Chapter 5

Double Infinitesimal Fourier Transform

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Additional information is available at the end of the chapter

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1. Introduction

For Fourier transform theory, one of the most important and difficult things is how to treat the Dirac delta function and how to define it. In 1930, the Dirac delta function was defined originally by Paul A.M. Dirac([1]) in order to create the quantum mechanical theory in Physics. In classical mechanics, there is the beautiful Newton's law. Under it, it is assumed that a particle is a point with a mass. For the investigation of a small world for example, elementary particles, it should be changed to the quantum mechanical theory where particles are not already only points as in Mathematics but also some area with infinitesimal length for us. They have some properties like waves. The Dirac delta function is defined to be realized the image of the particle in the small world. The particle changed to be the moving wave, and it becomes a set of such waves. It is called field in Physics and we need the second quantization . The quantum mechanics is developed to the quantum field theory where the delta function is much complicated to treat in the standard mathematical theory.

The delta function is usually defined as the delta measure in the functional analysis. Under the basic definition, the functional analysis is developed in the functional space for example Banach space, Hilbert space. These theory is applied to the existence problem of solutions for the ordinary and partial differential equations. On the other hand, the delta function is also defined just as a function in the extension of the real number field ([3],[4],[5],[18]) in 1962. The idea is that firstly the real and complex number fields are extended to the nonstandard universe, secondly the delta function is defined as a function in the extended universe (cf. [3]).

In this chapter the real number field and complex number field are extended twice and a higher degree of delta function is defined as a function on the space of functions. By using the secondly extended delta function, the Fourier transform theory is considered, that is called " double infinitesimal Fourier transform ". In the theory, the Poisson summation formula is also satisfied, and some important examples are calculated. The Fourier transforms of δ, δ^2, ... , and $\sqrt{\delta}$, ... can be calculated, which are constant functions as 1, infinite, ... , and infinitesimal, ...

Then the Fourier transform of the gaussian functional is also calculated. The gaussian functional means that the standard part of the image for $\alpha \in L^2$ is $\exp\left(-\pi\xi \int_{-\infty}^{\infty} \alpha^2(t)dt\right)$, for $\xi \in \mathbf{C}$ with $\mathrm{Re}(\xi) > 0$. The double infinitesimal Fourier transform is calculated as $C_\xi \exp\left(-\pi\xi^{-1} \int_{-\infty}^{\infty} \alpha^2(t)dt\right)$ for $\alpha \in L^2(\mathbf{R})$, in which C_ξ is a constant independent of α .

Finally a sort of functional is constructed in the theory that associates to Riemann's zeta function. The path integral is defined for the application in the theory, and it is shown that the path integral of the functional Z_s corresponds to Riemann's zeta function in the case that $\mathrm{Re}(s) > 1$. By using the Poisson summation formula for the functional, a relationship appears between the functional and Riemann's zeta function.

2. Infinitesimal Fourier transform

The usual universe is extended, in order to treat many stages of delta functions and functions on the space of functions. For the extension, there exists two methods, one is the second extension of the universe in the nonstandard analysis ([8],[9]) and the other is the Relative set theory in the axiomatic set theory ([13]). The first one ([8],[9]) is explained here, by using an ultrafilter .

2.1. First extension of the universe

Let Λ be an infinite set. Let F be a nonprincipal ultrafilter on Λ. For each $\lambda \in \Lambda$, let S_λ be a set. An equivalence relation $\sim$ is induced from F on $\prod_{\lambda\in\Lambda} S_\lambda$. For $\alpha = (\alpha_\lambda),\ \beta = (\beta_\lambda)$ $(\lambda \in \Lambda)$,

$$\alpha \sim \beta \Longleftrightarrow \{\lambda \in \Lambda \,|\, \alpha_\lambda = \beta_\lambda\} \in F. \tag{1}$$

The set of equivalence classes is called *ultraproduct* of S_λ for F with respect to $\sim$. If $S_\lambda = S$ for $\lambda \in \Lambda$, then it is called *ultraproduct* of S for F and it is written as ${}^*S$. The set S is naturally embedded in ${}^*S$ by the following mapping :

$$s\ (\in S) \mapsto [(s_\lambda = s), \lambda \in \Lambda]\ (\in {}^*S) \tag{2}$$

where [ ] denotes the equivalence class with respect to the ultrafilter F. The mapping is written as $*$, and call it naturally elementary embedding. From now on, we identify the image $*(S)$ as S.

Definition 2.1.1.

Let H $(\in {}^*\mathbf{Z})$ be an infinite even number. The infinite number H is even, when for $H = [(H_\lambda),\ \lambda \in \Lambda]$, $\{\lambda \in \Lambda \,|\, H_\lambda \text{ is even}\} \in F$. The number $\frac{1}{H}$ is written as ε. We define an infinitesimal lattice space $\mathbf{L}$, an infinitesimal lattice subspace L and a space of functions $R(L)$ on L as follows :

$\mathbf{L} := \varepsilon\,{}^*\mathbf{Z} = \{\varepsilon z \,|\, z \in {}^*\mathbf{Z}\}$,

$L := \left\{\varepsilon z \,\middle|\, z \in {}^*\mathbf{Z},\ -\frac{H}{2} \le \varepsilon z < \frac{H}{2}\right\}\ (\subset \mathbf{L})$,

$R(L) := \{\varphi \mid \varphi \text{ is an internal function from } L \text{ to } {}^*\mathbf{C}\}$.

The space $R(L)$ is extended to the space of periodic functions on $\mathbf{L}$ with period H. We write the same notation $R(L)$ for the space of periodic functions.

Gaishi Takeuchi([18]) introduced an infinitesimal δ function. Furthermore Moto-o Kinoshita ([4],[5]) constructed an infinitesimal Fourier transformation theory on $R(L)$. It is explained briefly.

Definition 2.1.2.

For $\varphi, \psi \in R(L)$, the infinitesimal δ function, the infinitesimal Fourier transformation $F\varphi$ ($\in R(L)$), the inverse infinitesimal Fourier transformation $\overline{F}\varphi$ ($\in R(L)$) and the convolution $\varphi * \psi$ ($\in R(L)$) are defined as follows :

$$\delta \in R(L),\ \ \delta(x) := \begin{cases} H & (x = 0) \\ 0 & (x \neq 0) \end{cases} \tag{3}$$

$$(F\varphi)(p) := \sum_{x \in L} \varepsilon \exp\left(-2\pi i p x\right) \varphi(x) \tag{4}$$

$$(\overline{F}\varphi)(p) := \sum_{x \in L} \varepsilon \exp\left(2\pi i p x\right) \varphi(x) \tag{5}$$

$$(\varphi * \psi)(x) := \sum_{y \in L} \varepsilon \varphi(x - y)\psi(y). \tag{6}$$

The definition implies the following theorem as same as the Fourier transform for the finite discrete abelian group.

Theorem 2.1.3.

(1) $\delta = F1 = \overline{F}1$, (2) F is unitary, $F^4 = 1, \overline{F}F = F\overline{F} = 1$,

(3) $f * \delta = \delta * f = f$, (4) $f * g = g * f$,

(5) $F(f * g) = (Ff)(Fg)$, (6) $\overline{F}(f * g) = (\overline{F}f)(\overline{F}g)$,

(7) $F(fg) = (Ff) * (Fg)$, (8) $\overline{F}(fg) = (\overline{F}f) * (\overline{F}g)$.

The definition implies the following proposition by the simple calculation:

Proposition 2.1.4.

If $l \in \mathbf{R}$, then

$$F\delta^l = (H)^{(l-1)}. \tag{7}$$

Examples of the infinitesimal Fourier transform for functions

The infinitesimal Fourier transforms of the gaussian function φ_ξ, $\varphi_{im} \in R(L)$ are calculated as follows: $\varphi_\xi(x) = \exp(-\xi\pi x^2)$, where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$,

$\varphi_{im}(x) = \exp(-im\pi x^2)$, where $m \in \mathbf{Z}$.

For φ_ξ, we obtain :

Proposition 2.1.5.

$(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$, where $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2)$ and p is an element of the lattice L.

If p is finite, then $\mathrm{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}}$.

Proof. The infinitesimal Fourier transforms of φ_ξ is :

$$\begin{aligned}(F\varphi_\xi)(p) &= \sum_{x\in L} \varepsilon \exp(-2\pi ipx)\exp(-\xi\pi x^2) \\ &= (\sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2))\exp(-\pi\frac{1}{\xi}p^2) = c_\xi(p)\varphi_\xi(\frac{p}{\xi}) \end{aligned} \tag{8}$$

where $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\xi\pi(x + \frac{i}{\xi}p)^2)$. If p is finite, then $\mathrm{st}(c_\xi(p))$
$= \int_{-\infty}^{\infty} \exp\left(-\xi\pi\left(t + \frac{i}{\xi}\mathrm{st}(p)\right)^2\right) dt = \frac{1}{\sqrt{\xi}}$.

Theorem 2.1.3 (8) implies the following for c_ξ :

Proposition 2.1.6.

$$\varphi_\xi(x') = \left(\overline{F}c_\xi(p) * \left(c_{\frac{1}{\xi}}(-x)\varphi_\xi(x)\right)\right)(x'). \tag{9}$$

Proof. It is obtained : $(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi})$, and put $\overline{F}$ to the above :

$(\overline{F}(F\varphi_\xi))(x) = (\overline{F}(c_\xi(p)\varphi_\xi(\frac{p}{\xi})))(x)$

$= (\overline{F}c_\xi(p) * \overline{F}\varphi_\xi(\frac{p}{\xi}))(x)$, that is, $\varphi_\xi(x) = (\overline{F}c_\xi(p) * \overline{F}\varphi_\xi(\frac{p}{\xi}))(x)$.

Now $(\overline{F}\varphi_\xi(\frac{p}{\xi}))(x) = \sum_{p\in L} \varepsilon \exp(-2\pi ipx)\exp(-\xi(\frac{p}{\xi})^2\pi)$

$= \sum_{p\in L} \varepsilon \exp(-\pi\frac{1}{\xi}(p^2 - 2\pi i\xi px)) = \left(\sum_{p\in L} \varepsilon \exp(-\frac{\pi}{\xi}(p - i\xi x)^2)\right)\varphi_\xi(x)$.

By the definition : $c_\xi(p) = \sum_{x\in L} \varepsilon \exp(-\pi\xi(x + i\frac{1}{\xi}p)^2)$, the sum
$\sum_{p\in L} \varepsilon \exp(-\frac{\pi}{\xi}(p - i\xi x)^2)$ is $c_{\frac{1}{\xi}}(-x)$. Hence

$$\varphi_\xi(x') = \left(\overline{F}c_\xi(p) * \left(c_{\frac{1}{\xi}}(-x)\varphi_\xi(x)\right)\right)(x'). \tag{10}$$

For the following proposition 2.1.7, the Gauss sum is recalled (cf.[15]) : For $z \in \mathbf{N}$, the Gauss sum $\sum_{l=0}^{z-1} \exp(-i\frac{2\pi}{z}l^2)$ is equal to $\sqrt{z}\frac{1+(-i)^z}{1-i}$.

Proposition 2.1.7. If $m|2H^2$ and $m|\frac{p}{\varepsilon}$, then

$$(F\varphi_{im})(p) = c_{im}(p)\exp(i\pi\frac{1}{m}p^2) \tag{11}$$

where $c_{im}(p) = \sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$ for positive m and $c_{im}(p) = \sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}$ for negative m.

Proof. $(F\varphi_{im})(p) = \sum_{x\in L}\varepsilon\exp(-im\pi x^2)\exp(-2\pi ixp)$
$= c_{im}(p)\exp(i\pi\frac{1}{m}p^2)$, where $c_{im}(p) = \sum_{x\in L}\varepsilon\exp(-im\pi(x+\frac{p}{m})^2)$.

Since $m|\frac{p}{\varepsilon}$, the element $\frac{p}{m}$ is in L. It is remarked that $\exp(-i\pi mx^2) = \exp(-i\pi m(x+H)^2)$. For positive m,

$$c_{im}(p) = \sum_{x\in L}\varepsilon\exp(-im\pi x^2) = \overline{\frac{m}{2}\left(\varepsilon\sqrt{\frac{2H^2}{m}}\frac{1+(-i)^{\frac{2H^2}{m}}}{1-i}\right)} \tag{12}$$

by the above Gauss sum. Hence $c_{im}(p) = \sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$. For negative m, the proof is as same as the above.

2.2. Second extension of the universe

To treat a *-unbounded functional f in the nonstandard analysis, we need a second nonstandardization. Let $F_2 := F$ be a nonprincipal ultrafilter on an infinite set $\Lambda_2 := \Lambda$ as above. Denote the ultraproduct of a set S with respect to F_2 by $^*S$ as above. Let F_1 be another nonprincipal ultrafilter on an infinite set Λ_1. Take the *-ultrafilter $^*F_1$ on $^*\Lambda_1$. For an internal set S in the sense of *-nonstandardization, let $^\star S$ be the *-ultraproduct of S with respect to $^*F_1$. Thus, a double ultraproduct $^\star(^*\mathbf{R})$, $^\star(^*\mathbf{Z})$, etc are defined for the set $\mathbf{R}$, $\mathbf{Z}$, etc. It is shown easily that

$$^\star(^*\mathbf{S}) = S^{\Lambda_1\times\Lambda_2}/F_1^{F_2}, \tag{13}$$

where $F_1^{F_2}$ denotes the ultrafilter on $\Lambda_1\times\Lambda_2$ such that for any $A\subset\Lambda_1\times\Lambda_2$, $A\in F_1^{F_2}$ if and only if

$$\{\lambda\in\Lambda_1 \,|\, \{\mu\in\Lambda_2 \,|\, (\lambda,\mu)\in A\}\in F_2\}\in F_1. \tag{14}$$

The work is done with this double nonstandardization. The natural imbedding $^\star S$ of an internal element S which is not considered as a set in *-nonstandardization is often denoted simply by S.

An infinite number in $\star({}^*\mathbf{R})$ is defined to be greater than any element in ${}^*\mathbf{R}$. We remark that an infinite number in ${}^*\mathbf{R}$ is not infinite in $\star({}^*\mathbf{R})$, that is, the word "an infinite number in $\star({}^*\mathbf{R})$" has a double meaning. An infinitesimal number in $\star({}^*\mathbf{R})$ is also defined to be nonzero and whose absolute value is less than each positive number in ${}^*\mathbf{R}$.

Definition 2.2.1.

Let $H(\in {}^*\mathbf{Z})$, $H'(\in \star({}^*\mathbf{Z}))$ be even positive numbers such that H' is larger than any element in ${}^*\mathbf{Z}$, and let $\varepsilon(\in {}^*\mathbf{R})$, $\varepsilon'(\in \star({}^*\mathbf{R}))$ be infinitesimals satifying $\varepsilon H = 1$, $\varepsilon' H' = 1$. We define as follows :

$\mathbf{L} := \varepsilon {}^*\mathbf{Z} = \{\varepsilon z \,|\, z \in {}^*\mathbf{Z}\}$, $\mathbf{L}' := \varepsilon' \star({}^*\mathbf{Z}) = \{\varepsilon' z' \,|\, z' \in \star({}^*\mathbf{Z})\}$,

$L := \left\{\varepsilon z \,\middle|\, z \in {}^*\mathbf{Z}, -\frac{H}{2} \leq \varepsilon z < \frac{H}{2}\right\} (\subset \mathbf{L})$,

$L' := \left\{\varepsilon' z' \,\middle|\, z' \in \star({}^*\mathbf{Z}), -\frac{H'}{2} \leq \varepsilon' z' < \frac{H'}{2}\right\} (\subset \mathbf{L}')$.

Here L is an ultraproduct of lattices

$L_\mu := \left\{\varepsilon_\mu z_\mu \,\middle|\, z_\mu \in \mathbf{Z}, -\frac{H_\mu}{2} \leq \varepsilon_\mu z_\mu < \frac{H_\mu}{2}\right\} \ (\mu \in \Lambda_2)$

in $\mathbf{R}$, and L' is also an ultraproduct of lattices

$L'_\lambda := \left\{\varepsilon'_\lambda z'_\lambda \,\middle|\, z'_\lambda \in {}^*\mathbf{Z}, -\frac{H'_\lambda}{2} \leq \varepsilon'_\lambda z'_\lambda < \frac{H'_\lambda}{2}\right\} \ (\lambda \in \Lambda_1)$

in ${}^*\mathbf{R}$ that is an ultraproduct of

$L'_{\lambda\mu} := \left\{\varepsilon'_{\lambda\mu} z'_{\lambda\mu} \,\middle|\, z'_{\lambda\mu} \in \mathbf{Z}, -\frac{H'_{\lambda\mu}}{2} \leq \varepsilon'_{\lambda\mu} z'_{\lambda\mu} < \frac{H'_{\lambda\mu}}{2}\right\} \ (\mu \in \Lambda_2)$.

A latticed space of functions X is defined as follows,

$$\begin{aligned} X &:= \{a \,|\, a \text{ is an internal function with double meanings, from } {}^\star L \text{ to } L'\} \\ &= \{[(a_\lambda),\, \lambda \in \Lambda_1] \,|\, a_\lambda \text{ is an internal function from } L \text{ to } L'_\lambda\} \end{aligned} \tag{15}$$

where $a_\lambda : L \to L'_\lambda$ is $a_\lambda = [(a_{\lambda\mu}),\, \mu \in \Lambda_2]$, $a_{\lambda\mu} : L_\mu \to L'_{\lambda\mu}$.

Three equivalence relations $\sim_H$, $\sim_{\star(H)}$ and $\sim_{H'}$ are defined on $\mathbf{L}$, $\star(\mathbf{L})$ and $\mathbf{L}'$:

$x \sim_H y \Longleftrightarrow x - y \in H {}^*\mathbf{Z}$, $x \sim_{\star(H)} y \Longleftrightarrow x - y \in \star(H) \star({}^*\mathbf{Z})$,

$x \sim_{H'} y \Longleftrightarrow x - y \in H' \star({}^*\mathbf{Z})$.

Then $\mathbf{L}/\sim_H$, $\star(\mathbf{L})/\sim_{\star(H)}$ and $\mathbf{L}'/\sim_{H'}$ are identified as L, $\star(L)$ and L'. Since $\star(L)$ is identified with L, the set $\star(\mathbf{L})/\sim_{\star(H)}$ is identified with $\mathbf{L}/\sim_H$. Furthermore X is represented as the following internal set :

$$\{a \,|\, a \text{ is an internal function from } \star(\mathbf{L})/\sim_{\star(H)} \text{ to } \mathbf{L}'/\sim_{H'}\}. \tag{16}$$

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The same notation is used as a function from ${}^{\star}(L)$ to L' to represent a function in the above internal set. The space A of functionals is defined as follows:

$$A := \{f \mid f \text{ is an internal function with a double meaning from } X \text{ to } {}^{\star}({}^{*}\mathbf{C})\}. \tag{17}$$

An infinitesimal delta function $\delta(a)(\in A)$, an infinitesimal Fourier transform of $f(\in A)$, an inverse infinitesimal Fourier transform of f and a convolution of f, $g(\in A)$, are defined by the following :

Definition 2.2.2. The delta function

$$\delta(a) := \begin{cases} (H')^{({}^{\star}H)^2} & (a = 0) \\ 0 & (a \neq 0) \end{cases} \tag{18}$$

and, with $\varepsilon_0 := (H')^{-({}^{\star}H)^2} \in {}^{\star}({}^{*}\mathbf{R})$,

$$(Ff)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i \sum_{k \in L} a(k)b(k)\right) f(a) \tag{19}$$

$$(\overline{F}f)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i \sum_{k \in L} a(k)b(k)\right) f(a) \tag{20}$$

$$(f * g)(a) := \sum_{a' \in X} \varepsilon_0 f(a - a')g(a'). \tag{21}$$

The inner product on A is defined as:

$$(f, g) := \sum_{b \in X} \varepsilon_0 \overline{f(b)} g(b), \tag{22}$$

where $\overline{f(b)}$ is the complex conjugate of $f(b)$. In the section 3, Riemann's zeta function is written down as a nonstandard functional in Definition 2.2.2. In general, $\sum_{k \in L} a^2(k)$ is infinite, and it is difficult to consider the meaning of F, $\overline{F}$ in Definition 2.2.2 as standard objects. They are defined only algebraically. In order to understand Definition 2.2.2 analytically for a standard one, we change the definition briefly, to Definition 2.2.3. By replacing the definitions of L', δ, ε_0, F, $\overline{F}$ in Definition 2.2.2 as the following, another type of infinitesimal Fourier transformation is defined later. The different point is only the definition of an inner product of the space of functions X. In Definition 2.2.2 , the inner product of a, $b(\in X)$ is $\sum_{k \in L} a(k)b(k)$, and in the following definition, it is ${}^{\star}\varepsilon \sum_{k \in L} a(k)b(k)$. **Definition 2.2.3.** $L' := \left\{\varepsilon' z' \;\middle|\; z' \in {}^{\star}({}^{*}\mathbf{Z}), -{}^{\star}H\frac{H'}{2} \leq \varepsilon' z' < {}^{\star}H\frac{H'}{2}\right\}$,

$$\delta(a) := \begin{cases} ({}^{\star}H)^{\frac{({}^{\star}H)^2}{2}} H'^{({}^{\star}H)^2} & (a = 0), \\ 0 & (a \neq 0) \end{cases} \tag{23}$$

and, with $\varepsilon_0 := ({}^{\star}H)^{-\frac{({}^{\star}H)^2}{2}} H'^{-({}^{\star}H)^2}$

$$(Ff)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(-2\pi i\, {}^{\star}\varepsilon \sum_{k \in L} a(k)b(k)\right) f(a) \tag{24}$$

$$(\overline{F}f)(b) := \sum_{a \in X} \varepsilon_0 \exp\left(2\pi i\, {}^{\star}\varepsilon \sum_{k \in L} a(k)b(k)\right) f(a). \tag{25}$$

Then the lattice $L'_{\lambda\mu}$ is an abelian group for each $\lambda\mu$. The following theorem is obtained as same as the case of the discrete abelian group :

Theorem 2.2.4.

(1) $\delta = F1 = \overline{F}1$, (2) F is unitary, $F^4 = 1, \overline{F}F = F\overline{F} = 1$,

(3) $f * \delta = \delta * f = f$, (4) $f * g = g * f$,

(5) $F(f * g) = (Ff)(Fg)$, (6) $\overline{F}(f * g) = (\overline{F}f)(\overline{F}g)$,

(7) $F(fg) = (Ff) * (Fg)$, (8) $\overline{F}(fg) = (\overline{F}f) * (\overline{F}g)$.

The definition directly implies the following proposition :

Proposition 2.2.5. If $l \in \mathbf{R}^+$, then

$$F\delta^l = (H')^{(l-1)({}^{\star}H)^2}. \tag{26}$$

If there exists α, $\beta \in L^2(\mathbf{R})$ so that $a = {}^{*}\alpha|_L$, $b = {}^{*}\beta|_L$, that is, $a(k) = \star({}^{*}\alpha(k))$, $b(k) = \star({}^{*}\beta(k))$, then $\mathrm{st}(\mathrm{st}({}^{\star}\varepsilon \sum_{k \in L} a(k)b(k))) = \int_{-\infty}^{\infty} \alpha(x)b(x)dx$. Definition 2.2.3 is easier understanding than Definition 2.2.2 for a standard meaning in analysis. For the reason, we consider mainly Definition 2.2.3 about several examples. However Definition 2.2.2 is also treated algebraically, as algebraically defined functions are not always L^2-functions on $\mathbf{R}$. The two types of Fourier transforms are different in a standard meaning.

Examples of the double infinitesimal Fourier transform

It is defined: an equivalence relation $\sim_{{}^{\star}HH'}$ in $\mathbf{L}'$ by $x \sim_{{}^{\star}HH'} y \Leftrightarrow x - y \in {}^{\star}HH'\,{}^{\star}({}^{*}\mathbf{Z})$. The quotient space $\mathbf{L}'/\sim_{{}^{\star}HH'}$ is defined with L'. Let

$X_{H,{}^{\star}HH'} := \{a' \mid a'$ is an internal function with a double meaning, from ${}^{\star}\mathbf{L}/\sim_{\star(H)}$ to $\mathbf{L}'/\sim_{{}^{\star}HH'}\}$

and let $\mathbf{e}$ be a mapping from X to $X_{H,{}^{\star}HH'}$, defined by $(\mathbf{e}(a))([k]) = [a(\hat{k})]$, where [ ] on the left-hand side represents the equivalence class for the equivalence relation $\sim_{\star(H)}$ in ${}^{\star}\mathbf{L}$, $\hat{k}$ is a representative in ${}^{\star}(L)$ satisfying $k \sim_{\star(H)} \hat{k}$, and [ ] on the right-hand side represents the equivalence class for the equivalence relation $\sim_{{}^{\star}HH'}$ in $\mathbf{L}'$. Furthermore $f(a')$ is identified to be $f(\mathbf{e}^{-1}(a'))$.

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The double infinitesimal Fourier transform of $\exp\left(-\pi\, {}^\star\varepsilon\xi \sum_{k\in L} a^2(k)\right)$

The double infinitesimal Fourier transform of

$$g_\xi(a) = \exp\left(-\pi\, {}^\star\varepsilon\xi \sum_{k\in L} a^2(k)\right), \tag{27}$$

where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$,

is calculated in the space A of functionals, for Definition 2.2.3. It is identified ${}^\star({}^*\xi) \in \mathbf{C}$ with $\xi \in \mathbf{C}$.

Theorem 2.2.6. $F(g_\xi)(b) = C_\xi(b) g_\xi(\frac{b}{\xi})$, where $b \in X$ and

$$C_\xi(b) = \sum_{a\in X} \varepsilon_0 \exp\left(-\pi\, {}^\star\varepsilon\xi \sum_{k\in L}(a(k) + i\frac{1}{\xi}b(k))^2\right). \tag{28}$$

Proof. The infinitesimal Fourier transform of $g_\xi(a)$ is done.

$F(g_\xi)(b) = F\left(\exp\left(-\pi\, {}^\star\varepsilon\xi \sum_{k\in L} a^2(k)\right)\right)(b)$

$= \sum_{a\in X} \varepsilon_0 \exp\left(-2i\pi\, {}^\star\varepsilon \sum_{k\in L} a(k)b(k)\right) \exp\left(-\pi\, {}^\star\varepsilon\xi \sum_{k\in L} a^2(k)\right)$

$= C_\xi(b) g_\xi(\frac{b}{\xi})$.

Let $\star\circ* : \mathbf{R} \to {}^\star({}^*\mathbf{R})$ be the natural elementary embedding and let $\mathbf{st}(c)$ for $c \in {}^\star({}^*\mathbf{R})$ be the standard part of c with respect to the natural elementary embedding $\star\circ*$. Let $\mathrm{st}(c)$ be the standard part of c with respect to the natural elementary embedding $\star$ and $*$. Then $\mathbf{st} = st \circ st$.

Theorem 2.2.7. If the image of b $(\in X)$ is bounded by a finite value of ${}^*\mathbf{R}$, that is, there exists $b_0 \in {}^*\mathbf{R}$ such that $k \in L \Rightarrow |b(k)| \leq \star(b_0)$, then

$$\mathrm{st}(C_\xi(b)) = \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2} (\in {}^*\mathbf{R}),\ \mathbf{st}\left(\frac{C_\xi(b)}{\star\left(\left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)}\right) = 1. \tag{29}$$

Proof. $\mathrm{st}(C_\xi(b)) = \mathrm{st}(\sum_{a\in X} \prod_{k\in L} \sqrt{\varepsilon}\varepsilon' \exp\left(-\pi\xi\{\sqrt{\varepsilon}(a(k)) + i\sqrt{\varepsilon}\frac{1}{\xi}(b(k))\}^2\right))$

$= \prod_{k\in L} \int_{-{}^*\infty}^{{}^*\infty} \exp\left(-\pi\xi\{x + i\sqrt{\varepsilon}\frac{1}{\xi}\mathrm{st}_2(b(k))\}^2\right) dx$

$= \prod_{k\in L} \int_{-{}^*\infty}^{{}^*\infty} \exp\left(-\pi\xi x^2\right) dx.$

The argument is same about the infinitesimal Fourier transform of $g'_\xi(a) = \exp(-\pi\xi \sum_{k\in L} a^2(k))$, for Definition 2.2.2, as the above.

Theorem 2.2.8.

$$F(g'_\xi)(b) = B_\xi(b) g'_\xi(\frac{b}{\xi}), \tag{30}$$

where $b \in X$ and

$B_\xi(b) = \sum_{a\in X} \varepsilon_0 \exp\left(-\pi\xi \sum_{k\in L}(a(k) + i\frac{1}{\xi}b(k))^2\right)$. Furthermore, if the image of b $(\in X)$ is bounded by a finite value of ${}^*\mathbf{R}$, that is, $\exists b_0 \in {}^*\mathbf{R}$ s.t. $k \in L \Rightarrow |b(k)| \le \star(b_0)$ then

$$\mathrm{st}(B_\xi(b)) = \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2} \ (\in {}^*\mathbf{R}),\ \mathbf{st}\left(\frac{B_\xi(b)}{\star\left(\left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right)}\right) = 1. \tag{31}$$

The double infinitesimal Fourier transform of $\exp(-i\pi m\,{}^\star\varepsilon \sum_{k\in L} a^2(k))$

The double infinitesimal Fourier transform of $g_{im}(a) = \exp(-i\pi m\,{}^\star\varepsilon \sum_{k\in L} a^2(k))$, where $m \in \mathbf{Z}$, is calculated for Definition 2.2.3.

Proposition 2.2.9. $F(g_{im})(b)$ is written as $C_{im}(b) g_{\frac{1}{im}}(b)$.

If $m | 2\,{}^\star H H'^2$ and $m | \frac{b(k)}{\varepsilon'}$ for an arbitrary k in L, then $F(g_{im})(b) = C_{im}(b) g_{\frac{1}{im}}(b)$, where $C_{im}(b) = \left(\sqrt{\frac{m}{2}} \frac{1+i^{\frac{2\,{}^\star HH'^2}{m}}}{1+i}\right)^{({}^\star H)^2}$ for a positive m and

$C_{im}(b) = \left(\sqrt{\frac{-m}{2}} \frac{1+(-i)^{\frac{2\,{}^\star HH'^2}{-m}}}{1-i}\right)^{({}^\star H)^2}$ for a negative m.

Proof.

$F(g_{im})(b) = C_{im}(b) g_{\frac{1}{im}}(b)$, where $C_{im}(b) = \sum_{a\in X} \varepsilon_0 \exp(-i\pi m\,{}^\star\varepsilon \sum_{k\in L}(a(k) + \frac{1}{m}b(k))^2)$.

When $a(k)$, $b(k)$ are denoted as $\varepsilon' n'$, $\varepsilon' l'$,

$\sum_{-{}^\star H\frac{H'^2}{2} \le a(k) < {}^\star H\frac{H'^2}{2}} \exp(-i\pi m\,{}^\star\varepsilon \sum_{k\in L}(a(k) + \frac{1}{m}b(k))^2$

$$= \sum_{-{}^\star H\frac{H'^2}{2} \le \varepsilon' n' < {}^\star H\frac{H'^2}{2}} \exp(-i\pi m\,{}^\star\varepsilon \sum_{k\in L}(\varepsilon' n' + \varepsilon' \frac{n'}{m})^2). \tag{32}$$

Since $m | \frac{b(k)}{\varepsilon'}$, for a positive m, it is equal to

$$\sum_{-{}^\star H\frac{H'^2}{2} \le \varepsilon' n' < {}^\star H\frac{H'^2}{2}} \exp(-i\pi m\,{}^\star\varepsilon\varepsilon'^2 n'^2) = \frac{m}{2}\sqrt{\frac{2\,{}^\star HH'^2}{m}} \frac{1+i^{\frac{2\,{}^\star HH'^2}{m}}}{1+i} \tag{33}$$

by Proposition 2.1.5. Hence $C_{im} = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2^\star HH'^2}{m}}}{1+i}\right)^{(^\star H)^2}$ for a positive m. For a negative m, the proof is as same as the above.

The argument for the infinitesimal Fourier transform of $g'_{im}(a) = \exp(-i\pi m \sum_{k\in L} a^2(k))$, for Definition 2.2.2, is as same as the above one of g_{im} for Definition 2.2.3.

Proposition 2.2.10. If $m|2^\star HH'^2$ and $m|\frac{b(k)}{\varepsilon'}$ for an arbitrary k in L, then $(F((g'_{im}))(b) = B_{im}(b)g'_{\frac{1}{im}}(b)$, where $B_{im}(b) = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{(^\star H)^2}$ for a positive m and $B_{im}(b) = \left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{(^\star H)^2}$ for a negative m.

2.3. The meaning of the double infinitesimal Fourier transform

There exists a natural injection from a space of standard functions to X as

$$\alpha \mapsto (a : k \in L \mapsto \star(^*\alpha(k)) \in L'). \tag{34}$$

Hence a space of standard functions is embedded in X through the natural injection. If there is no confusion, standard functions are identified as nonstandard functions by the natural injection.

For a standard functional f, if the domain of $^\star(^*f)$ is in X, we can define a Fourier transform $F(^\star(^*f))$. Since st(st($F(^\star(^*f)$))) is a standard functional as st(st($F(^\star(^*f)$)))(α) =st(st($F(^\star(^*f))(a)$)) for $a : k \in L \to \star(^*\alpha(k)) \in L'$, such standard functional has a Fourier transform st(st($F(^\star(^*f)$))).

Similarly to the case of functions, the following subspace $\mathcal{L}^2(A)$ of A is defined:

Definition 2.3.1.

$$\mathcal{L}^2(A) := \{f \in A | \text{ there exists } c \in {}^*\mathbf{R} \text{ so that } (\frac{1}{c}\sum_{a\in X}\varepsilon_0|f(a)|^2) < +\infty\}. \tag{35}$$

The standard part st($\sum_{a\in X}\varepsilon_0|f(a)|^2$) is a $*-$ norm in $\mathcal{L}^2(A)$. Theorem 2.1.3 (2) implies the following proposition.

Proposition 2.3.2. The Fourier transform F and the inverse $\overline{F}$ preserve the space $\mathcal{L}^2(A)$.

Hence if f is a standard functional so that $^\star(^*f)$ is an element of $\mathcal{L}^2(A)$, the Fourier transformation $F(^\star(^*f))$, $\overline{F}(^\star(^*f))$ are also in $\mathcal{L}^2(A)$. Now there is no theory of Fourier transform for functionals in "standard analysis", and it is well-known that there is no nontrivial translation-invariant measure on an infinite-dimensional separable Banach space. In fact, on the infinite-dimensional Banach space there is an infinite sequence of pairwise disjoint open balls of same sizes in a larger ball. The measure is translation-invariant, the measure of the small balls are same, but the measure of the larger ball is finite, it is contradiction. By the reason we do not argue a relationship between our Fourier transform and standard Fourier transform, any more.

Here number fields are extended twice to realize the delta function for functionals. The extended real number field divided to very small infinitesimal lattices. These lattices are too small for normal real number field and the first extended real number field to observe them. Axiomatically, the double extended number field can be treat in a large universe, that is, relative set theory ([13],[14]). The concept of observable and relatively observable are formulated, and two kinds of delta functions are defined. The Fourier transform theory is developed, which is called divergence Fourier transform . It is applied to solve an elementary ordinary differential equation with a delta function(cf.[12]).

3. Poisson summation formula

The Poisson summation formula is a fundamental formula for each Fourier transform theory. In this section, it is explained about the Kinoshita's Fourier transform and our double infinitesimal Fourier transform . Some examples of the gaussian type functions are calculated for the applications of the Poisson summation formula.

3.1. Poisson summation formula for infinitesimal Fourier transform

The Poisson summation formula of finite group is extended to Kinoshita's infinitesimal Fourier transform.

Formulation

Theorem 3.1.1. Let S be an internal subgroup of L. Then the following formula is obtained, for $\varphi \in R(L)$,

$$|S^{\perp}|^{-\frac{1}{2}} \sum_{p \in S^{\perp}} (F\varphi)(p) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi(x) \tag{36}$$

where $S^{\perp} := \{p \in L \,|\, \exp(2\pi i p x) = 1 \text{ for } \forall x \in S\}$.

Since L is an internal cyclic group, the group S is also an internal cyclic group. The generator of L is ε. The generator of S is written as εs ($s \in {}^*\mathbf{Z}$). Since the order of L is H^2, so s is a factor of H^2.

The following lemma is prepared for the proof of Theorem 3.1.1.

Lemma 3.1.2. $S^{\perp} = < \varepsilon \frac{H^2}{s} >$.

Proof of Lemma 3.1.2. For $p \in S^{\perp}$, we write $p = \varepsilon t$. Then the following is obtained:

$$\exp(2\pi i p \varepsilon s) = 1 \Longleftrightarrow \exp(2\pi i \varepsilon t \varepsilon s) = 1 \Longleftrightarrow \exp(2\pi i t \frac{s}{H^2}) = 1 \Longleftrightarrow t \frac{s}{H^2} \in {}^*\mathbf{Z}. \tag{37}$$

Hence the generater of $S^{\perp}$ is $\varepsilon\frac{H^2}{s}$.

Proof of Theorem 3.1.1. By Lemma 2.1.2, $|S| = \frac{H^2}{s}$ and $|S^{\perp}| = s$. If $x \notin S$, then $\varepsilon\frac{H^2}{s}xs = \varepsilon H^2 x \in {}^*\mathbf{Z}$, and $\left(\exp\left(2\pi i\varepsilon\frac{H^2}{s}x\right)\right)^s = 1$. For $x \in L$,

$$\sum_{p\in S^{\perp}} \exp(2\pi ipx) = \begin{cases} \frac{\exp(2\pi i(-\frac{H}{2})x)(1-(\exp(2\pi i\varepsilon\frac{H^2}{s}x)^s))}{1-\exp(2\pi i\varepsilon\frac{H^2}{s}x)} & (x \notin S) \\ \sum_{p\in S^{\perp}} 1 & (x \in S) \end{cases}$$
$$= \begin{cases} 0 & (x \notin S) \\ s & (x \in S) \end{cases}. \tag{38}$$

Hence

$\sum_{p\in S^{\perp}}(F\varphi)(p) = \sum_{p\in S^{\perp}} \varepsilon(\sum_{x\in L}\varphi(x)\exp(2\pi ipx))$

$$= \varepsilon\sum_{x\in L}\varphi(x)(\sum_{p\in S^{\perp}}\exp(2\pi ipx)) = \frac{s}{H}\sum_{x\in S}\varphi(x). \tag{39}$$

Thus

$$\frac{1}{\sqrt{s}}\sum_{p\in S^{\perp}}(F\varphi)(p) = \sqrt{\frac{s}{H^2}}\sum_{x\in S}\varphi(x) \tag{40}$$

hence

$$|S^{\perp}|^{-\frac{1}{2}}\sum_{p\in S^{\perp}}(F\varphi)(p) = \frac{1}{|S|^{\frac{1}{2}}}\sum_{x\in S}\varphi(x)\cdots(\sharp_1). \tag{41}$$

Proposition 3.1.3. Especially if s is equal to H, then $(\sharp_1)$ implies that $\sum_{p\in S^{\perp}}(F\varphi)(p) = \sum_{x\in S}\varphi(x)$. The standard part of it is $\mathrm{st}(\sum_{p\in S^{\perp}}(F\varphi)(p)) = \mathrm{st}(\sum_{x\in S}\varphi(x))$.

If there exists a standard function $\varphi' : \mathbf{R} \to \mathbf{C}$ so that $\varphi = {}^*\varphi'|_L$, then the right hand side is equal to $\sum_{-\infty<x<\infty}\varphi'(x)$, that is, $\sum_{-\infty<x<\infty}\mathrm{st}(\varphi)(x)$. Furthermore if εs is infinitesimal and φ' is integrable on $\mathbf{R}$, then

$\mathrm{st}(\varepsilon s\sum_{x\in S}\varphi(x)) = \int_{-\infty}^{\infty}\varphi'(x)dx$.

Since $(\sharp_1)$ implies that

$\sum_{p\in S^{\perp}}(F\varphi)(p) = \varepsilon s\sum_{x\in S}\varphi(x)$,

it is obtained $\mathrm{st}(\sum_{p\in S^{\perp}}(F\varphi)(p)) = \int_{-\infty}^{\infty}\varphi'(x)dx$, that is, $\int_{-\infty}^{\infty}\mathrm{st}(\varphi)(x)dx$.

The even number H is decomposed to prime factors $H = p_1^{l_1} p_2^{l_2} \cdots p_m^{l_m}$, where $p_1 = 2$, $p_1 < p_2 < \cdots < p_m$, each p_i is a prime number, $0 < l_i$. Since S is a subgroup of L, the number s is a factor of H^2. When we write s as $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, the order of S is equal to $p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdots p_m^{2l_m-k_m}$ and the order of $S^\perp$ is $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Hence (27) is

$$(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m})^{-\frac{1}{2}} \sum_{p \in S^\perp} (F\varphi)(p)) = (p_1^{2l_1-k_1} p_2^{2l_2-k_2} \cdots p_m^{2l_m-k_m})^{-\frac{1}{2}} \sum_{x \in S} \varphi(x). \tag{42}$$

Examples

Theorem 3.1.1 is applied to the following two kinds of functions :

$$1.\varphi_i(x) = \exp(-i\pi x^2) \tag{43}$$

$$2.\varphi_\xi(x) = \exp(-\xi\pi x^2) \tag{44}$$

where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$. Then the infinitesimal Fourier transforms are :

$$1.(F\varphi_i)(p) = \exp(-i\frac{\pi}{4})\overline{\varphi_i(p)} \cdots (\sharp_2) \tag{45}$$

$$2.(F\varphi_\xi)(p) = c_\xi(p)\varphi_\xi(\frac{p}{\xi}), \tag{46}$$

where $\mathrm{st}(c_\xi(p)) = \frac{1}{\sqrt{\xi}}$, if p is finite. Hence the following formulas are obtained :

$$1.|S^\perp|^{-\frac{1}{2}} \exp(-i\frac{\pi}{4}) \sum_{p \in S^\perp} \overline{\varphi_i(p)} = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi_i(x), \tag{47}$$

$$2.|S^\perp|^{-\frac{1}{2}} \sum_{p \in S^\perp} c_\xi(p)\varphi_\xi(\frac{p}{\xi}) = |S|^{-\frac{1}{2}} \sum_{x \in S} \varphi_\xi(x). \tag{48}$$

When the generator of S is εs, this is written as the following, explicitly :

$$1.H \exp(-i\frac{\pi}{4}) \sum_{p \in S^\perp} \exp(i\pi p^2) = s \sum_{x \in S} \exp(-i\pi x^2) \tag{49}$$

$$2.H \sum_{p \in S^\perp} c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2) = s \sum_{x \in S} \exp(-\xi\pi x^2). \tag{50}$$

The following proposition is obtained:

Proposition 3.1.4.

(i) If $s = H$, then the generator of S is 1 and $S = S^\perp = L \cap {}^*\mathbf{Z}$. Hence

$$1.\exp(-i\frac{\pi}{4})\sum_{p\in L\cap {}^*\mathbf{Z}}\exp(i\pi p^2)=\sum_{x\in L\cap {}^*\mathbf{Z}}\exp(-i\pi x^2) \tag{51}$$

$$2.\sum_{p\in L\cap {}^*\mathbf{Z}}c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2)=\sum_{x\in L\cap {}^*\mathbf{Z}}\exp(-\xi\pi x^2). \tag{52}$$

Taking their standard parts, we obtain :

$$\begin{aligned}2.st(\sum_{p\in L\cap {}^*\mathbf{Z}}c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2)) &= st(\sum_{x\in L\cap {}^*\mathbf{Z}}\exp(-\xi\pi x^2))\\ &= \sum_{-\infty<n<\infty}\exp(-\xi\pi n^2)=\theta(i\xi)\end{aligned} \tag{53}$$

where $\theta(z)$ is a θ-function, defined by $\theta(z)=\sum_{-\infty<n<\infty}\exp(i\pi z n^2)$.

(ii) If εs is infinitesimal, then the equation:

$H\sum_{p\in S^\perp}c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2)=s\sum_{x\in S}\exp(-\xi\pi x^2)$ implies the following:

$$\begin{aligned}st(\sum_{p\in S^\perp}c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2)) &= st(\varepsilon s\sum_{x\in S}\exp(-\xi\pi x^2))\\ &= \int_{-\infty}^{\infty}\exp(-\xi\pi x^2)dx=\frac{1}{\sqrt{\xi}}.\end{aligned} \tag{54}$$

It is known that $\mathrm{st}(c_\xi(p))=\frac{1}{\sqrt{\xi}}$, and $\sum_{-\infty<x<\infty}\exp(-\xi\pi x^2)$ in the formula 2 of (i) is equal to $\frac{1}{\sqrt{\xi}}\sum_{-\infty<p<\infty}\exp(-\frac{1}{\xi}\pi p^2)$ by the standard Poisson summation formula. Hence, by 2 of (i), we obtain $\mathrm{st}(\sum_{p\in S^\perp}c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2))=\sum_{-\infty<p<\infty}\mathrm{st}(c_\xi(p)\exp(-\frac{1}{\xi}\pi p^2))$.

The formula ($\sharp_2$) in 1 for $\varphi_i(x)$ is extended to $\varphi_{im}(x)=\exp(-im\pi x^2)$, for an integer m so that $m|2H^2$. If $m|\frac{p}{\varepsilon}$, we recall

$(F\varphi_{im})(p)=c_{im}(p)\exp(i\pi\frac{1}{m}p^2)$,

where $c_{im}(p)=\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}$ for a positive m and $c_{im}(p)=\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}$ for a negative m.

Hence $|S^\perp|^{-\frac{1}{2}}\sum_{p\in S^\perp}c_{im}(p)\varphi_{\frac{1}{im}}(p)=|S|^{-\frac{1}{2}}\sum_{x\in S}\varphi_{im}(x)$. When the generator $\varepsilon s'$ of $S^\perp$ satifies $m|s'$, that is, the generator εs of S satifies $m|\frac{H^2}{s}$, it reduces to the following:

$$H\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H^2}{m}}}{1+i}\sum_{p\in S^\perp}\exp(i\pi\frac{1}{m}p^2)=s\sum_{x\in S}\exp(-im\pi x^2) \tag{55}$$

for a positive m,

$$H\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H^2}{-m}}}{1-i}\sum_{p\in S^{\perp}}\exp(i\pi\frac{1}{m}p^2)=s\sum_{x\in S}\exp(-im\pi x^2) \tag{56}$$

for a negative m.

3.2. Poisson summation formula for Definition 2.2.2

Poisson summation formula of finite group is extended to the double infinitesimal Fourier transform for Definition 2.2.2 on the space of functionals.

Formulation

Theorem 3.2.1. Let Y be an internal subgroup of X. Then the following is obtained, for $f \in A$,

$$|Y^{\perp}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp}}(Ff)(b)=|Y|^{-\frac{1}{2}}\sum_{a\in Y}f(a) \tag{57}$$

where $Y^{\perp} := \{b \in X \mid \exp(2\pi i < a, b >) = 1 \text{ for } \forall a \in X\}$ and $< a, b >:= \sum_{k\in L} a(k)b(k)$.

Lemma 3.2.2. $|Y^{\perp}| = \frac{|X|}{|Y|}$.

Proof of Lemma 3.2.2. For $k \in L$, we denote $Y_k := \{a(k) \in L' \mid a \in Y\}$.

$b \in Y^{\perp} \iff \forall a \in Y,\ \exp(2\pi i \sum_{k\in L} a(k)b(k)) = 1$

$\iff \forall k \in L,\ b(k) \in Y_k^{\perp}$

$\iff b : L \to L',\ \forall k \in L,\ b(k) \in Y_k^{\perp}$.

Hence $|Y^{\perp}| = \prod_{k\in L} |Y_k^{\perp}|$. Lemma 3.1.2 implies $|Y_k^{\perp}| = \frac{H'^2}{|Y_k|}$. Thus

$$|Y^{\perp}|=\prod_{k\in L}\left(\frac{H'^2}{|Y_k|}\right)=\frac{H'^{2*H^2}}{\prod_{k\in L}|Y_k|}=\frac{|X|}{|Y|} \tag{58}$$

Proof of Theorem 3.2.1.

$$|Y^{\perp}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp}}(Ff)(b)=|Y^{\perp}|^{-\frac{1}{2}}\sum_{a\in X}\varepsilon_0(\sum_{b\in Y^{\perp}}\exp(-2\pi i<a,b>))f(a). \tag{59}$$

Since $\sum_{b\in Y^{\perp}} \exp(-2\pi i < a, b >) = \begin{cases} 0 & (a \notin Y) \\ |Y^{\perp}| & (a \in Y) \end{cases}$, the above is equal to

$$|Y^{\perp}|^{-\frac{1}{2}}\varepsilon_0|Y^{\perp}| \sum_{a\in Y} f(a) = |Y^{\perp}|^{\frac{1}{2}} H'^{-*H^2} \sum_{a\in Y} f(a) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f(a). \tag{60}$$

In the special case where $f(a) = \prod_{k\in L} f_k(a(k))$,
$(Ff)(b) = \sum_{a\in X} \varepsilon_0 \exp(-2\pi i \sum_{k\in L} a(k)b(k)) \prod_{k\in L} f_k(a(k))$

$$= \prod_{k\in L}(\sum_{a(k)\in L'} \varepsilon' \exp(-2\pi i a(k)b(k)) f_k(a(k)). \tag{61}$$

Namely, the Fourier transform in functional space is the product of those in function space.

Corollary 3.2.3.

(i) If each generator of Y_k is equal to 1, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\sum_{-\infty<n<\infty} \mathrm{st}(f_k)(n)$ converges, then

$$st(\sum_{b\in Y^{\perp}} (Ff)(b)) = \prod_{k\in L}(\sum_{-\infty<n<\infty} st(f_k)(n)). \tag{62}$$

(ii) If each generator of Y_k is infinitesimal, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$ and $\mathrm{st}(f_k)$ is L_1-integrable on $\mathbf{R}$, then

$$st(\sum_{b\in Y^{\perp}} (Ff)(b)) = \prod_{k\in L} \int_{-\infty<t<\infty} st(f_k)(t)dt. \tag{63}$$

Examples

Theorem 3.2.1 is applied to the following two kinds of functionals :

$$1. f_i(a) = \exp(-i\pi \sum_{k\in L} a(k)^2) \tag{64}$$

$$2. f_\xi(a) = \exp(-\xi\pi \sum_{k\in L} a(k)^2), \tag{65}$$

where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$.

The infinitesimal Fourier transforms of the functionals are :

$$1.(Ff_i)(b) = (-1)^{\frac{H}{2}}\overline{f_i(b)} \cdots (\sharp_3) \tag{66}$$

$$2.(Ff_\xi)(b) = B_\xi(b) f_\xi(\frac{b}{\xi}), \tag{67}$$

hence the followings are obtained :

$$1.|Y^\perp|^{-\frac{1}{2}}(-1)^{\frac{H}{2}} \sum_{b\in Y^\perp} \overline{f_i(b)} = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_i(a) \tag{68}$$

$$2.|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} B_\xi(b) f_\xi(\frac{b}{\xi}) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_\xi(a). \tag{69}$$

These are written as the following, explicitly :

$$1.|Y^\perp|^{-\frac{1}{2}}(-1)^{\frac{H}{2}} \sum_{b\in Y^\perp} \exp(-i\pi \sum_{k\in L} b(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} \exp(-i\pi \sum_{k\in L} a(k)^2), \tag{70}$$

$$2.|Y^\perp|^{-\frac{1}{2}} \sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} \exp(-\xi\pi \sum_{k\in L} a(k)^2). \tag{71}$$

Corollary 3.2.3 implies the following proposition 3.2.4.

Proposition 3.2.4.

(i) If each generator of Y_k is equal to 1, then

$$1.(-1)^{\frac{H}{2}} st(\sum_{b\in Y^\perp} \exp(-i\pi \prod_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-i\pi n^2))^{H^2} \tag{72}$$

$$2.st(\sum_{b\in Y^\perp} B_\xi(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-\xi\pi n^2))^{H^2} \tag{73}$$

$$\left(= (\theta(i\xi))^{H^2}\right).$$

(ii) If each generator of Y_k is equal to a natural number m_k, then

$$1.(-1)^{\frac{H}{2}} st(\sum_{b\in Y^{\perp}} \exp(-i\pi \prod_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-i\pi m_k^2 n^2)) \tag{74}$$

$$2.st(\sum_{b\in Y^{\perp}} B_{\xi}(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-\xi\pi m_k^2 n^2)) \tag{75}$$

$$\left(= \prod_{k\in L}(m_k\theta(im_k^2\xi))\right).$$

(iii) If each generator of Y_k is infinitesimal, then

$$2.st(\sum_{b\in Y^{\perp}} B_{\xi}(b) \exp(-\frac{1}{\xi}\pi \sum_{k\in L} b(k)^2)) = (\int_{-\infty}^{\infty} \exp(-\xi\pi t^2)dt)^{H^2} \tag{76}$$

$$\left(= \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right).$$

The above formula $(\sharp_3)$ for $f_i(a)$ is extended to $f_{im}(a) = \exp(-im\pi \sum_{k\in L} a^2(k))$, for an integer m so that $m|2H'^2$. If $m|\frac{b(k)}{\varepsilon'}$, we recall

$$(Ff_{im})(b) = B_{im}(b) f_{\frac{1}{im}}(b) \tag{77}$$

where $B_{im}(b) = \left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{({}^*H)^2}$ for a positive m , $B_{im}(b) = \left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{({}^*H)^2}$ for a negative m.

Hence $|Y^{\perp}|^{-\frac{1}{2}} \sum_{b\in Y^{\perp}} B_{im}(b) f_{\frac{1}{im}}(b) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f_{im}(a)$. When each generator $\varepsilon' s'_k$ of $Y_k^{\perp}$ satisfies $m|s'_k$, that is, each generator $\varepsilon' s_k$ of Y_k satisfies $m|\frac{H'^2}{s_k}$, it reduces to the following :

$$H'^{({}^*H)^2}\left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{({}^*H)^2} \sum_{b\in Y^{\perp}} \exp(i\pi\frac{1}{m}\sum_{k\in L} b(k)^2) = \prod_{k\in L} s_k \sum_{a\in Y} \exp(-im\pi \sum_{k\in L} a(k)^2) \tag{78}$$

for a positive m, and

$$H'^{(^\star H)^2}\left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{(^\star H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)$$
$$=\prod_{k\in L}s_k\sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2) \quad (79)$$

for a negative m.

If $s_k = H'$ and $m|H'$, then

$$\left(\sqrt{\frac{m}{2}}\frac{1+i^{\frac{2H'^2}{m}}}{1+i}\right)^{(^\star H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)=\sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2) \quad (80)$$

for a positive m, and

$$\left(\sqrt{\frac{-m}{2}}\frac{1+(-i)^{\frac{2H'^2}{-m}}}{1-i}\right)^{(^\star H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)=\sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2) \quad (81)$$

for a negative m, that is,

$$\left(\sqrt{m}\exp(-i\frac{\pi}{4})\right)^{(^\star H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)=\sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2) \quad (82)$$

for a positive m, and

$$\left(\sqrt{-m}\exp(i\frac{\pi}{4})\right)^{(^\star H)^2}\sum_{b\in Y^\perp}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)=\sum_{a\in Y}\exp(-im\pi\sum_{k\in L}a(k)^2) \quad (83)$$

for a negative m.

3.3. Poisson summation formula for Definition 2.2.3

Poisson summation formula of finite group is extended to the double infinitesimal Fourier transformation for Definition 2.2.3 on the space of functionals originally defined in [8].

Formulation

The following theorem for Definition 2.2.3 is obtained as the argument in the section 3.2.

Theorem 3.3.1. Let Y be an internal subgroup of X. Then the following is obtained, for $f \in A$,

$$|Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{b\in Y^{\perp\varepsilon}} (Ff)(b) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f(a) \tag{84}$$

where $< a,b >_\varepsilon := {}^\star\varepsilon \sum_{k\in L} a(k)b(k)$ and $Y^{\perp\varepsilon} := \{b \in X \mid \exp(2\pi i < a,b >_\varepsilon) = 1 \text{ for } \forall a \in Y\}$.

Lemma 3.3.2. $|Y^{\perp\varepsilon}| = \frac{|X|}{|Y|}$.

Proof of Lemma 3.3.2. For $k \in L$, it is denoted $Y_k := \{a(k) \in L' \mid a \in Y\}$.

$b \in Y^{\perp\varepsilon} \Longleftrightarrow \forall a \in Y,\ \exp(2\pi i {}^\star\varepsilon \sum_{k\in L} a(k)b(k)) = 1$

$\Longleftrightarrow \forall k \in L,\ {}^\star\varepsilon b(k) \in Y_k^\perp$.

For $k \in L$, generators defined by the following are written as m, n :

$Y_k =< \varepsilon' m >,\ \{b(k) \in L' \mid {}^\star\varepsilon b(k) \in Y_k^\perp\} =< \varepsilon' n >$.

Now

$$\exp(2\pi i {}^\star\varepsilon\varepsilon' m\varepsilon' n) = 1 \Longleftrightarrow {}^\star\varepsilon\varepsilon' m\varepsilon' n = 1. \tag{85}$$

It is written $Y_k^{\perp\varepsilon} := \{b(k) \in L' \mid {}^\star\varepsilon b(k) \in Y_k^\perp\}$. Then $|Y_k^{\perp\varepsilon}| = m$. This is equal to $\frac{{}^\star HH'^2}{{}^\star HH'^2/m} = \frac{|L'|}{|Y_k|}$. Hence

$$|Y^{\perp\varepsilon}| = \prod_{k\in L} |Y_k^{\perp\varepsilon}| = \frac{|X|}{|Y|}. \tag{86}$$

Proof of Theorem 3.3.1.

$$|Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{b\in Y^{\perp\varepsilon}} (Ff)(b) = |Y^{\perp\varepsilon}|^{-\frac{1}{2}} \sum_{a\in X} \varepsilon_0 (\sum_{b\in Y^{\perp\varepsilon}} \exp(-2\pi i < a,b >_\varepsilon)) f(a). \tag{87}$$

Since $\sum_{b\in Y^{\perp\varepsilon}} \exp(-2\pi i < a,b >_\varepsilon) = \begin{cases} 0 & (a \notin Y) \\ |Y^{\perp\varepsilon}| & (a \in Y) \end{cases}$, the above is equal to

$$|Y^{\perp\varepsilon}|^{-\frac{1}{2}} \varepsilon_0 |Y^{\perp\varepsilon}| \sum_{a\in Y} f(a) = |Y|^{-\frac{1}{2}} \sum_{a\in Y} f(a). \tag{88}$$

The following is obtained:

Corollary 3.3.3.

(i) If each generator of Y_k is equal to 1, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\sum_{-\infty<n<\infty}\mathrm{st}(f_k)(n)$ converges, then

$$H^{\frac{H^2}{2}} st(\sum_{b\in Y^\perp}(Ff)(b)) = \prod_{k\in L}(\sum_{-\infty<n<\infty} st(f_k)(n)). \tag{89}$$

(ii) If each generator of Y_k is infinitesimal, f is written as $\prod_{k\in L} f_k$, $f_k = {}^*(\mathrm{st}(f_k))|_{L'}$, and $\mathrm{st}(f_k)$ is L_1-integrable on $\mathbf{R}$, then

$$H^{\frac{H^2}{2}} st(\sum_{b\in Y^\perp}(Ff)(b)) = \prod_{k\in L}\int_{-\infty}^{\infty} st(f_k)(t)dt. \tag{90}$$

Examples Theorem 3.3.1 is applied to the following two functionals :

$$1.g_i(a) = \exp(-i\pi{}^\star\varepsilon\sum_{k\in L} a(k)^2) \tag{91}$$

$$2.g_\xi(a) = \exp(-\xi\pi{}^\star\varepsilon\sum_{k\in L} a(k)^2) \tag{92}$$

where $\xi \in \mathbf{C}$, $\mathrm{Re}(\xi) > 0$. The infinitesimal Fourier transforms are :

$$1.(Fg_i)(b) = (-1)^{\frac{H}{2}}\overline{g_i(b)}\cdots(\sharp_4) \tag{93}$$

$$2.(Fg_\xi)(b) = C_\xi(b)g_\xi(\frac{b}{\xi}) \tag{94}$$

hence the following formulas are obtained :

$$1.|Y^{\perp\varepsilon}|^{-\frac{1}{2}}(-1)^{\frac{H}{2}}\sum_{b\in Y^{\perp\varepsilon}}\overline{g_i(b)} = |Y|^{-\frac{1}{2}}\sum_{a\in Y} g_i(a) \tag{95}$$

$$2.|Y^{\perp\varepsilon}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp\varepsilon}} C_\xi(b)g_\xi(\frac{b}{\xi}) = |Y|^{-\frac{1}{2}}\sum_{a\in Y} g_\xi(a). \tag{96}$$

These are written as the following, explicitly :

$$1.|Y^{\perp\varepsilon}|^{-\frac{1}{2}}(-1)^{\frac{H}{2}}\sum_{b\in Y^{\perp\varepsilon}}\exp(-i\pi{}^\star\varepsilon\sum_{k\in L} b(k)^2) = |Y|^{-\frac{1}{2}}\sum_{a\in Y}\exp(-i\pi{}^\star\varepsilon\sum_{k\in L} a(k)^2) \tag{97}$$

$$2.|Y^{\perp\varepsilon}|^{-\frac{1}{2}}\sum_{b\in Y^{\perp\varepsilon}} C_\xi(b)\exp(-\frac{1}{\xi}\pi{}^\star\varepsilon\sum_{k\in L} a(k)^2) = |Y|^{-\frac{1}{2}}\sum_{a\in Y}\exp(-\xi\pi{}^\star\varepsilon\sum_{k\in L} a(k)^2). \tag{98}$$

Corollaly 3.3.3 implies the following proposition 3.3.4.

Proposition 3.3.4.

(i) If each generator of Y_k is equal to 1, then the standard parts are :

$$1. H^{\frac{H^2}{2}}(-1)^{\frac{H}{2}} st(\sum_{b\in Y_\varepsilon^\perp} \exp(-i\pi\varepsilon \sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-i\pi\varepsilon n^2))^{H^2} \tag{99}$$

$$2. H^{\frac{H^2}{2}} st(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b) \exp(-\frac{1}{\xi}\pi\varepsilon \sum_{k\in L} b(k)^2)) = (\sum_{-\infty<n<\infty} \exp(-\xi\pi\varepsilon n^2))^{H^2} \tag{100}$$

$$\left(= (\theta(i\xi))^{H^2}\right).$$

(ii) If each generator of Y_k is equal to a natural number m_k, then

$$1. H^{\frac{H^2}{2}}(-1)^{\frac{H}{2}} st(\sum_{b\in Y_\varepsilon^\perp} \exp(-i\pi\varepsilon \sum_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-i\pi\varepsilon m_k^2 n^2)) \tag{101}$$

$$2. H^{\frac{H^2}{2}} st(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b) \exp(-\frac{1}{\xi}\pi\varepsilon \sum_{k\in L} b(k)^2)) = \prod_{k\in L}(m_k \sum_{-\infty<n<\infty} \exp(-\xi\pi\varepsilon m_k^2 n^2)) \tag{102}$$

$$\left(= \prod_{k\in L}(m_k\theta(im_k^2\xi))\right).$$

(iii) If each generator of Y_k is infinitesimal, then

$$2. st(\sum_{b\in Y_\varepsilon^\perp} C_\xi(b) \exp(-\frac{1}{\xi}\pi\varepsilon \sum_{k\in L} b(k)^2)) = (\int_{-\infty}^{\infty} \exp(-\xi\pi t^2)dt)^{H^2} \tag{103}$$

$$\left(= \left(*\left(\frac{1}{\sqrt{\xi}}\right)\right)^{H^2}\right).$$

The above formulation ($\sharp_4$) of $g_i(a)$ is extended to $g_{im}(a) = \exp(-im\pi\,{}^\star\varepsilon \sum_{k\in L} a^2(k))$, for an integer m so that $m|2\,{}^\star HH'^2$. If $m|\frac{b(k)}{\varepsilon'}$ for an arbitrary $k \in L$, it is recalled

$(Fg_{im})(b) = C_{im}(b) g_{\frac{1}{im}}(b)$, where $C_{im}(b) = \left(\sqrt{\frac{m}{2}} \frac{1+i^{\frac{2^\star HH'^2}{m}}}{1+i}\right)^{{}^\star H^2}$ for a positive m and $C_{im}(b) = \left(\sqrt{\frac{-m}{2}} \frac{1+(-i)^{\frac{2^\star HH'^2}{-m}}}{1-i}\right)^{{}^\star H^2}$ for a negative m.

Hence $|Y^{\perp_\varepsilon}|^{-\frac{1}{2}} \sum_{b \in Y^\perp} C_{im}(b) g_{\frac{1}{im}}(b) = |Y|^{-\frac{1}{2}} \sum_{a \in Y} g_{im}(a)$. When each generator $\varepsilon' s'_k$ of $Y_k^{\perp_\varepsilon}$ satisfies $m | s'_k$, that is, each generator $\varepsilon' s_k$ of Y_k satisfies $m | \frac{{}^\star H H'^2}{s_k}$, it reduces to the following:

$$H^{\frac{H^2}{2}} H'^{({}^\star H)^2} \left(\sqrt{\frac{m}{2}} \frac{1 + i^{\frac{2 {}^\star H H'^2}{m}}}{1+i} \right)^{({}^\star H)^2} \sum_{b \in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} {}^\star\varepsilon \sum_{k \in L} b(k)^2)$$

$$= \prod_{k \in L} s_k \sum_{a \in Y} \exp(-im\pi {}^\star\varepsilon \sum_{k \in L} a(k)^2) \tag{104}$$

for a positive m, and

$$H^{\frac{H^2}{2}} H'^{({}^\star H)^2} \left(\sqrt{\frac{-m}{2}} \frac{1 + (-i)^{\frac{2 {}^\star H H'^2}{-m}}}{1-i} \right)^{({}^\star H)^2} \sum_{b \in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} {}^\star\varepsilon \sum_{k \in L} b(k)^2)$$

$$= \prod_{k \in L} s_k \sum_{a \in Y} \exp(-im\pi {}^\star\varepsilon \sum_{k \in L} a(k)^2) \tag{105}$$

for a negative m. If $s_k = H'$ and $m | H'$, then

$$H^{\frac{H^2}{2}} \left(\sqrt{\frac{m}{2}} \frac{1 + i^{\frac{2 {}^\star H H'^2}{m}}}{1+i} \right)^{({}^\star H)^2} \sum_{b \in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im\pi {}^\star\varepsilon \sum_{k \in L} a(k)^2) \tag{106}$$

for a positive m, and

$$H^{\frac{H^2}{2}} \left(\sqrt{\frac{-m}{2}} \frac{1 + (-i)^{\frac{2 {}^\star H H'^2}{-m}}}{1-i} \right)^{({}^\star H)^2} \sum_{b \in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2)$$
$$= \sum_{a \in Y} \exp(-im\pi {}^\star\varepsilon \sum_{k \in L} a(k)^2) \tag{107}$$

for a negative m, that is,

$$H^{\frac{H^2}{2}} \left(\sqrt{m} \exp(-i\frac{\pi}{4}) \right)^{({}^\star H)^2} \sum_{b \in Y^{\perp_\varepsilon}} \exp(i\pi \frac{1}{m} \sum_{k \in L} b(k)^2) = \sum_{a \in Y} \exp(-im\pi {}^\star\varepsilon \sum_{k \in L} a(k)^2) \tag{108}$$

for a positive m, and

$$H^{\frac{H^2}{2}}\left(\sqrt{-m}\exp(i\frac{\pi}{4})\right)^{({}^{\star}H)^2}\sum_{b\in Y^{\perp_\varepsilon}}\exp(i\pi\frac{1}{m}\sum_{k\in L}b(k)^2)=\sum_{a\in Y}\exp(-im\pi\,{}^{\star}\varepsilon\sum_{k\in L}a(k)^2) \quad (109)$$

for a negative m.

4. Quantum field theory and Zeta function

In this section the quantum field theory is developed by using the double infinitesimal Fourier transform. The propagator for a system of the harmonic oscillators is considered in the quantum field theory.

4.1. Path integral in the quantum field theory

Definition4.1.1. A path integral of $f(\in A)$ is defined as follows:

$$\sum_{a\in X}\varepsilon_0 f(a) \quad (110)$$

with $\varepsilon_0 := (H')^{-({}^{\star}H)^2} \in {}^{\star}({}^{*}\mathbf{R})$.

It is briefly explained that the complexification of the propagator for the harmonic oscillator is represented as the following path integral. In Feynman's formulation of quantum mechanics([2]), the propagator of the one-dimensional harmonic oscillator is the following path integral: $K(q,q_0,t)$

$$=\lim_{n\to\infty}\int_{\mathbf{R}^n}(\frac{m}{2\pi i\hbar\epsilon})^{(n+1)/2}\exp\Big(\frac{i\epsilon}{\hbar}\sum_{j=1}^{n+1}(\frac{m}{2}(\frac{x_j-x_{j-1}}{\epsilon})^2-\frac{m}{2}\lambda^2x_j^2)\Big)dx_1dx_2\cdots dx_n \quad (111)$$

where $x_0=q_0$, $x_{n+1}=q$, $\epsilon=\frac{t}{n}$. In nonstandard analysis, it is known that, for a sequence an,

$$\lim_{n\to\infty}a_n=a \quad iff \quad {}^{*}a_w\approx a \quad (112)$$

for any infinite natural number $\omega\in^{*}\mathbf{N}-\mathbf{N}$, where ${}^{*}a_n$ is the $*$ extension of $\{a_n\}_{n\in N}$, and $\approx$ means that ${}^{*}a_w-a$ is infinitesimal, that is, the standard part of a_w is a, usually denoted by $st({}^{*}a_w)=a$. The standard part of the nonstandard path integral is written as

$$st\int_{{}^{*}\mathbf{R}^w}(\frac{m}{2\pi i\hbar\epsilon})^{(\omega+1)/2}\exp\Big(\frac{i\epsilon}{\hbar}\sum_{j=1}^{w+1}(\frac{m}{2}(\frac{x_j-x_{j-1}}{\epsilon})^2-\frac{m}{2}\lambda^2x_j^2)\Big)dx_1dx_2\cdots dx_w. \quad (113)$$

By extending t to a complex number, the path integral is complexified to the following :

$$st\int_{^*\mathbf{R}^w}(\frac{m}{2\pi i\hbar\epsilon})^{(\omega+1)/2}\exp\Big(\frac{i\epsilon}{\hbar}\sum_{j=1}^{w+1}(\frac{m}{2}(\frac{x_j-x_{j-1}}{\epsilon})^2-\frac{m}{2}\lambda^2x_j^2)\Big)dx_1dx_2\cdots dx_w. \qquad (114)$$

Theorem 4.1.2. Let $t\in\mathbf{R}$, $t\neq st(\pm\frac{\sqrt{2}n}{\lambda}\sqrt{1-\cos(\frac{k\pi}{\omega+1})}), k=1,2,\cdots,\omega$, or for $t\in\mathbf{C}$, whose imaginary part is negative. The complexified one-dimensional harmonic oscillator standard functional integral is given by $(\frac{m}{2\pi i\hbar})^{\frac{1}{2}}\sqrt{\frac{\lambda}{\sin(\lambda t)}}\exp(\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}((q_0^2+q^2)\cos\lambda t-2qq_0))$.

Proof. If $t\neq st(\pm\frac{\sqrt{2}n}{\lambda}\sqrt{1-\cos(\frac{k\pi}{\omega+1})}), k=1,2,\cdots,\omega$, then

$$t\neq\pm\frac{\sqrt{2}n}{\lambda}\sqrt{1-\cos(\frac{k\pi}{\omega+1})}, k=1,2,\cdots,\omega \qquad (115)$$

for arbitrary infinite number ω . The theorem is followed from the discrete calculation using the matrix representation of the operator(cf. [7]).

It corresponds to the well-known real propagator for one dimensional harmonic oscillator. For the d-dimensional harmonic oscillator, d-dimensional vectors are written as $\mathbf{q_0},\mathbf{q}$, the square norms are $|\mathbf{q_0}|^2, |\mathbf{q}|^2$, and the inner product of $\mathbf{q_0}$, $\mathbf{q}$ is $\mathbf{q_0}\mathbf{q}$. We have :

Corollary 4.1.3. For the complexified d-dimensional harmonic oscillator standard functional integral, the complexified propagator is given by

$$(\frac{m}{2\pi i\hbar})^{\frac{d}{2}}(\frac{\lambda}{\sin(\lambda t)})^{\frac{d}{2}}\exp(\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}((|\mathbf{q_0}|^2+|\mathbf{q}|^2)\cos\lambda t-2\mathbf{q}\mathbf{q}_0)). \qquad (116)$$

Proof. By factorizing Theorem 4.1.2 into a product on d dimensional, the corollary is obtained.

The trace of the compiexified propagator is calculated for one dimensional harmonic oscillator.

Since

$$\int_{-\infty}^{\infty}(\frac{m}{2\pi i\hbar})^{\frac{1}{2}}\sqrt{\frac{\lambda}{\sin(\lambda t)}}\exp(\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}2((\cos\lambda t-1)q^2))dq=\frac{1}{2i\sin(\lambda t/2)} \qquad (117)$$

the following is obtained (cf.[6],[7]):

Theorem 4.1.4. Let $t\in\mathbf{R}$, $t\neq\pm st(\frac{\sqrt{2}\omega}{\lambda}\sqrt{1-\cos(\frac{k\pi}{\omega+1})}), k=1,2,\cdots,\omega$, or $t\in\mathbf{C}$, whose imaginary part is negative. The trace of the complexified one-dimensional harmonic oscillator standard functional integral is given by $\frac{1}{2i\sin(\lambda t/2)}$.

Proof. By putting $q_0 = q$ in $(\frac{m}{2\pi i\hbar})^{\frac{1}{2}}\sqrt{\frac{\lambda}{\sin(\lambda t)}}\exp(\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}((q_0^2+q^2)\cos\lambda t - 2qq_0))$, the trace is the following integral :

$$\int_{-\infty}^{\infty}(\frac{m}{2\pi i\hbar})^{\frac{1}{2}}\sqrt{\frac{\lambda}{\sin(\lambda t)}}\exp(\frac{im}{\hbar}\frac{\lambda}{\sin\lambda t}2((\cos\lambda t-1)q^2))dq = \frac{1}{2i\sin(\lambda t/2)}. \tag{118}$$

Corollary 4.1.5. If the potential is modified to $V(q) = \frac{m}{2}(\frac{\lambda^2}{2}|q|^2 - \frac{\lambda}{2})$, then the trace is $\frac{1}{2i\sin(\lambda t/2)}\exp(\frac{\lambda t}{2}))$.

For the d-dimensional harmonic oscillator, the following is obtained :

Corollary 4.1.6. For the trace of the modified complexified propagator for d-dimensional harmonic oscillator, the trace is $(\frac{1}{2i\sin(\lambda t/2)}\exp(\frac{\lambda t}{2})))^d$.

In the next section, Corollaries 4.1.5 and 4.1.6 are used to treat an infinite dimensional harmonic oscillator.

4.2. Representation of the zeta function.

Corollary 4.1.5 is extended to an infinite dimensional harmonic oscillator using nonstandard analysis. For it the three types of extension $^*\mathbf{R}$,$^{**}\mathbf{R}$, $^{\#**}\mathbf{R}$ of $\mathbf{R}$ are prepared corresponding to Definition 2.2.2 , then the three stages of infinite numbers exist. In these three extension fields, we fix infinite natural numbers $H_F \in {}^*\mathbf{N}$,$H_T \in {}^{**}\mathbf{N}$, $H'' \in 2^{\#**}\mathbf{N}$. Let T be a positive standard real number and let ϵ_T , ϵ'' be infinitesimals in $^{**}\mathbf{R}$, $^{\#**}\mathbf{R}$ defined by $\frac{T}{H_T}$, $\frac{1}{H''}$. A lattice L'' and two function space X, A are defined as the following:

$L'' := \left\{\varepsilon''z'' \,\middle|\, z'' \in {}^{\#**}\mathbf{Z}, -\frac{\mathbf{H}''}{\mathbf{2}} \leq {}''''\mathbf{z}'' < \frac{\mathbf{H}''}{\mathbf{2}}\right\})$.

$X := \{\alpha :^{\#*} \{0,1,\cdots .H_F-1\} \to L'', internal\}$,

$A := \{a :^{\#} \{0,1,\cdots .H_T\} \to X, internal\}$.

Then an element a of A is written as the component $(a_j^k, 0 \leq j \leq H_T, 0 \leq k \leq H_F-1)$. All prime numbers are ordered as $p(1) = 2$, $p(2) = 3$, ... , $p(n) < p(n+1)$, ... , that is, p is a mapping from $\mathbf{N}$ to the set of prime numbers, $p : \mathbf{N} \to \{\text{prime number}\}$. Let λ_k be $\ln^* p(k)$ for each k , $0 \leq k \leq H_F - 1$. A potential $V_k :^{\#**} \mathbf{R} \to^{\#**} \mathbf{R}$ is defined for each k , $0 \leq k \leq H_F - 1$, as $V_k(q) = \frac{\lambda_k^2}{2}|q|^2 - \frac{\lambda_k}{2}$. An element α of X is written as the component $\alpha = (\alpha^k, 0 \leq k \leq H_F - 1)$.

Let V be a global potential as the following:

$$V(\alpha) = \sum_{k=0}^{H_F-1} V_k(\alpha^k)\left(= \sum_{k=0}^{H_F-1}\left(\frac{\lambda_k^2}{2}\left|\alpha^k\right|^2 - \frac{\lambda_k}{2}\right)\right). \tag{119}$$

In order to transport t to later, the element $-\frac{\lambda_k}{2}$ is put in the usual potential for harmonic oscillators. It is considered the following summation $K(a,b,t)$ depending of $a,b \in X$:

$$K(a,b,t) = \sum_{a \in A, a_0 = a.a_{H_T} = b} ((\epsilon')^{H_F})^{H_T} ((\frac{1}{2\pi\epsilon_T})^{H_F})^{H_T} \exp(\epsilon_T(\frac{1}{2}\sum_{j=1}^{H_T} |\frac{a_j - a_{j-1}}{\epsilon_T}|^2 - V(a_j))). \quad (120)$$

Then $K(a,b,t)$ is calculated,

$$K(a,b,t) = (\frac{1}{2\pi\epsilon_T})^{H_T} \prod_{k=1}^{H_F-1} \sum_{a_j^k \in L', 0 \le j \le H_T-1} (\epsilon')^{H_T} (\frac{1}{2\pi\epsilon_T})^{H_T} \exp(\epsilon_T(\frac{1}{2}\sum_{j=1}^{H_T} |\frac{a_j^k - a_{j-1}^k}{\epsilon_T}|^2 - V(a_j^k))), \quad (121)$$

where $a_0 = a, a_{H_T} = b$.

The summation $\sum_{a \in X} (\epsilon')^{H_T} K(a,a.t)$ is denoted by $tr(K(a,a,t))$. Three correspondences putting standard parts are written as $st_\# : {}^{\#\star\star}\mathbf{R} \to {}^{\star\star}\mathbf{R}$, $st_\star : {}^{\star\star}\mathbf{R} \to {}^{*}\mathbf{R}$, $st_* : {}^{*}\mathbf{R} \to \mathbf{R}$. When there are no confusion, they are simply written as st. The composition $st_* \circ st_\star \circ st_\# : {}^{\#\star\star}\mathbf{R} \to \mathbf{R}$ is denoted also by st for simplicity.

Theorem 4.2.1. If the real part of t is greater than 1, the standard part $st(tr(K(a,a,t)))$ of $tr(K(a,a,t))$ corresponds to Riemann's zeta function $\zeta(t)$.

Proot. The standard parts of $tr(K(a,a,t))$ as follows.

$$st_\#(tr(K(a,a,t))) =$$

$$\prod_{k=1}^{H_F-1} \int\int \cdots \int (\frac{1}{2\pi\epsilon_T})^{H_T} \exp(\epsilon_T(\frac{1}{2}\sum_{j=1}^{H_T} (\frac{q_j^k - q_{j-1}^k}{\epsilon_T})^2 - V(q_j^k))) dq_0^k dq_1^k \cdots dq_{H_T-1}^k \quad (122)$$

$$= \prod_{k=1}^{H_F-1} \int \{ \int \cdots \int (\frac{1}{2\pi\epsilon_T})^{H_T} \exp(\epsilon_T(\frac{1}{2}\sum_{j=1}^{H_T} (\frac{q_j^k - q_{j-1}^k}{\epsilon_T})^2 - V(q_j^k))) dq_1^k \cdots dq_{H_T-1}^k \} dq_0^k \quad (123)$$

by Fubini's theorem. Furthermore,

$$st_\star st_\#(tr(K(a,a,t))) =$$

$$= \prod_{k=1}^{H_F-1} st_\star \{ \int \{ \int \cdots \int (\frac{1}{2\pi\epsilon_T})^{H_T} \exp(\epsilon_T(\frac{1}{2}\sum_{j=1}^{H_T} (\frac{q_j^k - q_{j-1}^k}{\epsilon_T})^2 - V(q_j^k))) dq_1^k \cdots dq_{H_T-1}^k \} dq_0^k \} \quad (124)$$

by the same calculation of Theorem 4.1.2 (cf.[6],[7]) ,

$$= \prod_{k=0}^{H_F-1} \left(\frac{1}{2i\sin(\frac{\lambda_k t}{2i})}\exp(\frac{\lambda_k t}{2})\right) = \prod_{k=0}^{H_F-1} \frac{1}{1-p_k^{-t}}. \quad (125)$$

By Lebesque's convergence theorem, $(st_*(st_\star(st_\#(tr(K(a,a,t))))) = \prod_{k=0}^{\infty} \frac{1}{1-p_k^{-t}} = \zeta(t)$, if the real part of t is positive .

4.2. Another representation of the zeta function

In this section, both $st_*, st_\star$ and $st_\#$ are denoted as st for the simplification. A functional is defined on X, and a relationship between the functional and Riemann's zeta function is shown later. The nonstandard extension ${}^*p : {}^*\mathbf{N} \to {}^*\{\text{prime number}\}$ is written as ${}^*p([l_\mu]) = [p(l_\mu)]$, and a mapping $\tilde{p} : {}^*\mathbf{N} \to {}^\star({}^*\{\text{prime number}\})$ is defined as $\tilde{p}([l_\mu]) = {}^\star[p(l_\mu)]$. For $s \in \mathbf{C}$, $Z_s(\in A)$ is defined as the following :

$$Z_s(a) := \prod_{k\in L} \tilde{p}(H(k+\frac{H}{2})+1)^{(-s(a(k)+\frac{H'}{2}))}. \quad (126)$$

Now $H(k+\frac{H}{2})+1$ is an element of ${}^*\mathbf{N}$ and $a(k)+H'/2$ is an element of ${}^\star({}^*\mathbf{N})$. Then $Z_s(a)$ is calculated as $\exp(-s\sum_{k\in L}\log(\tilde{p}(H(k+\frac{H}{2})+1))a(k))\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)^{-s\frac{H'}{2}}$. The following theorem is obtained for the Fourier transform of Z_s for Definition 2.2 1:

Theorem 4.3.1.

$$(F((Z_s))(b) = \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$$

$$\cdot \prod_{k\in L} \varepsilon' \frac{\sinh((2\pi i b(k) + s\log\tilde{p}(H(k+\frac{H}{2})+1))\frac{H'}{2})}{\exp(-\frac{\varepsilon'}{2}(2\pi i b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))\sinh(\frac{\varepsilon'}{2}(2\pi i b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1)}. \quad (127)$$

Proof.

$$(F((Z_s))(b) = \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$$

$$\cdot \sum_{a\in X}\varepsilon_0 \exp(-s\sum_{k\in L}\log\tilde{p}(H(k+\frac{H}{2})+1)a(k))\exp(-2\pi i\sum_{k\in L}a(k)b(k))$$

$$= \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}} \cdot \sum_{a\in X}\varepsilon_0\exp(-(2\pi i\, b(k)+s\log\tilde{p}(H(k+\frac{H}{2})+1))a(k))$$

$$= \left(\prod_{k\in L}\tilde{p}(H(k+\frac{H}{2})+1)\right)^{-s\frac{H'}{2}}$$

$$\cdot \prod_{k\in L} \varepsilon' \frac{\sinh((2\pi i\, b(k) + s\log \tilde{p}(H(k+\frac{H}{2})+1))\frac{H'}{2})}{\exp(-\frac{\varepsilon'}{2}(2\pi i\, b(k) + s\log \tilde{p}(H(k+\frac{H}{2})+1))\sinh(\frac{\varepsilon'}{2}(2\pi i\, b(k) + s\log \tilde{p}(H(k+\frac{H}{2})+1))}.$$

Riemann's zeta function $\zeta(s)$ is defined by $\zeta(s) = \prod_{l=1}^{\infty} \frac{1}{1-p(l)^{-s}}$ for $\mathrm{Re}(s) > 1$. Let $Y_{\mathbf{Z}}$ be a subgroup of X so that each generator of $(Y_{\mathbf{Z}})_k$ is equal to 1. Then the following theorem is obtained :

Theorem 4.3.2. If $\mathrm{Re}(s) > 1$, then $\mathrm{st}(\mathrm{st}(\sum_{a\in Y_{\mathbf{Z}}}(Z_s))(a))) = \zeta(s)$.

Proof. $\mathrm{st}(\mathrm{st}(\sum_{a\in Y_{\mathbf{Z}}}(Z_s)(a))) = \mathrm{st}\Big(\mathrm{st}\Big(\Big(\prod_{k\in L} \tilde{p}(H(k+\frac{H}{2})+1)\Big)^{(-s(a(k)+\frac{H'}{2}))}\Big)\Big)$

$$= st\Big(st\Big(\prod_{k\in L} \frac{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-sH'}}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\Big)\Big) = st\Big(\prod_{k\in L} \frac{1}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\Big) = \zeta(s). \quad (128)$$

Furthermore, Poisson summation formula and Theorem 4.3.2 imply the following :

Corollary 4.3.3.

$$st(\sum_{b\in Y_{\mathbf{Z}}^{\perp}} (F(Z_s)(b)) = st\Big(\prod_{k\in L} \frac{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-sH'}}{1-\tilde{p}(H(k+\frac{H}{2})+1)^{-s}}\Big). \quad (129)$$

Hence we obtain :

$$st(st(\sum_{b\in Y_{\mathbf{Z}}^{\perp}} (F(Z_s)(b)))) = \zeta(s) \quad (130)$$

for $\mathrm{Re}(s) > 1$.

In general, the physical theory has variables for position, time, and fields. Especially there are many kinds of variables in quantum field theory. The function depends on such variables mixed as $f(\mathbf{q}, t, \mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\mathbf{q}$ is position, t is time, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are fields. When the function is treated for such mixed variables, the Kinoshita's infinitesimal Fourier transform and our double infinitesimal Fourier transform are applied in the double extended number field. The two kinds of Fourier transforms can be used for one function. In the theory, the delta functions for variable and for fields have different infinitesimals and infinite values. The delta function for fields has an infinitesimal much smaller and much bigger infinite number. However they can be treat in the double extended number field. Two kinds of delta functions are defined with another degrees. One delta function has an infinitesimal of the first degree and the other delta function has an infinitesimal of the second degree. The infinitesimal of the second one can not be observable with respect to the first one.

5. Conclusion

The real and complex number fields are extended to the larger number fields where there are many infinitesimal and infinite numbers. A lattice of infinitesimal width is included in the extended real number field. An infinitesimal Fourier transform theory is constructed on the infinitesimal lattice. These extended number fields are furthermore extended to much higher generalized fields where there exist much higher infinitesimal and infinite numbers. A double infinitesimal Fourier transform theory is developed on these double extended number fields. The usual formulae for Fourier theory are satisfied in the theory, especially the Poisson summation formula. The Fourier theory is based on the integral theory for functionals corresponding to the path integral in the physics. The theory is associated to the physical theory in the quantum field theory which is mathematically rigorous . For an application for the double infinitesimal calculation, Riemann's zeta function is represented as such an integral for the propagator of an infinite dimensional harmonic oscillator.

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