Chapter 12

General Stability in Viscoelasticity

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Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/64217

Abstract

In this chapter, we consider a problem which describes the motion of a viscoelastic body and investigate the effect of the dissipation induced by the viscoelastic (integral) term on the solution. Precisely, we show that, under reasonable conditions on the relaxation function, the system stabilizes to a stationary state. We also obtain a general decay estimate from which the usual exponential and polynomial decay rates are only special cases.

Keywords: general decay, memory, relaxation function, stability, viscoelasticity

1. Introduction

Elastic materials, when subjected to a suddenly applied loading state held constant thereafter, respond instantaneously with a state of deformation which remains constant. On the other hand, Newtonian viscous fluids respond to a suddenly applied state of uniform shear stress by a steady flow process. However, there exist materials for which any suddenly applied and maintained state of uniform stress produces an instantaneous deformation followed by a flow process which might or might not be limited in magnitude as time grows. Such materials exhibit both instantaneous elasticity effects and creep characteristics. Obviously, such a behavior cannot be described by either an elasticity theory or a viscosity theory only but it combines features of each. The most interesting examples of such materials are polymers, which can display all the intermediate range of properties (glassy, brittle solid or an elastic rubber or a viscous liquid) depending on temperature and the experimentally chosen time scale. Such materials are said to possess memories.

Many scientists, such as Maxwell, Kelvin, Voigt, and Boltzmann, have contributed in modeling these phenomena. Boltzmann, in 1874, supplied the first formulation of a three-dimensional theory of isotropic viscoelasticity. He elaborated the model of a "linear" viscoelastic solid on a basic assumption which states that at any (fixed) point x of the body, the stress at any time t depends on the strain at all the proceeding times. In addition, if the strain at all preceding times is in the same direction, then the effect is to decrease the corresponding stress. The influence of a previous strain on the stress depends on the time elapsed since that strain occurred and is weaker than those strains that occurred long ago. Such properties make the model of solid, elaborated by Boltzmann, a material with (fading) memory. These memory effects are expressed by the dependence on the deformation gradient. Therefore, for these "viscoelastic" materials, the stress at each point and at each instant does not depend only on the present value of the deformation gradient but on the entire temporal prehistory of the motion. In addition, Boltzmann made the assumption that a superposition of the influence of previous strains holds, which means that the stress-strain relation is linear. Mathematically, this is interpreted by the time convolution of a "relaxation" function with the Laplacian of the solution. As a consequence, a subtle damping effect is produced. The types of equations we intend to discuss in this chapter are of the form:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\,ds = f(x,t), \quad x \in \Omega, t > 0,$$

where Ω is a bounded domain with regular boundary, g is a nonincreasing positive function, referred to as the relaxation function which describes the viscoelastic material in consideration, f is an external force, and $u(x, t)$ is the position of a point x "in the reference configuration" at a time t.

In early 1970s, Dafermos [1, 2] discussed a one-dimensional viscoelastic model, where he proved, for smooth monotonically decreasing relaxation functions, various existence and asymptotic stability results. However, no rate of decay has been given. After that, viscoelastic problems have attracted the attention of many researchers and many results of existence and long-time behavior have been established. To the best of our knowledge, the first work that studied the uniform decay of solutions was presented by Dassios and Zafirapoulos [3]. In their work, Dassios and Zafirapoulos presented a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay for exponentially decaying kernels. In 1994, Muñoz Rivera [4] considered, in $\mathbb{R}^n$ and in bounded domains, equations for linear isotropic homogeneous viscoelastic solids, with exponentially decaying memory kernels and showed that, in the absence of body forces, solutions decay exponentially for the bounded-domain case, whereas, for the whole space case, the decay is of a polynomial rate. After that, Cabanillas and Muñoz Rivera [5] studied problems, where the kernels are of algebraic (but not exponential) decay rates and showed that the decay of solutions is algebraic at a rate which can be determined by the rate of the decay of the relaxation function and the regularity of solutions. This result was later improved by Barreto et al. [6], where equations related to linear viscoelastic plates were treated. For viscoelastic systems with localized frictional dampings, Cavalcanti et al. [7] considered the following problem:

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$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{m-2}u, \quad in\ \Omega \times (0,\infty), \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0 \\ \qquad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 1$) with a smooth boundary ∂Ω, g is a positive nonincreasing function satisfying, for two positive constants, the conditions:

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

and $a(x) \geq a_0 > 0$ in a subdomain $\omega \subset \Omega$, with $meas(\omega) > 0$ and satisfying some geometry restrictions. They established an exponential rate of decay. Berrimi and Messaoudi [8] improved Cavalcanti's result by weakening the conditions on both a and g. In particular, the function a can vanish on the whole domain Ω and consequently the geometry condition is no longer needed. This result has been later extended to a situation, where a source is competing with the viscoelastic dissipation, by Berrimi and Messaoudi [9]. Also, Cavalcanti et al. [10] have studied a quasilinear equation, in a bounded domain, of the form:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau - \gamma\Delta u_t = 0,$$

with $\rho > 0$, and a global existence result for $\gamma \geq 0$, as well as an exponential decay for $\gamma > 0$, have been established. Messaoudi and Tatar [11,12] discussed the situation when $\gamma = 0$ and established polynomial and exponential decay results in the presence, as well as in the absence, of a nonlinear source term. Fabrizio and Polidoro [13] studied a homogeneous viscoelastic equation in the presence of a linear frictional damping (au_t, $a > 0$) and showed that the exponential decay of the relaxation function g is a necessary condition for the exponential decay of the solution energy of the solution. In other words, the presence of the memory term, with a non-exponentially decaying relaxation function, may prevent the exponential decay even if the frictional damping is linear. He also obtained a similar result for the polynomial decay case.

For more general decaying kernels, Messaoudi [14,15] considered

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{m-2}u, \quad in\ \Omega \times (0,\infty), \\ u(x,t) = 0, \qquad x \in \partial\Omega, \qquad t \geq 0 \\ \qquad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega \end{cases} \tag{1.1}$$

with $b = 0$ and $b = 1$ and for relaxation functions satisfying

$$g'(t) \le -\xi(t) g(t), \ \forall t \ge 0, \tag{1.2}$$

where $\xi: \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function. He showed that the rate of the decay of the energy is exactly the rate of decay of g, which is not necessarily of exponential or polynomial decay type. After that, a series of papers using Eq. (1.2) have appeared. See for instance, Han and Wang [16], Liu [17,18], Park and Park [19], and Xiaosen and Mingxing [20].

In this work, we intend to study the following problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a|u_t|^{m-2}u_t = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \ge 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{1.3}$$

where Ω is a bounded and regular domain of $\mathbb{R}^n$, $a > 0$ is a constant, and g is a positive nonincreasing function satisfying Eq. (1.2). We will establish some general decay results depending on the behavior of g and m.

2. Preliminary

In this section, we present some material needed in the proof of our result and state a global existence result which can be proved using the well-known Galerkin method. See, for example, [2,3]. In order to prove our main result, we make the following assumptions:

(A_1) $g: \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded differentiable function such that

$$g(0) > 0, \ g'(t) \le -\gamma(t) g(t), \ 1 - \int_0^\infty g(s)\mathrm{d}s = l > 0,$$

where $\gamma(t)$ is a differentiable function satisfying

$$\gamma(t) > 0, \ \gamma'(t) \le 0, \text{ and } \int_0^{+\infty} \gamma(t)\mathrm{d}t = +\infty.$$

(A_2) Concerning the nonlinearity in the damping, we assume that

http://dx.doi.org/10.5772/64217

$$1 < m \le \frac{2n}{n-2}, \text{ if } n > 2 \quad \text{and} \quad m > 1, \text{ if } n = 1,2$$

Remark 2.1. Examples of functions satisfying (A_1) are

$$g_1(t) = \frac{a}{(1+t)^v}, \ v > 1, \ g_2(t) = ae^{-b(t+1)p}, \ 0 < p \le 1$$

$$g_3(t) = \frac{a}{(1+t)\left[\ln(1+t)\right]^v}, \ v > 1,$$

for a and b constants to be chosen properly.

Proposition 2.1. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that (A_1), (A_2) hold. Then problem (1.3) has a unique global solution:

$$u \in C\left([0,\infty); \ H_0^1(\Omega)\right)$$

$$u_t \in C\left([0,\infty); \ L^2(\Omega)\right) \cap L^m\left(\Omega \times (0, \infty)\right). \tag{2.1}$$

Proposition 2.2. [21] Let E: $\mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and φ: $\mathbb{R}_+ \to \mathbb{R}_+$ be an increasing C^2-function such that

$$\varphi(0) = 0 \ \text{ and } \ \varphi(t) \to +\infty \ \text{ as } \ t \to +\infty.$$

Assume that there exist $q \ge 0$ and $A > 0$ such that

$$\int_S^{+\infty} E^{q+1}(t)\varphi'(t)\mathrm{d}t \le AE(S), \ 0 \le S < +\infty,$$

then we have

$$E(t) \le cE(0)\left(1+\varphi(t)\right)^{\frac{-1}{q}}, \ \forall t \ge 0, \ \text{if } q > 0,$$

$$E(t) \le cE(0)e^{-\omega\varphi(t)},\ \forall t \ge 0, \text{ if } q = 0,$$

where c and ω are positive constants independent of the initial energy $E(0)$.

Next, we introduce the "modified energy":

$$\mathcal{E}(t) : \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(go\nabla u)(t), \tag{2.2}$$

where

$$(gov)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 \,\mathrm{d}\tau.$$

Remark 2.2. By multiplying Eq. (1.3) by u_t and integrating over Ω, using integration by parts and hypotheses (A_1), (A_2), we get, after some manipulations, as in [3,20],

$$\begin{aligned} \mathcal{E}'(t) &\le -\left(a\int_\Omega |u_t|^m \,\mathrm{d}x - \frac{1}{2}(g'o\nabla u)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2\right) \\ &\le -a\int_\Omega |u_t|^m \,\mathrm{d}x + \frac{1}{2}(g'o\nabla u)(t) \le 0. \end{aligned} \tag{2.3}$$

3. Decay of solutions

In order to state and prove our main result, we set

$$F(t) := \varepsilon(t) + \varepsilon_1\Psi(t) + \varepsilon_2\mathcal{X}(t) \tag{3.1}$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\Psi(t) := \int_\Omega u u_t \mathrm{d}x, \quad \chi(t) := -\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau))\mathrm{d}\tau\,\mathrm{d}x.$$

Lemma 3.1. For ε_1 and ε_2 so small, we have

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t) \tag{3.2}$$

holds for two positive constants α_1 and α_2.

Proof. It is straightforward to see that

$$\begin{aligned} F(t) &\le E(t) + \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \frac{\varepsilon_1}{2}\int_\Omega |u|^2\,\mathrm{d}x \\ &+ \frac{\varepsilon_2}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \frac{\varepsilon_2}{2}\int_\Omega\left(\int_0^t g(t-\tau)\big(u(t)-u(\tau)\big)\mathrm{d}\tau\right)^2\mathrm{d}x \\ &\le E(t) + \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x + \frac{\varepsilon_1}{2}C_p\int_\Omega |\nabla u|^2\,\mathrm{d}x \\ &+ \frac{\varepsilon_2}{2}\int_\Omega |u_2|^2\,\mathrm{d}x + \frac{\varepsilon_2}{2}C_p(1-l)(go\nabla u)(t) \le \alpha_2\mathcal{E}(t), \end{aligned}$$

where C_p is the Poincaré constant. In the other hand,

$$\begin{aligned} &F(t) \ge E(t) - \frac{\varepsilon_1}{2}\int_\Omega |u_t|^2\,\mathrm{d}x - \frac{\varepsilon_1}{2}\int_\Omega |u|^2\,\mathrm{d}x - \frac{\varepsilon_2}{2}\int_\Omega |u_t|^2\,\mathrm{d}x - \frac{\varepsilon_2}{2}C_p \\ &(1-l)(go\nabla u)(t) \ge \frac{1}{2}l\|\nabla u(t)\|_2^2 + \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(go\nabla u)(t) + \frac{1}{\gamma+2}\|u\|_{\gamma+2}^{\gamma+2} \\ &-\left[\frac{\varepsilon_1+\varepsilon_2}{2}\right]\int_\Omega |u_t|^2\,\mathrm{d}x - \frac{\varepsilon_1}{2}C_p\int_\Omega |\nabla u|^2\,\mathrm{d}x - \frac{\varepsilon_2}{2}C_p(1-l)(go\nabla u)(t) \\ &\ge \alpha_1\mathcal{E}(t), \end{aligned}$$

for ε_1 and ε_2 small enough. Thus, Eq. (3.2) is established.

Lemma 3.2. Assume that $m \ge 2$ and assumptions (A_1), (A_2) hold. Then, the functional $\Psi(t)$ satisfies, along the solution of Eq. (1.3), the estimate:

$$\Psi'(t) \le \int_\Omega u_t^2\mathrm{d}x - \frac{1}{4}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1-l}{2l}(go\nabla u)(t) + C\int_\Omega |u_t|^m\,\mathrm{d}x, \tag{3.3}$$

where C is a “generic” positive constant independent of t.

Proof. By using Eq. (1.3), we easily see that

$$\Psi'(t) = \int_\Omega u_t^2 \mathrm{d}x - \int_\Omega |\nabla u|^2 \mathrm{d}x - a\int_\Omega |u_t|^{m-2} u_t\ u\ \mathrm{d}x + \int_\Omega \nabla u(t) \bullet \int_0^t g(t-\tau)\nabla u(\tau)\mathrm{d}\tau\,\mathrm{d}x. \tag{3.4}$$

We now estimate the third term of the right-hand side of Eq. (3.4), using Young's inequality and (A_2). Thus, we get

$$\begin{aligned}&\int_\Omega |u_t|^{m-2} u_t\ u\ \mathrm{d}x \le \delta \int_\Omega |u|^m \mathrm{d}x + c_\delta \int_\Omega |u_t|^m \mathrm{d}x \le \delta C \|\nabla u\|_2^{m-2} \|\nabla u\|_2^2 + \\ &c_\delta \int_\Omega |u_t|^m \mathrm{d}x \le \delta C \mathcal{E}^{\frac{m-2}{2}}(0)\|\nabla u\|_2^2 + c_\delta \int_\Omega |u_t|^m \mathrm{d}x\end{aligned} \tag{3.5}$$

where c_δ is a constant depending on δ. For the fourth term of the right-hand side of Eq. (3.4), we get

$$\begin{aligned}&\int_\Omega \nabla u(t) \bullet \int_0^t g(t-\tau)\nabla u(\tau)\mathrm{d}\tau \mathrm{d}x \le \frac{1}{2}\int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \\ &\frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)|\nabla u(\tau)|\mathrm{d}\tau\right)^2 \mathrm{d}x \le \frac{1}{2}\int_\Omega |\nabla u(t)|^2 \mathrm{d}x \\ &+\frac{1}{2}\int_\Omega \left(\int_0^t g(t-\tau)\left(|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|\right)\mathrm{d}\tau\right)^2 \mathrm{d}x.\end{aligned} \tag{3.6}$$

We then use Cauchy-Schwarz inequality, Young's inequality, and the fact that

$$\int_0^t g(\tau)\mathrm{d}\tau \le \int_0^{+\infty} g(\tau)\mathrm{d}\tau = 1 - l,$$

to obtain, for any $\eta > 0$,

$$\begin{aligned}
&\int_\Omega\left(\int_0^t g(t-\tau)\left(|\nabla u(\tau)-\nabla u(t)|+|\nabla u(t)|\right)\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\le \int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|\mathrm{d}\tau\right)^2 \mathrm{d}x+\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)|\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&+2\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|\mathrm{d}\tau\right) \mathrm{d}x\left(\int_0^t g(t-\tau)|\nabla u(t)|\mathrm{d}\tau\right)\mathrm{d}x \\
&\le (1+\eta)\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)|\mathrm{d}\tau\right)^2 dx+ \\
&\left(1+\frac{1}{\eta}\right)\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\le \left(1+\frac{1}{\eta}\right)\int_\Omega\int_0^t g(t-\tau)\mathrm{d}\tau\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\mathrm{d}\tau\mathrm{d}x+ \\
&(1+\eta)\int_\Omega|\nabla u(t)|^2\left(\int_0^t g(t-\tau)\mathrm{d}\tau\right)^2 \mathrm{d}x\le(1+\eta)(1-l)^2\int_\Omega|\nabla u(t)|^2\mathrm{d}x \\
&+\left(1+\frac{1}{\eta}\right)(1-l)\int_\Omega\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\mathrm{d}\tau\mathrm{d}x
\end{aligned} \tag{3.7}$$

By combining Eqs. (3.4)–(3.7), we arrive at

$$\begin{aligned}
\Psi'(t)&\le\int_\Omega u_t^2\mathrm{d}x-\frac{1}{2}\int_\Omega|\nabla u(t)|^2\mathrm{d}x+\delta C\mathcal{E}^{\frac{m-2}{2}}(0)\|\nabla u\|_2^2 \\
&+\frac{1}{2}(1+\eta)(1-l)^2\int_\Omega|\nabla u(t)|^2\mathrm{d}x+c_\delta\int_\Omega|u_t|^m\mathrm{d}x+\frac{1}{2}(1+\eta)(1-l)\int_\Omega\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)|^2\mathrm{d}\tau\mathrm{d}x \\
&\le\int_\Omega u_t^2\mathrm{d}x-\frac{1}{2}\left[1-(1+\eta)(1-l)^2-2\delta C\mathcal{E}^{\frac{m-2}{2}}(0)\right]\int_\Omega|\nabla u(t)|^2\mathrm{d}x \\
&+\frac{1}{2}(1+\eta)(1-l)(go\nabla u)(t)+c_\delta\int_\Omega|u_t|^m\mathrm{d}x.
\end{aligned}$$

By choosing $\eta = l/(1-l)$ and $\delta = l/4C\mathcal{E}^{\frac{(m-2)}{2}}(0)$, estimate (3.3) is established.

Lemma 3.3. Assume that $m \ge 2$ and assumptions (A_1), (A_3) hold. Then, the functional $\mathcal{X}(t)$ satisfies, along the solution of Eq. (1.3) and for any $\delta > 0$, the estimate

$$\mathcal{X}'(t) \le \delta\left[1+2(1-l)^2\right]\|\nabla u\|_2^2 + c_\delta(go\nabla u)(t)$$
$$+c_\delta\int_\Omega |u_t|^m\,\mathrm{d}x + \left(\delta - \int_0^t g(s)\mathrm{d}s\right)\int_\Omega u_t^2\mathrm{d}x + \frac{g(0)}{4\delta}C_p\left(-(g'o\nabla u)(t)\right). \tag{3.8}$$

Proof: By using Eq. (1.3), we easily see that

$$\begin{aligned}
\mathcal{X}'(t) &= \int_\Omega \nabla u(t)\cdot\left(\int_0^t g(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)\mathrm{d}\tau\right)\mathrm{dx}\\
&-\int_\Omega\left(\int_0^t g(t-\tau)\nabla u(\tau)\mathrm{d}\tau\right)\cdot\left(\int_0^t g(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)\mathrm{d}\tau\right)\mathrm{dx}\\
&+a\int_\Omega |u_t|^{m-2}u_t(t)\int_0^t g(t-\tau)\left(u(t)-u(\tau)\right)\mathrm{dzdx}\\
&-\int_\Omega u_t\int_0^t g'(t-\tau)\left(u(t)-u(\tau)\right)\mathrm{d}\tau\mathrm{dx} - \left(\int_0^t g(s)\mathrm{d}s\right)\int_\Omega u_t^2\mathrm{dx}
\end{aligned} \tag{3.9}$$

Similarly to Eq. (3.4), we estimate the right-hand side terms of Eq. (3.9). So for any $\delta > 0$, we have

$$\begin{aligned}
&-\int_\Omega \nabla u(t)\cdot\left(\int_0^t g(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)\mathrm{d}\tau\right)\mathrm{d}x\\
&\le \delta\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1-l}{4\delta}(g\circ\nabla u)(t),
\end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&\int_\Omega \left(\int_0^t g(t-s)\nabla u(s)\,\mathrm{d}s \right) \cdot \left(\int_0^t g(t-s)\left(\nabla u(t)-\nabla u(s)\right)\mathrm{d}s \right)\mathrm{d}x \\
&\le \delta \int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(s)\right|\mathrm{d}s \right)^2 \mathrm{d}x \\
&+\frac{1}{4\delta}\int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|\mathrm{d}s \right)^2 \mathrm{d}x \\
&\le \delta \int_\Omega \left(\int_0^t g(t-s)\left(\left|\nabla u(t)-\nabla u(s)\right|+\left|\nabla u(t)\right|\right)\mathrm{d}s \right)^2 \mathrm{d}x \\
&+\frac{1}{4\delta}\left(\int_0^t g(t-s)\,\mathrm{d}s \right)\int_\Omega\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 \mathrm{d}s\,\mathrm{d}x \\
&\le 2\delta \int_\Omega \left(\int_0^t g(t-s)\left|\nabla u(t)-\nabla u(s)\right|\mathrm{d}s \right)^2 \mathrm{d}x \\
&+2\delta(1-l)^2 \int_\Omega \left|\nabla u\right|^2 \mathrm{d}x + \frac{1}{4\delta}(1-l)(g\circ\nabla u)(t) \\
&\le \left(2\delta + \frac{1}{4\delta} \right)(1-l)(g\circ\nabla u)(t) + 2\delta(1-l)^2 \int_\Omega \left|\nabla u\right|^2 \mathrm{d}x,
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\int_\Omega \left|u_t\right|^{m-2} u_t(t)\int_0^t g(t-\tau)\left(u(t)-u(\tau)\right)\mathrm{d}\tau\,\mathrm{d}x \\
&\le \delta \int_\Omega \left| \int_0^t g(t-\tau)\left(u(t)-u(\tau)\right)\mathrm{d}\tau \right|^m \mathrm{d}x + c_\delta \int_\Omega \left|u_t\right|^m \mathrm{d}x \\
&\le \delta \left(\int_0^t g(\tau)\,\mathrm{d}\tau \right)^{m-1} \int_\Omega\int_0^t g(t-\tau)\left|u(t)-u(\tau)\right|^m \mathrm{d}\tau\,\mathrm{d}x + \\
&c_\delta \int_\Omega \left|u_t\right|^m \mathrm{d}x \le \delta(1-l)^{m-1} C\int_0^t g(t-\tau)\left\|\nabla u(t)-\nabla u(\tau)\right\|_2^m \mathrm{d}\tau \\
&+c_\delta \int_\Omega \left|u_t\right|^m \mathrm{d}x \le \delta(1-l)^{m-1} C_p \left(\frac{2\mathcal{E}(0)}{l} \right)^{\frac{m-2}{2}} \int_0^t g(t-\tau)\left\|\nabla u(t)-\nabla u(\tau)\right\|_2^2 \mathrm{d}\tau \\
&+c_\delta \int_\Omega \left|u_t\right|^m \mathrm{d}x \le \delta(1-l)^{m-1} C_p \left(\frac{2\mathcal{E}(0)}{l} \right)^{\frac{m-2}{2}} (g\circ\nabla u)(t) + c_\delta \int_\Omega \left|u_t\right|^m \mathrm{d}x,
\end{aligned}
\tag{3.12}
$$

$$-\int_{\Omega} u_t \int_0^t g'(t-\tau)\big(u(t)-u(\tau)\big)\mathrm{d}\tau\,\mathrm{d}x \le \delta \int_{\Omega} |u_t|^2\,\mathrm{d}x \\ + \frac{g(0)}{4\delta} C_p \int_{\Omega}\int_0^t - g'(t-s)\left|\nabla u(t)-\nabla u(s)\right|^2 \mathrm{d}s\mathrm{d}x. \tag{3.13}$$

A combination of Eqs. (3.9)–(3.13), then, yields Eq. (3.8).

Theorem 3.4. Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given. Assume that (A_1), (A_2) hold. Then, for any $t_0 > 0$, there exist positive constants K and λ such that the solution of Eq. (1.3) satisfies

$$\mathcal{E}(t) \le K e^{-\lambda \int_{t_0}^{t} \gamma(s)\mathrm{d}s}, \quad \forall t \ge t_0, \quad \text{if } m \ge 2 \tag{3.14}$$

$$\mathcal{E}(t) \le K\big(1+\gamma(t)\big)^{\frac{-(2m-2)}{2-m}}, \quad \forall t \ge t_0, \quad \textit{if } 1 < m < 2. \tag{3.15}$$

Proof: We start with the case $m \ge 2$. Since $g(0) > 0$, then there exists $t_0 > 0$ such that

$$\int_0^t g(s)\mathrm{d}s \ge \int_0^{t_0} g(s)\mathrm{d}s = g_0 > 0, \quad \forall t \ge t_0.$$

By using Eqs. (2.3), (3.1), (3.3), and (3.8), we obtain

$$F'(t) \le -\big(1-(\varepsilon_1+\varepsilon_2)c_\delta\big)a\int_{\Omega}|u_t|^m\,\mathrm{d}x \\ -\left[\varepsilon_2(g_0-\delta)-\varepsilon_1\right]\int_{\Omega} u_t^2\mathrm{d}x - \left[\frac{\varepsilon_1 l}{4} - \varepsilon_2\delta\left(1+2(1-l)^2\right)\right]\|\nabla u\|_2^2 \\ + \left[\frac{1}{2} - \varepsilon_2 \frac{g(0)}{4\delta} C_p\right](g' o \nabla u)(t) + \\ \left[\varepsilon_1 \frac{1-l}{2l} + \varepsilon_2 c_\delta\right](g o \nabla u)(t).$$

At this point, we choose δ so small that

$$g_0 - \delta > \frac{1}{2} g_0, \quad \frac{4}{l}\delta\left(1+2(1-l)^2\right) < \frac{1}{4} g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2 \tag{3.16}$$

makes

$$k_1 = \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0, \;\; k_2 = \frac{\varepsilon_1 l}{4} - \varepsilon_2\delta\left(1 + 2(1-l)^2\right) > 0.$$

We then pick ε_1 and ε_2 so small that Eqs. (3.2) and (3.16) remain valid and, further,

$$1 - (\varepsilon_1 + \varepsilon_2)c_\delta > 0, \;\; \frac{1}{2} - \varepsilon_2\frac{g(0)}{4\delta}C_p > 0.$$

Therefore, we arrive at

$$F'(t) \le -\beta\varepsilon(t) + c(go\nabla u)(t), \quad \forall t \ge t\ 0, \tag{3.17}$$

for two constants $c, \beta > 0$. We multiply (3.17) by $\gamma(t)$ and use Eqs. (1.2) and (2.3), to get

$$\begin{aligned}&\gamma(t)F'(t) \le -\beta\gamma(t)\mathcal{E}(t) - c(g'o\nabla u)(t)\\&\le -\beta\gamma(t)\mathcal{E}(t) - c\mathcal{E}'(t), \;\; \forall t \ge t_0.\end{aligned}$$

This implies that

$$\gamma(t)F'(t) + c\mathcal{E}'(t) \le -\beta\gamma(t)\mathcal{E}(t), \;\; \forall t \ge t_0.$$

Hence,

$$\left(\gamma(t)F(t) + c\mathcal{E}(t)\right)' - \gamma'(t)F(t) \le -\beta\gamma(t)\mathcal{E}(t), \;\; \forall t \ge t_0.$$

Again, by using the fact that $\gamma'(t) \le 0$, letting

$$\mathcal{L}(t) = \gamma(t)F(t) + c\mathcal{E}(t),$$

and noting that $\mathcal{L}\sim\mathcal{E}$, we arrive at

$$\mathcal{L}'(t) \le -\beta\gamma(t)\mathcal{E}(t) \le -\lambda\gamma(t)\mathcal{L}(t), \ \forall t \ge t_0. \tag{3.18}$$

A simple integration of Eq. (3.18) over (t_0, t) leads to

$$\mathcal{L}(t) \le L(t_0) e^{-\lambda \int_{t_0}^{t} \gamma(s) \mathrm{d}s}, \quad \forall t \ge t_0.$$

We obtain, then, Eq. (3.14) by virtue of equivalence of $\mathcal{E}$ and $\mathcal{L}$.

To establish Eq. (3.15), we re-estimate Eqs. (3.5) and (3.12), for $m < 2$, as follows

$$\begin{aligned} \int_\Omega |u_t|^{m-2} u_t u \mathrm{d}x &\le \delta \int_\Omega |u|^2 \mathrm{d}x + c_\delta \int_\Omega |u_t|^{2m-2} \mathrm{d}x \\ &\le \delta C_p \|\nabla u\|_2^2 + c_\delta C \left(\int_\Omega |u_t|^m \mathrm{d}x \right)^{\frac{2m-2}{m}}. \end{aligned} \tag{3.19}$$

Similarly, we have

$$\begin{aligned} &\int_\Omega |u_t|^{m-2} u_t(t) \int_0^t g(t-\tau)\big(u(t)-u(\tau)\big) \mathrm{d}\tau \mathrm{d}x \\ &\le \delta C_p (go\nabla u)(t) + c_\delta \left(\int_\Omega |u_t|^m \mathrm{d}x \right)^{\frac{2m-2}{m}}. \end{aligned} \tag{3.20}$$

By repeating all above steps and using Eqs. (3.19), (3.20) instead of Eqs. (3.5), (3.12), we arrive at

$$F'(t) \le -\beta\mathcal{E}(t) + c(go\nabla u)(t) + c_1 \left(\int_\Omega |u_t|^m \mathrm{d}x \right)^{\frac{2m-2}{m}}, \quad \forall t \ge t_0;$$

which gives

$$\mathcal{E}(t) \leq -\beta_1 F'(t) + c(go\nabla u)(t) + c_1 \left(\int_\Omega |u_t|^m \, dx \right)^{\frac{2m-2}{m}}, \quad \forall t \geq t_0 \tag{3.21}$$

By multiplying Eq. (3.21) by $\gamma(t)\mathcal{E}^q(t)$, for $q > 0$ to be specified later, and using (A_1), Eq. (2.3), and Young's inequality, we get

$$\begin{aligned} &\gamma(t)\mathcal{E}^{q+1}(t) \leq -\beta_2 \gamma(t) F^q F'(t) + c\mathcal{E}^q(t)\left(-\mathcal{E}'(t)\right) + \\ &c\gamma(t)\mathcal{E}^q(t)\left(-\mathcal{E}'(t)\right)^{\frac{2m-2}{m}} \leq \frac{-\beta_2 \gamma(t)}{q+1} \frac{dF^{q+1}(t)}{dt} - \frac{c}{2} \frac{d\mathcal{E}^{q+1}(t)}{dt} \\ &+ \mu \mathcal{E}^{\frac{qm}{2-m}}(t) + c_\mu\left(-\mathcal{E}'(t)\right). \end{aligned} \tag{3.22}$$

By choosing $q = (2 - m)/(2m - 2)$ (hence, $qm/(2 - m) = q + 1$] and taking μ small enough, Eq. (3.22) yields

$$\begin{aligned} &\gamma(t)\mathcal{E}^{q+1}(t) \leq \frac{-\beta_2}{q+1} \gamma(t) \frac{\mathrm{d}F^{q+1}(t)}{\mathrm{d}t} \\ &-\frac{c}{2} \frac{\mathrm{d}\mathcal{E}^{q+1}(t)}{\mathrm{d}t} + c\left(-\mathcal{E}(t)\right) \leq \frac{-\beta_2}{q+1} \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma(t) F^{q+1}(t)\right) \\ &+ \frac{\beta_2 \gamma'(t)}{q+1} - \frac{c}{q+1} \frac{\mathrm{d}\mathcal{E}^{q+1}(t)}{\mathrm{d}t} + c\left(-\mathcal{E}'(t)\right). \end{aligned} \tag{3.23}$$

By recalling that $\gamma'(t) \leq 0$ and integrating (3.23) over (S, T), $S \geq t_0$, we get

$$\int_S^T \gamma(t)\mathcal{E}^{q+1}(t)\mathrm{d}t \leq \frac{\beta_2}{q+1}\gamma(S)F^{q+1}(S) + \frac{c}{q+1}\mathcal{E}^{q+1}(S) + 2\mathcal{E}(S) \leq A\mathcal{E}(S), \tag{3.24}$$

for some positive constant A. Therefore, Proposition 2.2 gives (3.15). This completes the proof.

Remark 3.1. Estimates (3.14) and (3.15) also hold for $t \in [0, t_0]$ by virtue of continuity and boundedness of $\mathcal{E}$ and γ.

Remark 3.2. This result generalizes and improves many results in the literature. In particular, it allows some relaxation functions which satisfy

$$g'(t) \leq -ag^\rho(t), \quad 1 \leq \rho < 2,$$

instead of the usual assumption $1 \leq \rho < 3/2$.

Remark 3.3. Note that the exponential and the polynomial decay estimates, given in early works, are only particular cases of Eq. (3.14). More precisely, we obtain exponential decay for $\gamma(t) \equiv a$ and polynomial decay for $\gamma(t) = a(1 + t)^{-1}$, where $a > 0$ is a constant.

Acknowledgements

The author would like to express his sincere thanks to King Fahd University of Petroleum and Minerals for its support. This work has been funded by KFUPM under project # IN151002.

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