \sum_{j\in S\setminus\{i\}} z_j - r_2 z_i \leq d - r_1\left\lceil \frac{d}{c} \right\rceil + r_1 z_i - r_2 z_i = c\left\lfloor \frac{d}{c} \right\rfloor + r_1 - r_1\left\lceil \frac{d}{c} \right\rceil + (r_1 - r_2)z_i$$
$$\leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + (c - r_1) = \left\lceil \frac{d}{c} \right\rceil (c - r_1).$$

Then assume $y = a$ where $2 \leq a \leq U$. Since $2r_1 \geq c$, so $ar_1 \geq c$ and hence we have $r_a = ar_1 - \lfloor \frac{ar_1}{c} \rfloor c$ and $\lfloor \frac{ad}{c} \rfloor = a\lfloor \frac{d}{c} \rfloor + \lfloor \frac{ar_1}{c} \rfloor$, for $2 \leq a \leq U$. The following cases are considered.

Case I: If $\sum_{j\in S\setminus\{i\}} z_j + z_i \leq \lfloor \frac{ad}{c} \rfloor$. So

$$\sum_{j\in S\setminus\{i\}} (x_j - r_1 z_j) + (x_i - r_2 z_i) \leq \sum_{j\in S\setminus\{i\}} (c - r_1)z_j + (c - r_2)z_i = (c - r_1)\sum_{j\in S} z_j + (c - r_1)z_i$$
$$\leq \left\lfloor \frac{ad}{c} \right\rfloor (c - r_1) + (c - r_1) = a\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + \left\lfloor \frac{ar_1}{c} \right\rfloor (c - r_1) + (c - r_1)$$
$$= a\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + \left\lceil \frac{ar_1}{c} \right\rceil (c - r_1) \leq a\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + a(c - r_1) = a\left\lceil \frac{d}{c} \right\rceil (c - r_1),$$

where the last inequality is obtained by applying $\lceil \frac{ar_1}{c} \rceil \leq a \lceil \frac{r_1}{c} \rceil = a$.

Case II: If $\sum_{j \in S \setminus \{i\}} z_j \geq \lceil \frac{ad}{c} \rceil - z_i$. Then

$$\sum_{j \in S} x_j - r_1 \sum_{j \in S \setminus \{i\}} z_j - r_2 z_i \leq ad - r_1 \Big\lceil \frac{ad}{c} \Big\rceil + r_1 z_i - r_2 z_i = ad - r_1 \Big(\Big\lfloor \frac{ad}{c} \Big\rfloor + 1 \Big)$$
$$+ (r_1 - r_2) z_i \leq ac \Big\lfloor \frac{d}{c} \Big\rfloor + ar_1 - ar_1 \Big\lfloor \frac{d}{c} \Big\rfloor - r_1 \Big\lfloor \frac{ar_1}{c} \Big\rfloor - r_1 + (c - r_1) = a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) + r_a$$
$$+ c \Big\lfloor \frac{ar_1}{c} \Big\rfloor - r_1 \Big\lfloor \frac{ar_1}{c} \Big\rfloor - r_1 + (c - r_1) = a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) + \Big\lceil \frac{ar_1}{c} \Big\rceil (c - r_1) + r_a - r_1, \quad (3.32)$$

Note that if $r_a - r_1 \leq 0$ then the validity of inequality (3.28) is satisfied using $\lceil \frac{ar_1}{c} \rceil \leq a$. So let $r_a - r_1 > 0$. Then we claim that $\lceil \frac{ar_1}{c} \rceil \leq a - 1$. Assume to the contrary $\lceil \frac{ar_1}{c} \rceil = a$. Then

$$r_a = ar_1 - \Big\lfloor \frac{ar_1}{c} \Big\rfloor c = ar_1 - (a - 1)c = -a(c - r_1) + c < -(c - r_1) + c = r_1$$

which implies $r_a - r_1 < 0$ that is a contradiction. Thus, applying $\lceil \frac{ar_1}{c} \rceil \leq a - 1$ for inequality (3.32) gives

$$\sum_{j \in S} x_j - r_1 \sum_{j \in S \setminus \{i\}} z_j - r_2 z_i \leq a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) + \Big\lceil \frac{ar_1}{c} \Big\rceil (c - r_1) + r_a - r_1$$
$$< a \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) + (a - 1)(c - r_1) + (c - r_1) = a \Big\lceil \frac{d}{c} \Big\rceil (c - r_1),$$

which completes the proof of validity.

Then, we justify that inequality (3.28) is facet-defining. We apply the same technique used in part (i) by introducing the following feasible points belonging to K.

(1) $y = 0, \; x_j = 0, z_j = 0, j \in N$,

(2) $\forall S_1 \subset S \setminus \{i\}, |S_1| = \lfloor \frac{d}{c} \rfloor$,

$$y = 1, \; x_j = \begin{cases} c \;, \; j \in S_1, \\ r_1 \;, \; for \; i, \\ 0 \;, \; otherwise, \end{cases} \;, \; z_j = \begin{cases} 1 \;, \; j \in S_1, \\ 1 \;, \; for \; i, \\ 0 \;, \; otherwise, \end{cases}$$

(3) $\forall S_1 \subset S \setminus \{i\}, |S_1| = \lfloor \frac{d}{c} \rfloor - 1$,

$$y = 1, \; x_j = \begin{cases} c \;, \; j \in S_1, \\ c \;, \; for \; i, \\ 0 \;, \; otherwise, \end{cases} \;, \; z_j = \begin{cases} 1 \;, \; j \in S_1, \\ 1 \;, \; for \; i, \\ 0 \;, \; otherwise, \end{cases}$$

(4) $\forall S_1 \subset S \setminus \{i\}, |S_1| = \lfloor \frac{d}{c} \rfloor - 1, \forall k \in S \setminus (\{i\} \cup S_1)$,

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ c\ ,\ for\ i, \\ r_1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases} \quad ,\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ i, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(5) $\forall S_1 \subset S \setminus \{i\}, |S_1| = \lfloor \frac{d}{c} \rfloor - 1, \forall k \in N \setminus S,$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ c\ ,\ for\ i, \\ r_1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases} \quad ,\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ i, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(6) $\forall S_1 \subset S \setminus \{i\}, |S_1| = \lfloor \frac{d}{c} \rfloor - 1, \forall k \in N \setminus S,$

$$y = 1,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ c\ ,\ for\ i, \\ 0\ ,\ otherwise, \end{cases} \quad ,\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ i, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

(7) $\forall S_1 \subseteq S \setminus \{i\}, |S_1| = 2\lfloor \frac{d}{c} \rfloor,$

$$y = 2,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ c\ ,\ for\ i, \\ 0\ ,\ otherwise, \end{cases} \quad ,\ z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ i, \\ 0\ ,\ otherwise. \end{cases}$$

Similar to the proof of part (i), we can prove that inequality (3.28) defines a facet. □

3.4 Lifted Inequalities

In this section we discuss the lifting of set-up inequalities given in Proposition 3.3.8. In Section 3.4.1 we discuss simultaneous lifting of such inequalities while in Section 3.4.2 we study superadditive lifting. With this discussion we aim to derive new facet-defining inequalities for $P_{integer}$ and to provide some insight on the difficulty of providing the full polyhedral description of $P_{integer}$ in the original space of variables.

3.4.1 Simultaneous Lifting

In this section we generate some facet-defining valid inequalities for $P_{integer}$ using simultaneous lifting, following [28].

We select $C_1 \subset N$ such that $|C_1| = \lceil \frac{d}{c} \rceil$ and $C_2 \subseteq N \setminus C_1$. By setting $x_j = 0$, $z_j = 0$, for $j \in N \setminus C_1$, we obtain the following restricted set.

$$Y = \Big\{(x,z,y) \in \mathbb{R}_+^{|C_1|} \times \mathbb{B}^{|C_1|} \times \mathbb{Z}_+ \mid \sum_{j\in C_1} x_j \le dy, x_j \le cz_j, \\ z_j \le y, j \in C_1, y \in \{0,1,\dots,U\}\Big\}.$$

Proposition 3.3.8, case $k = \lfloor \frac{d}{c} \rfloor$, states that the set-up flow cover inequality

$$\sum_{j\in C_1}(x_j - r_1 z_j) \le \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1)y, \tag{3.33}$$

defines a facet of the convex hull of Y.

Then, the lifting function ϕ associated with valid inequality (3.33) is the following.

$$\phi(u) = \min \quad \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1)y - \sum_{j\in C_1}(x_j - r_1 z_j) \tag{3.34}$$

$$s.t. \quad \sum_{j\in C_1} x_j \le dy - u, \tag{3.35}$$

$$0 \le x_j \le cz_j,\ j \in C_1, \tag{3.36}$$

$$z_j \in \{0,1\}, j \in C_1, \tag{3.37}$$

$$y \in \{1,\dots,U\}, \tag{3.38}$$

where $u \in [0, Ud]$. Notice that we have replaced condition $\{0,\dots,U\}$ by (3.38) and removed constraints $z_j \le y, j \in C_1$ from the above-mentioned program because y can be zero only for $u = 0$ (otherwise the foregoing program becomes infeasible). As $\phi(0)$ can be computed by setting $y = 0, x_j = z_j = 0, j \in C_1$ or alternatively $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. Hence, we can exclude the solution with $y = 0$ from the foregoing mixed integer program.

Proposition 3.4.1. *The lifting function ϕ can be written on $[0, Ud]$ as follows (see Figure 3.1).*

$$\phi(u) = \begin{cases} k\lfloor \frac{d}{c} \rfloor (c - r_1), & k(\lfloor \frac{d}{c} \rfloor c + r_1) \le u < k\lfloor \frac{d}{c} \rfloor c + (k+1)r_1, \\ u - (k\lfloor \frac{d}{c} \rfloor + k + p + 1)r_1, & (k\lfloor \frac{d}{c} \rfloor + p)c + (k+1)r_1 \le u < (k\lfloor \frac{d}{c} \rfloor + p + 1)c + kr_1, \\ (k\lfloor \frac{d}{c} \rfloor + m)(c - r_1), & (k\lfloor \frac{d}{c} \rfloor + m)c + kr_1 \le u < (k\lfloor \frac{d}{c} \rfloor + m)c + (k+1)r_1, \\ \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)(c - r_1), & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + kr_1 \le u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1, \\ u - (k+1)\lceil \frac{d}{c} \rceil r_1, & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 \le u \le (k+1)(\lfloor \frac{d}{c} \rfloor c + r_1), \end{cases}$$

where $k \in \{0,\dots,U-1\}, p \in \{0,\dots,\lfloor \frac{d}{c} \rfloor - 2\}$, and $m \in \{1,\dots,\lfloor \frac{d}{c} \rfloor - 2\}$.

Proof. To compute the lifting function, for each u, we set $y = y_0$ where $y_0 \in \{\lceil \frac{u}{d} \rceil, \ldots, U\}$ and then minimize $\lfloor \frac{d}{c} \rfloor (c - r_1) y_0 - \sum_{j \in C_1} (x_j - r_1 z_j)$ under constraints (3.35)–(3.37). To achieve the minimum value in (3.34), x_j must be equal to cz_j for as many j as possible. We provide the lifting function on $[0, d]$ as follows.

The greatest value of u such that $\phi(u) = 0$ is r_1 where $\phi(r_1)$ is obtained by taking $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. The function ϕ increases for $u \in [r, c]$ and $\phi(c) = c - r_1$ which can be computed by setting $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor - 1, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. Other cases can be obtained similarly for $u \in [c, (\lfloor \frac{d}{c} \rfloor - 1)c]$ with $\phi(u) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. In order to find $\phi(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$, one can check that the minimum is found by setting $y = 2, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor + 1, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$ and so $\phi(u) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. Thus, the lifting function is constant on $[(\lfloor \frac{d}{c} \rfloor - 1)c, (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1]$. Function ϕ is increasing on interval $[(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1, d]$ where $\phi(d) = \lfloor \frac{d}{c} \rfloor (c - r_1)$ obtained by taking $y = 1, x_j = z_j = 0, j \in C_1$ or $y = 2, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$.

Note that the lifting function can be computed similarly on the other intervals. □

An important particular case is where y is binary, that is $U = 1$. This case was considered in [6]. In this case, the lifting function ϕ has the same pattern as the integer case with $U > 1$ for $u \leq (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$, but differs for u greater than that value. The lifting function ϕ on $[0, d]$ is shown in Figure 3.1. The dark line represents the case $U > 1$ while the case $U = 1$, that differs from the general case only for $u \in [(\lfloor \frac{d}{c} \rfloor - 1)c + r_1, d]$ is shown by dotted lines.

Next we explain the simultaneous lifting of (3.33) in detail. We lift variable pairs $(x_j, z_j), j \in C_2$. We attribute coefficients (λ_j, μ_j) to $(x_j, z_j), j \in C_2$ in such a way that the inequality

$$\sum_{j \in C_1} (x_j - r_1 z_j) + \sum_{j \in C_2} (\lambda_j x_j + \mu_j z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y, \tag{3.39}$$

is valid for $X_{integer}$ restricted to $x_j = z_j = 0, j \in N \setminus (C_1 \cup C_2)$, which we denote by $X_{C_1 \cup C_2}$. Let

$$X^{feasible} = \Big\{ (x, z) \in \mathbb{R}_+^{|C_2|} \times \mathbb{B}^{|C_2|} \mid 0 \leq x_j \leq cz_j, j \in C_2, z_j \in \{0, 1\}, j \in C_2 \Big\},$$

and

$$\Pi = \Big\{ (\lambda, \mu) \in \mathbb{R}^{|C_2| + |C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j) \;:\; (x, z) \in X^{feasible} \Big\}.$$

Then each coefficient vector $(\lambda, \mu) \in \Pi$ gives a valid inequality (3.39) for $X_{C_1 \cup C_2}$. Note that the constraints $z_j \leq y, j \in C_2$ are omitted in the description of Π because $y \in \{1, \ldots, U\}$.

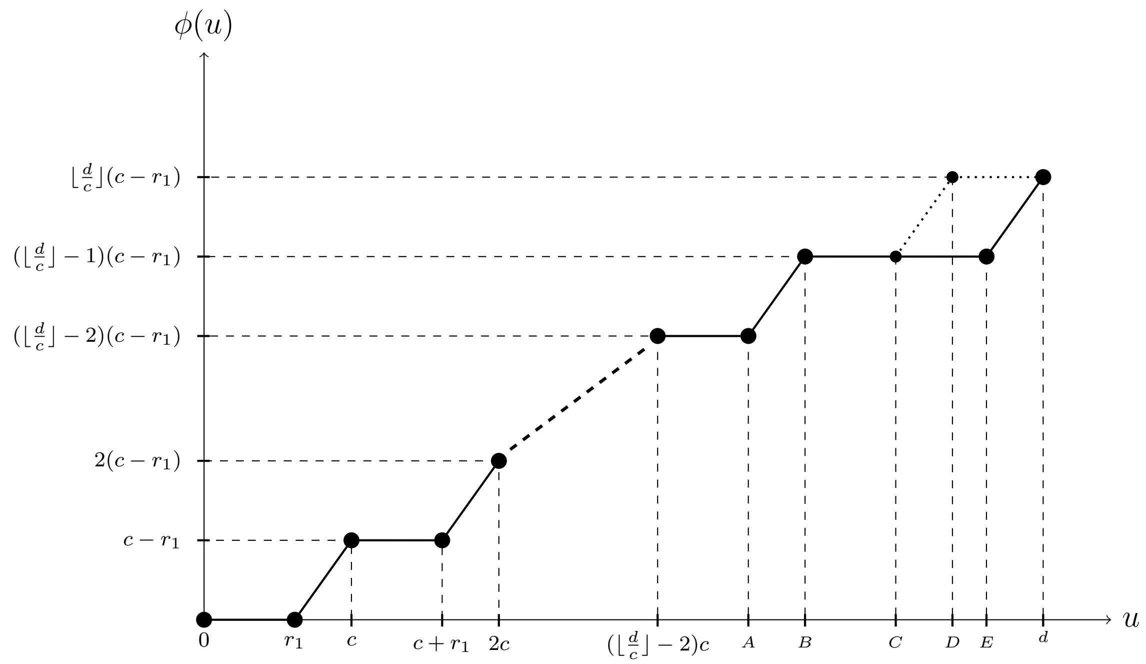

Figure 3.1: The lifting function ϕ on $[0, d]$ where $A = (\lfloor \frac{d}{c} \rfloor - 2)c + r_1$, $B = (\lfloor \frac{d}{c} \rfloor - 1)c$, $C = (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$, $D = \lfloor \frac{d}{c} \rfloor c$, and $E = (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$.

Since for all $j \in N$, x_j and z_j are bounded, then $X^{feasible}$ is bounded as well. Note that for any $u \in \mathbb{R}_+$, there exists $(x, z, y) \in \mathbb{R}^{|C_1|} \times \mathbb{B}_+^{|C_1|} \times \mathbb{Z}_+$ satisfying (3.35)–(3.38), so $\phi(u)$ is finite for all $u \in \mathbb{R}_+$. It follows from this result that Π is bounded.

Next we construct Π by splitting the interval $[0, Ud]$ to smaller intervals as follows.

Definition 3.4.2. *Let*

$$X_{[u_1,u_2]} = conv \left\{ X^{feasible} \bigcap \left\{ (x, z) \in \mathbb{R}_+^{|C_2|} \times \mathbb{B}^{|C_2|} \mid u_1 \leq \sum_{j \in C_2} x_j \leq u_2 \right\} \right\}$$
$$= conv \left\{ (x^1, z^1), \ldots, (x^q, z^q) \right\},$$

where (x^i, z^i), $i \in \{1, \ldots, q\}$, are the extreme points of the polyhedron $X_{[u_1,u_2]}$ and define

$$\Pi_{[u_1,u_2]} = \left\{ (\lambda, \mu) \in \mathbb{R}^{|C_2|+|C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j) \ , \ (x, z) \in X_{[u_1,u_2]} \right\}.$$

Lemma 3.4.3. *Under Definition 3.4.2,*

$$\Pi_{[u_1,u_2]} = \left\{ (\lambda, \mu) \in \mathbb{R}^{|C_2|+|C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j), (x, z) \text{ vertex of } X_{[u_1,u_2]} \right\}.$$

Proof. Since ϕ is piecewise linear, then for $u \in [u_1, u_2]$, we have $\phi(u) = au + b$, where a and b are constant. Now suppose that $(\tilde{x}, \tilde{z})$ be an arbitrary point in $X_{[u_1,u_2]}$ and $(x^i, z^i), i \in \{1, \dots, q\}$ are the extreme points of this polyhedron. Then $(\tilde{x}, \tilde{z}) = \sum_{i=1}^{q} \nu_i (x^i, z^i)$ such that $\nu_i \geq 0, \forall i \in \{1, \dots, q\}$ and $\sum_{i=1}^{q} \nu_i = 1$. Let $(\lambda, \mu) \in \Pi_{[u_1,u_2]}$. So

$$\sum_{j \in C_2} \lambda_j x_j^i + \sum_{j \in C_2} \mu_j z_j^i \leq \phi(\sum_{j \in C_2} x_j^i) = a(\sum_{j \in C_2} x_j^i) + b, i = 1, \dots, q. \tag{3.40}$$

Multiplying inequalities (3.40) by corresponding ν_i for all $i = 1, \dots, q$ and then summing them imply

$$\sum_{i=1}^{q} \sum_{j \in C_2} \nu_i \lambda_j x_j^i + \sum_{i=1}^{q} \sum_{j \in C_2} \nu_i \mu_j z_j^i \leq \sum_{i=1}^{q} \sum_{j \in C_2} a \nu_i x_j^i + \sum_{i=1}^{q} \nu_i b = \sum_{i=1}^{q} \sum_{j \in C_2} a \nu_i x_j^i + b,$$

and so

$$\sum_{j \in C_2} \lambda_j \tilde{x}_j + \sum_{j \in C_2} \mu_j \tilde{z}_j \leq a(\sum_{j \in C_2} \tilde{x}_j) + b = \phi(\sum_{j \in C_2} \tilde{x}_j),$$

which shows that the inequality is satisfied for $(\tilde{x}, \tilde{z})$. □

Observation 3.4.4. $\Pi = \Pi_{[0,r]} \bigcap \Pi_{[r,c]} \bigcap \dots \bigcap \Pi_{[(U \lfloor \frac{d}{c} \rfloor - 1)c + (U+1)r_1, Ud]}$.

Observation 3.4.5. Π *is a polyhedron.*

The following Lemma will be used to characterize Π.

Lemma 3.4.6. *If* (λ, μ) *is a vertex of* Π, *then* $\lambda_j \geq 0$, $j \in C_2$.

Proof. Let (λ, μ) be an extreme point of Π. Suppose to the contrary that $\lambda_k < 0$, for some $k \in C_2$. First, we show that $x_k = 0$, for all $(x, z) \in X^{feasible}$. So let $(x, z) \in X^{feasible}$ and assume to the contrary that $x_k > 0$. Since (λ, μ) is an extreme point of Π, so there exist defining inequalities of Π such that

$$\sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j = \phi(\sum_{j \in C_2} x_j). \tag{3.41}$$

Now consider a small enough $\epsilon > 0$ such that $x_k - \epsilon > 0$. Then we generate a new point $(x^*, z^*) \in X^{feasible}$ where $x_j^* = x_j, j \in C_2 \setminus \{k\}, x_k^* = x_k - \epsilon, z_j^* = z_j, j \in C_2$. Thus we have

$$\sum_{j \in C_2} \lambda_j x_j - \epsilon \lambda_k + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j - \epsilon).$$

Substituting equality (3.41) in the latter inequality gives

$$\epsilon \lambda_k \geq \phi(\sum_{j \in C_2} x_j) - \phi(\sum_{j \in C_2} x_j - \epsilon),$$

which is a contradiction, since $\epsilon\lambda_k < 0$ and $\phi(\sum_{j\in C_2} x_j) - \phi(\sum_{j\in C_2} x_j - \epsilon) \geq 0$. Therefore $x_k = 0, k \in C_2$, for all $(x, z) \in X^{feasible}$ such that equality (3.41) holds.

Now we define two points (λ^1, μ) and (λ^2, μ) as follows.

$$\lambda_i^1 = \lambda_i^2 = \lambda_i \ , \ i \neq k, \lambda_k^1 = \lambda_k + \epsilon, \lambda_k^2 = \lambda_k - \epsilon.$$

This definition implies if equality (3.41) is satisfied at extreme point (λ, μ) , then it is satisfied at (λ^1, μ) and (λ^2, μ) as well. It can be seen as a consequence of $x_k = 0$ that remaining defining inequalities of Π such that

$$\sum_{j\in C_2} \lambda_j x_j + \sum_{j\in C_2} \mu_j z_j < \phi(\sum_{j\in C_2} x_j),$$

are valid for (λ^1, μ) and (λ^2, μ). Therefore, (λ, μ) can be written as a convex combination of two points of Π which is a contradiction with the fact that (λ, μ) is a vertex of Π. □

Our approach to find the lifting coefficients is to apply Observation 3.4.4, Lemma 3.4.3, and Lemma 3.4.6 to find the characterization of the polyhedron Π. Then we compute the vertices of Π which are the lifting coefficients. In addition, since the set Y is full-dimensional, the initial inequality (3.33) is facet-defining, exact lifting function ϕ is used to define Π, and extreme points of Π are used as the lifting coefficients, then the lifted inequality is facet-defining for $P_{integer}$ (see [28]).

Below we discuss theoretically how to find valid inequalities which are required to describe Π in interval $[0, d]$. Note that the calculations to obtain the required valid inequalities to describe Π in other intervals can be done similarly.

Firstly, take interval $[0, r_1]$ and compute the extreme points of $X_{[0,r_1]}$ which are (i) $x_j = 0, j \in C_2; z_j \in \{0, 1\}, j \in C_2$, and (ii) $x_j = r_1$, for some $j \in C_2; x_i = 0, i \in C_2 \setminus \{j\}; z_j = 1; z_i \in \{0, 1\}, i \in C_2 \setminus \{j\}$. From Lemma 3.4.3, the following inequalities are valid for $\Pi_{[0,r_1]}$.

$$\sum_{i\in S} \mu_j \leq 0, \ \ S \subseteq C_2,$$
$$r_1\lambda_j + \mu_j + \sum_{i\in S} \mu_j \leq 0, \ \ j \in C_2, S \subseteq C_2 \setminus \{j\}.$$

Lemma 3.4.6 implies that the non-dominated inequalities are of the following format.

$$\mu_j \leq 0, j \in C_2, \tag{3.42}$$
$$r_1\lambda_j + \mu_j \leq 0, j \in C_2. \tag{3.43}$$

Secondly, we consider interval $[r_1, c]$ and compute $\Pi_{[r_1,c]}$ similarly. Then

$$c\lambda_j + \mu_j \leq c - r_1, \ \forall j \in C_2, \tag{3.44}$$

is the only non-dominated inequality. Then it can be readily checked that for $\Pi_{[kc,kc+r_1]}$ and $\Pi_{[kc+r_1,(k+1)c]}$ where $1 \leq k \leq \lfloor \frac{d}{c} \rfloor - 2$, and $\Pi_{[(\lfloor \frac{d}{c} \rfloor -1)c+2r_1,d]}$ there does not exist any non-dominated inequality.

Lastly, we consider the interval $[(\lfloor \frac{d}{c} \rfloor - 1)c, (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1]$. In order to describe $\Pi_{[(\lfloor \frac{d}{c} \rfloor - 1)c, (\lfloor \frac{d}{c} \rfloor - 1)c+2r_1]}$, we consider two cases as follows.
Case 1. If $2r_1 < c$. Then one can check that the only non-dominated inequality is the following.

$$\sum_{j\in S}(c\lambda_j + \mu_j) + 2r_1\lambda_k + \mu_k \leq \Big(\Big\lfloor \frac{d}{c} \Big\rfloor - 1\Big)(c - r_1), S \subseteq C_2, |S| = \Big\lfloor \frac{d}{c} \Big\rfloor - 1, k \in C_2 \setminus S.$$

Case 2. If $2r_1 \geq c$. Then it can be checked easily that the following inequality is non-dominated.

$$\sum_{j\in S}(c\lambda_j + \mu_j) \leq \Big(\Big\lfloor \frac{d}{c} \Big\rfloor - 1\Big)(c - r_1), S \subseteq C_2, |S| = \Big\lfloor \frac{d}{c} \Big\rfloor. \tag{3.45}$$

Note that concerning interval $[d, 2d]$, we need to consider cases (i) $3r_1 < c$, (ii) $c \leq 3r_1 < 2c$, and (iii) $2c \leq 3r_1 < 3c$ to describe $\Pi_{[(2\lfloor \frac{d}{c} \rfloor - 1)c, (2\lfloor \frac{d}{c} \rfloor - 1)c+3r_1]}$ which can be continued similarly for intervals $[kd, (k+1)d], 2 \leq k \leq U - 1$. Following this pattern, we obtain a wide range of inequalities which cannot be aggregated into a same family.

In the following, we consider a particular case where all required inequalities to describe Π are provided. Then we compute the corresponding lifting coefficients and finally give the lifted inequalities which are facet-defining for $P_{integer}$.

We define the set $\mathcal{A}$ as follows.

$$\mathcal{A} = \Big\{k \in \mathbb{Z}_+ \mid k \geq 1 \wedge |C_2|c \geq \Big(k\Big\lfloor \frac{d}{c} \Big\rfloor - 1\Big)c + (k+1)r_1\Big\}.$$

Proposition 3.4.7. *Assume* $|C_2| > \lfloor \frac{d}{c} \rfloor \geq 2$. *If* $kc \leq (k+1)r_1$, *for* $k \in \mathcal{A}$, *then inequalities* (3.42)–(3.45) *suffice to describe* Π.

In the next proposition, we express the extreme points of Π defined by Proposition 4.3.2.

Proposition 3.4.8. *The following points are the extreme points of* Π *described by inequalities* (3.42)–(3.45).

(i) $\lambda_j = 0, \mu_j = 0, j \in C_2$;

(ii) $\lambda_j = 1, \mu_j = -r_1, j \in S \subseteq C_2, 1 \leq |S| \leq \lfloor \frac{d}{c} \rfloor - 1, \lambda_j = \mu_j = 0, j \in C_2 \setminus S$;

(iii) $\lambda_j = \frac{\lfloor \frac{d}{c} \rfloor - 1}{\lfloor \frac{d}{c} \rfloor}, \mu_j = -r_1\frac{\lfloor \frac{d}{c} \rfloor - 1}{\lfloor \frac{d}{c} \rfloor}, j \in S \subseteq C_2, \lceil \frac{d}{c} \rceil \leq |S| \leq |C_2|, \lambda_j = \mu_j = 0, j \in C_2 \setminus S$;

(iv) $\lambda_j = 1, \mu_j = -r_1, j \in S_1 \subset C_2, 1 \leq |S_1| \leq \lfloor \frac{d}{c} \rfloor - 1, \lambda_j = \frac{\lfloor \frac{d}{c} \rfloor - |S_1| - 1}{\lfloor \frac{d}{c} \rfloor - |S_1|}, \mu_j = -r_1\frac{\lfloor \frac{d}{c} \rfloor - |S_1| - 1}{\lfloor \frac{d}{c} \rfloor - |S_1|}$, $j \in S \subseteq C_2 \setminus S_1, \lceil \frac{d}{c} \rceil - |S_1| \leq |S| \leq |C_2| - |S_1|, \lambda_j = 0, \mu_j = 0, j \in C_2 \setminus (S \cup S_1)$.

In the following proposition we state the lifted inequalities obtained by applying the lifting coefficients of Proposition 4.2.3 in inequality (3.39).

Proposition 3.4.9. *Under the conditions of Proposition 4.3.2, the following inequalities define a facet of* $P_{integer}$.

$$(i) \sum_{j \in C_1 \cup S} (x_j - r_1 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y,$$

where $S \subseteq C_2$ *and* $0 \leq |S| \leq \lfloor \frac{d}{c} \rfloor - 1$.

$$(ii) \sum_{j \in C_1} (x_j - r_1 z_j) + \sum_{j \in S} \Big(\frac{\lfloor \frac{d}{c} \rfloor - 1}{\lfloor \frac{d}{c} \rfloor} \Big) (x_j - r_1 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y, \tag{3.46}$$

where $S \subseteq C_2$ *and* $\lceil \frac{d}{c} \rceil \leq |S| \leq |C_2|$.

$$(iii) \sum_{j \in C_1 \cup S_1} (x_j - r_1 z_j) + \sum_{j \in S} \Big(\frac{\lfloor \frac{d}{c} \rfloor - |S_1| - 1}{\lfloor \frac{d}{c} \rfloor - |S_1|} \Big) (x_j - r_1 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y,$$

where $S_1 \subset C_2, 1 \leq |S_1| \leq \lfloor \frac{d}{c} \rfloor - 1, S \subseteq C_2 \setminus S_1,$ *and* $\lceil \frac{d}{c} \rceil - |S_1| \leq |S| \leq |C_2| - |S_1|$.

Since describing Π completely is outside of the scope of this dissertation, we express some of the lifted inequalities corresponding to some specific cases in Table 3.1.

3.4.2 Superadditive Lifting

In this section, we underestimate the lifting function ϕ by a superadditive function. We remind the reader the following concepts.

Definition 3.4.10. *A function* $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *is superadditive on* A *if* $f(x_1) + f(x_2) \leq f(x_1 + x_2)$ *for all* $x_1, x_2, x_1 + x_2 \in A$.

Definition 3.4.11. *A function* ψ *is said to be a superadditive valid lifting function if* ψ *is superadditive and* $\psi(u) \leq \phi(u)$ *for all* $u \in [0, Ud]$.

As ϕ, in general, is not superadditive, we aim to construct superadditive valid lifting function. Applying a superadditive lifting function in the lifting procedure leads to simplifying the process and obtaining sequence-independent lifting coefficients.

The following proposition states that the lifting function ϕ is superadditive if $\lfloor \frac{d}{c} \rfloor = 1$.

Proposition 3.4.12. *Assume* $\lfloor \frac{d}{c} \rfloor = 1$. *Then the lifting function* ϕ *is superadditive on* $[0, Ud]$.

Table 3.1: Lifted inequalities under different conditions.

Conditions		Lifted Inequalities
$\lfloor \frac{d}{c} \rfloor = 1$	$2r_1 < c$	$\sum_{j\in C_1}(x_j - r_1 z_j) + \sum_{j\in S}(x_j - 2r_1 z_j) \leq (c - r_1)y, S \subseteq C_2, \lvert S\rvert \geq 0$
	$2r_1 \geq c$	$\sum_{j\in C_1}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y$
$\lfloor \frac{d}{c} \rfloor \geq 2$	$\lvert C_2\rvert \leq \lfloor \frac{d}{c} \rfloor - 1$	$\sum_{j\in C_1\cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \lvert S\rvert \geq 0$
$\lvert C_2\rvert = \lfloor \frac{d}{c} \rfloor \geq 2$	$2r_1 < c$	$\sum_{j\in C_1}(x_j - r_1 z_j) + \sum_{j\in C_2}(\frac{(\lfloor \frac{d}{c} \rfloor - 1)(c-r_1)}{(\lfloor \frac{d}{c} \rfloor - 1)(c-r_1)+r_1})(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y$ $\sum_{j\in C_1\cup S}(x_j - r_1 z_j) + (x_i - 2r_1 z_i) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \lvert S\rvert = \lfloor \frac{d}{c} \rfloor - 1, i \in C_2 \setminus S$ $\sum_{j\in C_1\cup S}(x_j - r_1 z_j) + \sum_{j\in C_2\setminus S}(\frac{c-r_1}{c})(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \lvert S\rvert = \lfloor \frac{d}{c} \rfloor - 2$ $\sum_{j\in C_1\cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, 0 \leq \lvert S\rvert \leq \lfloor \frac{d}{c} \rfloor - 1$
	$c \leq 2r_1 < 2c$	$\sum_{j\in C_1\cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, 0 \leq \lvert S\rvert \leq \lfloor \frac{d}{c} \rfloor - 1$

Proof. First, note that ϕ can be written as follows.

$$\phi(u) = \begin{cases} k(c - r_1), & kd \leq u < kd + 2r_1, \\ u - 2(k+1)r_1, & kd + 2r_1 \leq u \leq (k+1)d, \end{cases}$$

where $k \in \{0, \ldots, U-1\}$. Then let $u_1, u_2 \in [0, Ud]$. We consider the following cases:

Case 1: Let $k_1 d \leq u_1 \leq k_1 d + 2r_1$ and $k_2 d + 2r_1 \leq u_2 \leq (k_2 + 1)d$ where $k_1 \leq k_2$ and $k_1, k_2 \in \{0, \ldots, U-1\}$. So $u_1 = k_1 d + \delta_1$ such that $0 \leq \delta_1 \leq 2r_1$ and $u_2 = k_2 d + 2r_1 + \delta_2$ where $0 \leq \delta_2 \leq c - r_1$. It follows that $u_1 + u_2 = (k_1 + k_2)d + 2r_1 + \delta_1 + \delta_2$ which implies $(k_1 + k_2)d + 2r_1 \leq u_1 + u_2 \leq (k_1 + k_2 + 1)d$. Thus, $d = c + r_1$ and $\delta_1 \geq 0$ imply

$$\begin{aligned} \phi(u_1 + u_2) &= (k_1 + k_2)d + 2r_1 + \delta_1 + \delta_2 - 2(k_1 + k_2 + 1)r_1 = (k_1 + k_2)(c - r_1) + \delta_1 + \delta_2 \\ &\geq (k_1 + k_2)(c - r_1) + \delta_2 = \phi(u_1) + \phi(u_2). \end{aligned}$$

Case 2: Let $k_1 d \leq u_1 \leq k_1 d + 2r_1$ and $k_2 d \leq u_2 \leq k_2 d + 2r_1$ where $k_1 \leq k_2$ and $k_1, k_2 \in \{0, \ldots, U-1\}$. Then $u_1 = k_1 d + \delta_1$ such that $0 \leq \delta_1 \leq 2r_1$, $u_2 = k_2 d + \delta_2$ where $0 \leq \delta_2 \leq 2r_1$ and so $u_1 + u_2 = (k_1 + k_2)d + \delta_1 + \delta_2$. Since ϕ is non-decreasing and $u_1 + u_2 \geq (k_1 + k_2)d$, so

$$\phi(u_1 + u_2) \geq \phi((k_1 + k_2)d) = (k_1 + k_2)(c - r_1) = \phi(u_1) + \phi(u_2).$$

Case 3: Assume $k_1 d + 2r_1 \leq u_1 \leq (k_1 + 1)d$ and $k_2 d + 2r_1 \leq u_2 \leq (k_2 + 1)d$ where $k_1 \leq k_2$ and $k_1, k_2 \in \{0, \ldots, U-1\}$. So $u_1 = k_1 d + 2r_1 + \delta_1$ such that $0 \leq \delta_1 \leq c - r_1$ and $u_2 = k_2 d + 2r_1 + \delta_2$ where $0 \leq \delta_2 \leq c - r_1$. In addition, $\phi(u_1) = k_1(c - r_1) + \delta_1$ and $\phi(u_2) = k_2(c - r_1) + \delta_2$. Now let $\delta = \delta_1 + \delta_2$ and so $0 \leq \delta \leq 2(c - r_1)$. We consider two following subcases.

Subcase *i*: $0 \leq \delta \leq c - r_1$. Then

$$u_1 + u_2 = (k_1 + k_2)d + 4r_1 + \delta \geq (k_1 + k_2)d + 2r_1 + \delta.$$

Since ϕ is non-decreasing, we have

$$\begin{aligned} \phi(u_1 + u_2) &\geq \phi((k_1 + k_2)d + 2r_1 + \delta) = (k_1 + k_2)d + 2r_1 + \delta - 2(k_1 + k_2 + 1)r_1 \\ &= (k_1 + k_2)(c - r_1) + \delta = \phi(u_1) + \phi(u_2). \end{aligned}$$

Subcase *ii*: $c - r_1 < \delta \leq 2(c - r_1)$ which implies $\delta = (c - r_1) + \delta'$ where $0 < \delta' \leq c - r_1$. So

$$\begin{aligned} u_1 + u_2 &= (k_1 + k_2)d + 4r_1 + \delta = (k_1 + k_2)d + 4r_1 + (c - r_1) + \delta' \\ &= (k_1 + k_2 + 1)d + 2r_1 + \delta'. \end{aligned}$$

Thus

$$\begin{aligned} \phi(u_1 + u_2) &= \phi((k_1 + k_2 + 1)d + 2r_1 + \delta') = (k_1 + k_2 + 1)d + 2r_1 + \delta' - 2(k_1 + k_2 + 2)r_1 \\ &= (k_1 + k_2 + 1)(c - r_1) + \delta' = (k_1 + k_2)(c - r_1) + \delta = \phi(u_1) + \phi(u_2). \end{aligned}$$

□

Note that the lifted inequalities where $\lfloor \frac{d}{c} \rfloor = 1$ are presented in Table 3.1. Let $\lfloor \frac{d}{c} \rfloor \geq 2$ and consider the following function f where $u \in [kd, (k+1)d], k \in \{0, \ldots, U-1\}$.

$$f(u) = \begin{cases} k\lfloor \frac{d}{c} \rfloor (c - r_1), & kd \leq u < kd + r_1, \\ \frac{(c-r_1)(u-(k+1)r_1)}{c}, & kd + r_1 \leq u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1; \\ \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)(c - r_1), & \\ \quad \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1 \leq u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 & \\ u - (k+1)r_1\lceil \frac{d}{c} \rceil, & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 \leq u \leq (k+1)d. \end{cases}$$

Proposition 3.4.13. *The function f is a superadditive valid lifting function.*

Proof. Clearly $f(u) \leq \phi(u)$, for $u \in [0, Ud]$. Next, we show that function f is superadditive. We start by proving that f has the following property. If $x = kd + v, 0 \leq v < d$ such that $k \in \mathbb{Z}_+$ and $v \geq 0$, then $f(x) = k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v)$. It is clear that this equality holds true for $k = 0$. Assume $k \geq 1$. Then we have the following cases.

Case 1: If $kd \leq kd + v \leq kd + r_1$. It implies $0 \leq v \leq r_1$ and so $f(v) = 0$. Thus, $f(kd + v) = k\lfloor \frac{d}{c} \rfloor (c - r_1) = k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v)$.

Case 2: If $kd + r_1 < kd + v \leq \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1$. Then we get $r_1 < v \leq (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$ and so $f(v) = \frac{(c-r_1)(v-r_1)}{c}$. Therefore

$$\begin{aligned} f(kd + v) &= \frac{(c - r_1)(kd + v - (k+1)r_1)}{c} = \frac{(c - r_1)(k\lfloor \frac{d}{c} \rfloor c + v - r_1)}{c} = k\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) \\ &+ \frac{(c - r_1)(v - r_1)}{c} = k\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + f(v). \end{aligned}$$

Case 3: If $\big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1 < kd + v \leq \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1$. Then $(\lfloor \frac{d}{c} \rfloor - 1)c + r_1 < v \leq (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$ and so $f(v) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. Thus

$$\begin{aligned} f(kd + v) &= \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)(c - r_1) = k\lfloor \frac{d}{c} \rfloor (c - r_1) + (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1) \\ &= k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v). \end{aligned}$$

Case 4: If $\big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 < kd + v \leq (k+1)d$. So $(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1 < v \leq d$ and then $f(v) = v - r_1\lceil \frac{d}{c} \rceil$. We get

$$\begin{aligned} f(kd + v) &= kd + v - (k+1)r_1\lceil \frac{d}{c} \rceil = k\lfloor \frac{d}{c} \rfloor (c - r_1) + v - r_1\lceil \frac{d}{c} \rceil \\ &= k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v), \end{aligned}$$

and it completes the proof of the first step.

Now we assume that $x_1 = k_1 d + v_1, x_2 = k_2 d + v_2$ such that $0 \le v_1, v_2 < d$. Then using the foregoing property implies

$$f(x_1) + f(x_2) = k_1\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1) + f(v_1) + k_2\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1) + f(v_2),$$

and

$$f(x_1 + x_2) = (k_1 + k_2)\Big\lfloor\frac{d}{c}\Big\rfloor(c - r_1) + f(v_1 + v_2).$$

Therefore, we have $f(x_1) + f(x_2) \le f(x_1 + x_2)$ if and only if $f(v_1) + f(v_2) \le f(v_1 + v_2)$ where $0 \le v_1, v_2 < d$. So in order to prove superadditivity of f in $[0, Ud]$, it suffices to prove f is superadditive on $[0, d]$.

Now we prove superadditivity on $[0, d]$. So consider the following cases.

Case i : If $0 \le x_1 \le r_1$ and $0 \le x_2 \le d$. Then $f(x_1) = 0$. Since $x_1 + x_2 \ge x_2$ and f is non-decreasing so $f(x_1 + x_2) \ge f(x_2) = f(x_1) + f(x_2)$.

Case ii : If $r_1 \le x_1 \le (\lfloor\frac{d}{c}\rfloor - 1)c + r_1$ and $r_1 \le x_2 \le (\lfloor\frac{d}{c}\rfloor - 1)c + r_1$. So $f(x_1) = \frac{(c-r_1)(x_1-r_1)}{c}$ and $f(x_2) = \frac{(c-r_1)(x_2-r_1)}{c}$. We have the following subcases. If $x_1 + x_2 \le (\lfloor\frac{d}{c}\rfloor - 1)c + r_1$ then $f(x_1 + x_2) = \frac{(c-r_1)(x_1+x_2-r_1)}{c}$. Thus

$$\begin{aligned} f(x_1 + x_2) = \frac{(c - r_1)(x_1 + x_2 - r_1)}{c} &= \frac{(c - r_1)(x_1 - r_1) + (c - r_1)x_2}{c} \ge \frac{(c - r_1)(x_1 - r_1)}{c} \\ &+ \frac{(c - r_1)(x_2 - r_1)}{c} = f(x_1) + f(x_2). \end{aligned}$$

If $(\lfloor\frac{d}{c}\rfloor - 1)c + r_1 < x_1 + x_2 \le (\lfloor\frac{d}{c}\rfloor - 1)c + 2r_1$, then $f(x_1 + x_2) = (\lfloor\frac{d}{c}\rfloor - 1)(c - r_1)$. Moreover, $x_1 + x_2 \le (\lfloor\frac{d}{c}\rfloor - 1)c + 2r_1$ implies $\lfloor\frac{d}{c}\rfloor - 1 \ge \frac{x_1+x_2-2r_1}{c}$. Thus

$$f(x_1 + x_2) = \Big(\Big\lfloor\frac{d}{c}\Big\rfloor - 1\Big)(c - r_1) \ge \frac{x_1 + x_2 - 2r_1}{c}(c - r_1) = f(x_1) + f(x_2).$$

If $(\lfloor\frac{d}{c}\rfloor - 1)c + 2r_1 < x_1 + x_2 \le d$, so $f(x_1 + x_2) = x_1 + x_2 - r_1\lceil\frac{d}{c}\rceil$. Then multiplying inequality

$$x_1 + x_2 > \Big(\Big\lfloor\frac{d}{c}\Big\rfloor - 1\Big)c + 2r_1 = c\Big\lceil\frac{d}{c}\Big\rceil - 2(c - r_1),$$

by $\frac{r_1}{c} = 1 - \frac{c-r_1}{c}$ implies

$$(x_1 + x_2)\Big(1 - \frac{c - r_1}{c}\Big) \ge r_1\Big\lceil\frac{d}{c}\Big\rceil - 2r_1\frac{c - r_1}{c},$$

which gives $f(x_1 + x_2) \ge f(x_1) + f(x_2)$.

Case iii : If $r_1 \le x_1 \le (\lfloor\frac{d}{c}\rfloor - 1)c + r_1$ and $(\lfloor\frac{d}{c}\rfloor - 1)c + r_1 \le x_2 \le (\lfloor\frac{d}{c}\rfloor - 1)c + 2r_1$. So $f(x_1) = \frac{(c-r_1)(x_1-r_1)}{c}$ and $f(x_2) = (\lfloor\frac{d}{c}\rfloor - 1)(c - r_1)$. This selection for x_1 and x_2 implies that $x_1 + x_2 \geq (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$ and so $f(x_1 + x_2) = x_1 + x_2 - r_1 \lceil \frac{d}{c} \rceil$. Therefore,

$$f(x_1 + x_2) = x_1 + x_2 - r_1 \left\lceil \frac{d}{c} \right\rceil \geq x_1 + \left(\left\lfloor \frac{d}{c} \right\rfloor - 1 \right) c + r_1 - r_1 \left\lceil \frac{d}{c} \right\rceil = (x_1 - r_1) + \left(\left\lfloor \frac{d}{c} \right\rfloor - 1 \right)(c - r_1) \geq (x_1 - r_1)\frac{c - r_1}{c} + \left(\left\lfloor \frac{d}{c} \right\rfloor - 1 \right)(c - r_1) = f(x_1) + f(x_2).$$

Case iv : If $r_1 \leq x_1 \leq (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$ and $(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1 \leq x_2 \leq d$. Then $f(x_1) = \frac{(c-r_1)(x_1-r_1)}{c}$ and $f(x_2) = x_2 - r_1 \lceil \frac{d}{c} \rceil$. Those intervals imply $x_1 + x_2 > (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$ and so $f(x_1 + x_2) = x_1 + x_2 - r_1 \lceil \frac{d}{c} \rceil$. Thus

$$f(x_1 + x_2) = x_1 + x_2 - r_1 \left\lceil \frac{d}{c} \right\rceil \geq (x_1 - r_1) + x_2 - r_1 \left\lceil \frac{d}{c} \right\rceil \geq \frac{(c - r_1)(x_1 - r_1)}{c} + x_2 - r_1 \left\lceil \frac{d}{c} \right\rceil = f(x_1) + f(x_2).$$

Note that these are the only cases where the sum of two variables belongs to $[0, d]$ and it completes the proof. □

Now replacing the lifting function ϕ (see Section 3.4.1) by the superadditive function f in the description of Π, one can show that the following inequalities suffice to describe Π.

$$\begin{cases} \mu_j \leq 0, j \in C_2, \\ r_1\lambda_j + \mu_j \leq 0, j \in C_2, \\ c\lambda_j + \mu_j \leq \frac{(c-r_1)^2}{c}, j \in C_2. \end{cases}$$

In addition, points $\lambda_j = \frac{c-r_1}{c}, \mu_j = -r_1\frac{c-r_1}{c}, j \in S \subseteq C_2, 0 \leq |S| \leq |C_2|$, $\lambda_i = 0, \mu_i = 0, i \in C_2 \setminus S$ are the extreme points of Π which shows that the following inequality is valid for $X_{integer}$.

$$c \sum_{j \in C_1} (x_j - r_1 z_j) + (c - r_1) \sum_{j \in S} (x_j - r_1 z_j) \leq c \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y, \tag{3.47}$$

where $S \subseteq C_2$ and $0 \leq |S| \leq |C_2|$. Notice that this inequality is the unique inequality obtained by lifting of (3.39).

3.5 Separation

In this section we study the separation problems associated with the families of valid inequalities we derived for $X_{integer}$ in the constant case. Consider a point $(x, z, y) \in \mathbb{R}_+^{2n+1}$. For each family $\mathcal{V}$ of valid inequalities the separation problem is to find an inequality in $\mathcal{V}$ that is violated by point (x, z, y) or show that there is no such inequality.

At first, we study the separation problem associated with inequality (3.16). In fact, we intend to find subset $S \subseteq N$ such that $\sum_{j \in S} x_j > r(S)y + (\mu(S) - 1)(d - r(S))$, or prove that such S does not exist.

Assume that $\mu(S)-1$ is fixed, namely, $\mu(S)-1=p$ where p is constant. Define binary variables $\alpha_j, j \in N$ where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise. Under these assumptions, $r(S)$ can be represented as $c\sum_{j\in N}\alpha_j - pd$ where $\lfloor \frac{pd}{c} \rfloor + 1 \leq \sum_{j\in N}\alpha_j \leq \lfloor \frac{(p+1)d}{c} \rfloor$. In order to separate inequality (3.16) we must find variables $\alpha_j, j \in N$ such that

$$\sum_{j\in N}\alpha_j x_j > \Big(c\sum_{j\in N}\alpha_j - pd\Big)y + p\Big(d - c\sum_{j\in N}\alpha_j + pd\Big).$$

Therefore, the separation problem of (3.16) amounts to solve the following binary integer program

$$\begin{aligned} \max \quad & \sum_{j\in N}(x_j + pc - cy)\alpha_j \\ s.t. \quad & \Big\lfloor \frac{pd}{c} \Big\rfloor + 1 \leq \sum_{j\in N}\alpha_j \leq \Big\lfloor \frac{(p+1)d}{c} \Big\rfloor, \\ & \alpha_j \in \{0,1\}, j \in N. \end{aligned} \tag{3.48}$$

Then for a fixed p, inequality (3.16) is violated if the optimal value of the foregoing maximization problem is strictly greater than $pd(p-y+1)$. In order to solve program (3.48), without loss of generality, assume that $x_1 \geq \cdots \geq x_n$. Then it follows from the structure of the optimal solution of problem (3.48) that subset $S \subseteq N$ can be generated as follows. Set $S_1 = \{1, \ldots, \lfloor \frac{pd}{c} \rfloor + 1\}$. Two cases can be considered: (i) $x_{\lfloor \frac{pd}{c} \rfloor + 2} + pc - cy \leq 0$, and (ii) $x_{\lfloor \frac{pd}{c} \rfloor + 2} + pc - cy > 0$. Let case (i) occurs. Then we set $S = S_1$. Next, assume case (ii) happens. Then

$$S = S_1 \cup \Big\{ j \in \Big\{ \Big\lfloor \frac{pd}{c} \Big\rfloor + 2, \ldots, \Big\lfloor \frac{(p+1)d}{c} \Big\rfloor \Big\} : x_j + pc - cy > 0 \Big\}.$$

Thus, corresponding to the generated set S, if $\sum_{j\in S}(x_j + pc - cy) > pd(p-y+1)$, then a violated inequality (3.16) is found. Otherwise, no such a violated inequality exists.

Note that since $0 \leq p \leq \lfloor \frac{nc}{d} \rfloor$ and the separation problem corresponding to each p can be solved in polynomial time, therefore the separation problem associated to inequality (3.16) can be solved in polynomial time.

Next, we discuss on the separation problem of inequality (3.19). Similar to the latter separation problem, we set $\mu(S)-1=p$ where p is constant and define binary variables $\alpha_j, j \in N$ where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$, otherwise. Then $r(S) = c\sum_{j\in N}\alpha_j - pd$ and $\bar{r}(S) = pd - \lfloor \frac{pd}{c} \rfloor c$ where $\lfloor \frac{pd}{c} \rfloor + 1 \leq \sum_{j\in N}\alpha_j \leq \lfloor \frac{(p+1)d}{c} \rfloor$. So for a fixed p, the separation problem associated to inequality (3.19) is equivalent to find variables $\alpha_j, j \in N$ such that

$$\begin{aligned} \sum_{j\in N}\alpha_j x_j - \Big(pd - \Big\lfloor \frac{pd}{c} \Big\rfloor c\Big)\sum_{j\in N}\alpha_j z_j + \Big(pd - \Big\lfloor \frac{pd}{c} \Big\rfloor c\Big)\sum_{j\in N}\alpha_j \\ + pc\sum_{j\in N}\alpha_j - cy\sum_{j\in N}\alpha_j > pd(p-y+1). \end{aligned}$$

which implies that the following binary integer program should be solved.

$$\begin{aligned} \max \quad & \sum_{j\in N}\Big(x_j - \big(pd - \Big\lfloor\frac{pd}{c}\Big\rfloor c\big)z_j + \big(pd - \Big\lfloor\frac{pd}{c}\Big\rfloor c\big) + pc - cy\Big)\alpha_j \\ s.t. \quad & \Big\lfloor\frac{pd}{c}\Big\rfloor + 1 \le \sum_{j\in N}\alpha_j \le \Big\lfloor\frac{(p+1)d}{c}\Big\rfloor, \\ & \alpha_j \in \{0,1\}, j \in N. \end{aligned} \tag{3.49}$$

In order to solve this maximization problem, assume $x_1-(pd-\lfloor\frac{pd}{c}\rfloor c)z_1 \ge \cdots \ge x_n-(pd-\lfloor\frac{pd}{c}\rfloor c)z_n$. Then subset S can be generated similar to what we applied in the separation problem of inequality (3.16). Thus, regarding the set S, for a fixed p, if $\sum_{j\in S}(x_j-(pd-\lfloor\frac{pd}{c}\rfloor c)z_j+(pd-\lfloor\frac{pd}{c}\rfloor c)+pc-cy) > pd(p-y+1)$, then a violated inequality (3.19) is found. Otherwise, there is no such an inequality. This separation problem can be solved in polynomial time as well.

Next we explain the separation problem corresponding to inequality (3.25) which is the generalization of inequality (3.23). We consider two cases.

Case 1. Assume $k \in \{1,\ldots,l_a\}$ where $l_a = \min\{\lfloor\frac{d}{c}\rfloor - 1, n - \lfloor\frac{ad}{c}\rfloor\}$. Then inequality (3.25) can be written as

$$\sum_{j\in S}(x_j - r_a z_j) - k(c-r_a)(y-a) \le a\Big\lfloor\frac{d}{c}\Big\rfloor(c-r_a),$$

where $|S| = a\lfloor\frac{d}{c}\rfloor + k$. Then the separation problem amounts to solve

$$\max_{S\subseteq N, |S|=a\lfloor\frac{d}{c}\rfloor+k, 1\le k\le l_a} \sum_{j\in S}(x_j - r_a z_j) - k(c-r_a)(y-a), \tag{3.50}$$

and so violation occurs if the optimal value of this maximization problem is strictly greater than $a\lfloor\frac{d}{c}\rfloor(c-r_a)$. Otherwise, there is no such a violated inequality. Notice that maximization problem (3.50) is equivalent to the following integer program.

$$\begin{aligned} \max \quad & \sum_{j\in N}(x_j - r_a z_j)\alpha_j - k(c-r_a)(y-a) \\ s.t. \quad & \sum_{j\in N}\alpha_j - k = a\Big\lfloor\frac{d}{c}\Big\rfloor, \\ & 1 \le k \le l_a, \\ & \alpha_j \in \{0,1\}, j\in N, k\in\mathbb{Z}_+, \end{aligned} \tag{3.51}$$

where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise.

It can be seen readily that the coefficient matrix corresponding to program (3.51) is totally unimodular and so the separation problem can be solved by solving the linear relaxation of program (3.51) which provides an optimal integer solution (see [36]).

Case 2. Let $k = \lfloor \frac{d}{c} \rfloor$. Then inequality (3.25) can be represented as

$$\sum_{j \in S}(x_j - r_a z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y,$$

where $|S| \leq (a+1)\lfloor \frac{d}{c} \rfloor$. Then a violated inequality is found if $\max_{S \subseteq N, |S| \leq (a+1)\lfloor \frac{d}{c} \rfloor}(x_j - r_a z_j)$ is strictly greater than $\lfloor \frac{d}{c} \rfloor (c - r_a)y$. The latter maximization problem corresponds to the following binary integer program.

$$\begin{aligned} \max \quad & \sum_{j \in N}(x_j - r_a z_j)\alpha_j \\ s.t. \quad & \sum_{j \in N} \alpha_j \leq (a+1)\left\lfloor \frac{d}{c} \right\rfloor, \\ & \alpha_j \in \{0,1\}, j \in N, \end{aligned}$$

where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise. In order to solve the above-mentioned binary integer program, without loss of generality, assume $x_1 - r_a z_1 \geq \cdots \geq x_n - r_a z_n$. Then we set $S = \big\{ j \in \{1, \ldots, (a+1)\lfloor \frac{d}{c} \rfloor\} : x_j - r_a z_j > 0 \big\}$. Thus, the separation problem associated with inequality (3.25) can be solved in polynomial time.

Next, we clarify the separation problem of inequality (3.27). Similar to the separation problem of inequality (3.25) (Case 1), the separation problem of inequality (3.27) is equivalent to solve the following binary integer program.

$$\begin{aligned} \max \quad & \sum_{j \in N}(x_j - r_1 z_j)\alpha_j + \sum_{j \in N}(x_j - r_2 z_j)\beta_j \\ s.t. \quad & \sum_{j \in N} \alpha_j = 2\left\lfloor \frac{d}{c} \right\rfloor, \\ & \alpha_j + \beta_j \leq 1, j \in N, \\ & \alpha_j \in \{0,1\}, \beta_j \in \{0,1\}, j \in N, \end{aligned} \tag{3.52}$$

where $\alpha_j = 1$ if and only if $j \in S_1$, and $\beta_j = 1$ if and only if $j \in S_2$. Thus, the violated inequality is obtained if the optimal value of the objective function of program (3.52) is strictly greater than $\lfloor \frac{d}{c} \rfloor (c - r_1)y$, and otherwise there is no violated inequality. Note that it can be seen that the coefficient matrix of program (3.52) is totally unimodular and hence it suffices to solve the linear relaxation of this program to obtain the optimal integer solution.

Lastly, we discuss on the separation problem associated to inequality (3.28). The separation problem can be stated similar to the separation of (3.27). So in order to separate inequality (3.28) it suffices to solve the following binary integer program.

$$\begin{aligned}
\max \quad & \sum_{j\in N}(x_j - r_1 z_j)\alpha_j + \sum_{j\in N}(x_j - r_2 z_j)\beta_j \\
s.t. \quad & \sum_{j\in N}\beta_j = 1, \\
& \alpha_j + \beta_j \leq 1, j \in N, \\
& \alpha_j \in \{0,1\}, \beta_j \in \{0,1\}, j \in N,
\end{aligned}$$

Thus, if the optimal value of this maximization problem is strictly greater that $\lceil \frac{d}{c} \rceil (c - r_1)y$, then the violated inequality (3.28) is obtained. Otherwise, such an inequality does not exist. Similarly, the coefficient matrix of the foregoing program is totally unimodular.

3.6 Computational Results

In this section we report some computational experiments to test the effectiveness of the inclusion of the inequalities introduced in Section 3.3 in solving randomly generated instances of the lot-sizing with supplier selection problem. In this experiment we compare these inequalities with default Xpress-Optimizer cuts. We consider instances of the following LSSP

$$\begin{aligned}
\min \quad & \sum_{t\in T} h_t s_t + \sum_{t\in T}\sum_{j\in N}(p_t + c_{jt})w_{jt} + \sum_{t\in T} f_t y_t + \sum_{t\in T}\sum_{j\in N} g_{jt} z_{jt} \\
s.t. \quad & s_{t-1} + x_t = d_t + s_t, \quad t \in T, \\
& x_t \leq d y_t, \quad t \in T, \\
& x_t = \sum_{j\in N} w_{jt}, \quad t \in T, \\
& w_{jt} \leq c z_{jt}, \quad j \in N, t \in T, \\
& s_0 = s_{|T|} = 0, \\
& x_t, s_t \geq 0, \quad t \in T, \\
& w_{jt} \geq 0, \quad j \in N, t \in T, \\
& y_t \in \{0, 1, \ldots, U\}, \quad t \in T, \\
& z_{jt} \in \{0,1\}, \quad j \in N, t \in T,
\end{aligned}$$

where T is the set of production periods, and N is the set of suppliers. $d_t > 0$ is the demand in period $t \in T$, h_t is the unit holding cost, f_t and p_t represent the production set-up cost and variable production cost in period t, respectively, and c_{jt} and g_{jt} are variable and fixed sourcing set-up costs for supplier j in period t. d and c are production and supplying capacities. In addition, several types of decision variables are defined. Let x_t be the quantity produced in period t; s_t be the stock level at the end of period $t \in T$; w_{jt} be the quantity sourced from supplier $j \in N$ in period $t \in T$; y_t is an integer variable indicating the number of batches produced in period t, and z_{jt} takes value 1 if and only if supplier j is selected in period t.

All computations are performed using the optimization software Xpress-Optimizer Version 23.01.03 with Xpress Mosel Version 3.4.0 [46], on a computer with processor Intel Core 2, 2.2 GHz and with 2 GB RAM.

We consider instances with $|T| = 20$ and $|N| = 10$. The test instances were generated randomly on the basis of the following data: $d \in \{40, 60, 80, 100\}$; $c \in \{9, 14, 19, 24\}$; d_t is randomly generated as an integer number in the intervals $[10, 20]$, $[10, 40]$, and $[10, 100]$; h_t is randomly generated in the interval $[0, 0.1)$; $p_t + c_{jt}$ is randomly selected in $\{0.5, 1.5\}$; f_t takes value in $\{100, 300\}$; g_{jt} is randomly generated as an integer number in the intervals $[100, 105]$ and $[300, 305]$.

The computational results are shown in Tables 3.2–3.6 where we provide average results for the LSSP on 12 instances generated for each pair (d, c).

Let $\mathcal{C}$ denote the set of inequalities containing (3.16), (3.19), (3.23), (3.25) with $k = \lfloor \frac{d}{c} \rfloor$, (3.27), (3.28), and (3.47) which are added to the LP relaxation as cutting planes. After solving the LP relaxation of an instance, the most violated inequality of each class is added to the formulation and finally the LP relaxation is solved again. The process is repeated until no new cuts are found. In Table 3.2, we present the integrality gap closed by Xpress cuts (GCX), integrality gap closed by cuts $\mathcal{C}$ (GC$\mathcal{C}$), and integrality gap closed by cuts $\mathcal{C}$ in addition to Xpress cuts (GC$\mathcal{C}$X). Closed gaps are calculated as $\frac{ILR-LR}{OPT-LR} \times 100$ where LR indicates the linear relaxation value, OPT denotes the optimal value of the problem, and ILR denotes the LP relaxation with default Xpress cuts for GCX, with inequalities belong to $\mathcal{C}$ for GC$\mathcal{C}$, and with inequalities belong to $\mathcal{C}$ in addition to Xpress cuts for GC$\mathcal{C}$X. It can be observed in Table 3.2 that for all instances the new cuts $\mathcal{C}$ in addition to Xpress cuts are more efficient in closing the integrality gap than Xpress cuts.

As a next step, we run the branch-and-bound algorithm during the time limit of 30 minutes with the default Xpress-Optimizer options. The results are reported in Table 3.3 where the second column (IG) is the initial integrality gap computed by running the branch-and-bound algorithm for 30 minutes and the third column (G$\mathcal{C}$) gives the integrality gap calculated by adding cuts $\mathcal{C}$ at the root node to the formulation, and then running the branch-and-bound algorithm. It can be concluded from Table 3.3 that adding our cuts to the formulation a priori is effective in improving the integrality gap.

Let SMALL, MEDIUM, and LARGE denote the sets of all instances whose $\lfloor \frac{d}{c} \rfloor$ belongs to $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 11\}$ respectively. Then the average closed gaps are classified in term of the value $\lfloor \frac{d}{c} \rfloor$ in Table 3.4. It can be concluded from Table 3.4 that as $\lfloor \frac{d}{c} \rfloor$ rises, the average closed gaps obtained by Xpress cuts and cuts $\mathcal{C}$ increase. Note that this property roughly holds for the average closed gaps obtained by cuts $\mathcal{C}$ in addition to Xpress cuts. In addition, the average integrality gaps classified in term of the value $\lfloor \frac{d}{c} \rfloor$ are shown in Table 3.5. This table shows that the best improvement of integrality gap is seen for those instances belonging to the set MEDIUM.

Finally we present the impact of simultaneous lifted inequalities (3.46) in Table 3.6. In this case, only the pair $(d, c) = (40, 14)$ from the above-mentioned instances satisfies the condition of proposition 4.3.2. So we add a new pair $(d, c) = (60, 16)$ which satisfies

Table 3.2: Average closed gaps on 192 randomly generated instances.

(d,c)	**GCX**	**G$\mathcal{C}$C**	**G$\mathcal{C}$CX**
(40,9)	33.3	47.20	54.44
(40,14)	22.78	29.99	40.29
(40,19)	50.66	24.63	63.68
(40,24)	22.12	5.39	23.12
(60,9)	28.1	46.27	57.11
(60,14)	42.87	45.76	55.09
(60,19)	46.88	32.00	66.59
(60,24)	33.51	7.71	35.45
(80,9)	48.47	55.37	65.83
(80,14)	30.67	36.66	53.64
(80,19)	61.99	44.95	68.52
(80,24)	37.92	17.49	48.09
(100,9)	52.39	43.66	53.95
(100,14)	48.58	27.01	51.63
(100,19)	57.6	40.05	71.40
(100,24)	56.37	28.11	59.25
Average	**42.14**	**33.27**	**54.26**

Table 3.3: Comparison of average integrality gaps.

(d,c)	**IG**	**G$\mathcal{C}$**
(40,9)	1.69	1.13
(40,14)	3.16	2.30
(40,19)	1.30	1.02
(40,24)	2.82	2.94
(60,9)	2.10	1.25
(60,14)	1.64	1.01
(60,19)	0.74	0.17
(60,24)	1.57	1.63
(80,9)	0.71	0.48
(80,14)	2.37	1.62
(80,19)	0.27	0.20
(80,24)	1.41	1.07
(100,9)	0.95	0.86
(100,14)	1.14	1.16
(100,19)	0.62	0.40
(100,24)	0.62	0.61
Average	1.44	1.11

Table 3.4: Classified average closed gaps in term of the value $\lfloor \frac{d}{c} \rfloor$.

(d,c)	GCX	GC$\mathcal{C}$	GC$\mathcal{C}$X
SMALL	35.65	19.54	46.2
MEDIUM	44.41	41.29	59.92
LARGE	49.81	42.01	57.14

Table 3.5: Classified average integrality gaps in term of the value $\lfloor \frac{d}{c} \rfloor$.

(d,c)	IG	G$\mathcal{C}$
SMALL	1.83	1.52
MEDIUM	1.33	0.89
LARGE	0.93	0.83

those conditions to run the tests over more instances. Thus, 24 instances are generated as explained before. We report the integrality gap closed by the cuts $\mathcal{C}$, denoted by (GC$\mathcal{C}$), and the integrality gap closed by cuts $\mathcal{C}$ in addition to the inequalities (3.46), denoted by (GC$\mathcal{C}^+$), in Table 3.6. It can be concluded that simultaneous lifted inequalities (3.46) have only a slight impact on improving the gap.

Table 3.6: Impact of Simultaneous Lifted Inequalities (3.46).

(d,c)	GC$\mathcal{C}$	GC$\mathcal{C}^+$
(40,14)	29.99	30.37
(60,16)	28.99	29.82
Average	**29.49**	**30.10**

3.7 Summary

The following is the summary of this chapter. We considered a set $X_{integer}$ that generalizes the single node fixed-charge network set and the single arc design set. For this set we obtained new inequalities that generalize the well-known flow cover inequalities and the arc residual capacity inequalities. For the constant capacitated case we derived an exact compact extended formulation, and some families of facet-defining inequalities in the original space of variables which give a partial description of the convex hull of $X_{integer}$. A preliminary computational study showed that these inequalities are effective in reducing the integrality gap of instances of the single-item lot-sizing with supplier selection problem. Furthermore, by lifting some basic inequalities we provide some insight on the difficulty of obtaining such a full polyhedral description on the general case. Preliminary computational results are presented.

Chapter 4

Valid Inequalities for a MIP Set with Conflict Between Variables

4.1 Introduction

It is well-known that the use of strong valid inequalities as cuts can be very effective in solving mixed integer problems. One classical approach to generate these valid inequalities is to study the polyhedral structure of simple sets which occur as relaxations of the feasible sets of those general problems. Two such successful examples are the use of MIR inequalities, derived from a basic mixed integer set [31, 40], and the use of valid inequalities for conflict graphs, resulting from logical relations between binary variables, for solving mixed integer programs [9].

In this chapter we investigate the polyhedral structure of the third mixed integer set that results from the intersection of the two well-known sets: a simple mixed integer set and the vertex packing set associated with a conflict graph.

Let X be the set of points $(s, x) \in \mathbb{R} \times \mathbb{Z}^n$ satisfying

$$s + c \sum_{i \in N_1} x_i \geq d, \tag{4.1}$$

$$x_i + x_j \leq 1, \ (i, j) \in E, \tag{4.2}$$

$$x_i \in \{0, 1\}, \ i \in N, \tag{4.3}$$

$$s \geq 0, \tag{4.4}$$

where $N = \{1, \ldots, n\}$ is the index set of binary variables, $E \subset N \times N$ is a set of index pairs, $N_1 \subseteq N$, and $c > 0$, $d > 0$. The graph $G = (N, E)$ in known as the conflict graph of pairwise conflicts between binary variables (see [2, 9]). We denote $N_0 = N \setminus N_1$.

Set X is the intersection of two sets: $X = X_{VP} \bigcap X_{SMI}$, where X_{VP} is the vertex packing set defined by (4.2)–(4.3), that results by considering the conflict graph $G = (N, E)$, and X_{SMI} is a simple mixed integer set defined by $\{(s, x) \in \mathbb{R} \times \mathbb{B}^{|N_1|}|$ satisfying (4.1) and (4.4)$\}$. The convex hull of X, X_{VP}, X_{SMI}, will be denoted by P, P_{VP}, P_{SMI}, respectively.

The set X_{SMI} has been intensively used as a relaxation of several mixed integer sets, see [40] for examples. It is well-known that in order to describe P_{SMI}, when $|N_1| \geq \lceil \frac{d}{c} \rceil$, suffices to add to the defining inequalities (4.1), (4.4), $x_i \geq 0$, $x_i \leq 1, i \in N_1$, the following MIR inequality

$$s + r \sum_{i \in N_1} x_i \geq r \left\lceil \frac{d}{c} \right\rceil, \tag{4.5}$$

where $r = d - c(\lceil \frac{d}{c} \rceil - 1)$.

On the contrary, a complete description of the convex hull of X_{VP} is not known and since optimizing a linear function over X_{VP} is a NP-hard problem, there is no much hope in finding such a description. Nevertheless, families of valid inequalities are known, see [18, 19, 35, 37].

The derivation of inequalities for integer programs based on conflict graphs have also been considered in the past, see [9].

Although the two sets X_{SMI} and X_{VP} have been intensively considered in the past, to the best of our knowledge, set X has never been studied before. The most related mixed integer set considered before is the mixed vertex packing set studied by Atamturk et al. [8].

Cuts from valid inequalities for X_{SMI} and X_{VP} are commonly used by researchers using MIP solvers, by identifying these sets as relaxations of the original feasible set. With the current research we aim at deriving new inequalities that can be used when those structures are present simultaneously. Such structures can be found in many mixed integer problems, such as, inventory routing, production planning, facility locations, network design, etc. In particular, by investigating the polyhedral structure of P, we generate valid inequalities that extend the well-known MIR inequalities to the case where incompatible constraints are imposed on pairs of binary variables. This will lead to new inequalities, some of them resembling MIR inequalities, that incorporate variables in N_0 that do not appear in the set X_{SMI}.

An practical example that motivated this research resulted from maritime Inventory Routing Problems (IRPs), see [3–5, 43]. IRPs combine the inventory management at each node with the routing of vehicles. Constraint (4.1) results from the relaxation of inventory constraints, where s is the stock level at a given location, d is the aggregated demand at that location during a set of periods, c is the vehicle capacity (when several vehicles are considered we may assume this capacity to be constant for all vehicles, otherwise we can take c as the maximum of these capacities) and x_i represents an arc traveled by a vehicle. N_1 is the index set of arcs entering to that particular node. Constraints (4.2) represent incompatible arcs, that is, arcs that cannot belong to the same route, for instance, due to time constraints.

Inequalities from such conflict graphs were used in [5] to tighten a formulation for a maritime short sea IRP. For such IRP few inequalities are known that combine the information from the routing with the information from the inventory. By studying set X, we intend to derive new inequalities that can be used to improve the integrality gaps of such problems. Consider a simple example of a maritime IRP, with two ports: A and B. Constraints (4.1) can be obtained as a relaxation of the inventory constraints at port A,

$x_i \in N_1$ may represent arcs entering into node A in different periods, and $x_i, i \in N \setminus N_1$ may represent arcs entering into node B. Valid inequalities for X including simultaneously nonnegative coefficients on s, $x_i, i \in N_1$ and $x_j, j \in N \setminus N_1$ relate visits to node B to the inventory at node A.

From the theoretical point of view, valid inequalities for X_{VP} and valid inequalities for X_{SMI} are valid for X. As, in general, P is strictly included in $P_{VP} \bigcap P_{SMI}$, there are fractional solutions that cannot be cut off by valid inequalities derived either for P_{VP} or P_{SMI}. Hence, here we focus on valid inequalities derived for P that take into account properties from the two sets, simultaneously.

The outline of this chapter is as follows. In Section 4.2 we discuss basic properties of P and relate them with P_{SMI} and P_{VP}. Furthermore, we establish the conditions for the MIR inequality and the defining inequality $s \geq 0$ to define facets of P. In Section 4.3 we introduce conflict MIR inequalities where the concept of conflict graph is combined with the set X_{SMI}, and then we derive several families of valid inequalities for X. In addition, we provide conditions for some of those inequalities to be facet-defining. In section 4.4 we discuss on the separation problems associated to those valid inequalities. In Section 4.5 preliminary computational experiments in improving the gap of the randomly generated instances of a single node fixed-charge set with conflicts on arcs are reported. Finally, in Section 4.6 we summarize this chapter.

4.2 Basic Polyhedral Results

In this section we provide some basic results on set X.

Proposition 4.2.1. *Polyhedron P is full-dimensional.*

Proof. It suffices to consider the following $n+2$ affinely independent points belonging to X.

(i) $v_1, \ldots, v_{n_1}$: for all $j \in N_1, x_j = 1; x_i = 0, i \in N \setminus \{j\}; s = d - c;$

(ii) $v_{n_1+1}, \ldots, v_n$: for all $j \in N_0, x_j = 1; x_i = 0, i \in N \setminus \{j\}; s = d;$

(iii) $v_{n+1} : x_i = 0, \forall i \in N; s = d;$

(iv) $v_{n+2} : x_i = 0, \forall i \in N; s = 2d;$

where $|N_1| = n_1$. In order to prove that the listed points are affinely independent we consider system $\sum_{j=1}^{n+2} \lambda_j v_j = \mathbf{0}$, and $\sum_{j=1}^{n+2} \lambda_j = 0$, for scalars $\lambda_j, j = 1, \ldots, n+2$. So obtaining $\lambda_1 = \cdots = \lambda_n = 0$ is straightforward and then $\lambda_{n+1} = \lambda_{n+2} = 0$ can be seen easily. □

Proposition 4.2.2. *Polyhedron P is unbounded with extreme ray $v = (1, \mathbf{0})$, where $\mathbf{0}$ is the null vector of dimension n.*

Proof. The characteristic cone of polyhedron P is the following.

$$\begin{aligned} char.cone(P) &= \Big\{(s,x) \mid s + c\sum_{i\in N_1} x_i \geq 0, x_i + x_j \leq 0, (i,j) \in E, s \geq 0, x_i = 0, i \in N\Big\} \\ &= \Big\{(s,x) \mid s \geq 0, x_i = 0, i \in N\Big\}. \end{aligned}$$

Hence, P has an extreme ray $(1, \mathbf{0})$. □

Proposition 4.2.3. *Inequality* (4.1) *defines a facet of* P.

Proof. It suffices to consider the first $n+1$ points given in the proof of Proposition 4.2.1. □

It is easy to check that the projection of X onto the space of x variables coincides with X_{VP}, which is stated in the following proposition.

Proposition 4.2.4. $Proj_x(X) = X_{VP}$.

The following result establishes a relation between facet-defining inequalities for P_{VP} and some facet-defining inequalities for P.

Proposition 4.2.5. *Every facet-defining inequality* $\sum_{i\in N} \alpha_i x_i \geq \delta$, *for* P_{VP} *is a facet-defining inequality for P. Conversely, every facet-defining inequality* $\sum_{i\in N} \alpha_i x_i + \beta s \geq \delta$, *for P with* $\beta = 0$, *is a facet-defining inequality of* P_{VP}.

Proof. Valid inequalities $\sum_{i\in N} \alpha_i x_i \geq \delta$ for X_{VP} are valid for X, since X includes all the constraints defining X_{VP}. As $(\mathbf{1}, 0)$ is a ray of P, then each facet-defining inequality of P_{VP} defines also a facet of P.

Conversely, as $proj_x(X) = X_{VP}$, valid inequalities $\sum_{i\in N} \alpha_i x_i + \beta s \geq \delta$, for X with $\beta = 0$, are valid for X_{VP}. To show that if $\sum_{i\in N} \alpha_i x_i \geq \delta$ defines a facet of P, then it also defines a facet of P_{VP}, assume not. That is, assume that all the points in P_{VP} satisfying $\sum_{i\in N} \alpha_i x_i = \delta$ also satisfy an equation $\pi x = \pi_0$. Then, all the points in the corresponding facet of P would also satisfy $\pi x = \pi_0$, which is a contradiction. □

As a consequence of Proposition 4.2.5 we conclude that all the inequalities we are interested in, which are those that combine the structure of the vertex packing set with the simple mixed integer set, must include the continuous variable.

We use the following notation throughout the chapter. Consider graph $G = (N, E)$. For $j \in N$, $N(j) = \{i \in N \mid (i,j) \in E\}$ is set of vertices in N which are in conflict with node j, $N_1(j) = \{i \in N_1 \mid (i,j) \in E\}$, and $N_0(j) = \{i \in N_0 \mid (i,j) \in E\}$. In addition, for $S \subseteq N$, $N_1(S) = \bigcup_{j\in S} N_1(j)$, $\tilde{N}_1(S) = \bigcap_{j\in S} N_1(j)$, and $N_0(S) = \bigcup_{j\in S} N_0(j)$. Notice that if S is a singleton then $\tilde{N}_1(S) = N_1(S)$. Moreover, $G[S]$ denotes the subgraph induced by set S and $\alpha(G[S])$ represents the independence number of the corresponding graph. For $A \subseteq N$ and $b \in \mathbb{Z}_+$, $\mathcal{I}(A)$ denotes the set of all independent sets of $G[A]$ which includes the empty set, and $\mathcal{I}_b(A)$ denotes the set of all independent sets of $G[A]$ with cardinality equal to b.

A class of well-known clique inequalities (see [35, 37]) for set X_{VP} is given next.

Theorem 4.2.6. *An inequality $\sum_{i \in K} x_i \leq 1$, where $K \subseteq N$, is a facet of convex hull of X_{VP} if and only if K is a maximal clique in the conflict graph G.*

Proposition 4.2.5 and Theorem 4.2.6 imply that inequality $\sum_{i \in K} x_i \leq 1$, where $K \subseteq N$ is a maximal clique in G defines a facet of P. In particular, as corollary of Theorem 4.2.6 we present the trivial facet-defining inequalities of P by the following proposition.

Proposition 4.2.7. *(i)* $x_i \geq 0, i \in N$ *is facet-defining for* P.
(ii) $x_i \leq 1, i \in N$ *defines a facet of* P *if and only if* $N(i) = \emptyset$.
(iii) $x_i + x_j \leq 1$ *defines a facet of* P *if and only if* $N(i) \cap N(j) = \emptyset$.

Proof. Proof of (i). First, let $i \in N_1$. Define $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies } x_i = 0\}$. Then we prove that inequality $x_i \geq 0$ is facet-defining by showing that whenever the inequality $\gamma s + \sum_{j \in N} \alpha_j x_j \geq \gamma_0$ is valid for X and satisfies the condition that

$$\gamma s + \sum_{j \in N} \alpha_j x_j = \gamma_0, \forall (s,x) \in K, \tag{4.6}$$

then equality (4.6) is a multiple of $x_i = 0$. We introduce the following $n+1$ points belonging to K.

(1) for all $k \in N \setminus N_1, x_k = 1; x_j = 0, j \in N \setminus \{k\}; s = d$;

(2) for all $k \in N_1 \setminus \{i\}, x_k = 1; x_j = 0, j \in N \setminus \{k\}; s = d - c$;

(3) $x_i = 0, \forall i \in N; s = d$;

(4) $x_i = 0, \forall i \in N; s = 2d$.

Then replacing solutions (3) and (4) in equation (4.6) and subtracting the resultant equations imply $\gamma = 0$. Substituting points (1) and (3) in equation (4.6) and subtracting them give $\alpha_j = 0, j \in N \setminus N_1$. Applying the same technique with solutions (2) and (3) give $\alpha_j = 0, j \in N_1 \setminus \{i\}$. Lastly, substituting solution (3) in equation (4.6) implies $\gamma_0 = 0$ which shows that $\alpha x_i = 0$ is a multiple of $x_i = 0$.

Proof of (ii). Note that condition $N(i) = \emptyset$ ensures that $\{i\}$ is a maximal 1-vertex clique. Thus, Proposition 4.2.5 and Theorem 4.2.6 imply the result.

Proof of (iii). It can be done similar to the proof of part (ii). □

The following proposition provides necessary and sufficient conditions for the valid inequality $s \geq 0$ to be facet-defining.

Proposition 4.2.8. *Inequality* $s \geq 0$ *defines a facet of* P *if and only if*

$$\alpha\big(G\big[N_1 \setminus (N_1(j) \cup \{j\})\big]\big) \geq \left\lceil \frac{d}{c} \right\rceil, \forall j \in N. \tag{4.7}$$

Proof. Suppose (4.7) does not hold, that is, there is $j \in N$ such that $\alpha(G[N_1 \setminus (N_1(j) \cup \{j\})]) \leq \lfloor \frac{d}{c} \rfloor$. Hence, every point in the face $\{(s,x) \mid s = 0\}$ satisfies either $x_j = 1$ if $j \in N_1$ or $x_j = 0$ if $j \in N_0$. Thus $s \geq 0$ does not define a facet.

Now assume (4.7) holds. We define $K = P \cap \{(s,x) \mid s = 0\}$ and show that inequality $s \geq 0$ is facet-defining by showing that whenever the inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition that $\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K$, then $\gamma s + \sum_{i \in N} \beta_i x_i$ and s are identical linear forms up to positive multiple. For each $j \in N$, let $T_j \subseteq N_1 \setminus (N_1(j) \cup \{j\})$ be an independent set such that $|T_j| = \lceil \frac{d}{c} \rceil$. Consider the following points belonging to K.

(1) $s = 0; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus T_j;$

(2) $s = 0; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus (T_j \cup \{j\}); x_j = 1.$

Points (1) and (2) imply $\beta_i = 0, \forall i \in N$. Then, using one of these points it follows that $\gamma_0 = 0$. □

Next we establish sufficient conditions for the MIR inequality to be facet-defining for P. We follow the idea of constructing an auxiliary graph presented in [27] to prove that a family of valid inequalities defines facets.

Define the graph $G'_a = (N', E'), a \in \mathbb{Z}_+$, having N' as node set and whose edges are defined as follows: two nodes i and j are adjacent in G'_a if and only if there exists an independent set $I \in \mathcal{I}_a(N')$ such that $i \in I, j \notin I$, and $(I \setminus \{i\}) \cup \{j\} \in \mathcal{I}_a(N')$.

Proposition 4.2.9. *The MIR inequality* (4.5) *defines a facet of* P *if the following conditions hold.*

(i) $\alpha\big(G\big[N_1\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(ii) $G'_{\lfloor \frac{d}{c} \rfloor} = (N_1, E')$ *is connected.*

(iii) $\alpha\big(G\big[N_1 \setminus N_1(j)\big]\big) \geq \lfloor \frac{d}{c} \rfloor, \forall j \in N_0$.

Proof. Consider the equation

$$s + r \sum_{i \in N_1} x_i = r \left\lceil \frac{d}{c} \right\rceil. \tag{4.8}$$

Let us define $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies } (4.8)\}$. We prove that inequality (4.5) is facet-defining by showing that whenever the inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K, \tag{4.9}$$

then equality (4.9) is a multiple of (4.8). Consider the following feasible points belonging to K.

(1) $\forall\, T \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1), s = 0; x_i = 1, i \in T; x_i = 0, i \in N \setminus T;$

(2) $\forall\, T \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(N_1), s = r; x_i = 1, i \in T; x_i = 0, i \in N \setminus T;$

(3) $\forall j \in N_0, \forall\, T_j \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(N_1 \setminus N_1(j)), s = r; x_i = 1, i \in T_j; x_i = 0, i \in N \setminus T_j;$

(4) $\forall j \in N_0, \forall\, T_j \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(N_1 \setminus N_1(j)), s = r; x_i = 1, i \in T_j; x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\}).$

Notice that condition (i) ensures the existence of the points of type (1) and (2) while condition (iii) ensures the existence of the points of type (3) and (4). For each $j \in N_0$, substituting the points of type (3) and (4) corresponding to set T_j in equation (4.9) and subtracting the resultant equations imply $\beta_j = 0, \forall j \in N_0$.

Thus, equality (4.9) can be rewritten as

$$\gamma s + \sum_{i \in N_1} \beta_i x_i = \gamma_0. \tag{4.10}$$

Then take $i, j \in N_1$ and assume that they are adjacent in graph $G'_{\lfloor \frac{d}{c} \rfloor}$. So there exists an independent set I such that $I \subseteq N_1, i \in I, j \notin I, I' = (I \setminus \{i\}) \cup \{j\}$ is an independent set and $|I| = |I'| = \lfloor \frac{d}{c} \rfloor$. It follows that solutions of type (2) corresponding to sets I and I' belong to K. Thus, substituting the two solutions in (4.10) and then subtracting the corresponding equations gives $\beta_i = \beta_j$. It now follows easily from the connectivity of graph $G'_{\lfloor \frac{d}{c} \rfloor}$ that $\beta_i = \beta, \forall i \in N_1$.

It follows from replacing points (1) and (2) in equation (4.10) that $\beta \lceil \frac{d}{c} \rceil = \gamma_0$ and $\gamma r + \beta \lfloor \frac{d}{c} \rfloor = \gamma_0$, respectively. These equalities imply $\beta = \gamma r$ and $\gamma_0 = \gamma r \lceil \frac{d}{c} \rceil$ and so (4.9) is a multiple of (4.8). □

Conditions (i) and (iii) of Proposition 4.2.9 are necessary conditions for (4.8) to define a facet. The following example shows that condition (ii) is not a necessary condition.

Example 4.2.10. *Consider the set X with $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, \ldots, 6\}$, and the conflict graph which is shown in Figure 4.1. Then it can be seen that the graph $G'_{\lfloor \frac{d}{c} \rfloor}$ (see Figure 4.2) is not connected while MIR inequality* (4.5) *defines a facet of P.*

Now we discuss on the relation between d and c. If $c > d$, then the inequality $s + c\sum_{i \in N_1} x_i \geq d$ can be replaced by the stronger inequality $s + d\sum_{i \in N_1} x_i \geq d$. Thus, we assume henceforward $c \leq d$.

The following proposition shows that if $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$, then all nontrivial facet-defining inequalities for P are those from the vertex packing polytope.

Proposition 4.2.11. *Let $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$. If inequality*

$$\sum_{i \in N} \alpha_i x_i + \beta s \geq \gamma, \tag{4.11}$$

with $\beta \neq 0$, defines a facet of P, then inequality (4.11) *is a multiple of inequality* (4.1).

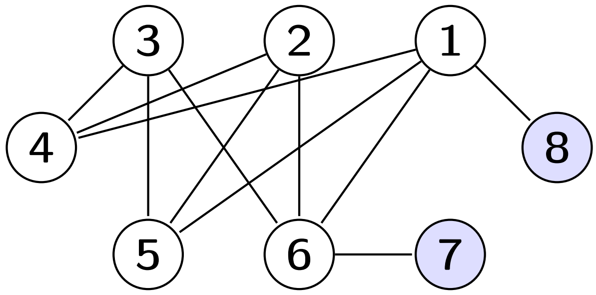

Figure 4.1: Conflict graph corresponding to Example 4.2.10.

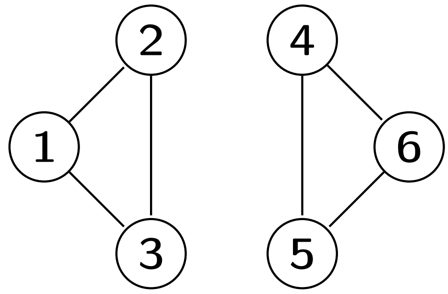

Figure 4.2: Graph $G'_{\lfloor \frac{d}{c} \rfloor}$ corresponding to Example 4.2.10.

Proof. First, note that since $(1, \mathbf{0})$ is an extreme ray, then we get $\beta \geq 0$. As $\beta \neq 0$, assume that $\beta > 0$. Then every point of X satisfying inequality (4.11) as equation also satisfies $s + c\sum_{i \in N_1} x_i = d$. Otherwise, if there exists a point $(s^*, x^*) \in X$ such that $s^* + c\sum_{i \in N_1} x_i^* > d$ and $\sum_{i \in N} \alpha_i x_i^* + \beta s^* = \gamma$, then condition $\alpha(G[N_1]) \leq \lfloor \frac{d}{c} \rfloor$ implies $s^* > 0$. So we create a new point $(s^* - \epsilon, x^*) \in X$ with $0 < \epsilon \leq s^* + \sum_{i \in N_1} x_i^* - d$ which violates inequality (4.11) which is a contradiction. □

Henceforward we assume $\alpha(G[N_1]) \geq \lceil \frac{d}{c} \rceil$.

4.3 Valid Inequalities

In this section we present new families of valid inequalities for X.

To generate the first family of inequalities consider $j \in N_0$, and a subset $\bar{S} \subset N_1 \setminus N_1(j)$ that cannot cover d in (4.1), that is, $\alpha(G[\bar{S}]) \leq p \leq \lfloor \frac{d}{c} \rfloor$. Then, if $x_j = 1$ the amount that is not covered by $\bar{S}$, $d - pc$, must be covered either from s or from $c\sum_{i \in N_1 \setminus (N_1(j) \cup \bar{S})} x_i$. Hence, the inequality $s \geq (d - pc)(x_j - \sum_{i \in S} x_i)$ is valid for X. Again, this inequality can be extended to any clique of N_0.

Proposition 4.3.1. *Let $S \subseteq N_0$ be a clique in G and $T \subseteq N_1 \setminus \tilde{N}_1(S)$ such that $\alpha(G[T]) \leq p \leq \lfloor \frac{d}{c} \rfloor$. Then the following inequality is valid for X.*

$$s \geq (d - pc)\Big(\sum_{i \in S} x_i - \sum_{i \in N_1 \setminus (\tilde{N}_1(S) \cup T)} x_i\Big). \tag{4.12}$$

Proof. First, assume $\sum_{i \in S} x_i = 0$. Then validity is implied by nonnegativity of s. Now let $\sum_{i \in S} x_i = 1$. Then the validity of (4.12) for $\sum_{i \in N_1 \setminus (\tilde{N}_1(S) \cup T)} x_i = 1$ is straightforward. So consider the case $\sum_{i\in N_1\setminus(\tilde{N}_1(S)\cup T)} x_i = 0$. So inequality (4.1) gives

$$s + c\sum_{i\in N_1} x_i = s + c\sum_{i\in N_1\setminus\tilde{N}_1(S)} x_i + c\sum_{i\in\tilde{N}_1(S)} x_i = s + c\sum_{i\in N_1\setminus\tilde{N}_1(S)} x_i = s + c\sum_{i\in N_1\setminus(\tilde{N}_1(S)\cup T)} x_i$$
$$+ c\sum_{i\in T} x_i = s + c\sum_{i\in T} x_i \geq d \iff$$
$$s \geq d - c\sum_{i\in T} x_i \geq d - pc = (d-pc)\Big(\sum_{i\in S} x_i - \sum_{i\in N_1\setminus(\tilde{N}_1(S)\cup T)} x_i\Big).$$

□

To derive other class of valid inequalities, notice that if $x_j = 1$ for some $j \in N$, then $x_i = 0, \forall i \in N_1(j)$. Hence it follows that

$$s \geq l_j x_j, \tag{4.13}$$

is valid for X, where $l_j = (d - \alpha\left(G[N_1 \setminus N_1(j)]\right) c)^+$. This inequality can be regarded as the lifting of inequality $s \geq 0$ when this inequality does not define a facet. Inequality (4.13) can be extended in two directions. One is to extend the right-hand side of the inequality for each clique. The other direction is to consider a subset of N_1 in the left-hand side. The following proposition gives the valid inequality for the general case.

Proposition 4.3.2. *Let $S \subseteq N$ be a clique in G and $T \subseteq N_1 \setminus S$. Then the following inequality is valid for X.*

$$s + c\sum_{i\in T} x_i \geq \sum_{i\in S}(d - p_i c)^+ x_i, \tag{4.14}$$

where $p_i = \alpha\left(G[N_1 \setminus (N_1(i) \cup T)]\right)$.

Proof. Let $(s, x) \in X$. Notice that since S is a clique then $\sum_{i\in S} x_i \leq 1$. If $\sum_{i\in S} x_i = 0$ then inequality (4.14) is implied by nonnegativity of $x_i, i \in T$ and s.

Assume $x_i = 1$ for some $i \in S$. This implies $x_j = 0, j \in N_1(i)$. If $(d - p_i c)^+ = 0$, then the inequality trivially holds. Hence, assume $d - p_i c > 0$. Then from (4.1) follows

$$s + c\sum_{i\in N_1} x_i = s + c\sum_{i\in T} x_i + c\sum_{i\in N_1(i)\setminus T} x_i + c\sum_{i\in N_1\setminus(N_1(i)\cup T)} x_i \geq d,$$

which implies

$$s + c\sum_{i\in T} x_i \geq d - c\sum_{i\in N_1\setminus(N_1(i)\cup T)} x_i \geq d - cp_i = (d - cp_i)^+ x_i = \sum_{i\in S}(d - p_i c)^+ x_i.$$

□

Next, we establish conditions on inequalities (4.14) to define facets.

Proposition 4.3.3. *If the following conditions hold, then inequality* (4.14) *defines a facet of* P.

(i) For each $i \in N_1 \setminus (T \cup S), \alpha\big(G\big[N_1 \setminus (T \cup S \cup N_1(i) \cup \{i\})\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(ii) For each $i \in N_0 \setminus S, \alpha\big(G\big[N_1 \setminus (T \cup S \cup N_1(i))\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(iii) For each $i \in T$*, there exists at least one* $j \in S$ *with* $(i, j) \notin E$*, and* $p_j < \lfloor \frac{d}{c} \rfloor$ *such that*

$$\alpha\big(G\big[N_1 \setminus (N_1(j) \cup T \setminus \{i\})\big]\big) \geq p_j + 1.$$

Proof. Without loss of generality we assume that $d - p_i c > 0, i \in S$. Consider the equality

$$s + c\sum_{i \in T} x_i = \sum_{i \in S}(d - p_i c)x_i, \tag{4.15}$$

and let $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies } (4.15)\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K. \tag{4.16}$$

So we show that equality (4.16) is a multiple of (4.15) by generating the following points belonging to K.

Conditions (i), (ii), (iii), and definition of p_j ensure the existence of the following points.

(1) $\forall\, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S)), s = 0; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus \overline{T}$;

(2) $\forall j \in N_1 \setminus (T \cup S), \forall\, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S \cup N_1(j) \cup \{j\})), s = 0; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\})$;

(3) $\forall j \in N_0 \setminus S, \forall\, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(N_1 \setminus (T \cup S \cup N_1(j))), s = 0; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\})$;

(4) $\forall j \in S, \forall\, \overline{T} \in \mathcal{I}_{p_j}(N_1 \setminus (T \cup N_1(j))), s = d - p_j c; x_i = 1, i \in \overline{T}; x_j = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j\})$;

(5) $\forall k \in T, \forall j \in S$ such that condition (iii) is satisfied, $s = d - (p_j + 1)c; x_i = 1, i \in \overline{T} \in \mathcal{I}_{p_j}(N_1 \setminus (T \cup N_1(j))); x_j = x_k = 1; x_i = 0, i \in N \setminus (\overline{T} \cup \{j, k\})$.

Points (1), (2) and (3) imply $\beta_i = 0, i \in N_1 \setminus (T \cup S), \beta_i = 0, i \in N_0 \setminus S$ and $\gamma_0 = 0$. Then substituting points (4) in equation (4.16) gives $\beta_i = -\gamma(d - p_i c), i \in S$. Finally, replacing points (5) in equation (4.16) implies $\beta_i = \gamma c, i \in T$. Hence, (4.16) is a multiple of (4.15). ☐

Facet-defining inequalities of type (4.14) are provided for the following example.

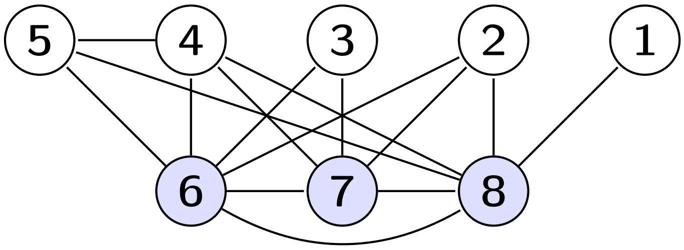

Figure 4.3: Conflict graph corresponding with Example 4.3.4.

Example 4.3.4. *Let* $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, 2, 3, 4, 5\}$ *and the conflict graph* G *shown in Figure 4.3. We can easily see that the following inequalities*

$$s + 9x_5 \geq 11x_6 + 11x_7 + 11x_8,$$
$$s \geq 11x_6 + 2x_7 + 11x_8,$$

define facets of P *with* $S = \{6, 7, 8\}, T = \{5\}$, *and* $S = \{6, 7, 8\}, T = \emptyset$, *respectively.*

Remark 4.3.5. *Consider valid inequality* (4.14) *by setting* $T = N_1 \setminus \tilde{N}_1(S)$. *Then one can check that* $p_i = 0, \forall i \in S$. *Thus, the following inequality is valid for* X.

$$s + c \sum_{i \in N_1 \setminus \tilde{N}_1(S)} x_i \geq d \sum_{i \in S} x_i. \tag{4.17}$$

For the particular case of $d - p_i c = r$, we have the following class of valid inequalities where S is not restricted to a clique.

Proposition 4.3.6. *Let* $S \subseteq N_0$, $T \subseteq N_1$ *such that*

$$\alpha\big(G[S]\big) \leq \left\lceil \frac{d}{c} \right\rceil,$$

and

$$\alpha\big(G[T \setminus N_1(\bar{S})]\big) \leq \left\lceil \frac{d}{c} \right\rceil - \left|\bar{S}\right|, \forall \bar{S} \in \mathcal{I}(S).$$

Then the following inequality is valid for X.

$$s + r \sum_{i \in N_1 \setminus T} x_i \geq r \sum_{i \in S} x_i. \tag{4.18}$$

Proof. If $\sum_{i \in S} x_i = 0$ then validity of (4.18) follows from nonnegativity of s and $x_i, i \in N_1 \setminus T$. Assume $\sum_{i \in S} x_i \geq 1$. Thus $\sum_{i \in S} x_i = \left|\bar{S}\right|$ where $\bar{S}$ is an independent set.

It follows from the MIR inequality that

$$s + r \sum_{i \in N_1 \setminus T} x_i \geq r\Big(\Big\lceil \frac{d}{c} \Big\rceil - \sum_{i \in T} x_i\Big) = r\Big(\Big\lceil \frac{d}{c} \Big\rceil - \sum_{i \in T \setminus N_1(\bar{S})} x_i\Big) \geq r\Big(\Big\lceil \frac{d}{c} \Big\rceil - \alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big)\Big)$$
$$\geq r\Big(\Big\lceil \frac{d}{c} \Big\rceil - \Big\lceil \frac{d}{c} \Big\rceil + |\bar{S}|\Big) = r \sum_{i \in S} x_i.$$

□

Next we show that under mild conditions inequalities (4.18) define facets of P.

Proposition 4.3.7. *Consider sets S and T as defined in the statement of Proposition 4.3.6. Suppose*

$$\mathcal{S} = \Big\{ \bar{S} \in \mathcal{I}(S) \mid \alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big) = \Big\lceil \frac{d}{c} \Big\rceil - |\bar{S}| \Big\} \neq \emptyset,$$

and consider the following two graphs:

$G' = (N_1 \setminus T, E')$, where $(i,j) \in E'$ if there exist $\bar{S} \in \mathcal{S}, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$, and an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))$ such that $|I| \in \{|\bar{S}|-1, |\bar{S}|\}$, $i \in I$, $j \notin I$, and $I' \cup \bar{S} \cup \bar{T}$ is an independent set where $I' = (I \setminus \{i\}) \cup \{j\}$;

$G'' = (S, E'')$, where $(i,j) \in E''$ if there exist $\bar{S} \in \mathcal{S}, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$, and an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))$ such that $|I| \in \{|\bar{S}|-1, |\bar{S}|\}$, $i \in \bar{S}$, $j \notin \bar{S}$, $\bar{S}' = (\bar{S} \setminus \{i\}) \cup \{j\} \in \mathcal{S}$ and sets $\bar{S} \cup \bar{T} \cup I$ and $\bar{S}' \cup \bar{T} \cup I$ are independent.

Then inequality (4.18) defines a facet of P if the following conditions hold.

(i) For each $i \in T, \alpha\big(G\big[T \setminus (N_1(i) \cup \{i\})\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(ii) For each $i \in N_0 \setminus S, \alpha\big(G\big[T \setminus N_1(i)\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(iii) For each $\bar{S} \in \mathcal{S}$ there exists $\bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that

$$\alpha\big(G\big[N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))\big]\big) \geq |\bar{S}|.$$

(iv) Graph $G' = (N_1 \setminus T, E')$ is connected.

(v) Graph $G'' = (S, E'')$ is connected.

Proof. Consider the equation

$$s + r \sum_{i \in N_1 \setminus T} x_i = r \sum_{i \in S} x_i, \tag{4.19}$$

and let $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies (4.19)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K. \tag{4.20}$$

So we justify that equality (4.20) is a multiple of (4.19). We create the points belonging to K as follows.

Condition (i) implies $\alpha(G[T]) \geq \lceil \frac{d}{c} \rceil$. So the following points exist and are in K.

(1) $\forall\, \overline{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T), s = 0; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus \overline{T}$.

In addition, condition (i) shows that for each $j \in T$, there exist $T_j \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T)$ such that $j \notin N_1(T_j)$. So the following points are in K.

(2) $\forall j \in T, s = 0; x_i = 1, i \in T_j; x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\})$.

Condition (ii) ensures the existence of the following points.

(3) $\forall j \in N_0 \setminus S, s = 0; x_i = 1, i \in T_j \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T); x_j = 1; x_i = 0, i \in N \setminus (T_j \cup \{j\})$.

From condition (iii) we get the following points.

(4) $\forall \bar{S} \in \mathcal{S}, \forall\, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S})), \forall\, I \in \mathcal{I}_{|\bar{S}|-1}(N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))), s = r; x_i = 1, i \in (\bar{S} \cup \bar{T} \cup I); x_i = 0, i \in N \setminus (\bar{S} \cup \bar{T} \cup I)$;

(5) $\forall \bar{S} \in \mathcal{S}, \forall\, \bar{T} \in \mathcal{I}_{\lceil \frac{d}{c} \rceil - |\bar{S}|}(T \setminus N_1(\bar{S})), \forall\, I \in \mathcal{I}_{|\bar{S}|}(N_1 \setminus (T \cup N_1(\bar{S}) \cup N_1(\bar{T}))), s = 0; x_i = 1, i \in (\bar{S} \cup \bar{T} \cup I); x_i = 0, i \in N \setminus (\bar{S} \cup \bar{T} \cup I)$.

Substituting points (1) and (2) in equation (4.20) and subtracting the resultant equations imply $\beta_j = 0, j \in T$. Similarly, using points (1) and (3) gives $\beta_j = 0, j \in N_0 \setminus S$. Then replacing any point (1) in equation (4.20) gives $\gamma_0 = 0$.

So equation (4.20) can be written as

$$\gamma s + \sum_{i \in N_1 \setminus T} \beta_i x_i + \sum_{i \in S} \beta_i x_i = 0. \tag{4.21}$$

Let $i, j \in N_1 \setminus T$ and assume that they are adjacent in $G' = (N_1 \setminus T, E')$. So condition (iv) implies that there exist $\bar{S} \in \mathcal{S}$ and an independent set $I \subseteq N_1 \setminus (T \cup N_1(\bar{S}))$ such that $|I| = |\bar{S}|$, $i \in I$, $j \notin I$, and $I' = (I \setminus \{i\}) \cup \{j\}$ is an independent set. Substituting points (4) or (5), depending on the cardinality of the independent set, corresponding to sets I and I' in equation (4.21) and subtracting them imply $\beta_i = \beta_j, i, j \in N_1 \setminus T$. It follows from connectivity of graph $G' = (N_1 \setminus T, E')$ that $\beta_i = \beta_1, i \in N_1 \setminus T$.

Similar to the justification of the foregoing part, one can check that, using condition (v), $\beta_i = \beta_2, i \in S$. Then replacing points (4) or (5) (depending on the cardinality of the independent set) in equation (4.21) imply $\beta_2 = -\beta_1$. Finally, substituting points (4) in equation (4.21) gives $\beta_1 = \gamma r$. □

Next we introduce a new family of valid inequalities.

Proposition 4.3.8. *Let $S \subseteq N_0$ such that $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$ and*

$$\alpha\big(G\big[N_1 \setminus N_1(\bar{S})\big]\big) \leq \left\lceil \frac{d}{c} \right\rceil - \left|\bar{S}\right|, \forall \bar{S} \in \mathcal{I}(S). \tag{4.22}$$

Then the following inequality is valid for X.

$$s + (c - r) \geq c \sum_{i \in S} x_i. \tag{4.23}$$

We omit the proof since we will provide the proof of a more general class later. Next we show that if $\tilde{N}_1(S) \neq \emptyset$, then (4.23) does not define a facet. Let

$$\mathcal{F} = \Big\{(s, x) \in X \mid s = c \sum_{i \in S} x_i - (c - r)\Big\}.$$

As $-(c-r) < 0$ and $s \geq 0$ then $\sum_{i \in S} x_i > 0, \forall (s, x) \in \mathcal{F}$. This implies that if $i \in \tilde{N}_1(S)$, then $x_i = 0, \forall (s, x) \in \mathcal{F}$. Thus, (4.23) does not define a facet when $\tilde{N}_1(S) \neq \emptyset$. In order to obtain a stronger inequality, we lift these variables which are zero for all points in $\mathcal{F}$. Consider $R \subseteq \tilde{N}_1(S)$ such that R is a clique in $G[\tilde{N}_1(S)]$. Hence, we want to find coefficients $l_i, i \in R$ such that inequality

$$s + (c - r) \geq c \sum_{i \in S} x_i + \sum_{i \in R} l_i x_i, \tag{4.24}$$

remains valid for X. If $x_i = 0, \forall i \in R$, then inequality (4.24) is trivially valid. So assume $x_j = 1$, for some $j \in R$. Notice that since R is a clique, then $x_j = 1$ implies $x_i = 0, \forall i \in R \setminus \{j\}$. Thus, in order for inequality

$$s + (c - r) \geq c \sum_{i \in S} x_i + l_j, \forall (s, x) \in X|_{x_j = 1},$$

to be valid, l_j must satisfy $l_j \leq s + (c - r) - c \sum_{i \in S} x_i, \forall (s, x) \in X|_{x_j=1}$. Since $j \in \tilde{N}_1(S)$, so $x_j = 1$ implies $x_i = 0, \forall i \in S$. Hence

$$l_j \leq s + (c - r), \forall (s, x) \in X|_{x_j = 1} \implies l_j \leq min_{(s,x) \in X|_{x_j=1}}\{s\} + (c - r).$$

The minimum value which s attains can be obtained by setting nonzero binary variables of N_1 equal to one as many as possible. Thus

$$l_j = (c - r) + \Big[d - \Big(\alpha\big(G\big[N_1 \setminus (j \cup N_1(j))\big]\big) + 1\Big)c\Big]^+. \tag{4.25}$$

Therefore, since R is a clique, inequality (4.24) is valid for X where $l_i, i \in R$ is defined by (4.25). Moreover, if condition $\alpha(G[N_1 \setminus (i \cup N_1(i))]) \geq \lfloor \frac{d}{c} \rfloor$ holds, then $s = 0$ implies $l_i = c - r, i \in R$.

Next we generalize inequality (4.23) as follows.

Proposition 4.3.9. *Let $S \subseteq N_0$ with $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$, and $T \subseteq N_1$ such that*

$$\alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big) \leq \left\lceil \frac{d}{c} \right\rceil - \left|\bar{S}\right|, \forall \bar{S} \in \mathcal{I}(S).$$

Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq c \sum_{i \in S} x_i. \tag{4.26}$$

Proof. Consider $(s, x) \in X$. If $\sum_{i \in S} x_i = 0$, then validity of (4.26) is implied by the nonnegativity of variables x_i and s. Assume $x_i = 1, i \in \bar{S} \subseteq S$ and $x_i = 0, i \in S \setminus \bar{S}$. From (4.1) it follows that $s + c \sum_{i \in N_1 \setminus T} x_i \geq d - c \sum_{i \in T} x_i$. Thus

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq d - c \sum_{i \in T} x_i + (c - r) \geq d - \alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big)c + (c - r)$$

$$\geq d - c\Big(\left\lceil \frac{d}{c} \right\rceil - |\bar{S}|\Big) + (c - r) = c\left\lfloor \frac{d}{c} \right\rfloor + r - c\Big(\left\lceil \frac{d}{c} \right\rceil - \left|\bar{S}\right|\Big) + (c - r) = c\left|\bar{S}\right| = c \sum_{i \in S} x_i.$$

□

Similarly to inequalities (4.23), inequalities (4.26) can be strengthened by lifting variables in $\tilde{N}_1(S)$. We lift these variables by taking $R \subseteq \tilde{N}_1(S)$ such that R is a clique. We aim to find lifting coefficients $l_i, i \in R$ such that inequality $s + (c - r) \geq c \sum_{i \in S} x_i - c \sum_{i \in N_1 \setminus T} x_i + \sum_{i \in R} l_i x_i$, remains valid for X. Following the forgoing steps to lift inequality (4.23), the more general family of valid inequalities is stated below.

Proposition 4.3.10. *Consider the sets $S \subseteq N_0$, $T \subseteq N_1$, and $R \subseteq \tilde{N}_1(S)$ such that $\alpha(G[S]) \leq \lceil \frac{d}{c} \rceil$,*

$$\alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big) \leq \left\lceil \frac{d}{c} \right\rceil - \left|\bar{S}\right|, \forall \bar{S} \in \mathcal{I}(S),$$

and R is a clique. Then following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + (c - r) \geq c \sum_{i \in S} x_i + \sum_{i \in R} l_i x_i,$$

where

$$l_i = (c - r) + \Big[d - \Big(\alpha\big(G\big[N_1(S) \setminus (i \cup N_1(i))\big]\big) + 1\Big)c\Big]^+, i \in T.$$

If $\alpha(G[N_1(S) \setminus (i \cup N_1(i))]) \geq \lfloor \frac{d}{c} \rfloor$, then we get $l_i = c - r, i \in T$.

4.3.1 Conflict MIR Inequalities

Next we introduce families of valid inequalities, called conflict MIR inequalities, that can be regarded as an extension of MIR inequalities to the case where a conflict graph representing incompatibilities between pairs of variables is considered. In order to introduce these families we first introduce a weaker MIR inequality, obtained from a restriction of set X. For each $T \subset N_1$, let $s' = s + c\sum_{i\in N_1\setminus T} x_i$. Then the following MIR inequality

$$s' + r\sum_{i\in T} x_i \geq r\left\lceil \frac{d}{c} \right\rceil,$$

is valid for X. When this inequality does not define facet (see Proposition 4.2.1) it must be lifted. In the following proposition we lift this inequality to obtain a new family of valid inequalities.

Proposition 4.3.11. *Consider $S \subseteq N_0$ with $\alpha(G[S]) \leq \lfloor \frac{d}{c} \rfloor$ and $T \subseteq N_1$ such that*

$$\alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right|, \ \forall \bar{S} \in \mathcal{I}(S). \tag{4.27}$$

Then the following inequality is valid for X.

$$s + c\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i \geq r\left\lceil \frac{d}{c} \right\rceil + (c-r)\sum_{i\in S} x_i. \tag{4.28}$$

Proof. Let $(s,x) \in X$. If $\sum_{i\in S} x_i = 0$, then the validity is implied by the MIR inequality (4.5) as follows.

$$s + c\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i \geq s + r\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i \geq r\left\lceil \frac{d}{c} \right\rceil.$$

Assume $\sum_{i\in S} x_i \geq 1$. So $\sum_{i\in S} x_i = \left|\bar{S}\right|$ where $\bar{S} \subseteq S$ is an independent set. Now let $\sum_{i\in T\setminus N_1(\bar{S})} x_i = \lfloor \frac{d}{c} \rfloor - \left|\bar{S}\right| - k$ where $0 \leq k \leq \lfloor \frac{d}{c} \rfloor - \left|\bar{S}\right|$.
As

$$\sum_{i\in N_1} x_i = \sum_{i\in N_1\setminus T} x_i + \sum_{i\in T\setminus N_1(\bar{S})} x_i = \sum_{i\in N_1\setminus T} x_i + \left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right| - k,$$

then, using inequality (4.1) gives

$$s + c\sum_{i\in N_1} x_i \geq d \iff s + c\sum_{i\in N_1\setminus T} x_i + c\sum_{i\in T\setminus N_1(\bar{S})} x_i \geq d.$$

Thus

$$s + c\sum_{i\in N_1\setminus T} x_i \geq d - c\Big(\left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right| - k\Big) \geq \left|\bar{S}\right| c + kc + r$$

$$\geq \left|\bar{S}\right| c + (k+1)r \geq r\left\lceil \frac{d}{c} \right\rceil + (c-r)\left|\bar{S}\right| - r\Big(\left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right| - k\Big).$$

Hence

$$s + c\sum_{i\in N_1\setminus T} x_i + r\Big(\left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right| - k\Big) = s + c\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T\setminus N_1(\bar{S})} x_i \geq r\left\lceil \frac{d}{c} \right\rceil + (c-r)\left|\bar{S}\right|.$$

□

Definition 4.3.12. *Let $S \subseteq N_0$ and $T \subseteq N$. $\bar{\alpha}(G[T \cup S])$ denotes the independence number of the subgraph induced by $T \cup S$ such that at least one node from set S appears in the corresponding independent set.*

In the following proposition we present sufficient conditions for inequality (4.28) to be facet-defining.

Proposition 4.3.13. *Consider S and T as defined in the statement of Proposition 4.3.11. Suppose*

$$\mathcal{S}_1 = \Big\{ \bar{S} \in \mathcal{I}(S) \mid \alpha\big(G\big[T \setminus N_1(\bar{S})\big]\big) = \left\lfloor \frac{d}{c} \right\rfloor - \left|\bar{S}\right| \Big\} \neq \emptyset,$$

and consider the following graph:

$G'' = (S, E'')$, where $(i,j) \in E''$ if there exists $J \in \mathcal{S}_1$ such that $i \in J$, $j \notin J$, and $J' = (J \setminus \{i\}) \cup \{j\} \in \mathcal{S}_1$.

Then inequality (4.28) is facet-defining for P if the following conditions hold.

(i) $\alpha\big(G\big[T\big]\big) \geq \lceil \frac{d}{c} \rceil$.

(ii) For each $i \in N_1 \setminus T$, $\bar{\alpha}\big(G\big[(T \cup S) \setminus N(i))\big]\big) \geq \lfloor \frac{d}{c} \rfloor$.

(iii) For each $i \in N_0 \setminus S$, $\bar{\alpha}\big(G\big[(T \cup S) \setminus N(i)\big]\big) \geq \lfloor \frac{d}{c} \rfloor$.

(iv) Graph $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$ *is connected.*

(vi) Graph $G'' = (S, E'')$ *is connected.*

Proof. In order to prove that inequality (4.28) defines a facet, consider the equation

$$s + c\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i = r\left\lceil \frac{d}{c} \right\rceil + (c-r)\sum_{i\in S} x_i, \tag{4.29}$$

and let $K = P \cap \{(s,x) \mid (s,x) \text{ satisfies } (4.29)\}$. Now assume inequality $\gamma s + \sum_{i\in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition

$$\gamma s + \sum_{i\in N} \beta_i x_i = \gamma_0, \forall (s,x) \in K. \tag{4.30}$$

So we justify that equality (4.30) is a multiple of (4.29). Consider the following points in K.

(1) $\forall\ T_1 \in \mathcal{I}_{\lceil \frac{d}{c} \rceil}(T), s = 0; x_i = 1, i \in T_1; x_i = 0, i \in N \setminus T_1;$

(2) $\forall\ T_2 \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor}(T), s = r; x_i = 1, i \in T_2; x_i = 0, i \in N \setminus T_2;$

(3) $\forall \bar{S} \in \mathcal{S}_1, \forall\ \overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S})), s = c\left|\bar{S}\right| + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T}).$

Note that condition (ii) implies that for each $k \in N_1 \setminus T$, there exist sets $\bar{S} \in \mathcal{S}_1$ and $\overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that $k \in N_1 \setminus (T \cup N_1(\overline{T} \cup \bar{S}))$. So the follwoing points are in K.

(4) $\forall k \in N_1 \setminus T, s = c(\left|\bar{S}\right| - 1) + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_k = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup \{j\}).$

In additon, it follows from condition (iii) that for each $k \in N_0 \setminus S$, there exist sets $\bar{S} \in \mathcal{S}_1$ and $\overline{T} \in \mathcal{I}_{\lfloor \frac{d}{c} \rfloor - |\bar{S}|}(T \setminus N_1(\bar{S}))$ such that $i \in N_0 \setminus (S \cup N_0(\overline{T} \cup \bar{S}))$. Thus, the next points belong to K.

(5) $\forall k \in N_0 \setminus S, s = c\left|\bar{S}\right| + r; x_i = 1, i \in \bar{S}; x_i = 1, i \in \overline{T}; x_k = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \overline{T} \cup \{j\}).$

Now let $i \in N_0 \setminus S$. So considering points of type (3) and (5) and substituting them in equation (4.30) and then subtracting the resultant equations imply $\beta_i = 0, i \in N_0 \setminus S$.

Thus, (4.30) can be written as follows.

$$\gamma s + \sum_{i \in N_1 \setminus T} \beta_i x_i + \sum_{i \in T} \beta_i x_i + \sum_{i \in S} \beta_i x_i = \gamma_0. \tag{4.31}$$

Consider $i, j \in T$ and suppose i and j are adjacent in $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$. So there exists an independent set $I \subseteq T$ such that $i \in I$, $j \notin I$, $|I| = \lfloor \frac{d}{c} \rfloor$, and $I' = (I \setminus \{i\}) \cup \{j\}$ is independent. Substituting solution (2) corresponding to sets I and I' in equation (4.31) and subtracting provide $\beta_i = \beta_j, i, j \in T$. It can be concluded from connectivity of $G'_{\lfloor \frac{d}{c} \rfloor} = (T, E')$ that $\beta_i = \beta_1, i \in T$.

Next we take $i, j \in S$ and assume that they are connected in $G'' = (S, E'')$. So there exists independent set J such that $J \subseteq S$, $\alpha(G[T \setminus N_1(J)]) = \lfloor \frac{d}{c} \rfloor - |J|$, $i \in J$, $j \notin J$, $J' = (J \setminus \{i\}) \cup \{j\}$ is an independent set, and $\alpha(G[T \setminus N_1(J')]) = \lfloor \frac{d}{c} \rfloor - |J|$. Substituting points (3) corresponding to J and J' in (4.31) and subtracting imply $\beta_i = \beta_j, i, j \in S$. It follows from connectivity of $G'' = (S, E'')$ that $\beta_i = \beta_2, i \in S$.

Then let $i \in N_1 \setminus T$. So substituting points of type (3) and (4) in equation (4.31) and subtracting them give $\beta_i = \gamma c, i \in N_1 \setminus T$.

It follows from replacing solutions (1) and (2) in equation (4.31) that $\gamma_0 = \beta_1 \lceil \frac{d}{c} \rceil$ and $\gamma r + \beta_1 \lfloor \frac{d}{c} \rfloor = \gamma_0$ which imply $\beta_1 = \gamma r$, $\gamma_0 = \gamma r \lceil \frac{d}{c} \rceil$. Finally, substituting points (3) in (4.31) gives $\beta_2 = -\gamma(c - r)$. $\square$

When $S \subseteq N_0$ is a clique, inequalities (4.28) can be strengthened as follows.

Proposition 4.3.14. *Let S is a clique in $G[N_0]$, and $T \subseteq N_1$ such that*

$$\alpha\big(G\big[T \setminus N_1(i)\big]\big) \le \Big\lfloor \frac{d}{c} \Big\rfloor - p_i, \forall i \in S,$$

where $p_i \in \big\{1, \dots, \lfloor \frac{d}{c} \rfloor\big\}, i \in S$. Then the following inequality is valid for X.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \ge r \Big\lceil \frac{d}{c} \Big\rceil + \sum_{i \in S} p_i (c-r) x_i. \tag{4.32}$$

Proof. Let $(s, x) \in X$. Assume $\sum_{i \in S} x_i = 0$. Then validity is implied by the MIR inequality (4.5) similarly to the proof of the same case given in Proposition 4.3.11.

Let $\sum_{i \in S} x_i = 1$. So assume $x_j = 1$, for some $j \in S$. Then $\sum_{i \in T \setminus N_1(j)} x_i = \lfloor \frac{d}{c} \rfloor - k_j$ where $p_j \le k_j \le \lfloor \frac{d}{c} \rfloor$. From (4.1), using $\sum_{i \in N_1(j)} x_i = 0$ and $\sum_{i \in T \setminus N_1(j)} x_i = \lfloor \frac{d}{c} \rfloor - k_j$, then

$$\begin{aligned}
& s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T \setminus N_1(j)} x_i + c \sum_{i \in N_1(j)} x_i \ge d \iff \\
& s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T \setminus N_1(j)} x_i \ge d - (c-r) \sum_{i \in T \setminus N_1(j)} x_i \ge d - (c-r)\Big(\Big\lfloor \frac{d}{c} \Big\rfloor - k_j\Big) \\
& = r + c\Big\lfloor \frac{d}{c} \Big\rfloor - (c-r)\Big(\Big\lfloor \frac{d}{c} \Big\rfloor - k_j\Big) = r\Big\lceil \frac{d}{c} \Big\rceil + (c-r)k_j \ge r\Big\lceil \frac{d}{c} \Big\rceil + (c-r)p_j.
\end{aligned}$$

□

Inequalities (4.32) can be lifted as follows.

Proposition 4.3.15. *Let $S \subseteq N_0$ define a clique in G, $k \in N_0 \setminus S$ such that $S \cup \{k\}$ does not define a clique, and $T \subseteq N_1$ such that*

$$\begin{aligned}
& \alpha\big(G\big[T \setminus N_1(i)\big]\big) \le \Big\lfloor \frac{d}{c} \Big\rfloor - p_i, \forall i \in S \cup k, \\
& \alpha\big(G\big[T \setminus N_1(\{k, j\})\big]\big) \le \Big\lfloor \frac{d}{c} \Big\rfloor - p_j - p_k, \forall j \in S_1,
\end{aligned}$$

where $p_i \in \big\{1, \dots, \lfloor \frac{d}{c} \rfloor\big\}, i \in S \cup \{k\}, 1 \le p_j + p_k \le \lfloor \frac{d}{c} \rfloor, j \in S_1 = \{j \in S : (j,k) \notin E\}$. Then the following inequality is valid.

$$s + c \sum_{i \in N_1 \setminus T} x_i + r \sum_{i \in T} x_i \ge r \Big\lceil \frac{d}{c} \Big\rceil + \sum_{i \in S} p_i (c-r) x_i + p_k (c-r) x_k. \tag{4.33}$$

Proof. If $x_k = 0$ or $x_k = 1$ and $\sum_{i \in S} x_i = 0$, then validity of (4.33) follows from validity of (4.32). The proof of case $x_k = 1$ and $\sum_{i \in S} x_i = 1$ is similar to the proof of validity of (4.32). □

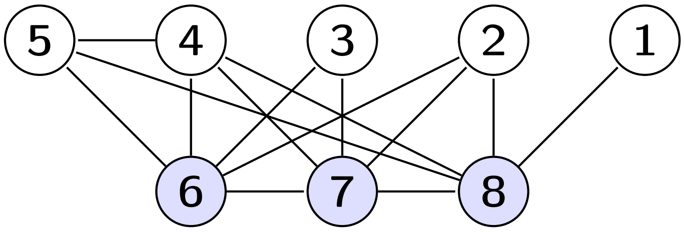

Figure 4.4: Conflict graph considered in Example 4.3.16.

The following example presents facet-defining inequalities of types (4.28), (4.32), and (4.33).

Example 4.3.16. *Assume* $d = 20, c = 9, N = \{1, \ldots, 8\}, N_1 = \{1, 2, 3, 4, 5\}$ *and the conflict graph* G *shown in Figure 4.4. Then it can be checked easily that condition* (4.27) *is satisfied for* $S = \{6, 7, 8\}$ *and* $T = \{2, 3, 4, 5\}$*. So the following inequality of type* (4.28) *is valid for* X*.*

$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \geq 6 + 7x_6 + 7x_7 + 7x_8.$$

One can check that the foregoing inequality as well as the following inequalities of type (4.28) *define facets of* P*.*

$$s + 2x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \geq 6 + 7x_6 + 7x_7 + 7x_8,$$
$$s + 2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \geq 6 + 7x_6 + 7x_8.$$

The following inequalities of type (4.32) *are facet-defining for* P*.*

$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \geq 6 + 14x_6 + 14x_7,$$
$$s + 2x_1 + 2x_2 + 9x_3 + 2x_4 + 9x_5 \geq 6 + 7x_7 + 14x_8,$$
$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 \geq 6 + 14x_6 + 7x_7,$$
$$s + 2x_1 + 2x_2 + 9x_3 + 2x_4 + 2x_5 \geq 6 + 14x_8.$$

The unique facet-defining inequality of type (4.33)*, is obtained with* $S = \{6, 7\}, k = \{8\}$ *and* $T = \{2, 3, 4\}$*, and is given by*

$$s + 9x_1 + 2x_2 + 2x_3 + 2x_4 + 9x_5 \geq 6 + 7x_6 + 14x_7 + 7x_8.$$

Next we generalize inequalities (4.28) as follows.

Proposition 4.3.17. *Let* $S \subseteq N_0$, $T \subseteq N_1$ *and let* $\{T_1, T_2\}$ *defines a partition of* T *such that*

$$\alpha\big(G[S]\big) \leq \Big\lfloor \frac{d}{c} \Big\rfloor + p,$$
$$\alpha\big(G[T_1 \setminus N_1(\bar{S})]\big) \leq \Big\lfloor \frac{d}{c} \Big\rfloor + p - |\bar{S}|, \ \forall \bar{S} \in \mathcal{I}(S), \tag{4.34}$$
$$\alpha\big(G[T_2 \setminus N_1(\bar{S})]\big) \leq \Big(p - |\bar{S}|\Big)^+, \ \forall \bar{S} \in \mathcal{I}(S). \tag{4.35}$$

Then the following inequality is valid for X.

$$s + c\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i \geq r\left\lceil\frac{d}{c}\right\rceil + (c-r)\Big(\sum_{i\in S} x_i - p + \sum_{i\in T_2} x_i\Big). \tag{4.36}$$

Proof. Let $(s,x)\in X$. Let $x_i = 1, i\in\bar{S}\subseteq S$, and $x_i = 0, i\in S\setminus\bar{S}$. If $|\bar{S}| < p$, then

$$\sum_{i\in S} x_i - p + \sum_{i\in T_2} x_i = |\bar{S}| - p + \sum_{i\in T_2} x_i \leq |\bar{S}| - p + \alpha\big(G\big[T_2\setminus N_1(\bar{S})\big]\big) \leq 0,$$

where the last inequality follows from (4.35). Hence inequality (4.36) is implied by the MIR inequality

$$s + r\sum_{i\in N_1\setminus T} x_i + r\sum_{i\in T} x_i \geq r\left\lceil\frac{d}{c}\right\rceil.$$

Now, let $|\bar{S}|\geq p$. Then, from (4.35) it follows that $x_i = 0, i\in T_2$. The proof is now similar to the proof of Proposition 4.3.11 for case $\sum_{i\in S} x_i \geq 1$. □

4.3.2 Valid Inequalities for Case $d = c$

Notice that all the inequalities discussed previously are valid for the case $d = c$. Here we introduce a new class of valid inequalities for X which define facets only when $d = c$.

Proposition 4.3.18. *Let $S\subseteq N_0$, $\alpha(G[S]) = p$ and define*

$$T = \bigcap_{\bar{S}\in\mathcal{I}_p(S)} N_1(\bar{S}),\quad \overline{T} = \bigcap_{\bar{S}\in\mathcal{I}_{p-1}(S)} N_1(\bar{S}).$$

Let $T'\subseteq\overline{T}$ such that T' defines a clique. The following inequality is valid for X.

$$s + c\sum_{i\in N_1\setminus T} x_i \geq c\Big(\sum_{i\in S} x_i - p + 1\Big) + c\sum_{i\in T'} x_i. \tag{4.37}$$

Proof. To prove validity of (4.37) we consider the following cases. Let $(s,x)\in X$.

Case I. Let $p = 1$. It implies that S is a clique, $T = \tilde{N}_1(S)$ and $\overline{T} = T' = \emptyset$. If $\sum_{i\in S} x_i = 0$ then the validity follows from nonnegativity of s and $x_i, i\in N_1\setminus T$. Assume $\sum_{i\in S} x_i = 1$. Then inequality (4.1) implies

$$\begin{aligned} s + c\sum_{i\in N_1} x_i &= s + c\sum_{i\in N_1\setminus T} x_i + c\sum_{i\in T} x_i = s + c\sum_{i\in N_1\setminus T} x_i + c\sum_{i\in\tilde{N}_1(S)} x_i \\ &= s + c\sum_{i\in N_1\setminus T} x_i \geq c = c\sum_{i\in S} x_i. \end{aligned}$$

Case II. Let $p \geq 2$. If $\sum_{i \in S} x_i = 0$ then validity of (4.37) is implied by nonnegativity of $s, x_i, i \in N_1 \setminus T$, and properties $\sum_{i \in T'} x_i \leq 1$ and $1 - p \leq -1$. Suppose $\sum_{i \in S} x_i = |\bar{S}|$ where $\bar{S}$ is an independent set. If $1 \leq |\bar{S}| \leq p - 2$, then

$$\sum_{i \in S} x_i - p + 1 = |\bar{S}| - p + 1 \leq p - 2 - p + 1 = -1,$$

which implies that $c\big(\sum_{i \in S} x_i - p + 1\big) + c \sum_{i \in T'} x_i \leq 0$. Thus, the validity is implies by nonnegativity of s and $x_i, i \in N_1 \setminus T$.

Now, let $p - 1 \leq |\bar{S}| \leq p$. Then it can be seen readily from the definition of T' that this condition implies $\sum_{i \in T'} x_i = 0$. So, for the case $|\bar{S}| = p - 1$, the validity follows from nonnegativity of s and $x_i, i \in N_1 \setminus T$. For $|\bar{S}| = p$, it can be concluded that $\sum_{i \in T} x_i = 0$. So inequality (4.1) implies

$$s + c \sum_{i \in N_1} x_i = s + c \sum_{i \in N_1 \setminus T} x_i + c \sum_{i \in T} x_i = s + c \sum_{i \in N_1 \setminus T} x_i \geq c = c\Big(\sum_{i \in S} x_i - p + 1\Big).$$

□

Sufficient conditions for inequality (4.37) to define a facet of P are presented as follows.

Proposition 4.3.19. *Let $S \subseteq N_0$ is an independent set. Inequality (4.37) is facet-defining for P if the following conditions hold.*

(i) For each $i \in T \setminus T'$, there exists at least one $\bar{S} \in \mathcal{I}_{p-1}(S)$ such that $i \in T \setminus (T' \cup N_1(\bar{S}))$.

(ii) For each $i \in T'$, there exists at least one $\bar{S} \in \mathcal{I}_{p-2}(S)$ such that $i \in T' \setminus N_1(\bar{S})$.

(iii) For each $i \in N_0 \setminus S$, there exists at least one $\bar{S} \in \mathcal{I}(S)$ where $p - 1 \leq |\bar{S}| \leq p$ such that $i \in N_0 \setminus (S \cup N_0(\bar{S}))$.

Proof. First, observe that since S is an independent set, so $T = N_1(S)$. Now consider an equation

$$s + c \sum_{i \in N_1 \setminus T} x_i = c \sum_{i \in S} x_i + c \sum_{i \in T'} x_i + c(1 - p), \tag{4.38}$$

and let $K = P \cap \{(s, x) \mid (s, x) \text{ satisfies (4.38)}\}$. Now assume inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is valid for X and satisfies the condition that

$$\gamma s + \sum_{i \in N} \beta_i x_i = \gamma_0, \forall (s, x) \in K. \tag{4.39}$$

We prove that equality (4.39) is a multiple of (4.38). We introduce the following points belonging to K.

(1) $s = c; x_i = 1, i \in S; x_i = 0, i \in N \setminus S;$

(2) $\forall j \in N_1 \setminus T, s = 0; x_j = 1; x_i = 1, i \in S; x_i = 0, i \in N \setminus (S \cup \{j\})$;

(3) $\forall \bar{S} \in \mathcal{I}_{p-1}(S), s = 0; x_i = 1, i \in \bar{S}; x_i = 0, i \in N \setminus \bar{S}$;

(4) $\forall \bar{S} \in \mathcal{I}_{p-1}(S), \forall j \in T \setminus T', s = 0; x_i = 1, i \in \bar{S}; x_j = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \{j\})$;

(5) $\forall \bar{S} \in \mathcal{I}_{p-2}(S), \forall j \in T', s = 0; x_i = 1, i \in \bar{S}; x_j = 1; x_i = 0, i \in N \setminus (\bar{S} \cup \{j\})$.

Now let $i \in T \setminus T'$. Then subtracting points of type (3) and (4) corresponding to set $\bar{S}$ in equation (4.39) and subtracting them imply $\beta_i = 0, i \in T \setminus T'$.

Take $i \in N_0 \setminus S$. Then it can be concluded from condition (iii) that points of type (1) or (3) in addition with $x_i = 1$ belong to K. Substituting this new point with points (1) or (3) in equation (4.39) and subtracting the resultant equations give $\beta_i = 0, i \in N_0 \setminus S$.

Substituting points (1) and (2) in equation (4.39) and subtracting the resultant equations imply $\beta_i = \gamma c, i \in N_1 \setminus T$. In addition, replacing points (1) and (3) in equation (4.39) and subtracting them give $\beta_i = -\gamma c, i \in S$.

Let $i, j \in T'$. As a consequence of condition (ii), there exist $\bar{S}_1, \bar{S}_2 \in \mathcal{I}_{p-2}(S)$ such that $i \in T' \setminus N_1(\bar{S}_1)$ and $j \in T' \setminus N_1(\bar{S}_2)$. Replacing solution (5) corresponding to subsets $\bar{S}_1$ and $\bar{S}_2$ in equation (4.39) imply $\beta_i = \beta_j, i, j \in T'$ and so $\beta_i = \beta, i \in T'$. Next, substituting points (1) in equation (4.39) give $\gamma_0 = \gamma c(1-p)$ and finally it can be obtained by replacing points (5) in equation (4.39) that $\beta = -\gamma c$. □

Observe that as we discussed in Section 4.2, inequality (4.37) under the foregoing conditions defines a facet of P if $c > d$.

4.4 Separation

In this section we study the separation problems associated with some families of valid inequalities presented in Section 4.3. Consider a point $(s, x) \in \mathbb{R}_+ \times [0, 1]^n$. Then for each family, $\mathcal{V}$, of valid inequalities the separation problem is: to find an inequality in $\mathcal{V}$ that is violated by the point (s, x) or show that there is no such inequality. All the separation problems discussed here are NP-hard since they include as subproblem the computation of the independence number of a graph.

For brevity, we discuss the separation problems only for inequalities (4.14) and (4.28).

First we consider inequalities (4.14). These inequalities can be written as follows.

$$\begin{aligned} & \sum_{i \in S} (d - p_i c)^+ x_i \leq s + c \sum_{i \in T} x_i \\ \iff \quad & \sum_{i \in S} (d - p_i c)^+ x_i + c \sum_{i \in N_1 \setminus T} x_i \leq s + c \sum_{i \in N_1} x_i. \end{aligned}$$

Hence, for a given solution (s^*, x^*), inequality (4.14) is violated if and only if the maximum of the LHS,

$$\max_{S\subseteq N, T\subseteq N_1\setminus S} \Big\{ \sum_{i\in S}(d-p_ic)^+x_i^* + c\sum_{i\in N_1\setminus T} x_i^* |\ S \text{ is a clique}\Big\}, \tag{4.40}$$

is greater than the constant $s^* + c\sum_{i\in N_1} x_i^*$. Recall that $p_i = \alpha(G[N_1 \setminus (N_1(i)\cup T)])$ and, therefore, it depends on the choice of set T.

In order to solve this separation problem to optimality, consider the binary variables $y_i, i \in N_1$ such that y_i is 1 if $i \in N_1 \setminus T$, and 0 otherwise, and consider the binary variables $z_i, i \in N$ indicating whether $i \in S$ or not. For each $i \in N$ we also define nonnegative integer variables γ_i which are 0 if $z_i = 0$ and are lower bounded by p_i if $z_i = 1$. The maximization problem (4.40) can be solved by solving the following MIP problem.

$$\max \sum_{i\in N_1} cx_i^*y_i + \sum_{i\in N} dx_i^*z_i - \sum_{i\in N} cx_i^*\gamma_i \tag{4.41}$$

$$z \text{ defines a clique in } N, \tag{4.42}$$

$$\gamma_i \geq \sum_{j\in I} y_jz_i, i\in N, I\in \mathcal{I}(N_1\setminus N_1(i)), \tag{4.43}$$

$$z_i \leq y_i, i\in N_1, \tag{4.44}$$

$$y_i \in \{0,1\},\ i\in N_1, \tag{4.45}$$

$$z_i \in \{0,1\},\ i\in N, \tag{4.46}$$

$$\gamma_i \in \mathbb{Z}_0^+,\ i\in N. \tag{4.47}$$

Constraints (4.42) can be modeled in many different ways. For a discussion and comparison of formulations for clique problems see [21]. Following [39], we define the variables $z_{ij}, (i,j) \in E$ indicating whether both nodes i and j belong to the clique. Then constraints (4.42) can be modeled as follows:

$$\begin{aligned}
& z_{ij} \leq z_i, z_{ij} \leq z_j,\ (i,j)\in E,\\
& z_i + z_j \leq 1 + z_{ij},\ (i,j)\in E,\\
& z_i + z_j \leq 1,\ (i,j)\notin E,\\
& z_{ij} \in \{0,1\},\ (i,j)\in E,\\
& z_i \in \{0,1\},\ i\in N.
\end{aligned}$$

Constraints (4.43) ensure that γ_i must be greater than the cardinality of each independent set defined by variables y, hence it must be greater than the maximum cardinality set. Clearly, in any optimal solution to (4.41)–(4.47), constraint (4.43) will be satisfied as equation, that is, $\gamma_i = p_i$. Since (4.43) are nonlinear, they can be linearized by introducing new binary variables $w_{ij} = y_jz_i$. For each $i \in N$, constraints (4.43) can be replaced by the following set of constraints.

$$\gamma_i \geq \sum_{j \in I} w_{ij}, I \in \mathcal{I}(N_1 \setminus N_1(i)), \tag{4.48}$$
$$w_{ij} \leq z_i,\ j \in N_1, \tag{4.49}$$
$$w_{ij} \leq y_j,\ j \in N_1, \tag{4.50}$$
$$w_{ij} \geq z_i + y_j - 1,\ j \in N_1, \tag{4.51}$$
$$w_{ij} \in \{0, 1\},\ j \in N_1. \tag{4.52}$$

Finally, constraints (4.44) impose that each element in S that also belongs to N_1 must be in $N_1 \setminus T$, that implies S and T are disjoint.

As the set of inequalities (4.48) is large (increases exponentially with the number of nodes of G) then for each $i \in N$, these inequalities can be added dynamically by determining the maximum independent set on the graph $G[N_1(W_i)]$, where $N_1(W_i) = \{j \in N_1 \setminus N_1(i)|\ w_{ij} = 1\}$.

Next we discuss on the separation of inequality (4.28). This inequality can be written as follows.

$$\begin{aligned}
& r\left\lceil \frac{d}{c} \right\rceil + (c-r)\sum_{i \in S} x_i \leq s + c\sum_{i \in N_1 \setminus T} x_i + r\sum_{i \in T} x_i \\
\iff\ & (c-r)\sum_{i \in S} x_i - c\sum_{i \in N_1 \setminus T} x_i - r\sum_{i \in T} x_i \leq s - r\left\lceil \frac{d}{c} \right\rceil \\
\iff\ & (c-r)\sum_{i \in S} x_i + (c-r)\sum_{i \in T} x_i \leq s - r\left\lceil \frac{d}{c} \right\rceil + c\sum_{i \in N_1} x_i \\
\iff\ & \sum_{i \in S} x_i + \sum_{i \in T} x_i \leq \frac{s - r\lceil \frac{d}{c} \rceil + c\sum_{i \in N_1} x_i}{c-r}.
\end{aligned} \tag{4.53}$$

Notice that condition (4.27) is equivalent to the following.

$$\bar{\alpha}\big(G[T \cup S]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor. \tag{4.54}$$

To find the most violated inequality we need to maximize the left-hand side of inequality (4.53) by determining S and T that satisfy condition (4.54):

$$\max_{S \subseteq N_0, T \subseteq N_1} \Big\{ \sum_{i \in S} x_i + \sum_{i \in T} x_i \mid \bar{\alpha}\big(G[T \cup S]\big) \leq \left\lfloor \frac{d}{c} \right\rfloor \Big\}.$$

Consider a fractional solution (s^*, x^*) and the graph G where the weight of node $i \in N$ is given by x_i^*. Therefore, the separation problem is equivalent to find the maximum-weight subset of N such that the maximum independence number of the subgraph induced by that subset is less than or equal to $\lfloor \frac{d}{c} \rfloor$, and this independent set must include at least one node from set N_0.

A possible approach to solve exactly this separation problem is to formulate it as a binary problem. Let us define the binary variables $z_i, i \in N$, that indicate, for $i \in N_1$, whether $i \in T$, and for $i \in N_0$, whether $i \in S$. Let $\mathcal{C}$ be the family of all subsets in N whose independence number is greater than $\lfloor \frac{d}{c} \rfloor$, that is $\mathcal{C} = \{C \subseteq N | \ \alpha(G[C]) > \lfloor \frac{d}{c} \rfloor\}$. Then the separation problem can be solved by solving the following binary problem.

$$\max \sum_{i \in N} x_i^* z_i \tag{4.55}$$

$$\sum_{j \in C} z_j \leq |C| - 1, \forall C \in \mathcal{C}, \tag{4.56}$$

$$\sum_{j \in N_0} z_j \geq 1, \tag{4.57}$$

$$z_i \in \{0, 1\}, \ i \in N. \tag{4.58}$$

Inequalities (4.57) increase exponentially with the size of the graph. Hence, these inequalities should must be included dynamically using a separation routine to find the maximum cardinality independent set.

Similar approach can be followed to separate inequalities (4.18) and (4.26).

4.5 Application to Single Node Fixed-Charge Set with Conflicts on Arcs

Applying the inequalities introduced here to general mixed integer problems raises several questions, namely, find the most efficient inequalities, find efficient separation algorithms, and test different relaxations of those problems since, for some problems as the ones discussed in [3], set X can be obtained through different relaxations. Given all these difficulties, we provide only preliminary computational tests for a set Y that can be seen as an intermediate set between those general mixed integer sets and set X. This set is a variant of the single node fixed-charge set where incompatibility between arcs are considered, and is defined as follows.

$$Y = \Big\{(s, y, x) \in \mathbb{R} \times \mathbb{R}^{|N_1|} \times \mathbb{B}^{|N|} \mid s + \sum_{i \in N_1} y_i \geq d, y_i \leq cx_i, i \in N_1,$$
$$x_i + x_j \leq 1, (i, j) \in E, s \geq 0, y_i \geq 0, i \in N_1\Big\},$$

where $N_1 \subset N$, and $E \subset N \times N$.

Set X is a restriction of Y by setting $y_i = cx_i, \forall i \in N_1$. Obviously, clique inequalities and odd hole inequalities (see [37]) are valid for Y.

Proposition 4.5.1. *Any valid inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ for X is also valid for Y.*

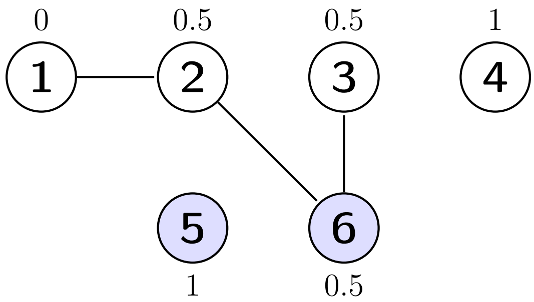

Figure 4.5: Conflict graph corresponding to the fractional solution given in Example 4.5.2.

Proof. Suppose not. That is, there exists $(s^*, y^*, x^*) \in Y$ such that $\gamma s^* + \sum_{i \in N} \beta_i x_i^* < \gamma_0$. Then the inequality is also violated by $(s^*, y', x^*) \in Y$ where $y'_i = cx_i^*$. Thus $(s^*, x^*) \in X$ and inequality $\gamma s + \sum_{i \in N} \beta_i x_i \geq \gamma_0$ is violated by this point, which is a contradiction. □

In the following example we consider set Y and provide a case where inequality (4.28) is violated.

Example 4.5.2. *Consider set Y with $N = \{1, \ldots, 6\}, N_1 = \{1, 2, 3, 4\}, d = 12, c = 5$ and $E = \{(1, 2), (2, 6), (6, 3)\}$. We consider the problem of minimizing an objective function over set Y. Then for a given objective function we get the following fractional solution of the linear relaxation of the problem.*

$$s = 2,\ y_1 = 0, y_2 = 2.5, y_3 = 2.5, y_4 = 5, y_5 = 5, y_6 = 2.5,$$
$$x_1 = 0, x_2 = 0.5, x_3 = 0.5, x_4 = 1, x_5 = 1, x_6 = 0.5.$$

The corresponding conflict graph is presented in figure 4.5 where the weight of node $i \in N$ is given by the value of x_i. In order to separate inequality (4.28), as explained in Section 4.4, we take $S = \{6\}$ and $T = \{2, 3, 4\}$ where $S \cup T$ is the maximum-weight subset of N satisfying condition (4.54). This gives 2.5 for the left-hand side of inequality (4.53), while the right-hand side is equal to 2, and so inequality (4.28) is violated for the proposed sets S and T.

4.5.1 Computational Experiment

In this section we report the result of computational experiments to test the effectiveness of the inclusion of those inequalities derived in Section 4.3 in improving the integrality gap of randomly generated instances of the single node fixed-charge set with conflicts on arcs.

All computations are performed using the optimization software Xpress-Optimizer Version 23.01.03 with Xpress Mosel Version 3.4.0 [46], on a computer with processor Intel Core 2, 2.2 GHz and with 2 GB RAM.

Let $\mathcal{C}_1$ denote the set containing inequalities (4.14), (4.18), (4.26), and (4.28), and $\mathcal{C}_2$ represent the set with the MIR inequality, clique inequalities, odd hole inequalities, inequalities (4.14), (4.18), (4.26), and (4.28). In order to test the impact of the inequalities introduced for X, in terms of integrality gap reduction, we generate different sets of instances considering a minimization problem and compute, for each set, the average initial integrality gap denoted by IG, the average closed gap by MIR, clique and odd hole inequalities denoted by GCMCO, and the average closed gap by inequalities of set $\mathcal{C}_1$ and $\mathcal{C}_2$ denoted by GC$\mathcal{C}_1$ and GC$\mathcal{C}_2$ respectively. Initial gaps are computed as $\frac{OPT-LR}{max\{|OPT|,|LR|\}} \times 100$ where OPT denotes the optimal value and LR indicates the linear relaxation value. Furthermore, closed gaps are calculated as $\frac{ILR-LR}{OPT-LR} \times 100$ where ILR denotes the value of the linear relaxation with MIR, clique and odd hole inequalities for GCMCO, with inequalities belonging to $\mathcal{C}_1$ for GC$\mathcal{C}_1$, and with inequalities belonging to $\mathcal{C}_2$ for GC$\mathcal{C}_2$. Observe that the MIR inequality is included a priori to the problem while clique and odd hole inequalities are introduced as cuts using the separation routines given in [34]. Moreover, for inequalities belonging to $\mathcal{C}_1$ we use the exact separation schemes discussed in Section 4.4.

We consider instances with $|N| = 20$. The test instances were generated randomly on the basis of the following data: $d \in \{55, 80, 95, 110, 130\}$; $c \in \{25, 35, 45\}$; conflict graph $G = (N, E)$ is generated randomly with graph density 50%; N_1 is generated using the uniform distribution on the interval [0,1]; coefficients of s in the objective function are randomly generated in the interval $[3, 5)$; coefficients of $y_i, i \in N_1$, in the objective function are randomly generated in the interval $[0, 1)$. coefficients of x_i are randomly generated in the interval $[0, 20)$ if $i \in N_1$, and in the interval $(-20, 0]$ otherwise.

For each pair (d, c) we generate 5 instances randomly. The computational results are reported in Table 4.1. It can be seen from this table that adding cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ to the linear relaxation of the problem is effective in improving the integrality gap of all generated instances.

4.6 Summary

In this chapter we considered a mixed integer set which results from the intersection of a simple mixed integer set with the vertex packing set. This set arises as a subproblem of more general mixed integer problems. We focused on deriving conflict mixed integer rounding inequalities where the incompatibility between binary variables is considered. We described families of strong valid inequalities that consider the structure of simple mixed integer set and the vertex packing set simultaneously and discussed on separation problems associated to those valid inequalities. A preliminary computational experiment was presented.

Table 4.1: Average integrality gaps and closed gaps on 75 randomly generated instances.

(d,c)	**IG**	**GCMCO**	**GC$\mathcal{C}_1$**	**GC$\mathcal{C}_2$**
(55,25)	105.46	87.64	65.96	95.78
(55,35)	73.04	85.81	37.82	89.03
(55,45)	69.53	83.67	43.07	85.33
(80,25)	142.43	87.75	73.93	99.34
(80,35)	99.79	92.19	69.12	96.57
(80,45)	69.21	73.64	18.92	79.21
(95,25)	138.44	78.86	70.28	90.26
(95,35)	116.20	87.78	57.48	90.56
(95,45)	92.91	80.25	51.96	91.27
(110,25)	103.73	86.26	62.02	98.96
(110,35)	141.95	88.16	87.28	98.79
(110,45)	113.13	90.31	64.03	92.68
(130,25)	93.10	90.90	59.45	94.90
(130,35)	175.50	79.25	59.94	88.55
(130,45)	132.18	89.02	59.33	95.96
Average	**111.10**	**85.43**	**58.69**	**92.48**

Chapter 5

Conclusions and Further Research

In this final section, we point out the main results obtained in this dissertation and we suggest some directions for further research.

In this thesis, three mixed integer sets which arise as a relaxation of complex inventory problems have been studied from a polyhedral point of view and several classes of strong valid inequalities for these sets have been derived in order to include them in the branch-and-cut framework to solve the main problems.

In Chapter 2 we study a new mixed integer set arising from inventory problems combined with supplier selection decisions. This set is of the form

$$X_{binary} = \Big\{(x, z, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{B} \mid \sum_{j \in N} x_j \leq dy, x_j \leq c_j z_j,\ j \in N\Big\},$$

where $N = \{1, \ldots, n\}$. Observe that the set X_{binary} can be obtained from the single node fixed-charge network set by replacing constraint $\sum_{j \in N} x_j \leq d$ by $\sum_{j \in N} x_j \leq dy$. The set-up binary variable y is associated to the node indicating whether the capacity of the node is installed or not. This study is motivated by the fact that the polyhedral structure of the set X_{binary} is richer than the polyhedral structure of the classical single node fixed-charge network set, i.e. in the presence of binary variable y, new facet-defining inequalities appear in the description of the convex hull of X_{binary}.

Chapter 2 contains the following new results. The well-know flow cover inequalities are generalized into the set-up flow cover inequalities and the extended set-up flow cover inequalities due to the presence of variable y. In the second part, the constant capacitated case ($c_j = c, \forall j \in N$) of X_{binary} is considered. In this case, the complete description of the convex hull of X_{binary} is described. Then we use the sequence independent lifting to strengthen the set-up flow cover inequalities. Furthermore, the lifting process is generalized for the cases where inequality $\sum_{j \in N} x_j \leq dy$ is replaced by $\sum_{j \in N} x_j - \sum_{j \in N^-} x_j \leq dy$ or by $\sum_{j \in N} x_j \leq dy + s$ with $s \geq 0$. Then a valid superadditive lifting function is provided for the latter case. Preliminary computational results have shown that the effectiveness of the set-up flow cover inequalities and the lifted inequalities in the reduction of the integrality gap of those randomly generated instances is considerable.

Future research directions corresponding to set X_{binary} include the study of fast separation heuristics for the set-up flow cover inequalities. The main goal of this line of research is to apply the new inequalities to more general mixed integer problems such as lot-sizing, inventory routing and network design problems.

In Chapter 3 we generate a new mixed integer set from the set X_{binary}, considered in Chapter 2, by imposing variable y to take integer and bounded values and adding the new constraints $z_j \leq y, j \in N$ to this set. The resultant mixed integer set can be represented as

$$X_{integer} = \Big\{(x, z, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{Z}_+ \mid \sum_{j \in N} x_j \leq dy,\ x_j \leq c_j z_j,$$
$$z_j \leq y, j \in N, y \in \{0, \dots, U\}\Big\},$$

where U is integer and $U \leq \left\lceil \frac{\sum_{j \in N} c_j}{d} \right\rceil$. This set can be regarded as a relaxation of lot-sizing and network design problems.

In Chapter 3 we have derived a class of valid inequalities which generalizes the well-known flow cover inequalities and the arc residual inequalities.

Next, we have studied the constant capacitated case. Using the concept of union of polyhedra, an extended compact formulation is derived for the convex hull of $X_{integer}$. Moreover, families of strong valid inequalities are generated. Next, we have applied the simultaneous lifting approach to strengthen a class of the derived inequalities and to provide some insight on the difficulty of providing the complete description of the convex hull of $X_{integer}$ in the original space of variables. We have reported a computational experiment to test the impact of the inclusion of those inequalities in solving instances of the lot-sizing with supplier selection problem. This experiment shows that adding these new inequalities to the formulation a priori is efficient in improving the integrality gap for those randomly generated instances.

As a future line of research it would be interesting to investigate separation heuristics for inequalities derived for the general case. Another line of research is to investigate the polyhedral structure of the convex hull of $X_{integer}$ in the case where constraints $z_j \leq y, j \in N$ are excluded from the definition of the set. This research direction is motivated by our preliminary investigation which shows that many new facet-defining inequalities appear for this case.

In Chapter 4 a new mixed integer set X is generated by taking the intersection of two well-known sets which are a simple mixed integer set X_{SMI} and the vertex packing set X_{VP}. This set arises as a substructure of general mixed integer problems, and more particularly inventory routing problems. Observe that valid inequalities for X_{SMI} and X_{VP} are valid for X as well. Thus, we have generated new valid inequalities for X that take into account the properties of the two sets X_{SMI} and X_{VP} simultaneously.

In this chapter we investigate the polyhedral structure of mixed integer set X. We have proved that the defining inequality $s \geq 0$ and the MIR inequality $s + r \sum_{i \in N_1} x_i \geq r\lceil \frac{d}{c} \rceil$, where conflicts between binary variables $x_j, j \in N$ are considered, are facet-defining under certain conditions. Furthermore, we have extended the MIR inequalities for X to the conflict MIR inequalities which define facets of the convex hull. Other families of strong valid inequalities are derived in this chapter. The impact of the proposed inequalities in improving the integrality gap on a set of randomly generated instances of the single node fixed-charge with conflicts on arcs is reported.

A research direction that is of interest to be followed in the future is to study the set of points $(s, x) \in \mathbb{R}^r \times \mathbb{Z}^n$ satisfying

$$s_k + c \sum_{i \in N_k} x_i \geq d_k, \;\; k \in R, \tag{5.1}$$

$$x_i + x_j \leq 1, \;\; (i, j) \in E, \tag{5.2}$$

$$x_i \in \{0, 1\}, \;\; i \in N, \tag{5.3}$$

$$s_k \geq 0, k \in R, \tag{5.4}$$

where $R = \{1, \ldots, r\}$, is the index set of continuous knapsack constraints, $N = \{1, \ldots, n\}$ is the index set of binary variables, $E \subset N \times N$ is a set of index pairs, and $N_k \subseteq N, k \in R$. Note that the set X is obtained from the foregoing constraints by setting $|R| = 1$. Investigating the set defined by constraints (5.1)–(5.4) is motivated by studying inventory routing problems where the inventory management at different nodes is combined with the routing decisions.

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