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# Using Dynamic Programming Based on Bayesian Inference in Selection Problems

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Mohammad Saber Fallah Nezhad

Additional information is available at the end of the chapter

<http://dx.doi.org/10.5772/57423>

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## 1. Introduction

An important subject in mathematical science that causes new improvements in data analysis is sequential analysis. In this type of analysis, the number of required observations is not fixed in advance, but is a variable and depends upon the values of the gathered observation. In sequential analysis, at any stage of data gathering process, to determine the number of required observations at the next stage, we analyze the data at hand and with respect to the obtained results, we determine how many more observations are necessary. In this way, the process of data gathering is cheaper and the information is used more effectively. In other words, the data gathering process in sequential analysis, in contrast to frequency analysis, is on-line. This idea caused some researches to conduct researches in various statistical aspects (Basseville and Nikiforov[1]).

In this chapter, using the concept of the sequential analysis approach, we develop an innovative Bayesian method designed specifically for the best solution in selection problem. The proposed method adopts the optimization concept of Bayesian inference and the uncertainty of the decision-making method in dynamic programming environment. The proposed algorithm is capable of taking into consideration the quality attributes of uncertain values in determining the optimal solution. Some authors have applied sequential analysis inference in combination with optimal stopping problem to maximize the probability of making correct decision. One of these researches is a new approach in probability distribution fitting of a given statistical data that Eshragh and Modarres [2] named it Decision on Belief (DOB). In this decision-making method, a sequential analysis approach is employed to find the best underlying probability distribution of the observed data. Moreover, Monfared and Ranaeifar [3] and Eshragh and Niaki [4] applied the DOB concept as a decision-making tool in some problems.

Since the idea behind the sequential analysis modeling is completely similar to the decision-making process of a human being in his life, it may perform better than available methods in decision-making problems. In these problems, when we want to make a decision, first we divide all of the probable solution space into smaller subspaces (the solution is one of the subspaces). Then based on our experiences, we assign a probability measure (belief) to each subspace, and finally we update the beliefs and make the decision.

## 2. An application to determine the best binomial distribution

In the best population selection problem, a similar decision-making process exists. First, the decision space can be divided into several subspaces (one for each population); second, the solution of the problem is one of the subspaces (the best population). Finally, we can assign a belief to each subspace where the belief denotes the performance of the population in terms of its parameter. Based upon the updated beliefs in iterations of the data gathering process, we may decide which population possesses the best parameter value.

Consider  $n$  independent populations  $P_1, P_2, \dots, P_n$ , where for each index  $i=1, 2, \dots, n$ , population  $P_i$  is characterized by the value of its parameter of interest  $p_i$ . Let  $p_{[1]} \leq \dots \leq p_{[n]}$  denote the ordered value of the parameters  $p_1, \dots, p_n$ . If we assume that the exact pairing between the ordered and the unordered parameter is unknown, then, a population  $P_i$  with  $p_i = p_{[n]}$  is called the best population.

There are many applications for the best population selection problem. As one application in supply chain environments, one needs to select the supplier among candidates that performs the best in terms of the quality of its products. As another example, in statistical analysis, we need to select a distribution among candidates that fits the collected observations the most. Selecting a production process that is in out-of-control state, selecting the stochastically optimum point of a multi-response problem, etc. are just a few of these applications.

The problem of selecting the best population was studied in papers by Bechhofer and Kulkarni [5] using the indifference zone approach and by Gupta and Panchapakesan [6] employing the best subset selection approach.

### 2.1. Belief and the approach of its improvement

Assume that there are  $n$  available *Binomial* populations and we intend to select the one with the highest probability of success. Furthermore, in each stage of the data gathering process and for each population, we take an independent sample of size  $m$ . Let us define  $\alpha'_{i,t}$  and  $\beta'_{i,t}$  to be the observed number of successes and failures of the  $i^{\text{th}}$  *Binomial* population in the  $t^{\text{th}}$  stage (sample) and  $\alpha_{i,k}$  and  $\beta_{i,k}$  to be the cumulative observed number of successes and failures of the  $i^{\text{th}}$  *Binomial* population up to the  $k^{\text{th}}$  stage (sample) respectively. In other words,  $\alpha_{i,k} = \sum_{t=1}^k \alpha'_{i,t}$  and  $\beta_{i,k} = \sum_{t=1}^k \beta'_{i,t}$ . Then, in the  $k^{\text{th}}$  stage defining  $\overline{p}_{i,k}$  to be the estimated probability

of success of the  $i^{\text{th}}$  population obtained by  $\frac{\alpha_{i,k}}{km}$ , referring to Jeffrey's prior (Nair et al.[7]), for  $\overline{p_{i,k}}$ , we take a *Beta* prior distribution with parameters  $\alpha_{i,0}=0.5$  and  $\beta_{i,0}=0.5$ . Then, using Bayesian inference, we can easily show that the posterior probability density function of  $\overline{p_{i,k}}$  is

$$f(\overline{p_{i,k}}) = \frac{\Gamma(\alpha_{i,k} + \beta_{i,k} + 1)}{\Gamma(\alpha_{i,k} + 0.5)\Gamma(\beta_{i,k} + 0.5)} \overline{p_{i,k}}^{\alpha_{i,k}-0.5} (1 - \overline{p_{i,k}})^{\beta_{i,k}-0.5} \quad (1)$$

At stage  $k$  of the data gathering process, after taking a sample and observing the numbers of failures and successes, we update the probability distribution function of  $\overline{p_{i,k}}$  for each population. To do this, define  $B(\alpha_{i,k}, \beta_{i,k})$  as a probability measure (called belief) of the  $i^{\text{th}}$  population to be the best one given  $\alpha_{i,k}$  and  $\beta_{i,k}$  as

$$B(\alpha_{i,k}, \beta_{i,k}) = \Pr\{i^{\text{th}} \text{ population is the best} \mid \alpha_{i,k}, \beta_{i,k}\} \quad (2)$$

We then update the beliefs based on the values of  $(\alpha_{i,k}, \beta_{i,k})$  for each population in iteration  $k$ . If we define  $B(\alpha_{i,k-1}, \beta_{i,k-1})$  as the prior belief for each population, in order to update the posterior belief  $B(\alpha_{i,k}, \beta_{i,k})$ , since we may assume that the data are taken independently in each stage, we will have

$$B(\alpha_{i,k}, \beta_{i,k}) = \frac{\Pr\{i^{\text{th}} \text{ Population is the best} \mid (\alpha_{i,k-1}, \beta_{i,k-1})\} \Pr\{(\alpha_{i,k}, \beta_{i,k}) \mid i^{\text{th}} \text{ Population is the best}\}}{\sum_{j=1}^n \left[ \Pr\{j^{\text{th}} \text{ Population is the best} \mid (\alpha_{j,k-1}, \beta_{j,k-1})\} \Pr\{(\alpha_{j,k}, \beta_{j,k}) \mid j^{\text{th}} \text{ Population is the best}\} \right]} \quad (3)$$

$$= \frac{B(\alpha_{i,k-1}, \beta_{i,k-1}) \Pr\{(\alpha_{i,k}, \beta_{i,k}) \mid i^{\text{th}} \text{ Population is the best}\}}{\sum_{j=1}^n \left[ B(\alpha_{j,k-1}, \beta_{j,k-1}) \Pr\{(\alpha_{j,k}, \beta_{j,k}) \mid j^{\text{th}} \text{ Population is the best}\} \right]}$$

From equation (3) we see that to update the beliefs, we need to evaluate  $\Pr\{(\alpha_{i,k}, \beta_{i,k}) \mid i^{\text{th}} \text{ Population is the best}\}$  ;  $i=1, 2, \dots, n$  in each decision-making stage. One way to do this is to use

$$\Pr\{(\alpha_{i,k}, \beta_{i,k}) \mid i^{\text{th}} \text{ Population is the best}\} = \frac{\overline{p_{i,k}}}{\sum_{j=1}^n \overline{p_{j,k}}} \quad (4)$$

Then, the probability given in equation (3) will increase when a better population is selected. In the next theorem, we will prove that when the number of decision-making stages goes to infinity this probability converges to one for the best population.

**Theorem 1**

If the  $i^{th}$  population is the best, then  $\lim_{k \rightarrow \infty} B(\alpha_{i,k}, \beta_{i,k}) = B_i = 1$ .

In order to prove the theorem first we prove the following two lemmas.

**Lemma 1:**

Define a recursive sequence  $\{R_{k,j}; j=1, 2, \dots, l\}$  as

$$R_{k,j} = \begin{cases} \frac{c_j R_{k-1,j}}{\sum_{i=1}^l c_i R_{k-1,i}} & \text{for } k=1,2,3,\dots \\ P_j & \text{for } k=0 \end{cases} \tag{5}$$

where  $c_1, c_2, \dots,$  and  $c_l$  are different positive constants,  $\sum_{j=1}^l P_j=1$ , and  $P_j>0$ . Then, if  $l_j = \lim_{k \rightarrow \infty} (R_{k,j})$ , there exist at most one non-zero  $l_j$ .

**Proof:**

Suppose there are two nonzero  $l_s > 0$  and  $l_t > 0$ . Taking the limit on  $R_{k,j}$  as  $k$  goes to infinity we have

$$\lim_{k \rightarrow \infty} (R_{k,j}) = l_j = \lim_{k \rightarrow \infty} \left( \frac{c_j R_{k-1,j}}{\sum_{i=1}^l c_i R_{k-1,i}} \right) = \frac{c_j l_j}{\sum_{i=1}^l c_i l_i} \tag{6}$$

Now since  $l_s > 0$  and  $l_t > 0$ , then by equation (6) we have

$$l_s = \frac{c_s l_s}{\sum_{i=1}^l c_i l_i} \Rightarrow c_s = \sum_{i=1}^l c_i l_i \text{ and } l_t = \frac{c_t l_t}{\sum_{i=1}^l c_i l_i} \Rightarrow c_t = \sum_{i=1}^l c_i l_i \tag{7}$$

In other words, we conclude  $c_s = c_t$ , which is a contradiction.

**Lemma 2:**

Sequence  $R_{k,j}$  converges to one for  $j=g$  and converges to zero for  $j \neq g$ , where  $g$  is an index for the maximum value of  $c_j$ .

**Proof**

From equation (6), we know that  $\sum_{j=1}^l l_j = 1$ . Then by lemma 1, we have  $l_i = 1$  for only one  $i$ . Now suppose that  $c_g = \max_{j \in \{1, \dots, m\}} \{c_j\}$  and  $g \neq i$ . We will show that this is a contradiction. Consider

$H_{k,i} = \frac{R_{k,g}}{R_{k,i}}$ . By equation (5), we have  $H_{k,i} = \frac{c_g}{c_i} H_{k-1,i}$ . Since  $H_{0,i} > 0$  we will have

$$H_{k,i} = \frac{c_g}{c_i} H_{k-1,i} = \left(\frac{c_g}{c_i}\right)^k H_{0,i} \Rightarrow \lim_{k \rightarrow \infty} (H_{k,i}) = \infty \tag{8}$$

That is a contradiction because  $\lim_{k \rightarrow \infty} (H_{k,i}) = \frac{\lim_{k \rightarrow \infty} (R_{k,g})}{\lim_{k \rightarrow \infty} (R_{k,i})} = \frac{l_g}{l_i} = 0$ . So  $l_g = 1$

Now we are ready to prove the convergence property of the proposed method. Taking limit on both sides of equation (3), we will have

$$\lim_{k \rightarrow \infty} B(\alpha_{i,k}, \beta_{i,k}) = B_i = \lim_{k \rightarrow \infty} \left[ \frac{B(\alpha_{i,k-1}, \beta_{i,k-1}) \Pr\{(\alpha_{i,k}, \beta_{i,k}) \mid i^{th} \text{ Population is the best}\}}{\sum_{j=1}^n [B(\alpha_{j,k-1}, \beta_{j,k-1}) \Pr\{(\alpha_{j,k}, \beta_{j,k}) \mid j^{th} \text{ Population is the best}\}]} \right] \tag{9}$$

From the law of large numbers, we know that  $\lim_{k \rightarrow \infty} \overline{p_{j,k}} = p_j$ , where  $p_j$  is the probability of success of the  $j^{th}$  population. Hence, using equation (7) we have  $B_i = \frac{B_i p_i}{\sum_{j=1}^n B_j p_j}$ . Then assuming population

$i$  is the best, i.e., it possesses the largest value of  $p_j$ 's, by lemma 1 and 2 we conclude that  $B_i = 1$  and  $B_j = 0$  for  $j \neq i$ . This concludes the convergence property of the proposed method.

In real-world applications, since there is a cost associated with the data gathering process we need to select the best population in a finite number of decision-making stages. In the next section, we present the proposed decision-making method in the form of a stochastic dynamic programming model in which there is a limited number of decision-making stages available to select the best population.

## 2.2. A dynamic programming approach

The proposed dynamic programming approach to model the decision-making problem of selecting the best Binomial population is similar to an optimal stopping problem.

Let us assume that to find the best population there is a limited number of stages ( $s$ ) available. Then, the general framework of the decision-making process in each stage is proposed as:

1. Take an independent sample of size  $m$  from each population.
2. Calculate the posterior beliefs in terms of the prior beliefs using Bayesian approach.
3. Select the two biggest beliefs.
4. Based upon the values of the two biggest beliefs calculate the minimum acceptable belief.
5. If the maximum belief is more than the minimum acceptable belief, then we can conclude that the corresponding subspace is the optimal one. Otherwise, go to step 1.

In step 3 of the above framework, let populations  $i$  and  $j$  be the two candidates of being the best populations (it means that the beliefs of populations  $i$  and  $j$  are the two biggest beliefs) and we have  $s$  decision-making stages. If the biggest belief is more than a threshold (minimum acceptable belief)  $d_{i,j}(s)$ , ( $0 \leq d_{i,j}(s) \leq 1$ ), we select the corresponding subspace of that belief as the solution. Otherwise, the decision-making process continues by taking more observations. We determine the value of  $d_{i,j}(s)$  such that the belief of making the correct decision is maximized. To do this suppose that for each population a new observation,  $(\alpha_{j,k}, \beta_{j,k})$ , is available at a given stage  $k$ . At this stage, we define  $V(s, d_{i,j}(s))$  to be the expected belief of making the correct decision in  $s$  stages when two populations  $i$  and  $j$  are the candidates for the optimal population. In other words, if we let  $CS$  denote the event of making the correct decision, we define  $V_{i,j}(s, d_{i,j}(s)) = E[B_{i,j}\{CS\}]$ , where  $B_{i,j}\{CS\}$  is the belief of making the correct decision. Furthermore, assume that the maximum of  $V_{i,j}(s, d_{i,j}(s))$  occurs at  $d_{i,j}^*(s)$ . Then, we will have

$$V_{i,j}(s, d_{i,j}^*(s)) = \text{Max}_{d_{i,j}(s)} \{V_{i,j}(s, d_{i,j}(s))\} = \text{Max} \{E[B_{i,j}\{CS\}]\} \quad (10)$$

We denote this optimal point by  $V_{i,j}^*(s)$ . In other words,  $V_{i,j}^*(s) = V_{i,j}(s, d_{i,j}^*(s))$ . Moreover, let us define  $S_i$  and  $S_j$  to be the state of selecting population  $i$  and  $j$  as the candidates for the optimal population, respectively, and  $NS_{i,j}$  as the state of choosing neither of these population. Then, by conditioning on the above states, we have

$$V_{i,j}^*(s) = \text{Max} \left\{ E[B_{i,j}\{CS\}] \right\} = \text{Max} \left\{ E[B_{i,j}\{CS|S_i\}B_{i,j}\{S_i\} + B_{i,j}\{CS|S_j\}B_{i,j}\{S_j\} + B_{i,j}\{CS|NS_{i,j}\}B_{i,j}\{NS_{i,j}\}] \right\} \quad (11)$$

In order to evaluate  $V_{i,j}^*(s)$ , in what follows we will find the belief terms of equation (11).



a.  $B_{i,j}\{CS | S_i\}$  and  $B_{i,j}\{CS | S_j\}$

These are the beliefs of making the correct decision if population  $i$  or  $j$  is selected as the optimal population, respectively. To make the evaluation easier, we denote these beliefs by  $B_{i,j}(i)$  and  $B_{i,j}(j)$ . Then, using equation (2) we have

$$B_{i,j}\{CS|S_i\} = B_{i,j}(i) = \frac{B(\alpha_{i,k-1}, \beta_{i,k-1}) \overline{p_{k,i}}}{B(\alpha_{i,k-1}, \beta_{i,k-1}) \overline{p_{k,i}} + B(\alpha_{j,k-1}, \beta_{j,k-1}) \overline{p_{k,j}}} \quad (12)$$

Similarly,

$$B_{i,j}\{CS|S_j\} = B_{i,j}(j) = \frac{B(\alpha_{j,k-1}, \beta_{j,k-1}) \overline{p_{k,j}}}{B(\alpha_{j,k-1}, \beta_{j,k-1}) \overline{p_{k,j}} + B(\alpha_{i,k-1}, \beta_{i,k-1}) \overline{p_{k,i}}} \quad (13)$$

b.  $B_{i,j}\{S_i\}$  and  $B_{i,j}\{S_j\}$

These are the beliefs of selecting population  $i$  or  $j$  as the optimal population, respectively. Regarding the decision-making strategy, we have:

$$B_{i,j}(i) = \max(B_{i,j}(i), B_{i,j}(j)) \text{ and } B_{i,j}(i) \geq d_{i,j}^*(s) \quad (14)$$

Hence, we define event  $S_i$  as

$$S_i \equiv \{B_{i,j}(i) = \max\{B_{i,j}(i), B_{i,j}(j)\}, B_{i,j}(i) \geq d_{i,j}^*(s)\} \quad (15)$$

Since  $B_{i,j}(i) + B_{i,j}(j) = 1$  and that the beliefs are not negative we conclude  $\max\{B_{i,j}(i), B_{i,j}(j)\} \geq 0.5$ . Furthermore, since the decision making is performed based upon the maximum value of the beliefs, without interruption of assumptions, we can change the variation interval of  $d_{i,j}^*(s)$  from  $[0,1]$  to  $[0.5,1]$ . Now by considering  $d_{i,j}^*(s) \geq 0.5$  implicitly, we have  $S_i \equiv \{B_{i,j}(i) \geq d_{i,j}^*(s)\}$ . By similar reasoning  $S_j \equiv \{B_{i,j}(j) \geq d_{i,j}^*(s)\}$ . Hence

$$B_{i,j}\{S_j\} = \Pr\{B_{i,j}(j) \geq d_{i,j}^*(s)\} = \Pr\left\{\frac{B(\alpha_{j,k-1}, \beta_{j,k-1}) \overline{p_{j,k}}}{B(\alpha_{i,k-1}, \beta_{i,k-1}) \overline{p_{i,k}} + B(\alpha_{j,k-1}, \beta_{j,k-1}) \overline{p_{j,k}}} \geq d_{i,j}^*(s)\right\} = \Pr\left\{\overline{p_{j,k}} \geq h(d_{i,j}^*(s)) \overline{p_{i,k}}\right\} = \Pr\left\{\frac{\overline{p_{j,k}}}{\overline{p_{i,k}}} \geq h(d_{i,j}^*(s))\right\} \quad (16)$$



In which,  $h(d_{i,j}^*(s)) = \frac{d_{i,j}^*(s)B(\alpha_{i,k-1}, \beta_{i,k-1})}{(1-d_{i,j}^*(s))B(\alpha_{j,k-1}, \beta_{j,k-1})}$ .

To evaluate  $\Pr\left\{\frac{\overline{p_{j,k}}}{\overline{p_{i,k}}} \geq h(d_{i,j}^*(s))\right\}$  in equation (16), let  $f_1(\overline{p_{j,k}})$  and  $f_2(\overline{p_{i,k}})$  to be the probability distributions of  $\overline{p_{j,k}}$  and  $\overline{p_{i,k}}$ , respectively. Then,

$$\begin{aligned} f_2(\overline{p_{i,k}}) &= \frac{\Gamma(\alpha_{i,k} + \beta_{i,k} + 1)}{\Gamma(\alpha_{i,k} + 0.5)\Gamma(\beta_{i,k} + 0.5)} \overline{p_{i,k}}^{\alpha_{i,k} - 0.5} (1 - \overline{p_{i,k}})^{\beta_{i,k} - 0.5} \\ f_1(\overline{p_{j,k}}) &= \frac{\Gamma(\alpha_{j,k} + \beta_{j,k} + 1)}{\Gamma(\alpha_{j,k} + 0.5)\Gamma(\beta_{j,k} + 0.5)} \overline{p_{j,k}}^{\alpha_{j,k} - 0.5} (1 - \overline{p_{j,k}})^{\beta_{j,k} - 0.5} \end{aligned} \tag{17}$$

Hence,

$$\begin{aligned} \Pr\left\{\overline{p_{j,k}} \geq h(d_{i,j}^*(s))\overline{p_{i,k}}\right\} &= \int_0^1 \int_{h(d_{i,j}^*(s))\overline{p_{i,k}}}^1 f_1(\overline{p_{j,k}}) f_2(\overline{p_{i,k}}) d\overline{p_{j,k}} d\overline{p_{i,k}} = \\ & \int_0^1 \int_{h(d_{i,j}^*(s))\overline{p_{i,k}}}^1 A_i \overline{p_{i,k}}^{\alpha_{i,k} - 0.5} (1 - \overline{p_{i,k}})^{\beta_{i,k} - 0.5} A_j \overline{p_{j,k}}^{\alpha_{j,k} - 0.5} (1 - \overline{p_{j,k}})^{\beta_{j,k} - 0.5} d\overline{p_{j,k}} d\overline{p_{i,k}} \end{aligned} \tag{18}$$

where

$$A_i = \frac{\Gamma(\alpha_{i,k} + \beta_{i,k} + 1)}{\Gamma(\alpha_{i,k} + 0.5)\Gamma(\beta_{i,k} + 0.5)}, A_j = \frac{\Gamma(\alpha_{j,k} + \beta_{j,k} + 1)}{\Gamma(\alpha_{j,k} + 0.5)\Gamma(\beta_{j,k} + 0.5)}. \tag{19}$$

By change of variables technique, we have:

$$\begin{aligned} U &= \frac{\overline{p_{j,k}}}{\overline{p_{i,k}}} \text{ and } V = \overline{p_{i,k}} \\ f(U) &= A_i A_j U^{\alpha_{i,k} - 0.5} \int_0^1 V^{\alpha_{i,k} + \alpha_{j,k}} (1 - V)^{\beta_{i,k} - 1} (1 - UV)^{\beta_{j,k} - 0.5} dV \\ \Pr\left\{\frac{\overline{p_{j,k}}}{\overline{p_{i,k}}} \geq h(d_{i,j}^*(s))\right\} &= 1 - \int_0^{h(d_{i,j}^*(s))} f(U) dU = 1 - F(h(d_{i,j}^*(s))) \end{aligned} \tag{20}$$

For  $B_{i,j}\{S_i\}$  we have

$$\begin{aligned} B_{i,j}\{S_i\} &= \Pr\{B_{i,j}(i) \geq d_{i,j}^*(s)\} = \\ \Pr\{1 - B_{i,j}(j) \geq d_{i,j}^*(s)\} &= \Pr\{B_{i,j}(j) \leq 1 - d_{i,j}^*(s)\} = F(h(1 - d_{i,j}^*(s))) \end{aligned} \tag{21}$$

c.  $B_{i,j}\{CS \mid NS_{i,j}\}$

$B_{i,j}\{CS \mid NS_{i,j}\}$  is the belief of making the correct decision when none of the subspaces  $i$  and  $j$  has been chosen as the optimal one. In other words, the maximum beliefs has been less than  $d_{i,j}^*(s)$  and the process of decision-making continues to the next stage. In terms of stochastic dynamic programming approach, the belief of this event is equal to the maximum belief of making the correct decision in  $(s-1)$  stages. Since the value of this belief is discounted in the current stage, using discount factor  $\alpha$ ,

$$B_{i,j}\{CS \mid NS_{i,j}\} = \alpha V_{i,j}^*(s-1) \tag{22}$$

Having all the belief terms of equation (11) evaluated in equations (12), (13), (14), (15), and (16), and knowing that by partitioning the state space we have  $B_{i,j}\{NS_{i,j}\} = 1 - (B_{i,j}\{S_i\} + B_{i,j}\{S_j\})$ , equation (11) can now be evaluated by substituting.

$$\begin{aligned} V_{i,j}^*(s) &= \max_{0.5 \leq d_{i,j}(s) \leq 1} \{B_{i,j}(i) \Pr\{B_{i,j}(i) \geq d_{i,j}(s)\} + B_{i,j}(j) \Pr\{B_{i,j}(j) \geq d_{i,j}(s)\} \\ &+ B_{i,j}\{NS_{i,j} \mid CS\} (1 - \Pr\{B_{i,j}(i) \geq d_{i,j}(s)\} - \Pr\{B_{i,j}(j) \geq d_{i,j}(s)\})\} \\ &= \max_{0.5 \leq d_{i,j}(s) \leq 1} \{B_{i,j}(i) \Pr\{B_{i,j}(i) \geq d_{i,j}(s)\} + B_{i,j}(j) \Pr\{B_{i,j}(j) \geq d_{i,j}(s)\} + \\ &\alpha V_{i,j}^*(s-1) (1 - \Pr\{B_{i,j}(i) \geq d_{i,j}(s)\} - \Pr\{B_{i,j}(j) \geq d_{i,j}(s)\})\} \end{aligned} \tag{23}$$

### 2.2.1. Making the decision

Assuming that for the two biggest beliefs we have  $B_{i,j}(i) \geq B_{i,j}(j)$ , equation (23) can be written as

$$\begin{aligned} V_{i,j}^*(s) &= (B_{i,j}(i) - \alpha V_{i,j}^*(s-1)) \Pr\{B_{i,j}(i) \geq d_{i,j}^*(s)\} + \\ &(B_{i,j}(j) - \alpha V_{i,j}^*(s-1)) \Pr\{B_{i,j}(j) \geq d_{i,j}^*(s)\} + \alpha V_{i,j}^*(s-1) \end{aligned} \tag{24}$$

For the decision-making problem at hand, three cases may happen

1.  $B_{i,j}(i) < \alpha V_{i,j}^*(s-1)$  :

In this case, both  $(B_{i,j}(i) - \alpha V_{i,j}^*(s-1))$  and  $(B_{i,j}(j) - \alpha V_{i,j}^*(s-1))$  are negative. Since we are maximizing  $V_{i,j}(s, d_{i,j}(s))$ , then the two probability terms in equation (24) must be minimized. This only happens when we let  $d_{i,j}^*(s) = 1$ , making the probability terms equal to zero. Now since  $B_{i,j}(i) < d_{i,j}^*(s) = 1$ , we continue to the next stage.

2.  $B_{i,j}(j) > \alpha V_{i,j}^*(s-1) :$

In this case,  $(B_{i,j}(i) - \alpha V_{i,j}^*(s-1))$  and  $(B_{i,j}(j) - \alpha V_{i,j}^*(s-1))$  are both positive and to maximize  $V_{i,j}(s, d_{i,j}(s))$  we need the two probability terms in equation (24) to be maximized. This only happens when we let  $d_{i,j}^*(s) = 0.5$ . Since  $B_{i,j}(i) > d_{i,j}^*(s) = 0.5$ , we select population  $i$  as the optimal subspace.

3.  $B_{i,j}(j) \leq \alpha V_{i,j}^*(s-1) \leq B_{i,j}(i) :$

In this case, one of the probability terms in equation (24) has positive coefficient and the other has negative coefficient. In this case, in order to maximize  $V_{i,j}(s, d_{i,j}(s))$  we take the derivative as follows.

Substituting equations (20) and (21) in equation (24) we have

$$V_{i,j}(s, d_{i,j}(s)) = (B_{i,j}(i) - \alpha V_{i,j}^*(s-1)) \{F(h(1 - d_{i,j}(s)))\} + (B_{i,j}(j) - \alpha V_{i,j}^*(s-1)) \{1 - F(h(d_{i,j}(s)))\} + \alpha V_{i,j}^*(s-1) \tag{25}$$

Thus following is obtained,

$$V_{i,j}^*(s) = (B_{i,j}(i) - \alpha V_{i,j}^*(s-1)) \Pr \left\{ \frac{\overline{p_{j,k}}}{p_{i,k}} \leq h(1 - d_{i,j}^*(s)) \right\} + (B_{i,j}(j) - \alpha V_{i,j}^*(s-1)) \Pr \left\{ \frac{\overline{p_{j,k}}}{p_{i,k}} \geq h(d_{i,j}^*(s)) \right\} + \alpha V_{i,j}^*(s-1) \tag{26}$$

For determining  $\Pr \left\{ \frac{\overline{p_{j,k}}}{p_{i,k}} \leq h(1 - d_{i,j}^*(s)) \right\}$ , first using an approximation, we assume that  $\overline{p_{i,k}}$  is a constant number equal to its mean, then we have:

$$\begin{aligned} \Pr \left\{ \frac{\overline{p_{j,k}}}{p_{i,k}} \leq h(1 - d_{i,j}^*(s)) \right\} &= \\ \Pr \left\{ \overline{p_{j,k}} \leq \overline{p_{i,k}} h(1 - d_{i,j}^*(s)) \right\} &= \int_0^{\overline{p_{i,k}} h(1 - d_{i,j}^*(s))} f_1(p_{j,k}) dp_{j,k} = \\ F_1(\overline{p_{i,k}} h(1 - d_{i,j}^*(s))) & \\ \frac{\partial F_1(\overline{p_{i,k}} h(1 - d_{i,j}^*(s)))}{\partial d_{i,j}^*(s)} &= \frac{B(\alpha_{i,k-1}, \beta_{i,k-1})}{(1 - d_{i,j}^*(s))^2 B(\alpha_{j,k-1}, \beta_{j,k-1})} f_1(\overline{p_{i,k}} h(1 - d_{i,j}^*(s))) \end{aligned} \tag{27}$$

Also  $\Pr\left\{\frac{\overline{p_{j,k}}}{p_{i,k}} \geq h(d_{i,j}^*(s))\right\}$  is obtained as follows,

$$\Pr\left\{\frac{\overline{p_{j,k}}}{p_{i,k}} \geq h(d_{i,j}^*(s))\right\} = \Pr\left\{\overline{p_{j,k}} \geq \overline{p_{i,k}}h(d_{i,j}^*(s))\right\} = \int_{\overline{p_{i,k}}h(d_{i,j}^*(s))}^1 f_1(p_{j,k}) dp_{j,k} =$$

$$\frac{1 - F_1(\overline{p_{i,k}}h(d_{i,j}^*(s)))}{\frac{\partial(1 - F_1(\overline{p_{i,k}}h(d_{i,j}^*(s))))}{\partial d_{i,j}^*(s)}} = \frac{-B(\alpha_{i,k-1}, \beta_{i,k-1})}{(d_{i,j}^*(s))^2 B(\alpha_{j,k-1}, \beta_{j,k-1})} f_1(\overline{p_{i,k}}h(d_{i,j}^*(s))) \quad (28)$$

Now following can be resulted,

$$\frac{\partial V_{i,j}(s, d_{i,j}(s))}{\partial d_{i,j}(s)} = 0 \Rightarrow$$

$$(B_{i,j}(i) - \alpha V_{i,j}^*(s-1)) \frac{B(\alpha_{i,k-1}, \beta_{i,k-1})}{(1 - d_{i,j}^*(s))^2 B(\alpha_{j,k-1}, \beta_{j,k-1})} f_1(\overline{p_{i,k}}h(1 - d_{i,j}^*(s))) =$$

$$-(B_{i,j}(j) - \alpha V_{i,j}^*(s-1)) \frac{B(\alpha_{i,k-1}, \beta_{i,k-1})}{(d_{i,j}^*(s))^2 B(\alpha_{j,k-1}, \beta_{j,k-1})} f_1(\overline{p_{i,k}}h(d_{i,j}^*(s))) \Rightarrow$$

$$\frac{(B_{i,j}(i) - \alpha V_{i,j}^*(s-1))}{-(B_{i,j}(j) - \alpha V_{i,j}^*(s-1))} = \left(\frac{1 - d_{i,j}^*(s)}{d_{i,j}^*(s)}\right)^{2(\alpha_{i,k-1} + \beta_{i,k-1})} \Rightarrow$$

$$d_{i,j}^1(s) = \frac{1}{\left(\frac{(B_{i,j}(i) - \alpha V_{i,j}^*(s-1))}{-(B_{i,j}(j) - \alpha V_{i,j}^*(s-1))}\right)^{\frac{1}{2(\alpha_{i,k-1} + \beta_{i,k-1})}} + 1} \quad (29)$$

Now the approximate value of  $d_{i,j}(s)$  say  $d_{i,j}^1(s)$  is determined.

Second using another approximation, we assume that  $\overline{p_{j,k}}$  is a constant number equal to its mean thus with similar reasoning, following is obtained:

$$d_{i,j}^2(s) = \frac{1}{\left(\frac{(B_{i,j}(j) - \alpha V_{i,j}^*(s-1))}{-(B_{i,j}(i) - \alpha V_{i,j}^*(s-1))}\right)^{\frac{1}{2(\alpha_{i,k-1} + \beta_{i,k-1})}} + 1} \quad (30)$$

Therefore the approximate optimal value of  $d_{i,j}^*(s)$  can be determined from following equation,

$$d_{i,j}^*(s) = \text{Max}\{d_{i,j}^1(s), d_{i,j}^2(s)\} \quad (31)$$

### 3. An application for fault detection and diagnosis in multivariate statistical quality control environments

#### 3.1. Introduction

In this section, a heuristic threshold policy is applied in phase II of a control charting procedure to not only detect the states of a multivariate quality control system, but also to diagnose the quality characteristic(s) responsible for an out-of-control signal. It is assumed that the in-control mean vector and in-control covariance matrix of the process have been obtained in phase I.

#### 3.2. Background

In a multivariate quality control environment, suppose there are  $m$  correlated quality characteristics whose means are being monitored simultaneously. Further, assume there is only one observation on the quality characteristics at each iteration of the data gathering process, where the goal is to detect the variable with the maximum mean shift. Let  $x_{ki}$  be the observation of the  $i^{\text{th}}$  quality characteristic,  $i = 1, 2, \dots, m$ , at iteration  $k$ ,  $k = 1, 2, \dots$ , and define the observation vector  $\mathbf{x}_k = [x_{k1}, x_{k2}, \dots, x_{km}]^T$  and observation matrix  $\mathbf{O}_k = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k)$ . After taking a new observation,  $\mathbf{x}_k$ , define  $B_i(\mathbf{x}_k, \mathbf{O}_{k-1})$ , the probability of variable  $i$  to be in an out-of-control state, as

$$B_i(\mathbf{x}_k, \mathbf{O}_{k-1}) = \Pr\{OOC_i | \mathbf{x}_k, \mathbf{O}_{k-1}\}, \quad (32)$$

where  $OOC$  stands for out-of-control. This probability has been called the belief of variable  $i$  to be in out-of-control condition given the observation matrix up to iteration  $k-1$  and the observation vector obtained at iteration  $k$ .

Assuming the observations are taken independently at each iteration, to improve the belief of the process being in an out-of-control state at the  $k^{\text{th}}$  iteration, based on the observation matrix  $\mathbf{O}_{k-1}$  and the new observation vector  $\mathbf{x}_k$ , we have

$$\Pr\{\mathbf{x}_k | OOC_i, \mathbf{O}_{k-1}\} = \Pr\{\mathbf{x}_k | OOC_i\} \quad (33)$$

Then, using the Bayesian rule the posterior belief is:

$$\begin{aligned}
 B_i(\mathbf{x}_k, \mathbf{O}_{k-1}) &= \Pr\{OOC_i | \mathbf{x}_k, \mathbf{O}_{k-1}\} = \frac{\Pr\{OOC_i, \mathbf{x}_k, \mathbf{O}_{k-1}\}}{\Pr\{\mathbf{x}_k, \mathbf{O}_{k-1}\}} = \\
 &= \frac{\Pr\{OOC_i, \mathbf{x}_k, \mathbf{O}_{k-1}\}}{\sum_{j=1}^m \Pr\{OOC_j, \mathbf{x}_k, \mathbf{O}_{k-1}\}} = \frac{\Pr\{OOC_i | \mathbf{O}_{k-1}\} \Pr\{\mathbf{x}_k | OOC_i, \mathbf{O}_{k-1}\}}{\sum_{j=1}^m \Pr\{OOC_j | \mathbf{O}_{k-1}\} \Pr\{\mathbf{x}_k | OOC_j, \mathbf{O}_{k-1}\}}
 \end{aligned} \tag{34}$$

Since the goal is to detect the variable with the maximum mean shift, only one quality characteristic can be considered out-of-control at each iteration. In this way, there are  $m-1$  remaining candidates for which  $m-1$  quality characteristics are in-control. Hence, one can say that the candidates are mutually exclusive and collectively exhaustive. Therefore, using the Bayes' theorem, one can write equation (34) as

$$B_i(\mathbf{x}_k, \mathbf{O}_{k-1}) = \frac{\Pr\{OOC_i | \mathbf{O}_{k-1}\} \Pr\{\mathbf{x}_k | OOC_i\}}{\sum_{j=1}^m \Pr\{OOC_j | \mathbf{O}_{k-1}\} \Pr\{\mathbf{x}_k | OOC_j\}} = \frac{B_i(\mathbf{x}_{k-1}, \mathbf{O}_{k-2}) \Pr\{\mathbf{x}_k | OOC_i\}}{\sum_{j=1}^m B_j(\mathbf{x}_{k-1}, \mathbf{O}_{k-2}) \Pr\{\mathbf{x}_k | OOC_j\}} \tag{35}$$

When the system is in-control, we assume the  $m$  characteristics follow a multinormal distribution with mean vector  $\boldsymbol{\mu} = [\mu_1, \mu_2, \dots, \mu_m]^T$  and covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdot & \sigma_{1m} \\ \sigma_{21} & \sigma_2^2 & \cdot & \sigma_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \sigma_{m1} & \sigma_{m2} & \cdot & \sigma_m^2 \end{bmatrix} \tag{36}$$

In out-of-control situations, only the mean vector changes and the probability distribution along with the covariance matrix remain unchanged. In latter case, equation (35) is used to calculate the probability of shifts in the process mean  $\boldsymbol{\mu}$  at different iterations. Moreover, in order to update the beliefs at iteration  $k$  one needs to evaluate  $\Pr\{\mathbf{x}_k | OOC_i\}$ .

The term  $\Pr\{\mathbf{x}_k | OOC_i\}$  is the probability of observing  $\mathbf{x}_k$  if only the  $i^{\text{th}}$  quality characteristic is out-of-control. The exact value of this probability can be determined using the multivariate normal density,  $A \exp\left(-\frac{1}{2}(\mathbf{x}_k - \boldsymbol{\mu}_{1i})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}_k - \boldsymbol{\mu}_{1i})\right)$ , where  $\boldsymbol{\mu}_{1i}$  denotes the mean vector in which only the  $i^{\text{th}}$  characteristic has shifted to an out-of-control condition and  $A$  is a known constant.

Since the exact value of the out-of-control mean vector  $\boldsymbol{\mu}_{1i}$  is not known a priori, two approximations are used in this research to determine  $\Pr\{\mathbf{x}_k \mid OOC_i\}$ . Note that we do not want to determine the exact probability. Instead, the aim is to have an approximate probability (a belief) on each characteristic being out-of-control. In the first approximation method, define  $IC_i$  to be the event that all characteristics are in-control, and let  $\Pr\{\mathbf{x}_k \mid IC_i\}$  be the conditional probability of observing  $\mathbf{x}_k$  given all characteristics are in-control. Further, let  $\mathbf{x}'_k = [\mu_{01}, \dots, x_{ki}, \mu_{0i+1}, \dots, \mu_{0m}]^T$  in the aforementioned multivariate normal density, so that  $\Pr\{\mathbf{x}_k \mid IC_i\}$  can be approximately evaluated using  $\Pr\{\mathbf{x}_k \mid IC_i\} = \Pr\{\mathbf{x}'_k \mid IC_i\}$ , where  $\Pr\{\mathbf{x}'_k \mid IC_i\} = A \exp\left(-\frac{1}{2}(\mathbf{x}'_k - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}'_k - \boldsymbol{\mu}_0)\right)$ . Note that this evaluation is proportional to  $\exp\left(-\frac{1}{2}\left(\frac{x_{ki} - \mu_{0i}}{\sigma_i}\right)^2\right)$ , and since it is assumed that characteristic  $i$  is under control, no matter the condition of the other characteristics, this approximation is justifiable.

In the second approximation method, we assume  $\Pr\{\mathbf{x}_k \mid OOC_i\} \propto \frac{1}{\Pr\{\mathbf{x}_k \mid IC_i\}}$ . Although it is obvious that  $\Pr\{\mathbf{x}_k \mid OOC_i\}$  is not equal to  $\frac{1}{\Pr\{\mathbf{x}_k \mid IC_i\}}$ , since we only need a belief function to evaluate  $\Pr\{\mathbf{x}_k \mid OOC_i\}$  and also we do not know the exact value of out-of-control mean vector, this approximation is just used to determine  $\Pr\{\mathbf{x}_k \mid OOC_i\}$ . Moreover, it can be easily seen that the closer the value of the  $i^{\text{th}}$  characteristic is to its in-control mean the smaller is  $\Pr\{\mathbf{x}_k \mid OOC_i\}$  as expected. We thus let

$$\Pr\{\mathbf{x}_k \mid OOC_i\} \propto \frac{1}{\Pr\{\mathbf{x}_k \mid IC_i\}} = R \exp\left(\frac{1}{2}\left(\frac{x_{ki} - \mu_{0i}}{\sigma_i}\right)^2\right) ; \quad i = 1, 2, \dots, m, \quad (37)$$

where  $R$  is a sufficiently big constant number to ensure the above definition is less than one.

The approximation to  $\Pr\{\mathbf{x}_k \mid OOC_i\}$  in equation (37) has the following two properties:

- It does not require the value of out-of-control means to be known.
- The determination of a threshold for the decision-making process (derived later) will be easier.

Niaki and Fallahnezhad [8] defined another equation for the above conditional probability and showed that if a shift occurs in the mean of variable  $i$ , then  $\lim_{k \rightarrow \infty} B_i(\mathbf{x}_k, \mathbf{O}_{k-1}) = B_i = 1$ . They proposed a novel method of detection and classification and used simulation to compare its performances with that of existing methods in terms of the average run length for different mean shifts. The results of the simulation study were in favor of their proposed method in almost all shift scenarios. Besides using a different equation, the main difference between the current research and Niaki and Fallahnezhad [8] is that the current work develops a novel heuristic threshold policy, in which to save sampling cost and time or when these factors are constrained, the number of the data gathering stages is limited.



### 3.3. The proposed procedure

Assuming a limited number of the data gathering stages,  $N$ , to detect and diagnose characteristic(s), a heuristic threshold policy-based model is developed in this Section. The framework of the proposed decision-making process follows.

#### Step I

Define  $i=1, 2, \dots, m$  as the set of indices for the characteristics, all of which having the potential of being out-of-control.

#### Step II

Using the maximum entropy principle, initialize  $B_i(O_0)=1/m$  as the prior belief of the  $i^{th}$  variable to be out-of-control. In other words, at the start of the decision-making process all variables have an equal chance of being out-of-control. Set the discount rate  $\alpha$ , the maximum probability of correct selection when  $N$  decision making stages remains  $V(N)$ , and the maximum number of decision making stages  $N$ .

#### Step III

Set  $k=0$

#### Step IV

Obtain an observation of the process.

#### Step V

Estimate the posterior beliefs,  $B_i(O_k)$  (for  $i=1, 2, \dots, m$ ), using equation (35).

#### Step VI

Obtain the order statistics on the posterior beliefs  $B_i(O_k)$  such that

$$B_{(1)}(O_k) < B_{(2)}(O_k) < \dots < B_{(m)}(O_k).$$

Furthermore, let  $B_{gr}(O_k) = B_{(m)}(O_k)$  and  $B_{sm}(O_k) = B_{(m-1)}(O_k)$ .

#### Step VII

Assume the variables with the indices  $i=gr$  and  $j=sm$  are the candidates of being out-of-control, where  $N$  decision-making steps are available. Define  $V(N, d_{i,j}(k))$  the probability of correct choice between the variables  $i$  and  $j$ , where  $d_{i,j}(k)$  is the acceptable belief. Also, define CS the event of correct selection and event  $E_{i,j}$  the existence of two out-of-control candidate variables  $i$  and  $j$ . Then, we have:

$$V(N, d_{i,j}(k)) = \Pr\{CS|E_{i,j}\} \triangleq \Pr_{i,j}\{CS\} \quad (38)$$

where " $\triangleq$ " means "defined as."

Assuming  $d_{i,j}^*(k)$  the maximum point of  $V(N, d_{i,j}(k))$ , called the minimum acceptable belief, we have

$$V(N, d_{i,j}^*(k)) \triangleq V_{i,j}^*(N) = \text{Max}_{d_{i,j}(k)} \{V(N, d_{i,j}(k))\} \triangleq \text{Max} \{Pr_{i,j} \{CS\}\} \quad (39)$$

Let  $S_i$  and  $S_j$  be the event of selecting  $i$  and  $j$  as the out-of-control variables, respectively, and  $NS_{i,j}$  be the event of not selecting any. Then, by conditioning on the probability, we have:

$$V_{i,j}^*(N) = \text{Max} \{Pr_{i,j} \{CS\}\} = \text{Max} \{Pr_{i,j} \{CS|S_i\} Pr_{i,j} \{S_i\} + Pr_{i,j} \{CS|S_j\} Pr_{i,j} \{S_j\} + Pr_{i,j} \{CS|NS_{i,j}\} Pr_{i,j} \{NS_{i,j}\}\} \quad (40)$$

At the  $k^{\text{th}}$  iteration, the conditional bi-variate distribution of the sample means for variables  $gr$  and  $sm$ , i.e,  $X_{k,j=gr,sm} | X_{k,j \neq gr,sm}$  is determined using the conditional property of multivariate normal distribution given in appendix 1. Moreover, knowing  $E(x_{k,j}) = \mu_j$  and evaluating the conditional mean and standard deviation (see appendix 1) results in

$$E(X_{k,i} | X_{k,j}) = \mu_i + \rho \frac{\sigma_i}{\sigma_j} (X_{k,j} - \mu_j) \quad (41)$$

and

$$E(X_{k,i} | X_{k,j}) = \mu_i + \rho \frac{\sigma_i}{\sigma_j} (X_{k,j} - \mu_j) \quad (42)$$

Based on the decomposition method of Mason et al. [9], define statistics  $T_{k,j}$  and  $T_{k,i|j}$  as

$$T_{k,j} = \left( \frac{X_{k,j} - \mu_j}{\sigma_j} \right) \quad (43)$$

$$T_{k,i|j} = \left( \frac{X_{k,i} - E(X_{k,i} | X_{k,j})}{\sigma_{X_{k,i} | X_{k,j}}} \right) \quad (44)$$

Thus, when the process is in-control, the statistics  $T_{k,j}$  and  $T_{k,i|j}$  follow a standard normal distribution [9].

Now, let  $B_{i,j}(i; x_k, O_{k-1})$  denote the probability of correct selection conditioned on selecting  $i$  as the out-of-control variable. Hence,

$$B_{i,j}(i; x_k, O_{k-1}) = \frac{B_i(O_{k-1})e^{(0.5T_{k,i})^2}}{B_j(O_{k-1})e^{(0.5T_{k,j})^2} + B_i(O_{k-1})e^{(0.5T_{k,i})^2}} \quad (45)$$

$$B_{i,j}(j; x_k, O_{k-1}) = \frac{B_j(O_{k-1})e^{(0.5T_{k,j})^2}}{B_j(O_{k-1})e^{(0.5T_{k,j})^2} + B_i(O_{k-1})e^{(0.5T_{k,i})^2}}$$

Then, the probability measure  $\Pr_{i,j}\{CS | S_i\}$  is calculated using the following equation,

$$\Pr_{i,j}\{CS | S_i\} = B_{i,j}(i; x_k, O_{k-1}) \quad (46)$$

The probability measure  $\Pr_{i,j}\{S_i\}$  is defined as the probability of selecting variable  $i$  to be out-of-control. Regarding to the explained strategy, we have:

$$S_i \equiv \left\{ B_{i,j}(i; x_k, O_{k-1}) = \text{Max} \left\{ \begin{matrix} B_{i,j}(i; x_k, O_{k-1}), \\ B_{i,j}(j; x_k, O_{k-1}) \end{matrix} \right\}, B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k) \right\} \quad (47)$$

Since  $B_{i,j}(i; x_k, O_{k-1}) + B_{i,j}(j; x_k, O_{k-1}) = 1$  and the value of beliefs are not negative, we conclude

$$\max \{ B_{i,j}(i; x_k, O_{k-1}), B_{i,j}(j; x_k, O_{k-1}) \} \geq 0.5 \quad (48)$$

Without interruption of assumptions, we can change the variation interval of  $d_{i,j}(k)$  from  $[0,1]$  to  $[0.5,1]$ . Hence,

$$S_i \equiv \{ B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k) \} \quad (49)$$

By similar reasoning, we have:

$$S_j \equiv \{ B_{i,j}(j; x_k, O_{k-1}) \geq d_{i,j}(k) \} \quad (50)$$

The term  $\Pr_{i,j}\{CS | N S_{i,j}\}$  denotes the probability of correct selection conditioned on excluding the candidates  $i$  and  $j$  as the solution. In other words, the maximum belief has been less than

the threshold (minimum acceptable belief)  $d_{i,j}^*(k)$  and the decision making process continues to the next stage. In terms of stochastic dynamic programming approach, the probability of this event is equal to the maximum probability of correct selection when there are  $N - 1$  stages remaining. The discounted value of this probability in the current stage using the discount factor  $\alpha$  equals to  $\alpha V_{i,j}(N - 1)$ . Further, since we partitioned the decision space into events  $\{NS_{i,j}; S_i; S_j\}$ , we have:

$$\Pr_{i,j}\{NS_{i,j}\} = 1 - (\Pr_{i,j}\{S_i\} + \Pr_{i,j}\{S_j\}) \tag{51}$$

Now we evaluate  $V_{i,j}^*(N)$  as follows,

$$\begin{aligned} V_{i,j}^*(N) &= \left\{ \begin{aligned} &B_{i,j}(i; x_k, O_{k-1})\Pr_{i,j}\{B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k)\} + \\ &B_{i,j}(j; x_k, O_{k-1})\Pr_{i,j}\{B_{i,j}(j; x_k, O_{k-1}) \geq d_{i,j}(k)\} + \\ &\Pr_{i,j}\{NS_{i,j}|CS\} \left( \begin{aligned} &1 - \Pr_{i,j}\{B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k)\} - \\ &\Pr\{B_{i,j}(j; x_k, O_{k-1}) \geq d_{i,j}(k)\} \end{aligned} \right) \end{aligned} \right\} \tag{52} \\ &= \left\{ \begin{aligned} &B_{i,j}(i; x_k, O_{k-1})\Pr_{i,j}\{B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k)\} + \\ &B_{i,j}(j; x_k, O_{k-1})\Pr_{i,j}\{B_{i,j}(j; x_k, O_{k-1}) \geq d_{i,j}(k)\} + \\ &\alpha V_{i,j}^*(N - 1) \left( \begin{aligned} &1 - \Pr_{i,j}\{B_{i,j}(i; x_k, O_{k-1}) \geq d_{i,j}(k)\} - \\ &\Pr\{B_{i,j}(j; x_k, O_{k-1}) \geq d_{i,j}(k)\} \end{aligned} \right) \end{aligned} \right\} \end{aligned}$$

In other words,

$$V_{i,j}^*(N) = \left\{ \begin{aligned} &B_{i,j}(i; O_k)\Pr_{i,j}\{B_{i,j}(i; O_k) \geq d_{i,j}(k)\} + \\ &B_{i,j}(j; O_k)\Pr_{i,j}\{B_{i,j}(j; O_k) \geq d_{i,j}(k)\} + \\ &\alpha V_{i,j}^*(N - 1) \left( \begin{aligned} &1 - \Pr_{i,j}\{B_{i,j}(i; O_k) \geq d_{i,j}(k)\} \\ &-\Pr\{B_{i,j}(j; O_k) \geq d_{i,j}(k)\} \end{aligned} \right) \end{aligned} \right\} \tag{53}$$

The method of evaluating the minimum acceptable belief  $d_{gr,sm}^*(k)$  is given in Appendix 2.

### Step VIII: The Decision Step

If the belief  $B_{gr,sm}(gr; x_k, O_{k-1})$  in the candidate set  $(sm, gr)$  is equal to or greater than  $d_{gr,sm}^*(k)$  then choose the variable with index  $gr$  to be out-of-control. In this case, the decision-making process ends. Otherwise, without having any selection at this stage, obtain another observation, lower the number of remaining decision-stages to  $N - 1$ , set  $k = k + 1$ , and return to step V above. The process will continue until either the stopping condition is reached or the number of stages is finished. The optimal strategy with  $N$  decision-making stages that maximizes the probability of correct selection would be resulted from this process.

In what follows, the procedure to evaluate  $V_{i,j}^*(N)$  of equation (53) is given in detail.

#### 3.4. Method of evaluating $V_{i,j}^*(N)$

Using  $d_{i,j}^*(k)$  as the minimum acceptable belief, from equation (53) we have

$$V_{i,j}^*(N) = (B_{i,j}(i; O_k) - \alpha V_{i,j}^*(N - 1)) \Pr\{B_{i,j}(i; O_k) \geq d_{i,j}^*(k)\} + (B_{i,j}(j; O_k) - \alpha V_{i,j}^*(N - 1)) \Pr\{B_{i,j}(j; O_k) \geq d_{i,j}^*(k)\} + \alpha V_{i,j}^*(N - 1) \quad (54)$$

Then, for the decision-making problem at hand, three cases may occur

1.  $B_{i,j}(i; O_k) < \alpha V_{i,j}^*(N - 1)$

In this case, both  $(B_{i,j}(i; O_k) - \alpha V_{i,j}^*(N - 1))$  and  $(B_{i,j}(j; O_k) - \alpha V_{i,j}^*(N - 1))$  are negative. Since we are maximizing  $V_{i,j}(N, d_{i,j}(k))$ , the two probability terms in equation (54) must be minimized. This only happens when  $d_{i,j}^*(k) = 1$ , making the probability terms equal to zero. In other words, since  $B_{i,j}(i; O_k) < d_{i,j}^*(k) = 1$ , we continue to the next stage.

2.  $B_{i,j}(j; O_k) > \alpha V_{i,j}^*(N - 1)$

In this case,  $(B_{i,j}(i; O_k) - \alpha V_{i,j}^*(N - 1))$  and  $(B_{i,j}(j; O_k) - \alpha V_{i,j}^*(N - 1))$  are both positive and to maximize  $V_{i,j}(N, d_{i,j}(k))$  we need the two probability terms in equation (54) to be maximized. This only happens when  $d_{i,j}^*(k) = 0.5$ . In other words, since  $B_{i,j}(j; O_k) > d_{i,j}^*(k) = 0.5$ , the variable with the index  $j$  is selected.

3.  $B_{i,j}(j; O_k) < \alpha V_{i,j}^*(N - 1) < B_{i,j}(i; O_k)$

In this case, one of the probability terms in equation (54) has a positive and the other a negative coefficient. Then, in order to maximize  $V_{i,j}(N, d_{i,j}(k))$ , the first derivative on  $d_{i,j}(k)$  must be equated to zero. To do this, define  $h(d_{gr,sm}(k))$  and  $r(d_{gr,sm}(k))$  as follows:

$$h(d_{gr,sm}(k)) = \frac{d_{gr,sm}(k)B_{gr}(gr, O_{k-1})}{(1-d_{gr,sm}(k))B_{sm}(sm, O_{k-1})} \quad (55)$$

$$r(d_{gr,sm}(k)) = \frac{d_{gr,sm}(k)B_{sm}(sm, O_{k-1})}{(1-d_{gr,sm}(k))B_{gr}(gr, O_{k-1})} \quad (56)$$

We first present the method of evaluating  $\Pr\{B_{gr,sm}(sm; O_k) \geq d_{gr,sm}(k)\}$  as follows.

$$\begin{aligned} & \Pr\{B_{gr,sm}(sm; O_k) \geq d_{gr,sm}(k)\} = \\ & \Pr\left\{\frac{B_{sm}(sm, O_{k-1})e^{(T_{k,gr|sm})^2}}{B_{sm}(sm, O_{k-1})e^{(T_{k,gr|sm})^2} + B_{gr}(gr, O_{k-1})e^{(T_{k,sm})^2}} \geq d_{gr,sm}(k)\right\} \\ & = \Pr\left\{e^{(T_{k,gr|sm})^2} \geq h(d_{gr,sm}(k))e^{(T_{k,sm})^2}\right\} \end{aligned} \quad (57)$$

Then, the method of evaluating probability terms in equation (57) is given in appendix 2.

With similar reasoning, we have,

$$\Pr\{B_{gr,sm}(gr; O_k) \geq d_{gr,sm}(k)\} = \Pr\left\{e^{(T_{k,gr|sm})^2} \geq r(d_{gr,sm}(k))e^{(T_{k,sm})^2}\right\} \quad (58)$$

The method of determining the minimum acceptable belief is given in appendix 2.

#### 4. An application for fault detection in uni-variate statistical quality control environments

In a uni-variate quality control environment, if we limit ourselves to apply a control charting method, most of the information obtained from data behavior will be ignored. The main aim of a control charting method is to detect quickly undesired faults in the process. However, we may calculate the belief for the process being out-of-control applying Bayesian rule at any iteration in which some observations on the quality characteristic are gathered. Regarding these beliefs and a stopping rule, we may find and specify a control threshold for these beliefs and when the updated belief in any iteration is more than this threshold, an out-of-control signal is observed.

In Decision on Beliefs, first, all probable solution spaces will be divided into several candidates (the solution is one of the candidates), then a belief will be assigned to each candidate consid-

ering our experiences and finally, the beliefs are updated and the optimal decision is selected based on the current situation. In a SPC problem, a similar decision-making process exists. First, the decision space can be divided into two candidates; an in-control or out-of-control production process. Second, the problem solution is one of the candidates (in-control or out-of-control process). Finally, a belief is assigned to each candidate so that the belief shows the probability of being a fault in the process. Based upon the updated belief, we may decide about states of the process (in-control or out-of-control process).

#### 4.1. Learning — The beliefs and approach for its improvement

For simplicity, individual observation on the quality characteristic of interest in any iteration of data gathering process was gathered. At iteration  $k$  of data gathering process,  $O_k = (x_1, x_2, \dots, x_k)$  was defined as the observation vector where resemble observations for previous iterations  $1, 2, \dots, k$ . After taking a new observation,  $O_{k-1}$  the belief of being in an out-of-control state is defined as  $B(x_k, O_{k-1}) = \Pr\{Out - of - control \mid x_k, O_{k-1}\}$ . At this iteration, we want to update the belief of being in out-of-control state based on observation vector  $O_{k-1}$  and new observation  $x_k$ . If we define  $B(O_{k-1}) = B(x_{k-1}, O_{k-2})$  as the prior belief of an out-of-control state, in order to update the posterior belief  $B(x_k, O_{k-1})$ , since we may assume that the observations are taken independently in any iteration, then we will have

$$\Pr\{x_k \mid Out - of - control, O_{k-1}\} = \Pr\{x_k \mid Out - of - control\} \tag{59}$$

With this feature, the updated belief is obtained using Bayesian rule:

$$\begin{aligned} B(x_k, O_{k-1}) &= \Pr\{Out - of - control \mid x_k, O_{k-1}\} = \frac{\Pr\{Out - of - control, x_k \mid O_{k-1}\}}{\Pr\{x_k \mid O_{k-1}\}} \\ &= \frac{\Pr\{Out - of - control \mid O_{k-1}\} \Pr\{x_k \mid Out - of - control, O_{k-1}\}}{\Pr\{x_k \mid O_{k-1}\}} \end{aligned} \tag{60}$$

Since in-control or out-of-control state partition the decision space, we can write equation (60) as

$$\begin{aligned} B(x_k, O_{k-1}) &= \frac{\Pr\{Out - of - control \mid O_{k-1}\} \Pr\{x_k \mid Out - of - control\}}{\Pr\{Out - of - control \mid O_{k-1}\} \Pr\{x_k \mid Out - of - control\} + \Pr\{In - control \mid O_{k-1}\} \Pr\{x_k \mid In - control\}} \\ &= \frac{B(O_{k-1}) \Pr\{x_k \mid Out - of - control\}}{B(O_{k-1}) \Pr\{x_k \mid Out - of - control\} + (1 - B(O_{k-1})) \Pr\{x_k \mid In - control\}} \end{aligned} \tag{61}$$

Assuming the quality characteristic of interest follows a normal distribution with mean  $\mu$  and variance  $\sigma^2$ , we use equation (61) to calculate both beliefs for occurring positive or negative shifts in the process mean  $\mu$ .



- Positive shifts in the process mean

The values of  $B^+(O_k)$ , showing the probability of occurring a positive shift in the process mean, will be calculated applying equation (61) recursively.  $\Pr\{x_k | In-control\}$  is defined by the following equation,

$$\Pr\{x_k | In-control\} = 0.5 \quad (62)$$

For positive shift, the probability of being a positive shift in the process at iteration  $k$ ,  $\Pr\{x_k | Out-of-control\}$ , is calculated using equation (63).

$$\Pr\{x_k | Out-of-control\} = \varphi(x_k) \quad (63)$$

where  $\varphi(x_k)$  is the cumulative probability distribution function for the normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Above probabilities are not exact probabilities and they are a kind of belief function to ascertain good properties for  $B^+(O_k)$

Therefore  $B^+(O_k)$  is determined by the following equation,

$$B^+(O_k) = \frac{B^+(O_{k-1})\varphi(x_k)}{B^+(O_{k-1})\varphi(x_k) + 0.5(1 - B^+(O_{k-1}))} \quad (64)$$

- Negative shifts in the process mean

The values of  $B^-(O_k)$  denotes the probability of being a negative shift in the process mean that is calculated using equation (61) recursively. In this case,  $\Pr\{x_k | In-control\}$  is defined by the following equation,

$$\Pr\{x_k | In-control\} = 0.5 \quad (65)$$

Also is  $\Pr\{x_k | Out-of-control\}$  calculated using equation (66).

$$\Pr\{x_k | Out-of-control\} = 1 - \varphi(x_k) \quad (66)$$

Thus  $B^-(O_k)$  is determined by the following equation,

$$B^-(O_k) = \frac{B^-(O_{k-1})(1 - \varphi(x_k))}{B^-(O_{k-1})(1 - \varphi(x_k)) + 0.5(1 - B^-(O_{k-1}))} \quad (67)$$

#### 4.2. A decision on beliefs approach

We present a decision making approach in terms of Stochastic Dynamic Programming approach. Presented approach is like an optimal stopping problem.

Suppose  $n$  stages for decision making is remained and two decisions are available.

- A positive shift is occurred in the process mean
- No positive shift is occurred in the process mean

Decision making framework is as follows:

- Gather a new observation.
- Calculate the posterior Beliefs in terms of prior Beliefs.
- Order the current Beliefs as an ascending form and choose the maximum.
- Determine the value of the minimum acceptable belief ( $d^+(n)$  is the minimum acceptable belief for detecting the positive shift and  $d^-(n)$  is the least acceptable belief for detecting the negative shift)
- If the maximum Belief was more than the minimum acceptable belief,  $d^+(n)$ , select the belief candidate with maximum value as a solution else go to step 1.
- In terms of above algorithm, the belief with maximum value is chosen and if this belief was more than a control threshold like  $d^+(n)$ , the candidate of that Belief will be selected as optimal candidate else the sampling process is continued. The objective of this model is to determine the optimal values of  $d^+(n)$ . The result of this process is the optimal strategy with  $n$  decision making stages that maximize the probability of correct selection.

Suppose new observation  $x_k$  is gathered. ( $k$  is the number of gathered observations so far).  $V(n, d^+(n))$  is defined as the probability of correct selection when  $n$  decision making stages are remained and we follow  $d^+(n)$  strategy explained above also  $V(n)$  denotes the maximum value of  $V(n, d^+(n))$  thus,

$$V(n) = \text{Max}_{d^+(n)} \{V(n, d^+(n))\} \quad (68)$$

CS is defined as the event of correct selection.  $S_1$  is defined as selecting the out-of-control condition (positive shift) as an optimal solution and  $S_2$  is defined as selecting the in-control condition as an optimal decision and  $NS$  is defined as not selecting any candidate in this stage.

Hence, using the total probability law, it is concluded that:

$$V(n, d^+(n)) = \text{Max}\{\text{Pr}\{CS\}\} = \text{Pr}\{CS|S_1\}\text{Pr}\{S_1\} + \text{Pr}\{CS|S_2\}\text{Pr}\{S_2\} + \text{Pr}\{CS|NS\}\text{Pr}\{NS\} \quad (69)$$

$\Pr\{CS \mid S_1\}$  denotes the probability of correct selection when candidate  $S_1$  is selected as the optimal candidate and this probability equals to its belief,  $B^+(O_k)$ , and with the same discussion, it is concluded that  $\Pr\{CS \mid S_2\} = 1 - B^+(O_k)$

$\Pr\{S_1\}$  is the probability of selecting out of control candidate (positive shift) as the solution thus following the decision making strategy, we should have  $B^+(O_k) = \max(B^+(O_k), 1 - B^+(O_k))$  and  $B^+(O_k) > d^+(n)$  that is equivalent to following,

$$\Pr\{S_1\} = \Pr\{B^+(O_k) > d^+(n)\}, d^+(n) \in [0.5, 1] \quad (70)$$

With the same reasoning, it is concluded that,

$$\Pr\{S_2\} = \Pr\{1 - B^+(O_k) > d^+(n)\}, d^+(n) \in [0.5, 1] \quad (71)$$

1.  $\Pr\{CS \mid NS\}$  denotes the probability of correct selection when none of candidates has been selected and it means that the maximum value of the beliefs is less than  $d^+(n)$  and the process of decision making continues to latter stage. As a result, in terms of Dynamic Programming Approach, the probability of this event equals to maximum of probability of correct selection in latter stage ( $n-1$ ),  $V(n-1)$ , but since taking observations has cost, then the value of this probability in current time is less than its actual value and by using the discounting factor  $\alpha$ , it equals  $\alpha V(n-1)$
2. Since the entire solution space is partitioned, it is concluded that  $\Pr\{CS \mid NS\} = 1 - (\Pr\{S_1\} + \Pr\{S_2\})$

By the above preliminaries, the function  $V(n)$  is determined as follows:

$$\begin{aligned} V(n) &= \max_{0.5 < d^+(n) < 1} \left[ \begin{aligned} &B^+(O_k) \Pr\{B^+(O_k) > d^+(n)\} + (1 - B^+(O_k)) \Pr\{1 - B^+(O_k) > d^+(n)\} \\ &+ \Pr\{CS \mid NS\} \left( 1 - \Pr\{B^+(O_k) > d^+(n)\} - \Pr\{1 - B^+(O_k) > d^+(n)\} \right) \end{aligned} \right] \quad (72) \\ &= \max_{0.5 < d^+(n) < 1} \left[ \begin{aligned} &B^+(O_k) \Pr\{B^+(O_k) > d^+(n)\} + (1 - B^+(O_k)) \Pr\{1 - B^+(O_k) > d(n)\} \\ &+ \alpha V(n-1) \left( 1 - \Pr\{B^+(O_k) > d^+(n)\} - \Pr\{1 - B^+(O_k) > d^+(n)\} \right) \end{aligned} \right] \end{aligned}$$

In terms of above equation,  $V(n, d^+(n))$  is obtained as follows:

$$V(n, d^+(n)) = \left[ \begin{aligned} &B^+(O_k) \Pr\{B^+(O_k) > d^+(n)\} + (1 - B^+(O_k)) \Pr\{(1 - B^+(O_k)) > d^+(n)\} \\ &+ \alpha V(n-1) \left( 1 - \Pr\{B^+(O_k) > d^+(n)\} - \Pr\{1 - B^+(O_k) > d^+(n)\} \right) \end{aligned} \right] \quad (73)$$

Calculation method for  $V(n, d^+(n))$  :

$B^+(gr, O_k)$  and  $B^+(sm, O_k)$  are defined as follows:

$$\begin{aligned} B^+(gr, O_k) &= \max\{B^+(O_k), 1 - B^+(O_k)\} \\ B^-(gr, O_k) &= \min\{B^+(O_k), 1 - B^+(O_k)\} \end{aligned} \quad (74)$$

Now equation (73) is rewritten as follows:

$$\begin{aligned} V(n, d^+(n)) &= (B^+(gr, O_k) - \alpha V(n-1)) \Pr\{B^+(gr, O_k) > d^+(n)\} + \\ & (B^+(sm, O_k) - \alpha V(n-1)) \Pr\{B^+(sm, O_k) > d^+(n)\} + \alpha V(n-1) \end{aligned} \quad (75)$$

There are three conditions:

1.  $B^+(gr, O_k) < \alpha V(n-1)$

In this condition, both  $B^+(gr, O_k) - \alpha V(n-1)$  and  $B^+(sm, O_k) - \alpha V(n-1)$  are negative, thus we should have  $d^+(n) = 1$  in order to maximize  $V(n, d^+(n))$ . Since  $B^+(gr, O_k) < d^+(n) = 1$ , we don't select any candidate in this condition and sampling process continues.

2.  $B^+(sm, O_k) > \alpha V(n-1)$

In this condition, both  $B^+(gr, O_k) - \alpha V(n-1)$  and  $B^+(sm, O_k) - \alpha V(n-1)$  are positive, thus we should have  $d^+(n) = 0.5$  in order to maximize  $V(n, d^+(n))$ . since  $B^+(gr, O_k) > d^+(n) = 0.5$ , we select the candidate of belief  $B^+(gr, O_k)$  as the solution.

3.  $B^+(sm, O_k) < \alpha V(n-1) < B^+(gr, O_k)$

In this condition, one of the probabilities in equation (10) has positive coefficient and one has negative coefficient, to maximize  $V(n, d^+(n))$ , optimality methods should be applied.

• *Definition:*  $h(d^+(n))$  is defined as follows:

$$h(d^+(n)) = \frac{d^+(n)(1 - B^+(O_{k-1}))}{(1 - d^+(n))B^+(O_{k-1})} \quad (76)$$

First the value of  $\Pr\{B^+(O_k) > d^+(n)\}$  is determined as follows:

$$\Pr\{B^+(O_k) > d^+(n)\} = \Pr\left\{\frac{\varphi(x_k)B^+(O_{k-1})}{\varphi(x_k)B^+(O_{k-1}) + (1-B^+(O_{k-1}))0.5} > d^+(n)\right\} =$$

$$\Pr\{\varphi(x_k) > h(d^+(n))0.5\} = 1 - 0.5h(d^+(n)) \quad (77)$$

Since  $\varphi(x_k)$  is a cumulative distribution function thus it follows a uniform distribution function in interval  $[0, 1]$ , thus the above equality is concluded.

With the same reasoning, it is concluded that:

$$\Pr\{1 - B^+(O_k) \geq d^+(n)\} = \Pr\{1 - d^+(n) \geq B^+(O_k)\} = 0.5h(1 - d^+(n)) \quad (78)$$

Now equation (73) can be written as follows:

$$V(n) = \max_{0.5 < d^+(n) < 1} \left[ \begin{aligned} & B^+(O_k)(1 - 0.5h(d^+(n))) + (1 - B^+(O_k))0.5h(1 - d^+(n)) \\ & + \alpha V(n-1)(1 - 0.5(1 - h(d^+(n)))) - 0.5h(1 - d^+(n)) \end{aligned} \right] \quad (79)$$

And equation (79) can be written as follows:

$$\begin{aligned} V(n, d^+(n)) &= (B^+(O_k) - \alpha V(n-1))(1 - h(d^+(n))0.5) + \\ & (1 - B^+(O_k) - \alpha V(n-1))0.5h(1 - d^+(n)) + \alpha V(n-1) \end{aligned} \quad (80)$$

Since  $V^*(n) = \text{Max}_{0.5 < d^+(n) < 1} [V(n, d^+(n))]$  thus it is sufficient to maximize the real value function  $V(n, d^+(n))$ , therefore; we should find the function value in points where first derivative is equated to zero as follows,

$$\frac{\partial V(n, d^+(n))}{\partial d^+(n)} = 0 \Rightarrow -\frac{(B^+(O_k) - \alpha V(n-1))}{(1 - B^+(O_k) - \alpha V(n-1))} = \frac{(1 - d^+(n))^2}{d^{+2}(n)}$$

$$\Rightarrow d^+(n) = \frac{1}{\sqrt{-\frac{(B^+(O_k) - \alpha V(n-1))}{(1 - B^+(O_k) - \alpha V(n-1))} + 1}} \quad (81)$$

The optimal threshold  $d^+(n)$  is determined by the above equation. Since the optimal value of  $d^+(n)$  should be in the interval  $[0.5, 1]$  thus it is concluded that the optimal value of  $d^+(n)$  will be determined as follows:

$$d^+(n) = \text{Max} \left[ \frac{1}{\sqrt{\frac{(B^+(O_k) - \alpha V(n-1))}{(1 - B^+(O_k) - \alpha V(n-1))} + 1}}, 0.5 \right] \quad (82)$$

The above method is presented for detecting the positive shifts in the process mean and can be adapted for detecting the negative shifts with the same reasoning.

The general decision making algorithm is summarized as follows:

1. Set  $k=0$  and the initial beliefs  $B^+(O_0)=0.5$ ,  $B^-(O_0)=0.5$ .
2. Gather an observation and set  $k=k+1$ ,  $n=n-1$ .
3. If  $n < 0$ , then no shift is occurred in the process mean and decision making stops.
4. Update the values for the beliefs  $B^-(O_k)$ ,  $B^+(O_k)$  by equation (61).
5. If  $\text{Min}(B^+(O_k), 1 - B^+(O_k)) > \alpha V(n-1)$ , then if  $\text{Max}(B^+(O_k), 1 - B^+(O_k)) = B^+(O_k)$ , it is concluded that a positive shift is occurred in the process mean and decision making stops, also if  $\text{Max}(B^+(O_k), 1 - B^+(O_k)) = 1 - B^+(O_k)$ , then no positive shift is occurred in the process mean and decision making stops.
6. If  $\text{Max}(B^+(O_k), 1 - B^+(O_k)) < \alpha V(n-1)$ , then data is not sufficient for detecting the positive shift and go to stage 2 after checking the occurrence of negative shift in the rest of the algorithm.
7. If  $\text{Min}(B^-(O_k), 1 - B^-(O_k)) > \alpha V(n-1)$  then if  $\text{Max}(B^-(O_k), 1 - B^-(O_k)) = B^-(O_k)$  it is concluded that a negative shift is occurred the process mean and decision making stops and if  $\text{Max}(B^-(O_k), 1 - B^-(O_k)) = 1 - B^-(O_k)$ , then no negative shift is occurred in the process mean and decision making stops.
8. If  $\text{Max}(B^-(O_k), 1 - B^-(O_k)) < \alpha V(n-1)$ , then data is not sufficient for detecting the negative shift and go to stage 2.
9. If  $\text{Max}(B^+(O_k), 1 - B^+(O_k)) > \alpha V(n-1) > \text{Min}(B^+(O_k), 1 - B^+(O_k))$ , then determine the value of  $d^+(n)$  (minimum acceptable belief for detecting the positive shift) by the following equation:

$$d^+(n) = \text{Max} \left( \frac{1}{\sqrt{\frac{(B^+(O_k) - \alpha V(n-1))}{(1 - B^+(O_k) - \alpha V(n-1))} + 1}}, 0.5 \right) \tag{83}$$

10. If  $\text{Max}(B^-(O_k), 1 - B^-(O_k)) > \alpha V(n-1) > \text{Min}(B^-(O_k), 1 - B^-(O_k))$ , then determine the value of  $d^-(n)$  (minimum acceptable belief for detecting the negative shift) by the following equation:

$$d^-(n) = \text{Max} \left( \frac{1}{\sqrt{\frac{(B^-(O_k) - \alpha V(n-1))}{(1 - B^-(O_k) - \alpha V(n-1))} + 1}}, 0.5 \right) \tag{84}$$

1. If  $B^+(O_k) > d^+(n)$ , then a positive shift is occurred and decision making stops, and if  $(1 - B^+(O_k)) > d^+(n)$ , then no positive shift is occurred and decision making stops, else go to stage 2 after checking the occurrence of negative shift in rest of the algorithm.
2. If  $B^-(O_k) > d^-(n)$ , then a negative shift is occurred and decision making stops, and If  $(1 - B^-(O_k)) > d^-(n)$ , then no negative shift is occurred and decision making stops, else go to stage 2.
3. The approximate value of  $\alpha V(n-1)$  based on the discount factor  $\alpha$  in the stochastic dynamic programming approach is  $\alpha^n V(0)$ .

## 5. Conclusion

In this chapter, we introduced a new approach to determine the best solution out of  $m$  candidates. To do this, first, we defined the belief of selecting the best solution and explained how to model the problem by the Bayesian analysis approach. Second, we clarified the approach by which we improved the beliefs, and proved that it converges to detect the best solution. Next, we proposed a decision-making strategy using dynamic programming approach in which there were a limited number of decision-making stages.



## Appendix 1

Conditional Mean and Variance of the Variables

Conditional mean of variables  $gr$  and  $sm$  can be evaluated using the following equation.

$$(\mu_{sm}, \mu_{gr} | (\mu_j)_{j \neq gr, sm}) = (\mu_{sm}, \mu_{gr}) + \mathbf{b}_2' ((X_{kj})_{j \neq gr, sm} - (\mu_j)_{j \neq gr, sm}) \quad (85)$$

where,  $\mathbf{b}_2' = \Sigma_{xX} \Sigma_{XX}^{-1}$

and

$$\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{xX} \\ \Sigma_{xX} & \Sigma_{xx} \end{bmatrix} \quad (86)$$

$\Sigma$ : The covariance matrix of the process

$\Sigma_{xx}$ : Submatrix of the covariance matrix  $\Sigma$  for variables  $j = gr, sm$

$\Sigma_{xX}$ : Submatrix of the covariance matrix  $\Sigma$  between variables  $j = gr, sm$  and  $j \neq gr, sm$

$\Sigma_{XX}$ : Submatrix of the covariance matrix  $\Sigma$  for variables  $j \neq gr, sm$

Further, the conditional covariance matrix of variables  $j = gr, sm$  on variables  $j \neq gr, sm$ , is obtained as  $\Sigma_{xx} - \Sigma_{xX}^T \Sigma_{XX}^{-1} \Sigma_{xX}$ .

## Appendix 2

Evaluating the Optimal Value of  $d_{gr,sm}(k)$

Assume  $(\mu_j)_{j \in \{1,2,\dots,m\}} = 0$  and  $(\sigma_j)_{j \in \{1,2,\dots,m\}} = 1$ . Then,

$$\begin{aligned} & \Pr \left\{ \left\{ e^{0.5(T_{k,gr|sm})^2} \geq h(d_{gr,sm}(k)) e^{0.5(T_{k,sm})^2} \right\} \right\} = \\ & \Pr \left\{ 0.5(T_{k,gr|sm})^2 \geq \ln(h(d_{gr,sm}(k))) + 0.5(T_{k,sm})^2 \right\} = \\ & \Pr \left\{ (T_{k,gr|sm})^2 - (T_{k,sm})^2 \geq 2 \ln(h(d_{gr,sm}(k))) \right\} \end{aligned} \quad (87)$$

Now since  $(T_{k,sm}, T_{k,gr|sm})$  follow a standard normal distribution  $(\mu_j)_{j \in \{gr, sm\}} = 0$  and  $(\sigma_j)_{j \in \{gr, sm\}} = 1$ , hence  $(T_{k,gr|sm})^2$  and  $(T_{k,sm})^2$  follow a  $\chi^2$  distribution with one degree of freedom. Then using an approximation, if we assume that  $(T_{k,sm})^2$  is approximately equal to its mean, we have

$$(T_{k,sm})^2 \propto E(T_{k,sm}^2) = E(T_{k,sm})^2 + \text{Var}(T_{k,sm}) = 1 \quad (88)$$

Thus,

$$\begin{aligned} & \Pr\left\{(T_{k,gr|sm})^2 - (T_{k,sm})^2 \geq 2\ln\left(h(d_{gr,sm}(k))\right)\right\} \propto \\ & \Pr\left\{(T_{k,gr|sm})^2 - E(T_{k,sm}^2) \geq 2\ln\left(h(d_{gr,sm}(k))\right)\right\} \\ & = \Pr\left\{(T_{k,gr|sm})^2 - 1 \geq 2\ln\left(h(d_{gr,sm}(k))\right)\right\} = \\ & \Pr\left\{(T_{k,gr|sm})^2 \geq 2\ln\left(h(d_{gr,sm}(k))\right) + 1\right\} \end{aligned} \quad (89)$$

Now, since  $(T_{k,gr|sm})^2 \propto \chi^2(1)$ , we have

$$\Pr\left\{(T_{k,gr|sm})^2 \geq 2\ln\left(h(d_{gr,sm}(k))\right) + 1\right\} = \int_{2\ln\left(h(d_{gr,sm}(k))\right)+1}^{\infty} \frac{e^{-\frac{t}{2}} t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} dt \quad (90)$$

Hence,

$$\Pr\left\{(T_{k,gr|sm})^2 - (T_{k,sm})^2 \geq 2\ln\left(h(d_{gr,sm}(k))\right)\right\} = \int_{2\ln\left(h(d_{gr,sm}(k))\right)+1}^{\infty} \frac{e^{-\frac{t}{2}} t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} dt \quad (91)$$

Similarly

$$\begin{aligned} & \Pr\left\{B_{gr,sm}(gr; O_k) \geq d_{gr,sm}(k)\right\} = \Pr\left\{e^{0.5(T_{k,gr|sm})^2} \geq r(d_{gr,sm}(k)) e^{0.5(T_{k,sm})^2}\right\} \approx \\ & \int_{2\ln\left(r(d_{gr,sm}(k))\right)+1}^{\infty} \frac{e^{-\frac{t}{2}} t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} dt \end{aligned} \quad (92)$$

Replacing the above equations in equation (53) results in

$$\begin{aligned} & V_{i,j}^*(N) \propto \left(B_{gr,sm}(gr; O_k) - \alpha V_{i,j}^*(N-1)\right) \int_{2\ln\left(r(d_{gr,sm}(k))\right)+1}^{\infty} \frac{e^{-\frac{t}{2}} t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} dt + \\ & \left(B_{gr,sm}(sm; O_k) - \alpha V_{i,j}^*(N-1)\right) \int_{2\ln\left(h(d_{gr,sm}(k))\right)+1}^{\infty} \frac{e^{-\frac{t}{2}} t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} dt + \alpha V_{i,j}^*(N-1) \end{aligned} \quad (93)$$

Now by solving the equation  $\frac{\delta V_{i,j}(N)}{\delta d_{gr,sm}(k)} = 0$ , the following equation is obtained.

$$\begin{aligned} & (B_{gr,sm}(gr;O_k) - \alpha V_{gr,sm}^*(N-1)) \frac{1}{\sqrt{(\ln(r(d_{gr,sm}(k))) + 1)}} = \\ & - (B_{gr,sm}(sm;O_k) - \alpha V_{gr,sm}^*(N-1)) \frac{1}{\sqrt{(\ln(h(d_{gr,sm}(k))) + 1)}} \end{aligned} \quad (94)$$

Finally, the approximate value of  $d_{gr,sm}(k)$  say  $d_{gr,sm}^1(k)$  is determined by solving this equation numerically or by a search algorithm.

Now using another approximation, if we assume that  $(T_{k,gr})^2$  is approximately equal to its mean, the approximate value of  $d_{gr,sm}(k)$  say  $d_{gr,sm}^2(k)$  is determined by solving following equation,

$$\begin{aligned} & (B_{gr,sm}(sm;O_k) - \alpha V_{gr,sm}^*(N-1)) \frac{1}{\sqrt{(\ln(r(d_{gr,sm}(k))) + 1)}} = \\ & - (B_{gr,sm}(gr;O_k) - \alpha V_{gr,sm}^*(N-1)) \frac{1}{\sqrt{(\ln(h(d_{gr,sm}(k))) + 1)}} \end{aligned} \quad (95)$$

The approximate optimal value of  $d_{gr,sm}(k)$  is obtained as follows,

$$d_{gr,sm}(k) = \max\{d_{gr,sm}^1(k), d_{gr,sm}^2(k)\} \quad (96)$$

## Author details

Mohammad Saber Fallah Nezhad\*

Address all correspondence to: [Fallahnezhad@yazd.ac.ir](mailto:Fallahnezhad@yazd.ac.ir)

Associate Professor of Industrial Engineering, Yazd University, Iran

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