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Henstock-Kurzweil Integral Transforms and the Riemann-Lebesgue Lemma

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Additional information is available at the end of the chapter

1. Introduction

Let f be a function defined on a closed interval $[a, b]$ in the extended real line $\overline{\mathbb{R}}$, its Fourier transform at $s \in \mathbb{R}$ is defined as

$$\hat{f}(s) = \int_a^b e^{-ixs} f(x) dx. \tag{1}$$

The classical Riemann-Lebesgue Lemma states that

$$\lim_{|s| \to \infty} \int_a^b e^{-ixs} f(x) dx = 0, \tag{2}$$

whenever $f \in L^1([a, b])$.

We consider important to study analogous results about this lemma due to the following reasons:

- The classical Riemann-Lebesgue Lemma is an important tool used when proving several results related with convergence of Fourier Series and Fourier transform. In turn, these theorems have applications in the Harmonic Analysis which has many applications in the physics, biology, engineering and others sciences. For example, it is directly applied to the study of periodic perturbations of a class of resonant problems.

- An important problem is to consider an orthogonal basis, different to the trigonometric basis, and study the Fourier expansion of a function with respect to this basis. In this case, it is obtained an expression in this way

$$\int_a^b h(xs)f(x)dx. \tag{3}$$

- In some cases the expression (1) exists and the expression (2) is true even if the function f is not Lebesgue integrable.

Thus, a variant of the Riemann-Lebesgue Lemma is to get conditions for the functions f and h which ensure that (3) is well defined and satisfies

$$\lim_{|s|\to\infty} \int_a^b h(xs)f(x)dx = 0. \tag{4}$$

Some results of this type and related results are found in [1], [2], [3], and [4].

In the space of Henstock-Kurzweil integrable functions over $\mathbb{R}$, $HK(\mathbb{R})$, the Fourier transform does not always exist. In [5] was proven that $e^{-i(\cdot)s}f$ is Henstock-Kurzweil integrable under certain conditions and that, in general, does not satisfy the Riemann-Lebesgue Lemma. Subsequently, it was shown in [6], [3] and [4] that the Fourier transform exists and the equation (2) is true when $-\infty = a$, $b = \infty$ and f belongs to $BV_0(\mathbb{R})$, the space of bounded variation functions that vanish at infinity. A special case arises when f is in the intersection of functions of bounded variation and Henstock-Kurzweil integrable functions.

There exist Henstock-Kurzweil (HK) integrable functions f which $f \in HK(I) \setminus L^1(I)$ such that (2) is not fulfilled, when I is a bounded interval. In [7], Zygmund exhibited Henstock-Kurzweil integrable functions such that their Fourier coefficients do not tend to zero. In [8] are given necessary and sufficient conditions in order to $\int_a^b f(x)g_n(x)dx \longrightarrow \int_a^b f(x)g(x)dx$, for all $f \in HK([a,b])$. Thus, we will prove that the Fourier transform has the asymptotic behavior:

$$\hat{f}(s) = o(s), \text{ as } |s| \to \infty,$$

where $f \in HK(I) \setminus L^1(I)$

Moreover in [9], Titchmarsh proved that it is the best possible approximation for functions with improper Riemann integral.

This chapter is divided into 5 sections; we present the main results we have obtained in recent years: [3], [4], [6] and [10]. In this section we introduce basic concepts and important theorems about the Henstock-Kurzweil integral and bounded variation functions. In the second part of this study we prove some generalizations about the convergence of integrals of products in the completion of the space $HK([a,b])$, $\widehat{HK([a,b])}$, where $[a,b]$ can be a bounded or unbounded interval. As a consequence, some results related to the Riemann-Lebesgue Lemma in the context of the Henstock-Kurzweil integral are proved over bounded intervals. Besides, for elements in the completion of the space of Henstock-Kurzweil integrable functions, we get a similar result to the Riemann-Lebesgue property for the Dirichlet Kernel, as well as the asymptotic behavior of the n-th partial sum of Fourier series.

In the third section, we consider a complex function g defined on certain subset of $\mathbb{R}^2$. Many functions on functional analysis are integrals of the form $\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t,s)dt$. We study the function Γ when f belongs to $BV_0(\mathbb{R})$ and $g(t,\cdot)$ is continuous for all t. The integral we use is Henstock-Kurzweil integral. There are well known results about existence, continuity and differentiability of Γ, considering the Lebesgue theory. In the HK integral context there are results about this too, for example, Theorems 12.12 and 12.13 from [11]. But they all need the stronger condition that the function $f(t)g(t,s)$ is bounded by a HK integrable function. We give more conditions for existence, continuity and differentiability of Γ. Finally we give some applications such as some properties about the convolution of the Fourier and Laplace transforms.

In section 4, we exhibit a family of functions in $HK(\mathbb{R})$ included in $BV_0(\mathbb{R}) \setminus L^1(\mathbb{R})$. At the last section we get a version of Riemann-Lebesgue Lemma for bounded variation functions that vanish at infinity. With this result we get properties for the Fourier transform of functions in $BV_0(\mathbb{R})$: it is well defined, is continuous on $\mathbb{R} - \{0\}$, and vanishes at $\pm\infty$, as classical results. Moreover, we obtain a result on pointwise inversion of the Fourier transform.

1.1. Basic concepts and nomenclature

We will refer to a finite or infinite interval if its Lebesgue measure is finite or infinite. Let $I \subset \mathbb{R}$ be a closed interval, finite or infinite. A *partition P* of I is a increasing finite collection of points $\{t_0, t_1, ..., t_n\} \subset I$ such that if I is a compact interval $[a, b]$, then $t_0 = a$ and $t_n = b$; if $I = [a, \infty)$, $t_0 = a$; and if $I = (-\infty, b]$ then $t_n = b$.

Let us consider $I \subset \mathbb{R}$ as a closed interval finite. A *tagged partition* of I is a set of ordered pairs $\{[t_{i-1}, ti], s_i\}_{i=1}^n$ where it is assigned a point $s_i \in [t_{i-1}, t_i]$, which is called a *tag* of $[t_{i-1}, t_i]$. With this concept we define the Henstock-Kurzweil integral on finite intervals in $\overline{\mathbb{R}}$.

Definition 1. *The function $f : [a, b] \to \mathbb{R}$ is Henstock-Kurzweil integrable if there exists $H \in \mathbb{R}$ which satisfies the following: for each $\varepsilon > 0$ exists a function $\gamma_\varepsilon : [a, b] \to (0, \infty)$ such that if $P = \{(\,[t_{i-1}, t_i], s_i)\}_{i=1}^n$ is a tagged partition such that*

$$[t_{i-1}, t_i] \subset [s_i - \gamma_\varepsilon(s_i), s_i + \gamma_\varepsilon(s_i)] \quad \text{for } i = 1, 2, ..., n., \tag{5}$$

then

$$|\Sigma_{i=1}^n f(s_i)(t_i - t_{i-1}) - H| < \varepsilon.$$

H is the integral of f over $[a, b]$ and it is denoted as

$$H = \int_a^b f = \int_a^b f\, dt$$

It is said that a tagged partition is called $\gamma_\varepsilon-$fine if satisfies (5).

This definition can be extended on infinite intervals as follows.

Definition 2. *Let* $\gamma : [a, \infty] \to (0, \infty)$ *be a function, we will say that the tagged partition* $P = \{([t_{i-1}, t_i], s_i)\}_{i=1}^{n+1}$ *is* $\gamma-$fine *if:*

(a) $t_0 = a$, $t_{n+1} = \infty$.

(b) $[t_{i-1}, t_i] \subset [s_i - \gamma_\varepsilon(s_i), s_i + \gamma_\varepsilon(s_i)]$ *for* $i = 1, 2, ..., n$.

(c) $[t_n, \infty] \subset [1/\gamma(\infty), \infty]$.

Put $f(\infty) = 0$ and $f(-\infty) = 0$. This allows us define the integral of f over infinite intervals.

Definition 3. *It is said that the function* $f : [a, \infty] \to \mathbb{R}$ *is Henstock-Kurzweil integrable if it satisfies the Definition 1, but the partition P must be* $\gamma_\varepsilon-$*fine according to Definition 2.*

For functions defined on $[-\infty, a]$ or $[-\infty, +\infty]$ the integral is defined analogously. We will denote the vector space of Henstock-Kurzweil integrable functions on I as $HK(I)$

The space of Henstock-Kurzweil integrable functions on the interval $I = [a, b]$, finite or infinite interval, is a semi-normed space with the Alexiewicz semi-norm
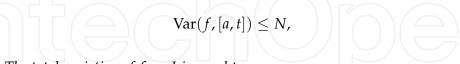
$$||f||_A = \sup_{a \leq x \leq b} \left| \int_a^x f(t)dt \right|. \tag{6}$$

We denote the space of functions in $HK([c, d])$ for each finite interval $[c, d]$ in I as $HK_{loc}(I)$.

Definition 4. *A function* $f : I \to \mathbb{R}$ *is a* bounded variation *function over I (finite interval) if exists a* $M > 0$ *such that*

$$\text{Var}(f, I) = \sup \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : P \text{ is a partition of } I \right\} < M.$$

Its total variation over I is $\text{Var}(f, I)$. *In case I is not finite, for example* $[a, \infty]$, *it is said that* $f : [a, \infty] \to \mathbb{R}$ *is a bounded variation function over I if there exists* $N > 0$ *such that*

$$\text{Var}(f, [a, t]) \leq N,$$

for all $t \geq a$. *The total variation of f on I is equal to*

$$\text{Var}(f, [a, \infty)) = \sup \{\text{Var}(f, [a, t]) : a \leq t\}. \tag{7}$$

For $I = (-\infty, b]$ *the considerations are analogous.*

The set of bounded variation functions over $[a, b]$ is denoted as $BV([a, b])$ and we will denote the space of functions f such that $f \in BV([c, d])$ for each compact interval $[c, d]$ in $\mathbb{R}$ as $BV_{loc}(\mathbb{R})$. We will refer to $BV_0(\mathbb{R})$ as the subspace of functions f belong to $BV(\mathbb{R})$ such that vanishing at $\pm\infty$.

At the Lemma 25 we prove that: $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset BV_0(\mathbb{R})$. It is not hard prove that $BV_0(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$ and $BV_0(\mathbb{R}) \not\subseteq HK(\mathbb{R})$. Furthermore, there are functions in $HK(\mathbb{R})$ or $L^1(\mathbb{R})$ but they are not in $BV_0(\mathbb{R})$. For example, the function $f(t)$ defined by 0 for $t \in (-\infty, 1)$ and $1/t$ for $t \in [1, \infty)$ belongs to $BV_0(\mathbb{R})$ but does not belong to $L^1(\mathbb{R})$, neither in $HK(\mathbb{R})$. In addition, other examples are the characteristic function of $\mathbb{Q}$ on a compact interval and $g(t) = t^2 \sin(\exp(t^2))$ are in $HK(\mathbb{R}) \setminus BV_0(\mathbb{R})$.

We consider the completion of $HK([a,b])$ as

$$\{[\{f_k\}] : \{f_k\} \text{ is a Cauchy sequence in } HK([a,b])\},$$

where the convergence is respect Alexiewicz norm, and will be denoted by $\widehat{HK([a,b])}$. It is possible to prove that $\widehat{HK([a,b])}$ is isometrically isomorphic to the subspace of distributions each of which is the distributional derivative of a continuous function, see [12]. The indefinite integral of $f = [\{f_k\}] \in \widehat{HK([a,b])}$ is defined as

$$\int_a^x f = \lim_{k \to \infty} \int_a^x f_k.$$

Thus, $\widehat{HK([a,b])}$ is a Banach space with the Alexiewicz norm (6). The completion is also defined in [13]. Besides, basic results of the integral continue being true on the completion. More details see [12].

To facilitate reading, we recall the following results. The first one is a well known result, and it can be found for example in [14] and [15].

Theorem 5. *Let f be a real function defined on $\mathbb{N} \times \mathbb{N}$. If $\lim_{n \to n_0} f(k, n) = \psi(k)$ exists for each k, and $\lim_{k \to k_0} f(k, n) = \varphi(n)$ converges uniformly on n, then*

$$\lim_{k \to k_0} \lim_{n \to n_0} f(k, n) = \lim_{n \to n_0} \lim_{k \to k_0} f(k, n).$$

Theorem 6. *[16, Theorem 33.1] Suppose X is a normed space, Y is a Banach space and that $\{T_n\}$ is a sequence of bounded linear operators from X into Y. Then the conditions: i) $\{\|T_n\|\}$ is bounded and ii) $\{T_n(x)\}$ is convergent for each $x \in Z$, where Z is a dense subset on X implies that for each $x \in X$, the sequence $(T_n(x))$ is convergent in Y and the linear operator $T : X \to Y$ defined by $T(x) = \lim_{n \to \infty} T_n(x)$ is bounded.*

Theorem 7. *[17] If g is a HK integrable function on $[a, b] \subseteq \mathbb{R}$ and f is a bounded variation function on $[a, b]$, then fg is HK integrable on $[a, b]$ and*

$$\left| \int_a^b fg \right| \leq \inf_{t \in [a,b]} |f(t)| \left| \int_a^b g(t)dt \right| + \|g\|_{[a,b]} \mathrm{Var}(f, [a, b]).$$

Theorem 8. *[11, Hake's Theorem] $\varphi \in HK([a, \infty])$ if and only if for each b, ε such that $b > a$ $b - a > \varepsilon > 0$, it follows that $\varphi \in HK([a + \varepsilon, b])$ and $\lim_{\varepsilon \to 0,\, b \to \infty} \int_{a+\varepsilon}^{b} \varphi(t)dt$ exists. In this case, this limit is $\int_{a}^{\infty} \varphi(t)dt$.*

Theorem 9. *[11, Chartier-Dirichlet's Test] Let f and g be functions defined on $[a, \infty)$. Suppose that*

1. *$g \in HK([a, c])$ for every $c \geq a$, and G defined by $G(x) = \int_{a}^{x} g$ is bounded on $[a, \infty)$.*
2. *f is of bounded variation on $[a, \infty)$ and $\lim_{x \to \infty} f(x) = 0$.*

Then $fg \in HK([a, \infty))$.

Moreover, by Multiplier Theorem, Hake's Theorem and Chartier-Dirichlet Test, we have the following lemma.

Lemma 10. *Let $f, g : [a, \infty] \to \mathbb{R}$. Suppose that $f \in BV_0([a, \infty])$, $\varphi \in HK([a, b])$ for every $b > a$, and $\Phi(t) = \int_{a}^{t} \varphi du$ is bounded on $[a, \infty)$. Then $\varphi f \in HK([a, b])$,*

$$\int_{a}^{\infty} \varphi f dt = - \int_{a}^{\infty} \Phi(t)df(t)$$

and

$$\left| \int_{a}^{\infty} \varphi f dt \right| \leq \sup_{a < t} |\Phi(t)| \operatorname{Var}(f, [a, \infty]).$$

Similar results are valid for the cases $[-\infty, \infty]$ and $[-\infty, a]$.

Let $I = [a, b]$ and $E \subset I$. We say that the function $F : I \to \mathbb{R}$ is in $AC_\delta(E)$ if for each $\epsilon > 0$ there exist $\eta_\varepsilon > 0$ and a gauge δ_ε on E such that if $\{(x_i, y_i)\}_{i}^{N}$ is a (δ_ε, E)−fine subpartition of E such that $\sum_{i}^{N}(y_i - x_i) < \eta_\varepsilon$, then $\sum_{1}^{N} |F(x_i) - F(y_i)| < \epsilon$. On the other hand, F belongs to the class $ACG_\delta(I)$ if there exists a sequence $\{E_n\}_{1}^{\infty}$ of sets in I such that $I = \cup_{n=1}^{\infty} E_n$ and $F \in AC_\delta(E_n)$ for each $n \in \mathbb{N}$. A characterization of this type of functions is the following.

Theorem 11. *A function $f \in HK(I)$ if and only if there exists a function $F \in ACG_\delta$ such that $F' = f$ a.e.*

Theorem 12. *[18, Theorem 4] Let $a, b \in \mathbb{R}$. If $h : \mathbb{R} \times [a, b] \to \mathbb{C}$ is such that*

1. *$h(t, \cdot)$ belongs to ACG_δ on $[a, b]$ for almost all $t \in \mathbb{R}$;*
2. *and $h(\cdot, s)$ is a HK integrable function on $\mathbb{R}$ for all $s \in [a, b]$.*

Then $H := \int_{-\infty}^{\infty} h(t, \cdot)dt$ belongs to ACG_δ on $[a, b]$ and $H'(s) = \int_{-\infty}^{\infty} D_2h(t, s)dt$ for almost all $s \in (a, b)$, iff,

$$\int_{s}^{t} \int_{-\infty}^{\infty} D_2h(t, s)dt ds = \int_{-\infty}^{\infty} \int_{s}^{t} D_2h(t, s)ds dt$$

for all $[s, t] \subseteq [a, b]$. In particular,

$$H'(s_0) = \int_{-\infty}^{\infty} D_2 h(t, s_0) dt$$

when $H_2 := \int_{-\infty}^{\infty} D_2 h(t, \cdot) dt$ is continuous at s_0.

2. Fourier coefficients for functions in the Henstock-Kurzweil completion.

For finite intervals, the Theorem 12.11 of [19] tells us that: In order that $\int_a^b f g_n \to \int_a^b f g$, $n \to \infty$, whenever $f \in HK([a, b])$, it is necessary and sufficient that: i) g_n is almost everywhere of bounded variation on $[a, b]$ for each n; ii) $\sup\{||g_n||_\infty + ||g_n||_{BV}\} < \infty$; iii) $\int_c^d g_n \to \int_c^d g$, $n \to \infty$, for each interval $(c, d) \subset (a, b)$. The Theorem 3 of [8] proves that above theorem is valid for infinite intervals. In this section we show that [19, Theorem 12.11] and [8, Theorem 3] are true for functions belonging to the completion of the Henstock-Kurzweil space. First, we need to prove the next lemma. The class of step functions on $[a, b]$ will be denoted as $K([a, b])$.

Lemma 13. *Let $[a, b]$ be an infinite interval. The set $K([a, b])$ is dense in $HK([a, b])$.*

Proof. Let $f \in HK([a, \infty])$ and $\epsilon > 0$ be given. By Hake's Theorem, exists $N \in \mathbb{N}$ such that for each $x \geq N$,

$$\left| \int_x^\infty f \right| < \frac{\epsilon}{2}. \tag{8}$$

Since $K([a, N])$ is dense in $HK([a, N])$, by Theorem 7 of [20], there exists a function $h \in K([a, N])$ such that

$$||f - h||_{A, [a, N]} = \sup_{x \in [a, N]} \left| \int_a^x (f - h) \right| < \frac{\epsilon}{2}. \tag{9}$$

Defining $h_0 \in K([a, \infty])$ as

$$h_0(x) = \begin{cases} h(x) \text{ if } x \in [a, N] \\ 0 \text{ if } x \in (N, \infty]. \end{cases}$$

It follows, by (8) and (9), that

$$||f - h_0||_A \leq \epsilon.$$

Similar arguments apply for intervals as $[-\infty, a]$ or $[-\infty, \infty]$. □

2.1. The convergence of integrals of products in the completion

The following result appears in [3]. Here, we present a detailed proof.

Theorem 14. *Let $[a, b] \subset \overline{\mathbb{R}}$. In order that*

$$\int_a^b f g_n \to \int_a^b f g, \quad n \to \infty, \tag{10}$$

whenever $f \in \widehat{HK([a, b])}$, it is necessary and sufficient that: i) g_n is almost everywhere of bounded variation on $[a, b]$ for each n; ii) $\sup\{||g_n||_\infty + ||g_n||_{BV}\} < \infty$; iii) $\int_c^d g_n \to \int_c^d g$, $n \to \infty$, for each interval $(c, d) \subset (a, b)$.

Proof. The necessity follows from [19, Theorem 12.11]. Now we will prove the sufficiency condition. Define the linear functionals $T, T_n : \widehat{HK([a, b])} \to \mathbb{R}$ by

$$T_n(f) = \int_a^b f g_n \quad \text{and} \quad T(f) = \int_a^b f g. \tag{11}$$

Supposing *i*) and *ii*), we have, by Multiplier Theorem, that the sequence $\{T_n\}$ is bounded by $\sup\{||g_n||_\infty + ||g_n||_{BV}\}$. Owing to Lemma 13, the space of step functions is dense in $\widehat{HK([a, b])}$, then considering the Theorem 6 it is sufficient to prove that $\{T_n(f)\}$ converge to $T(f)$, for each step function f. First, let $f(x) = \chi_{(c,d)}(x)$ be the characteristic function of $(c, d) \subset [a, b]$. Thus,

$$T_n(f) = \int_a^b \chi_{(c,d)} g_n = \int_c^d g_n,$$

by the hypothesis *iii*), we have that $\{T_n(\chi_{(c,d)})\}$ converges to $T(\chi_{(c,d)})$, as $n \to \infty$. Now, let f be a step function. Being that each T_n is a linear functional, then $\{T_n(f)\}$ converges to $T(f)$, as $n \to \infty$. Thus, the result holds. $\square$

Remark 15. *On $HK([a, b])$. The hypothesis iii) can be replaced by: g_n converges pointwise to g, then the result follows from Corollary 3.2 of [21]. The result on the completion holds by Theorem 6. Note that the conditions i), ii) and iii) do not imply converges pointwise from $\{g_n\}$ to g, see example 2 of [8].*

For the case of functions defined on a finite interval we get Theorem 16, and a lemma of Riemann-Lebesgue type for functions in the Henstock-Kurzweil space completion.

Theorem 16. *Let $[a, b]$ be a finite interval. If i) g_n converges to g in measure on $[a, b]$, ii) each g_n is equal to h_n almost everywhere, a normalized bounded variation function and iii) there is $M > 0$ such that $Var(h_n, [a, b]) \leq M$, $n \geq 1$, then for all $f \in \widehat{HK([a, b])}$,*

$$\int_a^b f g_n \to \int_a^b f g, \quad n \to \infty.$$

Proof. Let $f \in \widehat{HK}([a,b])$ be given, where $f = [\{f_k\}]$, we want to prove that

$$\lim_{n\to\infty} \lim_{k\to\infty} \int_a^b f_k g_n = \int_a^b fg.$$

Define $f(k,n) = \int_a^b f_k g_n$. By the hypothesis i) about (g_n) we have

$$\lim_{n\to\infty} \int_a^b f_k g_n = \int_a^b f_k g.$$

Moreover

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_a^b f_k g_n = \int_a^b fg,$$

by the integral definition on the completion. We will prove that $\lim_{k\to\infty} f(k,n) = \int_a^b fg_n$ converges uniformly on n. Let $\epsilon > 0$ be given, there exists k_0 such that $||f_k - f||_A \leq \epsilon$, if $k \geq k_0$. Besides, if $k \geq k_0$,

$$\left| \int_a^b f_k g_n - \int_a^b fg_n \right| \leq ||f_k - f||_A Var(g_n, [a,b])$$

$$\leq M\epsilon.$$

Therefore, by Theorem 5,

$$\lim_{n\to\infty} \lim_{k\to\infty} \int_a^b f_k g_n = \int_a^b fg.$$

$\square$

The following result is a "generalization" of Riemann-Lebesgue Lemma on the completion of the space $HK([a,b])$, over finite intervals, it also appears in [3].

Corollary 17. *If $\varphi : \mathbb{R} \to \mathbb{R}$ such that φ' exists, is bounded and $\varphi(s) = o(s)$, as $|s| \to \infty$, then for each $f \in \widehat{HK}([a,b])$ we have the next asymptotic behavior*

$$\int_a^b \varphi(st)f(t)dt = o(s), \quad as \ |s| \to \infty.$$

Proof. For each $s \neq 0$ define $\varphi_s : \mathbb{R} \to \mathbb{R}$ as $\varphi_s(t) = \varphi(st)/s$. In order to prove

$$\lim_{s\to\infty} \int_a^b \frac{\varphi(st)}{s} f(t)dt = 0,$$

it is sufficient to show that φ_s fulfills the hypothesis of Theorem 16. Now, we will check item by item. i) Because of $\varphi(s) = o(s)$, as $|s| \to \infty$ and the interval $[a,b]$ is finite, it follows that φ_s converges in measure to 0. ii) Owing to φ' is bounded then, by the Mean Value Theorem, we have that $\varphi_s \in BV([a,b])$. iii) $Var(\varphi_s, [a,b])$ is bounded uniformly by upper bound of φ' and $a - b$. $\square$

2.2. Riemann-Lebesgue Property

This property establishes that $\int_r^\pi f(t)D_n(t)dt \to 0$, for each $f \in L^1[-\pi, \pi]$ and $r \in (0, \pi]$, where $D_n(t) = \frac{\sin(n+1/2)t}{\sin(t/2)}$ denotes the n-th Dirichlet Kernel of order n. Now, we provide an analogous result concerning the Henstock-Kurzweil completion.

Theorem 18. *For any $f \in \widehat{HK([-\pi, \pi])}$, and $r \in (0, \pi]$,*

$$\lim_{n\to\infty} \frac{1}{n} \int_r^\pi f(t)D_n(t)dt = 0. \tag{12}$$

Proof. Note that the function $g(t) = 1/\sin(t/2)$ is in $BV([r, \pi])$. Moreover, by Multiplier Theorem we have $fg \in \widehat{HK([r, \pi])}$. Hence, by Corollary 17, we get

$$\int_r^\pi f(t)g(t)\sin(n+1/2)t\,dt = o(n), \quad |n| \to \infty.$$

$\square$

Considering an similar argument from above proof, it follows that

$$\int_r^\pi f(t)\frac{\sin(n+1/2)t}{t/2}dt = o(n), \quad |n| \to \infty. \tag{13}$$

For $n \in \mathbb{N} \cup \{0\}$, we define the function $\Phi_n(t) = \frac{\sin(n+1/2)t}{t/2}$ for $t \neq 0$ and $\Phi_n(0) = 2n+1$, it is called the discrete Fourier Kernel of order n. This kernel provides a very good approximation to the Dirichlet Kernel D_n for $|t| < 2$, but Φ_n decreases more rapidly than D_n, see [1].

Theorem 19. *Let $f \in \widehat{HK([0, \pi])}$ and $r \in (0, \pi]$. Then, assuming that any of next limits exist,*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^r f(t)D_n(t)dt = \lim_{n\to\infty} \frac{1}{n} \int_0^r f(t)\frac{\sin(n+1/2)t}{t/2}dt.$$

Proof. Define $g : [0, \pi] \to \mathbb{R}$ by

$$g(t) = \begin{cases} \frac{1}{\sin(t/2)} - \frac{1}{t/2} & \text{for } t \in (0, \pi] \\ \\ 0 & \text{for } t = 0. \end{cases}$$

Since $g \in BV([0, \pi])$, $fg \in \widehat{HK([0, \pi])}$. By Corollary 17, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\pi f(t) \left(\frac{1}{\sin(t/2)} - \frac{1}{t/2}\right) \sin\left(n+1/2\right) t \, dt = 0. \tag{14}$$

Now, by (14), we have

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\pi \left(f(t)D_n(t) - f(t)\frac{\sin(n+1/2)t}{t/2}\right) dt = 0.$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \left[\int_0^r \left(f(t)D_n(t) - f(t)\frac{\sin(n+1/2)t}{t/2}\right) dt\right.$$
$$\left. + \int_r^\pi \left(f(t)D_n(t) - f(t)\frac{\sin(n+1/2)t}{t/2}\right) dt\right] = 0.$$

By Theorem 18 and (13),

$$\lim_{n\to\infty} \frac{1}{n} \int_r^\pi \left(f(t)D_n(t) - f(t)\frac{\sin(n+1/2)t}{t/2}\right) dt = 0.$$

Therefore, assuming that any of the limits exist, we have

$$\lim_{n\to\infty} \frac{1}{n} \int_0^r f(t)D_n(t) dt = \lim_{n\to\infty} \frac{1}{n} \int_0^r f(t)\frac{\sin(n+1/2)t}{t/2} dt.$$

$\square$

The following result is a characterization of the asymptotic behavior of $n-th$ partial sum of the Fourier series, it can be found in [3].

Corollary 20. *Let* $f \in \widehat{HK([-\pi, \pi])}$ *be* $2\pi-$ *periodic. The* $n-th$ *partial sum of the Fourier series at t has the following asymptotic behavior* $S_n(f,t) = o(n)$, *when* $|n| \to \infty$ *iff*

$$\int_0^\pi [f(t+u) + f(t-u)]\frac{\sin(n+1/2)u}{u} du = o(n),$$

if $|n| \to \infty$.

Proof. Since $S_n(f,t) = \int_{-\pi}^\pi f(t+u)D_n(u) du$, realizing a change of variable (see section 6 of [13]), then by Theorem 19 we get the result. $\square$

3. Henstock-Kurzweil integral transform

The results in this section are based for functions in the vector space $BV_0(\mathbb{R})$, and they have to [10] as principal reference.

We will introduce some additional terminology in order to facilitate the following results.

If $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a function and $s_0 \in \mathbb{R}$, we say that s_0 fulfills hypothesis (**H**) relative to g if:

(**H**) there exist $\delta = \delta(s_0) > 0$ and $M = M(s_0) > 0$, such that, if $|s - s_0| < \delta$ then

$$\left| \int_u^v g(t,s)dt \right| \leq M,$$

for all $[u,v] \subseteq \mathbb{R}$.

This condition plays a significant role in the following results. Also, the next theorems can be found in [10].

Theorem 21. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions. Assume that $f \in BV_0(\mathbb{R})$, and $s_0 \in \mathbb{R}$ fulfills Hypothesis (**H**) relative to g, then*

$$\Gamma(s) = \int_{-\infty}^{\infty} f(t)g(t,s)dt$$

is well defined for all s in a neighborhood of s_0.

Proof. Applying Theorem 9 the result holds. □

Theorem 22. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be functions assume that*

1. *f belongs to $BV_0(\mathbb{R})$, g is bounded, and*
2. *$g(t, \cdot)$ is continuous for all $t \in \mathbb{R}$.*

*If $s_0 \in \mathbb{R}$ fulfills Hypothesis (**H**) relative to g, then the function Γ is continuous at s_0.*

Proof. By Hypothesis (**H**), there exist $\delta_1 > 0$ and $M > 0$, such that, if $|s - s_0| < \delta_1$ then

$$\left| \int_u^v g(t,s)dt \right| \leq M \tag{15}$$

for all $[u,v] \subseteq \mathbb{R}$. From Theorem 21, $\Gamma(s)$ exists for all $s \in B_{\delta_1}(s_0)$.

Let an arbitrary $\epsilon > 0$, by Hake's theorem, there exists $K_1 > 0$ such that

$$\left| \int_{|t| \geq u} f(t)g(t, s_0)dt \right| < \frac{\epsilon}{3} \tag{16}$$

for all $u \geq K_1$. On the other hand, as

$$\lim_{t \to -\infty} \mathrm{Var}(f, (-\infty, t]) = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathrm{Var}(f, [t, \infty)) = 0,$$

there is $K_2 > 0$ such that for each $t > K_2$,

$$\mathrm{Var}(f, (-\infty, -t]) + \mathrm{Var}(f, [t, \infty)) < \frac{\epsilon}{3M}.$$

Let $K = \max\{K_1, K_2\}$. From Theorem 7, it follows that for every $v \geq K$ and every $s \in B_{\delta_1}(s_0)$,

$$\left| \int_K^v f(t)g(t, s)dt \right| \leq \|g(\cdot, s)\|_{[K, v]} \left[\inf_{t \in [K, v]} |f(t)| + \mathrm{Var}(f, [K, v]) \right]$$
$$\leq M \left[|f(v)| + \mathrm{Var}(f, [K, \infty)) \right],$$

where the second inequality is true due to (15). This implies, since $\lim_{t \to \infty} |f(t)| = 0$, that

$$\left| \int_K^\infty f(t)g(t, s)dt \right| \leq M \cdot \mathrm{Var}(f, [K, \infty)).$$

Analogously we have that

$$\left| \int_{-\infty}^{-K} f(t)g(t, s)dt \right| \leq M \cdot \mathrm{Var}(f, (-\infty, -K]).$$

Therefore, for each $s \in B_{\delta_1}(s_0)$,

$$\left| \int_{|t| \geq K} f(t)g(t, s)dt \right| \leq M \left[\mathrm{Var}(f, (-\infty, -K])f + \mathrm{Var}(f, [K, \infty)) \right]$$
$$< M \frac{\epsilon}{3M} = \frac{\epsilon}{3}. \tag{17}$$

Since f is $L^1[-K, K]$, g is bounded and $g(t, \cdot)$ is continuous for all $t \in \mathbb{R}$. For example, using Theorem 12.12 of [11], it is easy to show that the function

$$\Gamma_K(s) = \int_{-K}^{K} f(t)g(t,s)dt, \ \ s \in \mathbb{R},$$

is continuous at s_0. This implies that there is $\delta_2 > 0$ such that for every $s \in B_{\delta_2}(s_0)$,

$$\left| \int_{-K}^{K} f(t)[g(t,s) - g(t,s_0)]dt \right| < \frac{\epsilon}{3}. \tag{18}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $s \in B_\delta(s_0)$,

$$|\Gamma(s) - \Gamma(s_0)| \le \left| \int_{-K}^{K} f(t)[g(t,s) - g(t,s_0)]dt \right|$$

$$+ \left| \int_{|t| \ge K} f(t)g(t,s)dt \right| + \left| \int_{|t| \ge K} f(t)g(t,s_0)dt \right|.$$

Thus, from (16), (17) and (18), $|\Gamma(s) - \Gamma(s_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$, for all $s \in B_\delta(s_0)$. $\qquad \square$

Theorem 23. *Let $a, b \in \mathbb{R}$. If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times [a, b] \to \mathbb{C}$ are functions such that*

1. $f \in BV_0(\mathbb{R})$, g is measurable, bounded and
*2. for all $s \in [a, b]$, s satisfies Hypothesis (**H**) relative to g.*

Then

$$\int_a^b \int_{-\infty}^{\infty} f(t)g(t,s)dtds = \int_{-\infty}^{\infty} \int_a^b f(t)g(t,s)dsdt$$

Proof. From (2) and since $[a, b]$ is compact, there exists $M > 0$ such that, for every $s \in [a, b]$ and for all $[u, v] \subseteq \mathbb{R} : \left| \int_u^v g(t,s)dt \right| \le M$.

For $r > 0$ and $s \in [a, b]$, let $\Gamma_r(s) = \int_{-r}^{r} f(t)g(t,s)dt$. By Theorem 7, we notice that

$$|\Gamma_r(s)| = \left| \int_{-r}^{r} f(t)g(t,s)dt \right|$$

$$\le \|g(\cdot,s)\|_{[-r,r]} \left[\inf_{t \in [-r,r]} |f(t)| + V_{[-r,r]}f \right]$$

$$\le M[|f(0)| + Vf]$$

for all $s \in [a, b]$.

So, for each $r > 0$, Γ_r is HK integrable on $[a, b]$ and is bounded for a fixed constant. Moreover, by Theorem 21 and Hake's theorem

$$\lim_{r \to \infty} \Gamma_r(s) = \Gamma(s)$$

for all $s \in [a, b]$.

Using the Lebesgue Dominated Convergence Theorem, we have that Γ is HK integrable on $[a, b]$ and

$$\int_a^b \Gamma(s)ds = \lim_{r \to \infty} \int_a^b \Gamma_r(s)ds.$$

Now, because of f is Lebesgue integrable on $[-r, r]$; g is measurable and bounded; and by Fubini's theorem, it follows that

$$\int_a^b \int_{-r}^r f(t)g(t,s)dtds = \int_{-r}^r \int_a^b f(t)g(t,s) \, dsdt.$$

Consequently

$$\lim_{r \to \infty} \int_{-r}^r \int_a^b f(t)g(t,s)dsdt = \lim_{r \to \infty} \int_a^b \Gamma_r(s)ds = \int_a^b \Gamma(s)ds.$$

So by Hake's theorem,

$$\int_{-\infty}^\infty \int_a^b f(t)g(t,s) \, dsdt = \int_a^b \Gamma(s)ds = \int_a^b \int_{-\infty}^\infty f(t)g(t,s)dtds.$$

$\square$

Theorem 24. *Let $f \in BV_0(\mathbb{R})$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a function such that its partial derivative D_2g is bounded and continuous on $\mathbb{R} \times \mathbb{R}$. If $s_0 \in \mathbb{R}$ is such that*

1. *there is $K > 0$ for which $\|g(\cdot, s_0)\|_{[u,v]} \leq K$ for all $[u, v] \subseteq \mathbb{R}$, and*

2. *s_0 satisfies Hypothesis (**H**) relative to D_2g.*

Then Γ is derivable at s_0, and

$$\Gamma'(s_0) = \int_{-\infty}^\infty f(t)D_2g(t,s_0)dt. \tag{19}$$

Proof. Using conditions (1) and (2) and the Mean Value theorem, there exist $\delta > 0$ and $M > 0$ such that, for each $s \in (s_0 - \delta, s_0 + \delta)$,

$$\left| \int_u^v D_2 g(t,s)dt \right| < M \text{ and } \left| \int_u^v g(t,s)dt \right| < M, \tag{20}$$

for all $[u,v] \subseteq \mathbb{R}$.

Let a, b be real numbers with $s_0 - \delta < a < s_0 < b < s_0 + \delta$. We use Theorem 12 to prove (19). The function $f(t)g(t,\cdot)$ is differentiable on $[a,b]$ for each $t \in \mathbb{R}$, therefore $f(t)g(t,\cdot)$ is ACG_δ on $[a,b]$ for all $t \in \mathbb{R}$. By (20) and Theorem 9, $f(\cdot)g(\cdot,s)$ is HK-integrable on $\mathbb{R}$ for all $s \in [a,b]$. Then

$$\Gamma'(s_0) = \int_{-\infty}^{\infty} f(t)D_2 g(t,s_0)dt$$

when, if

$$\Gamma_2 := \int_{-\infty}^{\infty} f(t)D_2 g(t,\cdot)dt$$

is continuous at s_0, and

$$\int_s^t \int_{-\infty}^{\infty} f(t)D_2 g(t,s)dtds = \int_{-\infty}^{\infty} \int_s^t f(t)D_2 g(t,s)dsdt$$

for all $[s,t] \subseteq [a,b]$. The first affirmation is true by (20) and Theorem 22, and the second affirmation is true due to (20) and Theorem 23 $\qquad\square$

3.1. Some applications

An important work about the Fourier transform using the Henstock-Kurzweil integral: existence, continuity, inversion theorems etc. was published in [5]. Nevertheless, there are some omissions in that results that use the Lemma 25 (a) of [5]. Also the authors of this book chapter in [6], [3] and [4] have studied existence, continuity and Riemann-Lebesgue lemma about the Fourier transform of functions belong to $HK(\mathbb{R}) \cap BV(\mathbb{R})$ and $BV_0(\mathbb{R})$. Following the line of [6], in Theorem 26 we include some results from them as consequences of theorems above section.

Let f and g be real-valued functions on $\mathbb{R}$. The convolution of f and g is the function $f * g$ defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

for all x such that the integral exists. Several conditions can be imposed on f and g to guarantee that $f * g$ is defined on $\mathbb{R}$. For example, if f is HK- integrable and g is of bounded variation.

Lemma 25. *For $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, $\lim\limits_{|x| \to \infty} f(x) = 0$.*

Proof. Since f is a bounded variation function on $\mathbb{R}$ then the limit of $f(x)$, as $|x| \to \infty$, exists. Suppose that $\lim\limits_{|x| \to \infty} f(x) = \alpha \neq 0$. Take $0 < \epsilon < |\alpha|$. There exists $A > 0$ such that $\alpha - \epsilon < f(x)$, for all $|x| > A$. Observe that $f(x) > 0$ on $[A, \infty)$, so $f \in L([A, \infty))$. Therefore the constant function $\alpha - \epsilon$ is Lebesgue integrable on $[A, \infty)$, which is a contradiction. $\square$

Observe, as consequence of above Lemma, we have that the vector space $HK(\mathbb{R}) \cap BV(\mathbb{R})$ is contained in $BV_0(\mathbb{R})$. So the next theorem is an immediately consequence of above section.

Theorem 26. *If $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then*

1. *$\widehat{f}$ exists on $\mathbb{R}$.*

2. *$\widehat{f}$ is continuous on $\mathbb{R} \setminus \{0\}$.*

3. *If $g(t) = tf(t)$ and $g \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then $\widehat{f}$ is differentiable on $\mathbb{R} \setminus \{0\}$, and*

$$\widehat{f}'(s) = -i\widehat{g}(s), \text{ for each } s \in \mathbb{R} \setminus \{0\}.$$

4. *For $h \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $\widehat{f * h}(s) = \widehat{f}(s)\widehat{h}(s)$ for all $s \in \mathbb{R}$.*

Proof. We observe that

$$\left| \int_u^v e^{-its} dt \right| \leq \frac{2}{|s|}, \tag{21}$$

for all $[u, v] \subseteq \mathbb{R}$. Thus, each $s_0 \neq 0$ satisfies Hypothesis (**H**) relative to e^{-its}.

(a) Theorem 21 implies that $\widehat{f}(s_0)$ exists for all $s_0 \neq 0$ and, since $f \in HK(\mathbb{R})$, $\widehat{f}(0)$ exists. Therefore $\widehat{f}$ exists on $\mathbb{R}$.

(b) By Theorem 22, $\widehat{f}$ is continuous at s_0, for all $s_0 \neq 0$.

(c) It follows by Theorem 12 in similar way to the proof of Theorem 24.

(d) Let $k(x, y) = f(y - x)e^{-iys}$, where s is a fixed real number. We get, for each $y \in \mathbb{R}$ and all $[u, v] \subseteq \mathbb{R}$,

$$\left| \int_u^v k(x, y) dx \right| = \left| \int_u^v f(y - x) dx \right|$$

$$= \left| \int_{y-u}^{y-v} f(z) dz \right| \leq \|f\|_A.$$

So, every real number y satisfies Hypothesis (**H**) relative to k. Now, observe that $h \in BV_0(\mathbb{R})$ and k is measurable and bounded. Thus, by Theorem 23,

$$\int_{-a}^{a} \int_{-\infty}^{\infty} h(x)k(x,y)dxdy = \int_{-\infty}^{\infty} \int_{-a}^{a} h(x)k(x,y)dydx, \tag{22}$$

for all $a > 0$.

On the other hand,

$$\left| h(x) \int_{-a}^{a} f(y - x)e^{-iys}dy \right| \leq |h(x)| \left| \int_{-a-x}^{a-x} f(z)e^{-izs}dz \right|$$

$$\leq |h(x)| \|f(\cdot)e^{-i(\cdot)s}\|_A.$$

Since $h \in L(\mathbb{R})$, using Dominated Convergence theorem, it follows that

$$\widehat{f}(s)\widehat{h}(s) = \int_{-\infty}^{\infty} h(x) \int_{-\infty}^{\infty} f(y - x)e^{-iys}dydx$$

$$= \lim_{a \to \infty} \int_{-\infty}^{\infty} h(x) \int_{-a}^{a} f(y - x)e^{-iys}dydx.$$

Moreover, from (22), we have

$$\widehat{f}(s)\widehat{h}(s) = \lim_{a \to \infty} \int_{-a}^{a} \int_{-\infty}^{\infty} h(x)f(y - x)e^{-iys}dxdy$$

$$= \lim_{a \to \infty} \int_{-a}^{a} (f * h)(y)e^{-iys}dy.$$

We conclude, by Hake's theorem, that

$$\widehat{f * h}(s) = \widehat{f}(s)\widehat{h}(s).$$

$\square$

Recall that the Laplace transform, at $z \in \mathbb{C}$, of a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$L(f)(z) = \int_{0}^{\infty} f(t)e^{-zt}dt.$$

Theorem 27. *If $f \in HK([0, \infty)) \cap BV([0, \infty))$, then*

1. *$L(f)(z)$ exists for all $z \in \mathbb{C}$.*
2. *If $F(x, y) = L(f)(x + iy)$, then $F(\cdot, y)$ is continuous on $\mathbb{R}$ for all $y \neq 0$, and $F(x, \cdot)$ is continuous on $\mathbb{R}$ for all $x \neq 0$.*

4. A set of functions in $HK(\mathbb{R}) \cap BV_0(\mathbb{R}) \setminus L^1(\mathbb{R})$

Taking into account Lemma 25, the set $HK(\mathbb{R}) \cap BV(\mathbb{R})$ is included in $BV_0(\mathbb{R})$ and does not have inclusion relations with $L^1(\mathbb{R})$. Since the step functions belong to $HK(\mathbb{R}) \cap BV(\mathbb{R})$, then by Lemma 13, we have that $HK(\mathbb{R}) \cap BV(\mathbb{R})$ is dense in $HK(\mathbb{R})$. In this section we exhibit a set of functions in $HK(\mathbb{R}) \cap BV_0(\mathbb{R}) \setminus L^1(\mathbb{R})$.

Proposition 28. *Let $b > a > 0$. Suppose that $f : [a, \infty) \to \mathbb{R}$ is not identically zero, is continuous and periodic with period $b - a$. Let $F(x) = \int_a^x f(t)dt$ be bounded on $[a, \infty)$. Moreover, assume that $\varphi : [a, \infty) \to \mathbb{R}$ is a nonnegative and monotone decreasing function which satisfies the next conditions:*

(i) $\lim_{t \to \infty} \varphi(t) = 0$,

(ii) $\varphi \notin HK([a, \infty))$.

Then the product $\varphi f \in HK([a, \infty)) \setminus L^1([a, \infty))$.

Proof. We take $t_o \in (a, b)$, $\delta_o > 0$ and $\gamma > 0$ such that

$$\gamma \leq |f(t)| \quad \text{for each} \quad t \in [t_o - \delta_o, t_o + \delta_o] \subset (a, b).$$

Periodicity of f gives

$$\gamma \leq |f(t)|$$

for each $t \in \bigcup_{k=0}^{\infty} [t_o - \delta_o + k(b - a), t_o + \delta_o + k(b - a)]$. Therefore,

$$
\begin{aligned}
\int_a^{b+n(b-a)} \varphi(t)|f(t)|dt &\geq \gamma \sum_{k=0}^{n} \int_{t_o-\delta_o+k(b-a)}^{t_o+\delta_o+k(b-a)} \varphi(t)dt \\
&\geq \gamma \sum_{k=0}^{n} \int_{t_o-\delta_o+k(b-a)}^{t_o+\delta_o+k(b-a)} \varphi(t_o + \delta_o + k(b-a))dt \\
&= \gamma(2\delta_o) \sum_{k=0}^{n} \varphi(t_o + \delta_o + k(b-a)).
\end{aligned}
\tag{23}
$$

Also,

$$
\begin{aligned}
\int_a^{b+n(b-a)} \varphi(t)dt &\leq \sum_{k=0}^{n} \int_{a+k(b-a)}^{b+k(b-a)} \varphi(t)dt \\
&\leq \sum_{k=0}^{n} \varphi(a + k(b-a)) \int_{a+k(b-a)}^{b+k(b-a)} dt \\
&\leq (b-a)\varphi(a) \\
&\quad + (b-a) \sum_{k=1}^{n} \varphi(t_o + \delta_o + (k-1)(b-a)).
\end{aligned}
\tag{24}
$$

Because of $\varphi \notin HK([a, \infty])$, we get $\lim_{n \to \infty} \int_a^{b+n(b-a)} \varphi(t)dt = \infty$. Thus, equations (23) and (24) imply $\varphi f \notin L^1([a, \infty)]$. On the other hand, by Chartier-Dirichlet's Test of [11], the function φf belongs to $HK([a, \infty))$. □

Corollary 29. *Let α, β be positive numbers such that $\alpha + \beta > 1$ with $\beta \leq 1$. Suppose $a > 0$ and $f : [a, \infty) \to \mathbb{R}$ obeys the hypotheses of Proposition 28. Then, the function $f_{\alpha, \beta} : [a^{1/\alpha}, \infty) \to \mathbb{R}$ defined by*

$$f_{\alpha, \beta}(t) = \frac{f(t^\alpha)}{t^\beta} \tag{25}$$

is in $HK([a^{1/\alpha}, \infty)) \setminus L^1([a^{1/\alpha}, \infty))$.

Proof. The change of variable $u = t^\alpha$ gives,

$$\int_{a^{\frac{1}{\alpha}}}^{\infty} \frac{f(t^\alpha)}{t^\beta} dt = \int_{a}^{\infty} \frac{f(u)}{u^{\frac{\beta-1}{\alpha}+1}} du. \tag{26}$$

The hypotheses for α, β imply that the function $\varphi(u) = u^{-[\frac{(\beta-1)}{\alpha}+1]}$ satisfies the conditions of Proposition 28. Then, $\varphi f \in HK([a, \infty) \setminus L^1([a, \infty))$, satisfying the statement of the corollary. □

Proposition 30. *Let $\beta > \alpha > 0$ be fixed with $\beta + \alpha > 1$. Suppose $f : [a, \infty) \to \mathbb{R}$ is a bounded and continuous function, with bounded derivative. Then the function $f_{\alpha, \beta} : [a^{1/\alpha}, \infty) \to \mathbb{R}$, defined by $f_{\alpha, \beta}(t) = f(t^\alpha)/t^\beta$, belongs to the space $BV([a^{1/\alpha}, \infty))$.*

Proof. Let M_1 and M_2 be bounds for f and f', respectively. We have,

$$f'_{\alpha, \beta}(t) = \frac{\alpha f'(t^\alpha)}{t^{\beta-\alpha+1}} - \frac{\beta f(t^\alpha)}{t^{\beta+1}},$$

which gives

$$\left| f'_{\alpha, \beta}(t) \right| \leq \frac{\alpha M_2}{t^{\beta-\alpha+1}} + \frac{\beta M_1}{t^{\beta+1}}.$$

Now, take $x > a^{\frac{1}{\alpha}}$. Since $\beta - \alpha > 0$, then

$$\frac{\alpha M_2}{t^{\beta-\alpha+1}} + \frac{\beta M_1}{t^{\beta+1}} \in L^1([a^{\frac{1}{\alpha}}, x)).$$

A straightforward application of the Theorem 7.7 of [11] implies $f'_{\alpha, \beta} \in L^1([a^{1/\alpha}, x))$. Moreover

$$\int_{a^{\frac{1}{\alpha}}}^{x} |f'_{\alpha,\beta}(t)| dt \le \alpha M_2 \int_{a^{\frac{1}{\alpha}}}^{x} t^{-\beta+\alpha-1} dt$$

$$+ \beta M_1 \int_{a^{\frac{1}{\alpha}}}^{x} t^{-\beta-1} dt$$

$$= -\frac{\alpha M_2}{\beta - \alpha} \left(\frac{1}{x^{\beta-\alpha}} - \frac{1}{a^{\frac{\beta-\alpha}{\alpha}}} \right)$$

$$- M_1 \left(\frac{1}{x^{\beta}} - \frac{1}{a^{\frac{\beta}{\alpha}}} \right)$$

$$\le \frac{\alpha M_2}{\beta - \alpha} \frac{1}{a^{\frac{\beta-\alpha}{\alpha}}} + \frac{M_1}{a^{\frac{\beta}{\alpha}}}.$$

These estimates together with the Theorem 7.5 of [11] imply,

$$V(f_{\alpha,\beta}; [a^{\frac{1}{\alpha}}, x)) \le \frac{\alpha M_2}{\beta - \alpha} \frac{1}{a^{\frac{\beta-\alpha}{\alpha}}} + \frac{M_1}{a^{\frac{\beta}{\alpha}}}. \tag{27}$$

If x tends to ∞, one gets $f_{\alpha,\beta} \in BV([a^{1/\alpha}, \infty))$. □

Corollary 29 and Proposition 30 provide us Henstock-Kurzweil integrable functions defined on unbounded intervals which are not Lebesgue integrable.

Corollary 31. *Let a, α, β be such that: $0 < a$, $0 < \alpha < \beta \le 1$ and $1 < \beta + \alpha$. Suppose that $f : [a, \infty) \to \mathbb{R}$ satisfies both the hypotheses of Corollary 29 and Proposition 30. Then, the function $f_{\alpha,\beta}$ belongs to $HK([a^{1/\alpha}, \infty)) \cap BV([a^{1/\alpha}, \infty)) \setminus L^1([a^{1/\alpha}, \infty))$.*

Taking into account the above functions we have the following corollary.

Corollary 32. *Let a, α, β be such that: $0 < a$, $0 < \alpha < \beta \le 1$ and $1 < \beta + \alpha$, and let h in $BV([-a^{1/\alpha}, a^{1/\alpha}])$. Suppose that $f : [a, \infty) \to \mathbb{R}$ satisfies both the hypotheses of Corollary 29 and Proposition 30. Then $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$g(t) = \begin{cases} h(t) & \text{if } t \in (-a^{1/\alpha}, a^{1/\alpha}), \\ \dfrac{f(|t|^{\alpha})}{|t|^{\beta}} & \text{if } f \in (-\infty, -a^{1/\alpha}] \cup [a^{1/\alpha}, \infty) \end{cases}$$

is in $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$.

Example 33. *Let us consider the trigonometric functions $\sin(t)$ and $\cos(t)$. Then the following family of functions satisfies the hypotheses of Theorem 34.*

$$\sin_{\beta}^{\alpha} : \mathbb{R} \to \mathbb{R}; \quad \sin_{\beta}^{\alpha}(t) = \chi_{1,\alpha}(t) \frac{\sin(t^{\alpha})}{t^{\beta}},$$

$$\cos_{\beta}^{\alpha} : \mathbb{R} \to \mathbb{R}; \quad \cos_{\beta}^{\alpha}(t) = \chi_{2,\alpha}(t) \frac{\cos(t^{\alpha})}{t^{\beta}}.$$

Here $\chi_{1,\alpha}$ and $\chi_{2,\alpha}$ are the characteristic functions of the intervals $[\pi^{1/\alpha}, \infty)$ and $[(\pi/2)^{1/\alpha}, \infty)$, respectively. The numbers α, β are taken as in Corollary 31.

From the above example belongs to $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$. By the Multiplier theorem it follows that $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset L^2(\mathbb{R})$, so the above function is in $BV_0(\mathbb{R}) \cap L^2(\mathbb{R}) \setminus L(\mathbb{R})$. Therefore, there exist functions in $L^2(\mathbb{R}) \setminus L(\mathbb{R})$ such that their Fourier transforms exist as in (1), as an integral in HK sense.

5. The Riemann-Lebesgue Lemma and the Dirichlet-Jordan Theorem for BV_0 functions

The Riemann-Lebesgue lemma is a fundamental result of the Harmonic Analysis. An novel aspect is the validity of this lemma for functions which are not Lebesgue integrable, since this fact could help to expand the space of functions where the inversion of the Fourier transform is possible. In this section we prove a generalization of the Riemann-Lebesgue Lemma for functions of bounded variation which vanish at infinity. As consequence, it is obtained a proof of the Dirichlet-Jordan theorem for this kind of functions. This theorem provides a pointwise inversion of the Fourier transform.

We observe that the implications 1 and 2 of Theorem 26 are particularizations of the next result.

Theorem 34 (Generalization of Riemann-Lebesgue Lemma). *Let $\varphi \in HK_{loc}(\mathbb{R})$ be a function such that $\Phi(t) = \int_0^t \varphi(x)dx$ is bounded function on $\mathbb{R}$. If $f \in BV_0(\mathbb{R})$, then the function $H(w) = \int_{-\infty}^{\infty} f(t)\varphi(wt)dt$ is defined on $\mathbb{R} \setminus \{0\}$, it is continuous and*

$$\lim_{|w| \to \infty} H(w) = 0.$$

Proof. Given $w \in \mathbb{R}$, we define $\varphi_w(t) = \varphi(wt)$. Because of $\varphi \in HK_{loc}(\mathbb{R})$, then φ and φ_w are in $HK([0,b])$, for $b > 0$. By Jordan decomposition, there exist functions f_1 and f_2 which are nondecreasing functions belonging to $BV_0(\mathbb{R})$ such that $f = f_1 - f_2$. Hence, by Chartier-Dirichlet's Test, $f\varphi_w \in HK([0,\infty])$. By applying the Multiplier Theorem and supposing $w \neq 0$, it follows

$$\int_0^{\infty} f(t)\varphi(wt)dt = -\int_0^{\infty} \frac{\Phi(wt)}{w}df(t)$$

$$= -\int_0^{\infty} \frac{\Phi(wt)}{w}df_1(t) \quad (28)$$

$$+ \int_0^{\infty} \frac{\Phi(wt)}{w}df_2(t),$$

where $df_i(t)$ is the Lebesgue-Sieltjes measure generated by f_i, $i = 1, 2$.

Let β a positive number and let M the upper bound of $|\Phi|$. For $w \in [\beta, \infty)$ we have that

$$\left| \frac{\Phi(wt)}{w} \right| \leq \frac{M}{\beta}. \tag{29}$$

Since $\Phi(wt)/w$ is continuous over $[\beta, \infty)$ and the measures $df_i(t)$ are finite, then by the Dominated Convergence Theorem applied to right side integrals in (28), it follows that

$$\lim_{w \to w_0} H(w) = H(w_0),$$

for each $w_0 \in [\beta, \infty)$. Since β is arbitrary, we obtain the continuity of H on $(0, \infty)$.

Moreover, by (28), we have for $w \in (0, \infty)$ that

$$\left| \int_0^\infty f(t)\varphi(wt)dt \right| \leq \frac{M}{|w|} Var(f; [0, \infty)).$$

Thus, we conclude that

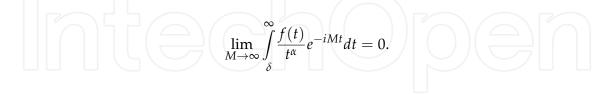$$\lim_{|w| \to \infty} \int_0^\infty f(t)\varphi(wt)dt = 0.$$

To complete the proof, we use similar arguments for the interval $(-\infty, 0]$.   □

The above theorem confirms that $H \in C_0(\mathbb{R} \setminus \{0\})$, for each $f \in BV_0(\mathbb{R})$. As corollary we have the Riemann-Lebesgue Lemma.

Corollary 35. *If $f \in BV_0(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R} \setminus \{0\})$.*

We know that if $g, h \in BV([a, \infty])$ then $gh \in BV([a, \infty])$. Employing this fact and Theorem 34 we get the following corollary.

Corollary 36. *Suppose that $\delta, \alpha > 0$ and $f \in BV(\mathbb{R})$, then*

$$\lim_{M \to \infty} \int_\delta^\infty \frac{f(t)}{t^\alpha} e^{-iMt} dt = 0.$$

The Sine Integral is defined as

$$Si(x) = \frac{2}{\pi} \int_0^x \frac{\sin t}{t} dt,$$

which has the properties:

1. $Si(0) = 0$, $\lim_{x \to \infty} Si(x) = 1$ and
2. $Si(x) \leq Si(\pi)$ for all $x \in [0, \infty]$.

We use the Sine Integral function in the proof of the following lemma.

Lemma 37. *Let $\delta > 0$. If $f \in BV_0(\mathbb{R})$, then*

$$\lim_{\varepsilon \to 0} \int_{\delta}^{\infty} f(t) \frac{\sin \varepsilon t}{t} dt = 0.$$

Proof. By Lemma 10 we have

$$\left| \int_{\delta}^{\infty} \frac{\sin \varepsilon t}{t} f(t) dt \right| \leq \left| \int_{\delta}^{\infty} \left(\int_{\delta \varepsilon}^{t \varepsilon} \frac{\sin u}{u} du \right) df(t) \right|. \tag{30}$$

Since for each $t \in [a, \infty)$: $\lim_{\varepsilon \to 0} \int_{\delta \varepsilon}^{t \varepsilon} \frac{\sin u}{u} du = 0$ and $\left| \int_{\delta \varepsilon}^{t \varepsilon} \frac{\sin u}{u} du \right| \leq \pi Si(\pi)$ for all $\varepsilon > 0$. Then, we obtain the result applying the Lebesgue Dominated Convergence theorem to the integral on the right in (30).   □

Lemma 38. *Suppose that $0 < \alpha < \beta$ or $\alpha < \beta < 0$. If $f \in BV_0(\mathbb{R})$, then for all $s \in [\alpha, \beta]$ we have*

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_{\alpha}^{\beta} e^{ixs} \int_{a}^{b} f(t) e^{-ist} dt ds = \int_{\alpha}^{\beta} e^{ixs} \int_{-\infty}^{\infty} f(t) e^{-ist} dt ds. \tag{31}$$

Proof. We will do the proof for $0 < \alpha < \beta$. Let $\widehat{f_{0b}}(s) = \int_{0}^{b} f(t) e^{-ist} dt$ and $\widehat{f_0}(s) = \int_{0}^{\infty} f(t) e^{-ist} dt$, which are continuous on $\mathbb{R} \setminus \{0\}$. Therefore the integrals in (31) exist. We know that there is $R > 0$ such that $|f(t)| \leq R$ for all $t \in \mathbb{R}$, and that for any $b > 0$: $V(f; [0, b]) \leq V(f; [0, \infty))$. For each $s \in [\alpha, \beta]$ the Multiplier theorem implies

$$\left| \widehat{f_{0b}}(s) \right| \leq \frac{2}{\alpha} \{ R + V(f; [0, \infty)) \} = N.$$

This inequality implies that for any $b > 0$ and all $s \in [\alpha, \beta]$: $\left| e^{ixs} \widehat{f_{0b}}(s) \right| \leq N$, for each $x \in \mathbb{R}$. Applying the theorem of Hake we have: $\lim_{b \to \infty} \widehat{f_{0b}}(s) = \widehat{f_0}(s)$. Then, by the Lebesgue Dominated Convergence theorem,

$$\lim_{b \to \infty} \int_{\alpha}^{\beta} e^{ixs} \widehat{f_{0b}}(s) ds = \int_{\alpha}^{\beta} e^{ixs} \widehat{f_0}(s) ds.$$

To conclude the proof, we follow a similar process over the interval $[a, 0]$ leading a to minus infinity.   □

To obtain the Dirichlet-Jordan theorem we state the following lemma [22, Theorem 11.8].

Lemma 39. *Let $\delta > 0$. If g is of bounded variation on $[0, \delta]$, then*

$$\lim_{M \to \infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin Mt}{t} dt = g(0+)$$

Theorem 40 (Dirichlet-Jordan Theorem). *If f is a function in $BV_0(\mathbb{R})$, then, for each $x \in \mathbb{R}$,*

$$\lim_{\substack{M \to \infty \\ \varepsilon \to 0}} \frac{1}{2\pi} \int_{\varepsilon < |s| < M} e^{ixs} \widehat{f}(s) ds = \frac{1}{2} \{ f(x+0) + f(x-0) \}. \tag{32}$$

Proof. Let $g(x,t) = f(x-t) + f(x+t)$ and suppose that $\delta > 0$. By Fubini's theorem for the Lebesgue integral [22, Theorem 15.7] at $[-M, -\varepsilon] \times [a,b]$ and $[\varepsilon, M] \times [a,b]$ and Lema 38, we have

$$\int_{\varepsilon < |s| < M} e^{ixs} \int_{-\infty}^\infty f(t) e^{-ist} dt ds = \lim_{\substack{a \to -\infty \\ b \to \infty}} \left(\int_{-M}^{-\varepsilon} + \int_\varepsilon^M \right) e^{ixs} \int_a^b f(t) e^{-ist} dt ds$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f(t) \left(\int_{-M}^{-\varepsilon} + \int_\varepsilon^M \right) e^{is(x-t)} ds dt$$

$$= \int_{-\infty}^\infty f(t) \left(\int_{-M}^{-\varepsilon} + \int_\varepsilon^M \right) e^{is(x-t)} ds dt$$

$$= 2 \int_0^\infty \frac{g(x,t)}{t} (\sin M\, t - \sin \varepsilon t) dt$$

$$= 2 \int_\delta^\infty \frac{g(x,t)}{t} (\sin M\, t - \sin \varepsilon t) dt$$

$$+ 2 \int_0^\delta \frac{g(x,t)}{t} (\sin M\, t - \sin \varepsilon t) dt.$$

In $[\delta, \infty]$, by Corollary 36 and Lemma 37, we get

$$\lim_{M \to \infty, \varepsilon \to 0} \int_\delta^\infty \frac{g(x,t)}{t} (\sin M\, t - \sin \varepsilon t) dt = 0. \tag{33}$$

In $[0, \delta]$, applying the Lebesgue Dominate Convergence theorem,

$$\lim_{\varepsilon \to 0} \int_0^\delta \frac{g(x,t)}{t} \sin \varepsilon t dt = 0. \tag{34}$$

Now, by Lemma 39,

$$\lim_{M\to\infty} \int_0^\delta g(x,t)\frac{\sin Mt}{t}dt = g(x,0+) = \frac{\pi}{2}\left[f(x-0) + f(x+0)\right].$$

We conclude the proof combining (33), (34) and the above expression. $\square$

We observe that the classical theorem of Dirichlet-Jordan on $L(\mathbb{R})$ is a particular case of Theorem 40. Taking into account that $HK(\mathbb{R}) \cap BV(\mathbb{R}) \subset BV_0(\mathbb{R})$, then from Theorem 34 and Theorem 40 we get that if $f \in HK(\mathbb{R}) \cap BV(\mathbb{R})$, then its Fourier transform $\widehat{f}(s)$ exists in each $s \in \mathbb{R}$; $\widehat{f} \in C_0(\mathbb{R}\setminus\{0\})$, and the expression (32) holds for each $x \in \mathbb{R}$.

Corollary 41. *There exist functions in $L^2(\mathbb{R}) \setminus L(\mathbb{R})$ such that their Fourier transforms exist as in (1) and, for each $x \in \mathbb{R}$, the expression (32) is true.*

6. Conclusions

We present theorems about convergence of integrals of products in the completion of $HK(I)$, which those we have a version of Riemann-Lebesgue Lemma (over compact intervals) and analogous results at Riemann-Lebesgue property, a characterization of behavior of n-th partial sum of the Fourier series. Moreover, we have gotten basic properties (existence as integral, continuity, asymptotic behavior) about Fourier transform using Henstock-Kurzweil Integral, for this was necessary to get a generalization of Riemann-Lebesgue Lemma over $BV_0(\mathbb{R})$, in particular those characteristics are valid over $HK(\mathbb{R}) \cap BV(\mathbb{R})$. This intersection does not have relation inclusion with Lebesgue integrable functions space, we give a set of functions such that it belongs to $HK(\mathbb{R}) \cap BV(\mathbb{R}) \setminus L(\mathbb{R})$. Finally we have a generalization of Dirichlet-Jordan over $BV_0(\mathbb{R})$.

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