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Chapter 14

Physical Realization of a Quantum Game

A.M. Kowalski, A. Plastino and M. Casas

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1. Introduction

Game theory is a mathematical methodology for analyzing calculated circumstances, such as it happens in games, where a person's success is based upon the choices of others. More formally, it is the study of mathematical models of conflict and cooperation between intelligent rational decision-makers [1]. Another way of describing this is interactive decision theory [2]. Game theory is mainly used in economics, political science, psychology, logic, and biology. The subject first addressed is called zero-sum games, such that one person's gains exactly equal net losses of the other participant(s). In our days game theory applies to a wide range of class relations, and has developed into an umbrella term for the logical side of science, including both human and non-humans (computers). Classic uses include a sense of balance in numerous games, where each participant develops a tactic that cannot successfully better his/her results.

Mathematical game theory began with E. Borel in his book *Applications aux Jeux de Hasard*. His results were somewhat limited, and the theory regarding the non-existence of blended-strategy equilibrium in two-player games was incorrect. Modern game theory began with an idea regarding the existence of mixed-strategy equilibria in two-person zero-sum games, proved by J. von Neumann, that used Brouwer's fixed-point theorem on continuous mappings into compact convex sets [3]. Game theory was later explicitly applied to biology in the 1970s. Game theory has been widely recognized as an important tool in many fields. Eight game-theorists have won the Nobel Prize in Economic Sciences, and John Maynard Smith was awarded the Crafoord Prize for his application of game theory to biology [4, 5]. As a method of applied mathematics, game theory has also been used to study a wide variety of human and animal behaviors. The use of game theory in the social sciences has expanded, and game theory has been applied to political, sociological, and psychological behaviors as well. Game-theoretic analysis was initially used to study animal behavior by R. Fisher in the 1930s (note that Charles Darwin made a few informal game-theoretic statements). Fisher's work predates the name "game theory", although it shares important features with this field. The developments in economics were later applied to biology largely by J. M. Smith in his book *Evolution and the Theory of Games*. In addition to being used to describe, predict, and explain behavior, game theory has also been employed to develop theories of ethical or normative behavior and to prescribe such behavior. In economics and philosophy, scholars have applied game theory to help in the understanding of good or proper behavior. Game-theoretic arguments of this type actually date back to Plato [4, 5].

The first known use of game theory attempted to describe and model how human populations behave. Some researcher believe that by finding the equilibria of games they can predict how actual human populations will behave when confronted with situations analogous to the game being studied. This particular view of game theory has come under recent criticism. First, it is criticized because the assumptions made by game theorists are often violated. Game theorists may assume players always act in a way to directly maximize their wins (the Homo *economicus* model), but in practice, human behavior often deviates from this model. Explanations of this phenomenon are many: irrationality, new models of deliberation, or even different motives (like that of altruism). Game theorists respond by comparing their assumptions to those used in physics. Thus while their assumptions do not always hold, they can treat game theory as a reasonable scientific ideal akin to the models used by physicists.

2. Introductory notions: representation of games

The processes studied in game theory are well-defined mathematical objects. A game consists of

- a set of players,
- a set of moves (or strategies) available to those players, and
- a specification of payoffs for each combination of strategies, where
- payoffs are numbers which represent the motivations of players. Payoffs may represent profit, quantity, "utility," or other continuous measures (cardinal payoffs), or may simply rank the desirability of outcomes (ordinal payoffs). In all cases, the payoffs must reflect the motivations of the particular player.

Most cooperative games are presented in a characteristic functional form, while so-called extensive and the normal forms are used to define noncooperative games.

The extensive game-form can be used to formalize games with a time sequencing of moves. Games here are played on trees. Each vertex (or node) of the tree represents a point of choice for a player. The player is specified by a number listed by the vertex. The lines out of the vertex represent a possible action for that player. The payoffs are specified at the bottom of the tree. The extensive form can be viewed as a multi-player generalization of a decision tree. If there are two players, player 1 moves first and chooses a specific move. Player 2 watches player 1's move and then chooses a response, etc. The extensive form can also capture simultaneous-move games and games with imperfect information.

The normal (or strategic form) game is usually represented by a matrix which shows the players, strategies, and payoffs. More generally it can be represented by any function that associates a payoff for each player with every possible combination of actions. For two players, one chooses the matrix' rows and the other the columns. Each player has a number of strategies, which are specified by the number of rows and the number of columns. The payoffs are provided in the matrix' interior.

A player's strategy in a game is a complete plan of action for whatever situation might arise; this fully determines the player's behavior. A player's strategy will determine the action the player will take at any stage of the game, for every possible history of play up to that stage. A strategy profile (sometimes called a strategy combination) is a set of strategies for each player which fully specifies all actions in a game. A strategy profile must include one and only one strategy for every player. The strategy concept is sometimes (wrongly) confused with that of a move. A move is an action taken by a player at some point during the play of a game (e.g., in chess, moving white's Knight). A strategy on the other hand is a complete algorithm for playing the game, telling a player what to do for every possible situation throughout the game.

A *pure* strategy provides a complete definition of how a player will play a game. In particular, it determines the move a player will make for any situation he/she could face. A player's strategy set is the set of pure strategies available to that player. A *mixed* strategy is an assignment of a probability to each pure strategy. This allows for a player to randomly select a pure strategy. Since probabilities are continuous, there are infinitely many mixed strategies available to a player, even if their strategy set is finite. Of course, one can regard a pure strategy as a degenerate case of a mixed strategy, in which that particular pure strategy is selected with unit probability and every other strategy with null probability. A totally mixed strategy is a mixed strategy in which the player assigns a strictly positive probability to every pure strategy.

The above elementary paragraphs on game theory miss an extremely important point, namely that players in a game maintain internal models, which in turn contain the models which they expect other players are using, and so on. This basic ingredient (infinite regress in mental model building) might be regarded as posing difficulties whenever comparisons with physical processes are made. In this respect we bypass the issue by following ideas arising already in the 19th Century with Maxwell. One considers the description of natural processes as a game between Nature and the Observer, which need not be a human being. A lucid and detailed example is provided in a celebrated book by Frieden, for instance [6]. We will here effect the associations

- players $\rightarrow$ sets of strategies/payoffs,
- classical probability distributions $\rightarrow$ mixed strategies,
- quantum states $\rightarrow$ quantal strategies,
- players using mixed strategies $\rightarrow$ classical players,
- players using quantum states' strategies $\rightarrow$ quantum players.

3. Elementary quantum notions

Quantum mechanics, also known as quantum physics or quantum theory (QT), is a physics' discipline that deals with physical phenomena where the action is of the order of Planck's constant. QT provides a mathematical description of much of the dual particle-like and wave-like behavior and interactions of energy and matter. It departs from classical mechanics primarily at the atomic and subatomic scales, the so-called quantum realm. In advanced topics of quantum mechanics, some of these behaviors are macroscopic and only emerge at very low or very high energies or temperatures. The name "quantum mechanics" was coined by Planck, and derives from the observation that some physical quantities can change only by discrete amounts, or quanta. For example, the angular momentum of an electron bound to an atom or molecule is quantized. In a quantal context, the wave particle duality of energy and matter and the uncertainty principle provide a unified view of the behavior of photons, electrons and other atomic-scale objects. The mathematical formulations of quantum mechanics are abstract. A mathematical function called the wave-function, or a matrix called the density matrix provide information about the probability amplitude of position, momentum, and other physical properties of a system. Mathematical manipulations of the wave-function (density matrix) involves the mathematics of Hilbert's space. Many of the results of quantum mechanics are not easily visualized in classical terms: For instance, the ground state in the quantum mechanical model is a non-zero energy state that is the lowest permitted energy state of a system, rather than a more traditional system that is thought of as simply being at rest with zero kinetic energy. Aa wave-function changes (density matrix) when a mathematical entity called an Operator is applied to it. In this vein, time-evolution of a systems is conceived as the temporal change of the wave-function (density matrix) caused by the action of a spacial kind of operators called Unitary Operators. These operators are functions of the Hamiltonian operator, that represents the system's energy. The trace of an operator $\hat{O}$, denoted by $Tr\hat{O}$ is an important quantity. Each operator is represented by a square matrix and the trace is the sum of the diagonal elements of that matrix.

Particles in Nature are either bosons or fermions. Bosons are subatomic particles that obey statistical rules called the Bose Einstein ones. Several bosons can occupy the same quantum state. The word boson derives from the name of the Indian physicist Satyendra Nath Bose. Bosons contrast with fermions, which obey a different set of statistical rules, called the Fermi Dirac ones. Two or more fermions cannot occupy the same quantum state. Since bosons with the same energy can occupy the same place in space, bosons are often force carrier particles. In contrast, fermions are usually associated with matter (although in quantum physics the distinction between the two concepts is not clear cut). Bosons may be either elementary, like photons, or composite, like mesons. All observed bosons have integer spin, as opposed to fermions, which have half-integer spin. This is in accordance with the spin-statistics theorem, which states that in any reasonable relativistic quantum field theory, particles with integer spin are bosons, while particles with half-integer spin are fermions.

4. The semi-classical approach to quantum mechanics

This is an approach in which one part of a system is described quantum-mechanically whereas the other is treated classically.

The semiclassical approach has had a long and distinguished history and is a very important weapon in the physics' armory. Indeed, semiclassical approximations to quantum mechanics remain an indispensable tool in many areas of physics and chemistry. Despite the extraordinary evolution of computer technology in the last years, exact numerical solution of the Schrödinger equation is still quite difficult for problems with more than a few degrees of freedom. Another great advantage of the semiclassical approximation lies in that it facilitates an intuitive understanding of the underlying physics, which is usually hidden in blind numerical solutions of the Schrödinger equation. Although semiclassical mechanics is as old as the quantum theory itself, the field is continuously evolving. There still exist many open problems in the mathematical aspects of the approximation as well as in the quest for new effective ways to apply the approximation to various physical systems (see, for instance, [7, 8] and references therein).

5. Attractors and fixed points

The branch of mathematics that studies fixed points and attractors for different systems is called the theory of dynamical systems. An attractor is a set towards which a variable moving according to the dictates of a dynamical system evolves over time. That is, points that get close enough to the attractor remain close even if slightly disturbed. The evolving variable may be represented algebraically as an n-dimensional vector. The attractor is a region in an n-dimensional space. In physical systems, the n dimensions may be, for example, two or three positional coordinates for each of one or more physical entities; in economic systems, they may be separate variables such as the inflation rate and the unemployment rate. If the evolving variable is two- or three-dimensional, the attractor of the dynamic process can be represented geometrically in two or three dimensions. An attractor can be a point, a finite set of points, a curve, a manifold, or even a complicated set with a fractal structure known as a strange attractor. If the variable is a scalar, the attractor is a subset of the real number line. Describing the attractors of chaotic dynamical systems has been one of the achievements of chaos theory. A trajectory of the dynamical system in the attractor does not have to satisfy any special constraints except for remaining on the attractor. The trajectory may be periodic or chaotic. If a set of points is periodic or chaotic, but the flow in the neighborhood is away from the set, the set is not an attractor, but instead is called a repeller.

A fixed point (also known as an invariant point) of a function is a point that is mapped to itself by the function. A set of fixed points is sometimes called a fixed set. Thus, c is a fixed point of the function $f(x)$ if and only if $f(c) = c$. Not all functions have fixed points: for example, if f is a function defined on the real numbers as $f(x) = x + 1$, then it has no fixed points, since x is never equal to $x + 1$ for any real number. In graphical terms, a fixed point means the point $(x, f(x))$ is on the line $y = x$. The example $f(x) = x + 1$ is a case where the graph and the line are a pair of parallel lines. Points which come back to the same value after a finite number of iterations of the function are known as periodic points. Thus, a fixed point is a periodic point with period equal to one. The stability of a fixed point is addressed in mathematics by the so-called stability theory. It investigates the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions.

6. Our goals

In recent times much attention has been paid to the task of extending game theory concepts [3] to quantum mechanics [9–19]. *Quantum games* may be of interest given that its classical counterpart (CGT) has had such a phenomenal success. Thus, much is to be expected from "quantizing" classical game theory. It is well known that various problems in physics can be usefully thought of as games. Quantum cryptography, for example, is easily reworded as a game between i) individuals who wish to communicate and ii) those who wish to eavesdrop [9]. Quantum cloning has been cast in the guise of a physicist playing a game against nature [10]. Even the cornerstone of physics, measurement processes themselves, may be approached in these terms.

Meyer [12] has pointed out that algorithms conceived for quantum computers may also be regarded as games between classical and quantum agents, and thus viewed in scenarios in which players are a quantum computer and its operator. It has been stated in Ref. [14] that *against this background, it is natural to seek a unified theory of games and quantum mechanics*. We wish here to add, within such philosophy, material to the notion of regarding quantum processes from a game theory viewpoint, *with emphasis on the semiclassical realm*.

Our ideas revolve around the picture developed in [18], un which an open quantum system corresponding to a biophysical Hamiltonian is regarded as a quantum game. Although our subject is quite different the underlying concepts are is the same. We shall regard the temporal evolution of a semiclassical system as a game. The concomitant process is then to be transcribed in term os strategies involving players hoping to optimize their chances.

7. The physical model and its associated game

As stated above, much quantum insight is gained from semiclassical viewpoints. Several methodologies are available (WKB, Born-Oppenheimer approach, etc.) Here we consider two interacting systems: one is classical and the other quantal. This can be done whenever the quantum effects of one of the two systems are negligible in comparison to those of the other one. Examples can be readily found. We just mention Bloch equations [20], two-level systems interacting with an electromagnetic field within a cavity and Jaynes-Cummings semiclassical model [21–26], collective nuclear motion [27], etc.

More recently [28–32], a special bipartite model has been employed with reference to problems in such fields as chaos, wave-function collapse, measurement processes, and cosmology [33]. In a related vein we consider the interaction between a quantum system and a classical one described by a Hamiltonian of the form [34–37]

$$H = H_q + H_{cl} + H_{cl}^q, \tag{1}$$

where H_q and H_{cl} stand for quantal and classical Hamiltonians, respectively, and H_{cl}^q is an interaction potential. The dynamical equations for the associated quantal variables are the canonical ones, i.e., any operator O evolves as

$$\frac{dO}{dt} = \frac{i}{\hbar}[\,H, O\,], \tag{2}$$

and the concomitant mean value as (Ehrenfest's theorem))

$$\frac{d\langle O\rangle}{dt} = \frac{i}{\hbar}\langle[\,H, O\,]\rangle. \tag{3}$$

The evolution of the system is dissipative. A dissipative system is a thermodynamically open system which is operating out of, and often far from, thermodynamic equilibrium in an environment with which it exchanges energy and/or matter. A dissipative structure, in turn, is a dissipative system that has a dynamical regime (here provided by Eqs. (3)) that is in some sense in a reproducible steady state. This reproducible steady state may be reached by natural evolution of the system, by artifice, or by a combination of these two.

We will see below that our classical variables obey dissipative equations because of the presence of an $\eta-$term. Without it, no dissipation occurs, because the resulting equations would conserve energy. Accordingly, if we take the classical variables to be a position X and a momentum P_X, we set [34, 35]

$$\frac{dX}{dt} = \frac{\partial \langle H \rangle}{\partial P_X}, \tag{4a}$$

$$\frac{dP_X}{dt} = -(\frac{\partial \langle H \rangle}{\partial X} + \eta P_X). \tag{4b}$$

The energy is taken here to coincide with the quantum expectation value of the Hamiltonian (1). Consequently, the classical equations of motion to be used here are well-defined ones [35].

As anticipated, the parameter $\eta > 0$ is a dissipative one. Through this parameter η, the classical variable is coupled to an appropriate impulse P_X-reservoir (that provides for P_X-growth) and energy is dissipated into this reservoir. Such indirect route allows for a dynamical description in which no quantum rules are violated [34–37]. The commutation-relations are trivially conserved for all time (the quantal evolution is the canonical one), so that one is able to avoid any quantum pitfall [34, 35]. The set of equations derived from (i) Eqs. (3) for variables belonging to the quantal system, and from (ii) (4) for the classical variables, give raise to an autonomous set of first-order coupled differential equations of the form [34–37]

$$\frac{d\vec{u}}{dt} = \vec{F}(\vec{u}), \tag{5}$$

where $\vec{u}$ a "vector" with both classical and quantum components. The reader may go back to Section V and realize that $\vec{F}$ is the function f of that section. If ones considers an arbitrary volume element V_S enclosed by a surface S in the space where evolves this vector, the dissipative η term induces a contraction of V_S [34, 35]

$$\frac{dV_S(t)}{dt} = -\eta V_S(t). \tag{6}$$

If the classical Hamiltonian adopts the general appearance $H_{cl} = \frac{1}{2M}P_X^2 + V(X)$, one easily ascertains that the temporal evolution for the total energy $\langle H \rangle$ is given by [34–37]

$$\frac{d\langle H \rangle}{dt} = -\frac{\eta}{M}P_X^2, \tag{7}$$

whose significance is to be appreciated in the light of Eq. (6).

7.1. Reformulation in game-theory language

According to the theory of dynamical systems, Eqs. (6) and (7) **guarantee the existence of attractors** [34, 35]. In translating the physical problem at hand into a game, our essential ingredients are these attractors.

We invite the reader to imagine a game whose results are in correspondence with the end-points of the possible trajectories that eventually reach one of these attractors:

possible results	→	different attractors
possible player's payoffs	→	appropiate quantum operators P

.

Bets are placed on which attractor will "prevail" [38]. One assumes that players are aware of the details of the underlying dynamical process, i.e., they are cognizant of (1). Thus, the only freedom of choice refers to the initial conditions(IC) for the system of differential equations (5). Remember such mathematical field an initial value problem (also called the Cauchy problem) reduces to an ordinary differential equation together plus the IC for an unknown function (here $\vec{F}$) at a given point in the domain of the solution (here the initial time). In physics or other sciences, modelling a system frequently amounts to solving an initial value problem. In such context, the differential equation is an evolution equation specifying how, given initial conditions, the system will evolve with time. A pivotal role is then played by the initial density matrix, which is the mathematical quantum object ρ containing the available physical information, that weights possible quantum states $|Q_j>$ of the system with the amounts p_j:

$$\rho = \sum_j \; |Q_j > \; p_j \; < Q_j|, \tag{8}$$

where $\sum_j p_j = 1$. We face here a complete-information game. Each player knows all possible strategies and payoffs. We will call "classical" those players who choose probabilities p_j as (mixed) strategies. Quantum players, instead, select quantum states $|Q_j>$ as their strategies [38]. These choices can be made following specified rules, each distinct set of rules leading to a different game, all of them for the same Hamiltonian.

A bit (a contraction of binary digit) is the basic unit of information in computing and telecommunications, being the amount of information stored by a digital device or other physical system that exists in one of two possible distinct states. These may be the two stable states of a flip-flop, two positions of an electrical switch, two distinct voltage or current levels allowed by a circuit, two distinct levels of light intensity, two directions of magnetization or polarization, the orientation of reversible double stranded DNA, etc. Instead, a qubit or quantum bit is a unit of quantum information: the quantum analogue of the classical bit, with additional dimensions associated to the quantum properties of a physical atom. A quantum computation is performed by initializing a system of qubits with a quantum algorithm. The qubit is described by a quantum state in a two-state quantum-mechanical system, which is formally equivalent to a two-dimensional vector space over the complex numbers (see next Section below). One example of a two-state quantum system is the polarization of a single photon: here the two states are vertical polarization and horizontal polarization. In a classical system, a bit would have to be in one state or the other, but quantum mechanics allows the qubit to be in a superposition of both states at the same time, a property which is fundamental to quantum computing.

Quantum players make use of qubits and *classical ones* of bits ([11], [12], [14]). In particular, Meyer [12] considers density matrices of the type (8). Moves that reflect quantum strategies are represented by unitary operators and classical strategies are of a mixed character. Expected payoffs are calculated via (8) using [38]

$$Tr(\rho \; P), \tag{9}$$

where P standing for a convenient ("payoff") operator associated to each player. The above calculation is performed at the attractors' locations for the pertinent initial values, as detailed below in Sect. 9.

8. Two level systems

In quantum mechanics, a two-state system (also known as a TLS or two-level system) is a system which has two possible states. More formally, the Hilbert space of a two-state system has two degrees of freedom, so a complete basis spanning the space must consist of two independent states. An example of a two-state system is the spin of a spin-1/2 particle such as an electron or proton, whose spin can have values 1/2, −1/2 in units of the Planck constant. The physics of a quantum mechanical two-state system is trivial if both states are degenerate, that is, if the states have the same energy. However, if there is an energy difference between the two states, nontrivial dynamics can ensue that often allow for deep insight into physical problems. Here, we consider the following two-level boson Hamiltonian [34, 35]

$$H = E_1 N_1 + E_2 N_2 + \frac{\omega}{2}(P_X{}^2 + X^2) + \gamma X\,(a_1^\dagger a_2 + a_2^\dagger a_1), \tag{10}$$

that represents matter interacting with a single-mode of a electromagnetic field within a cavity. One has $N_1 = a_1^\dagger a_1$ and $N_2 = a_2^\dagger a_2$, the population operators corresponding to the levels one and two, respectively, and we assume $E_2 > E_1$. Here $a_1^\dagger$, a_1 and $a_2^\dagger$, a_2, are the creation and annihilation operators of a boson in the levels "one" and "two", respectively. The electromagnetic field, regarded as classical, is represented by the variables (classical) X and P_X (X's conjugate momentum) [24, 25].

Taking the set $\{\Delta N = N_2 - N_1, O_- = i(a_1^\dagger a_2 - a_2^\dagger a_1), O_+ = (a_1^\dagger a_2 + a_2^\dagger a_1)\}$, where we have introduced the population difference operator ΔN, and applying (3) we obtain ($\hbar = 1$)

$$\frac{d\langle \Delta N\rangle}{dt} = 2\gamma X \langle O_-\rangle, \tag{11a}$$

$$\frac{d\langle O_-\rangle}{dt} = -2\gamma X \Delta N + \omega_0 \langle O_+\rangle, \tag{11b}$$

$$\frac{d\langle O_+\rangle}{dt} = -\omega_0 \langle O_-\rangle, \tag{11c}$$

with $\omega_0 = (E_2 - E_1)$. The mean value $\langle O_-\rangle$ represents a "current" vector and $\langle O_+\rangle$ is the expectation value of the quantal factor of the interaction potential. For the classical variables we obtain

$$\frac{dX}{dt} = \omega P_X, \tag{12a}$$

$$\frac{dP_X}{dt} = -(\omega X + \gamma \langle O_+\rangle + \eta P_X). \tag{12b}$$

Each level's population, $\langle N_1\rangle$ and $\langle N_2\rangle$, can be obtained in the fashion:

$$\langle N_2\rangle(t) = \frac{1}{2}(n + \langle \Delta N\rangle(t)), \tag{13a}$$

$$\langle N_1\rangle(t) = \frac{1}{2}(n - \langle \Delta N\rangle(t)), \tag{13b}$$

where $\langle N\rangle(t) = n$, with n the total number of particles, as $N = N_1 + N_2$ is an motion-invariant of the system. We can also define the Bloch-like quantity I_B as

$$I_B = \left(\Delta N^2 + \langle O_-\rangle^2 + \langle O_+\rangle^2\right)^{1/2}, \tag{14}$$

which is also an invariant of the motion. We consider the five-dimensional space determined by $u = (\langle \Delta N \rangle, \langle O_- \rangle, \langle O_+ \rangle, X, P_X)$. The fixed points or equilibrium points (labelled by the subindex f) of our system of non linear equations can be classified as being of type A or B, respectively, according to whether its X value vanishes or not. Using the invariant I_B we obtain [34, 35]

Type A:

$$\langle \Delta N \rangle_f = -\frac{\omega\,\omega_0}{2\gamma^2}, \tag{15a}$$

$$\langle O_- \rangle_f = 0, \tag{15b}$$

$$\langle O_+ \rangle_f = \pm({I_B}^2 - 4\frac{\omega^2\,\omega_0^2}{\gamma^4})^{1/2}, \tag{15c}$$

$$X_f = -\frac{\gamma}{\omega}\langle O_+ \rangle_f, \tag{15d}$$

$$P_{Xf} = 0, \tag{15e}$$

if $(\omega\,\omega_0)/2\,\gamma^2 < I_B$.

Type (B)

$$\langle \Delta N \rangle_f = \pm I_B, \tag{16a}$$

$$\langle O_- \rangle_f = 0, \tag{16b}$$

$$\langle O_+ \rangle_f = 0, \tag{16c}$$

$$X_f = 0, \tag{16d}$$

$$P_{Xf} = 0. \tag{16e}$$

Studying the stability of these fixed points we can ascertain that those of Type A are stable [34, 35], while those of Type B are stable only when $(\omega\,\omega_0)/2\,\gamma^2 \geq I_B$ together with $\langle \Delta N \rangle_f = -I_B$. The stable fixed points are the only attractors of the system (see the detailed investigation of [34]). For this case, the final population distribution is originated by a flux from the upper to the lower level, independently of the initial conditions, and of the values of the H-parameters. Instead, for the unstable solution, the flux runs towards the upper level, but for this to happen we need that at the initial time the system has to be already found at the fixed point, where of course it remains for ever.

Type B points minimize the quantum energy as well as the total energy. Instead, for Type A only the total energy is minimized, allowing for the quantum energy part to be either increased or not, depending on the initial conditions and on the parameter-values. This fact allows for the final boson-number of the upper level to be greater than the initial ones, i.e.,

$$\langle N_2 \rangle_f - \langle N_2 \rangle(0) = -\frac{1}{2}\left(\Delta N(0) + \frac{\omega\,\omega_0}{2\,\gamma^2}\right) \geq 0, \tag{17}$$

which can happen for

$$\frac{\omega\,\omega_0}{2\,\gamma^2} < -\Delta N(0), \tag{18}$$

with $\Delta N(0) < 0$.

9. Expected payoffs for two level games

On the basis of (17) we define a game with two options and two players: *the populations of each level either increase or decrease*, with the following expected payoffs [38]:

$$P_2 = \langle N_2 \rangle_f - \langle N_2 \rangle(0), \tag{19a}$$

$$P_1 = \langle N_1 \rangle_f - \langle N_1 \rangle(0), \tag{19b}$$

that can be recast in the form (9) as $P_i = Tr[\rho\ (N_i(t \to \infty) - N_i(0))]$ [35]. Using (13) we have

$$P_2 = -P_1 = \langle \Delta N \rangle_f - \langle \Delta N \rangle(0), \tag{20}$$

so that we face a **zero-sum game**, whose physical counterpart is boson-number conservation. Henceforth we need to fix attention only on P_2. According to the stable point character (A or B) we get

Type A

$$P_2 = -\frac{1}{2}\left(\Delta N(0) + \frac{\omega\,\omega_0}{2\,\gamma^2}\right), \tag{21}$$

if $(\omega\,\omega_0)/2\,\gamma^2 < I_B$. Of course, $P_2 \geq 0$, if (18) is verified.

Type B

$$P_2 = -\frac{1}{2}\left(\Delta N(0) + I_B\right), \tag{22}$$

if $(\omega\,\omega_0)/2\,\gamma^2 \geq I_B$. In the last case we always have $P_2 \leq 0$. We remark that, of course, if initially the system is at any fixed point, including those unstable of the type B, it will remain there and $P_2 = P_1 = 0$. Also, the validity-ranges and the payoffs do not depend on the values of the classical variables X and P_X. We proceed next to determine under which initial conditions the system ends-up in one or the other of the two attractors.

10. Game's strategies and initial conditions

The density matrix (8) may represent a game played i) by classical players if we keep fixed the $|Q_j > -$states), ii) between quantum players if $p_1 = 1$ and the remaining p_j vanish, and iii) between both classical and quantum players. If there are several players, the probabilities p_j are expressed as products (of probabilities) y the states $|Q_j >$ as tensor products of quantum states.

We now specialize (8) to the case of the two levels-system (Cf. Eq. 10)). We consider the illustrative instance in which a classical C-player and a quantum Q-one play with two different strategies: a mixed one for player C and a quantum strategy for player Q. This is as follows: *mixed*: select probability p_j, *quantum*: choice of $|Q_j >$ [38]. Matrix (8) expresses a situation in which *C-players* support with probability p_j (o reject with probability $1 - p_j$) the $|Q_j >$-strategy. This scenario motivates one to follow Meyer's approach [12] regarding quantum coins. In his game a coin is hidden within a box, initially heads-up. It can be in alternate fashion manipulated three times by two players C (just once) and Q (twice). The *Q-player* wins if the penny is head up when finally the open the box is open for all to see. C supports Q's strategy with probability p ("leave the coin untouched") and $1 - p$ ("set the coin in the "tails" state") (see [12]). We slightly generalize things in our game. Let the general initial state be

$$|Q>=\sum_{i=0}^{n}\alpha_i\,|n-i,i>, \tag{23}$$

with $\sum_{i=0}^{n}\alpha_i\,\alpha_i^* = 1$. Vectors $|n-i,i>$, represent states with $n-i$ bosons downstairs and i particles in the upper level. They are a basis in the pertinent Fock-space. If Q chooses the strategy $|Q_1>\equiv|Q>$ of (23), the alternative strategies are given by the set of vectors

$$|Q_j>=\pi_j\,|Q>, \tag{24}$$

with π_j operators that acting on $|Q>$ produce all possible permutations among the α_i, generating $(n+1)!$ quantal strategies. These operators can be written as $\pi_j=\prod_{lm}e_{lm}$, with the e_{lm} being "elemental" operators that exchange α_l with α_m. We are lead to

$$\rho=\sum_{j=1}^{(n+1)!}|Q_j>\;p_j\;<Q_j|, \tag{25}$$

with $\sum_{j=1}^{(n+1)!}p_j=1$. Here $\pi_1=I$, the identity permutation ($|Q_1>=|Q>$).

Eqs. (23) and (24) state that quantum strategies are represented by qubits for $n=1$ (as in [12]), qutrits for $n=2$, and, in general, by qu$n$-its if we deal with n bosons.

Neither Q nor C know what her rival plays. Although the game may be either of sequential or simultaneous nature, it is here more natural to regard it as sequential, with the *Q-player* making the first move.

The matrix version of (25), in terms of the matrix versions of the operators π_j read

$$\bar{\rho}=\sum_{j=1}^{(n+1)!}p_j\;\bar{\pi}_j\;\bar{\rho}_Q\;\bar{\pi}_j^{\dagger}\,, \tag{26}$$

where $\bar{\rho}_Q$ corresponds to the pure ρ_Q given by

$$\rho_Q=|Q><Q|. \tag{27}$$

Matrices $\bar{\pi}_j$ can in turn be cast in the fashion $\bar{\pi}_j=\prod_{lm}\bar{e}_{lm}$, i.e., in terms of the elemental matrices $\bar{e}_{lm}$ that arise out of interchanging rows l and m in the identity matrix. In (26) classical strategies are represented by the choice of the p_j, together with the operations implied by the matrices $\bar{\pi}_j$.

11. A bosonic game

Let us discuss in more detail the $n=1-$instance of one Q-player and one C-one, since this is already enlightening enough, as will be seen. The density operator is written

$$\rho=p_1\,|Q_1><Q_1|+p_2\,|Q_2><Q_2|, \tag{28}$$

with $p_1+p_2=1$ and

$$|Q_1>\;=\;|Q>=\alpha_0\,|1,0>+\alpha_1\,|0,1>, \tag{29a}$$

$$|Q_2>\;=\;\alpha_1\,|1,0>+\alpha_0\,|0,1>, \tag{29b}$$

where $|\alpha_0|^2 + |\alpha_1|^2 = 1$. $|1,0>$ stands for our particle being downstairs while it is upwards in $|0,1>$. Matrices $\bar{\pi}_j$ $(j = 1, 2)$ read

$$\bar{\pi}_1 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \bar{\pi}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

Thus, (26) adopts here the appearance

$$\bar{\rho} = \begin{pmatrix} p_1|\alpha_0|^2 + p_2|\alpha_1|^2 & p_1\alpha_0\alpha_1^* + p_2\alpha_1\alpha_0^* \\ p_1\alpha_1\alpha_0^* + p_2\alpha_0\alpha_1^* & p_1|\alpha_1|^2 + p_2|\alpha_0|^2 \end{pmatrix}. \tag{31}$$

We assume now that the *Q-player* places his bet on that the upper level will increase its population for $t \to \infty$. This is a priori the un-likeliest choice. The ensuing payoff will be P_2. Using (31) we find, associated to the type of fixed point (A or B) the payoffs (see (21) - (22)).

Type A

$$P_2 = \frac{1}{2}\left((2\alpha_0^2 - 1)\,(2p_1 - 1) - \frac{\omega\,\omega_0}{2\,\gamma^2}\right), \tag{32}$$

if $(\omega\,\omega_0)/2\,\gamma^2 < I_B$.

Type B

$$P_2 = \frac{1}{2}\left((2\alpha_0^2 - 1)\,(2p_1 - 1) - I_B\right), \tag{33}$$

if $(\omega\,\omega_0)/2\,\gamma^2 \geq I_B$. In this case I_B writes

$$I_B = \left((2\alpha_0^2 - 1)^2\,(2p_1 - 1)^2 + 4\alpha_0^2(1 - \alpha_0^2)\right)^{1/2}. \tag{34}$$

We have above taken α_0 y α_1 to be real, without loss of generality.

In order to gain intuitive understanding we would need to consider the game's version that uses only pure strategies [3]. The first player bets on one of the two levels and places a particle there. A third party (referee) asks the second player (who ignores what choice has been made before) whether she wishes to change or support her partner's bet. Afterwards, the system evolves and ends up in one of the two levels. If the first player bet on level 2, his payoffs would be 1, -1, or 0 according to whether the boson has climbed, descended, or remained in the original place. Since strategies (mixed ones) can be followed using betting-probabilities , these lead to an appropriate expected payoff [3]. Now, if one player follows a Q-strategy, in the $n = 1$ instance this strategy is represented by a qubit and we are led to the expected payoff computed according to (9) which leads to either (32) or (33).

11.1. Game's results

Some results concerning the just discussed issues are illustrated in Fig. 1 (the reader is also directed to Fig. 1 of Ref. [38]). Therein we display the regions corresponding to each type of fixed point (Type A and Type B), which are separated by the curve $I_B = \omega\,\omega_0/(2\gamma^2)$ (solid curve), with I_B given by (34). Zones in which the Q-player either wins or loses are also delimited. The dotted curve is in this case the "separator", being given by

$$(2\alpha_0^2 - 1)\,(2p_1 - 1) - \frac{\omega\,\omega_0}{2\,\gamma^2} = 0, \tag{35}$$

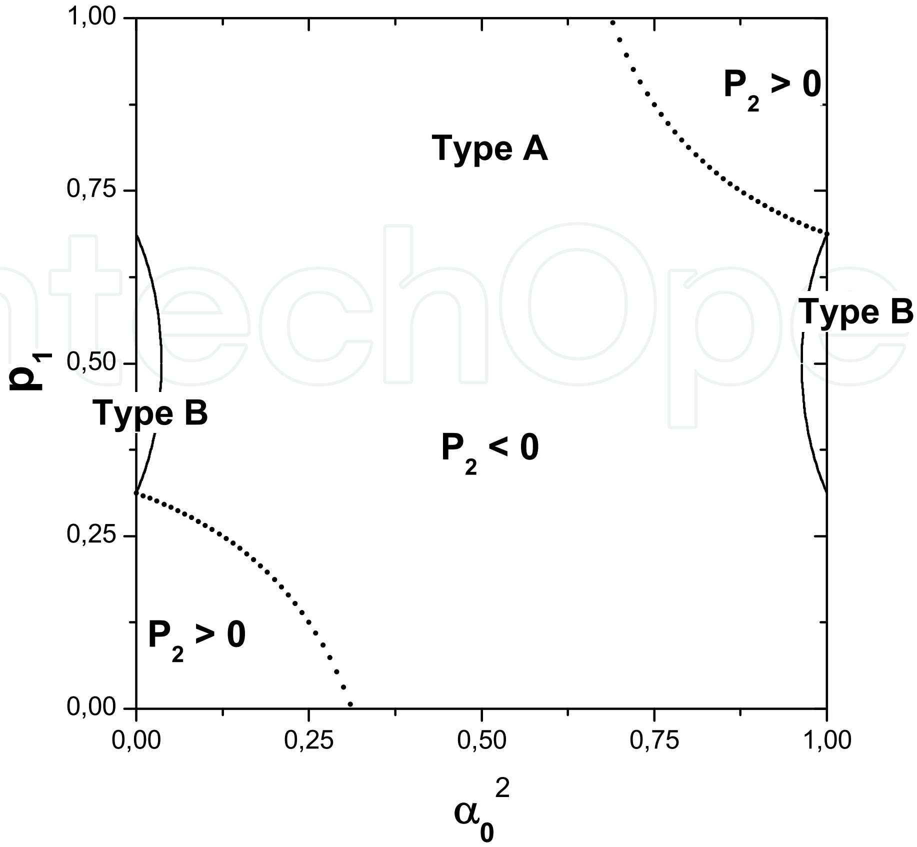

Figure 1. Classical and quantum probabilities p_1 and α_0^2. Regions corresponding to the fixed points of types A and B, separated by the curve $I_B = \omega\,\omega_0/(2\gamma^2)$ (solid line) are delineated (I_B is given by (34)). One sets $\omega\,\omega_0/\gamma^2 = 3/4$. We also depict the zones in which the C-player either wins or loses, represented by positive or negative payoffs, respectively, as given by the P_2 of Eq. (32) or (33). The dotted line separates the two zones and can be calculated using Eq. (35).

that one gets by setting $P_2 = 0$ in (32). We chose as independent parameters α_0^2 and p_1, setting $\omega\,\omega_0/\gamma^2 = 3/4$. Since we wish for the existence of the two types of fixed points for $n = 1$, one needs that $\omega\,\omega_0/\gamma^2 < 2$.

We indeed verify that the strategy corresponding to selecting p_1 in such manner that

$$\frac{1}{2} - \frac{\omega\,\omega_0}{4\,\gamma^2} \leq p_1 \leq \frac{1}{2} + \frac{\omega\,\omega_0}{4\,\gamma^2}\,, \tag{36}$$

guarantees that the C-player will win, independently of the quantal strategy. Moreover, if $p_1 = 1/2$, this holds for any choice of parameters' values. Such "happy" circumstances do not exist for the C-player, no matter its strategic choice.

We detect the existence of a Nash equilibrium-point of type A ("large" coupling) at $p_1 = 1/2$ and $\alpha_0 = 1/\sqrt{2}$. This result mimics the one found for a classical Meyer game of "Penny Flipover" [12], after two rounds. The C-player bets with equal probability on both the Q-strategy and its opposite one. At the same time, the *Q-player*'s strategy (unknown to the C-one) bets evenly on the two alternative options of placing the particle up- or downstairs. Here one has $P_2 = -\omega\,\omega_0/(2\,\gamma^2) < 0$, since $p_1 = 1/2$.

12. Conclusions

We have cast the physics of a semi-classical Hamiltonian (10) in terms of Game Theory. As stated above, this physics is associated to the interaction between matter and a single-mode of a electromagnetic field within a cavity. The concomitant Hamiltonian is a specialization of the Hamiltonian (1).

The interaction-agency is represented as a game between classical and quantum players who bet on initial conditions. The associated dynamics, via the physical system's attractors, is expressed in terms of the game's payoffs (positive o negative). This is clearly appreciated in Fig. 1 for a single-boson example.

In the present context the boson-number conservation is cast in the guise of a zero-sum game. This result can be generalized for any problem with invariants.

Here our game is not a mere abstraction. Given that it represents a physical interaction, we can speak of a "real" game (as far as a physical model can be considered real). In this context, any experimenter can be viewed as a is a classical player. In practical terms, if the experimenter's interest, lies, for instance, in ascertaining that the final ground state population be larger than that of the excited state, she must use a specific strategy to such an end. The best one is choosing to initially prepare the system (in the $n = 1$-case) in such a way that the concomitant level-probabilities are, respectively, p_1 and $1 - p_1$, with p_1 given by (36), depending, of course, on the values of the system's parameters, letting afterwards the system to evolve. Selecting $p_1 = 1/2$ he/she can even neglect possible errors in these values and ensure "victory", thus achieving her goal.

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Author details

A.M. Kowalski
La Plata Physics Institute, National University La Plata and Buenos Aires Scientific Research Commission (CIC)

A. Plastino
National University La Plata & CONICET IFLP-CCT, C.C. 727 - 1900 La Plata, Argentina
Universitat de les Illes Balears and IFISC-CSIC, 07122 Palma de Mallorca, Spain

M. Casas
Universitat de les Illes Balears and IFISC-CSIC, 07122 Palma de Mallorca, Spain

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