We are IntechOpen, the world's leading publisher of Open Access books Built by scientists, for scientists

4,800
Open access books available

122,000
International authors and editors

135M
Downloads

Our authors are among the

154
Countries delivered to

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

CLARIVATE ANALYTICS BOOK CITATION INDEX INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Chapter 30

A Novel Isospectral Deformation Chain in Supersymmetric Quantum Mechanics

Bjørn Jensen

Additional information is available at the end of the chapter

http://dx.doi.org/10.5772/53845

1. Introduction

Supersymmetric quantum mechanics (SUSYQM) has turned out to be surprisingly fertile field which is also able to successfully address challenges in traditional quantum mechanics and beyond. It has its roots in the works of Schrödinger, Infeld and Hull [1] on factorization methods of the Schrödinger equation. The term *supersymmetric* is due to a work by Witten [2] which brought these methods in contact with contemporary ideas in high energy physics. He showed in particular that the a factorized one-dimensional Schrödinger equation can accompany a super-Lie algebra thus providing a rich toy-model where features and concepts in supersymmetric quantum field theories can be studied in a greatly simplified context. A key ingredient in supersymmetric theories is that every bosonic state has a fermionic superpartner with all properties equal except the spin. In SUSYQM these states emerge as bosonic doublet states. The bosonic and the fermionic states are described in terms of the Schrödinger equation, but they interact with different physical potentials. These potentials are called partner potentials. Not completely surprising, knowing in advance the energy-eigenvalues and functions of the bosonic (fermionic) states the theory provides a map to the fermionic (bosonic) states with exactly the same energy-eigenvalues. Of key interest to us is that the physical partner potentials are expressed in terms of the same superpotential. These expressions are in general not unique. Different superpotentials can give rise to a particular physical potential in the fermionic (bosonic) sector. This does not imply that when the superpotential is changed (deformed) in such a way that the physical potential in the fermionic (bosonic) sector stays unchanged that the physical potential stays invariant in the bosonic (fermionic) sector. Whenever we deform a superpotential in the fermionic (bosonic) sector such that the fermionic (bosonic) potential is invariant the bosonic (fermionic) potential will generally change, but the theory nevertheless assures that the energy-eigenvalues in the bosonic (fermionic) sector stays the same. Such deformations are called isospectral deformations. They are the subject of this chapter.

Isospectral transformations in the context of SUSYQM has a long history exhibiting methods dating all the way back to Darboux [3]. The dominating approach is to study isospectral Hamilton operators. Different operator methods exist, but the main ones was brought under a single unifying principle by Pursey [4] with the use of isometric operators. A second approach to the study of isospectral transformations is what has been called deformation theory (see [5], e.g.). This is a more direct approach compared with the operator approach in that one studies deformations of the superpotential as described briefly above. It is rather surprising to note that this second approach has not been given much attention in the literature. To the knowledge of this author only one of the simplest deformations possible has been discussed to some extend. In a previous work [6] we initiated a work with the aim to remedy this situation. In [6] we showed that the isospectral deformation which has been considered in previous works is part of a more general deformation scheme. In this work we extend our results in [6]. We explicitly construct an in principle infinite recursively defined isospectral deformation chain where the deformation scheme in [6] emerges as the root of the chain.

This chapter is organized as follows. In the next section we very briefly review some of the basics of SUSYQM, mainly in order to fix notation. We define the notions of partner potentials, superpotential, isospectrality and supersymmetry. In section 2 we briefly remain ourselves about the results in [6]. In section 3 we define the recursive deformation scheme. We also discuss other various deformation schemes but show that a number of other canonical deformation schemes defined along the lines of our recursive scheme either do not allow a recursive structure or either reduces to our scheme. We apply our apparently rather unique recursive deformation scheme to the Coulomb potential and calculate several novel potentials. We summarize our findings and conclude in the last section. No attempt has been made to give an in depth review of the relevant literature due to its immense size. The works which have been acknowledged in the list of references have been so only because of their utility to this author.

2. SUSYQM - A very brief introduction

SUSYQM can in its most basic formulation be thought of as the following two factorizations of the Hamiltonian in the stationary Schrödinger equation in appropriate units

$$\begin{cases} (-\partial + W(x))(\partial + W(x)) \equiv A^+A^- \equiv H_- \,, \\ (\partial + W(x))(-\partial + W(x)) \equiv A^-A^+ \equiv H_+ \,. \end{cases} \tag{1}$$

Here ∂ is short hand for differentiation with respect to the single spatial coordinate x, and $W(x)$ is the so called superpotential. Both of these factorizations give rise to a Schrödinger equation, but with different potentials $V_-(x)$ and $V_+(x)$ (the so called superpotentials) given by

$$V_\pm(x) = W^2(x) \pm \partial W(x) \,. \tag{2}$$

Let us denote the energy-eigenvalues and eigenstates associated with $H_\pm$ by $E_n^\pm$ and $\psi_n^\pm(x)$, respectively. Let $n = 0$ denote the ground state. We note that

$$A^\pm \psi_0^\pm(x) = 0 \Rightarrow H_\pm \psi_0^\pm(x) = 0 \,. \tag{3}$$

http://dx.doi.org/10.5772/53845

The ground state eigenfunction is thus simply given by

$$\psi_0^{\pm}(x) \sim e^{\pm \int^x W(x)dx} \sim \frac{1}{\psi_0^{\mp}(x)}\,. \tag{4}$$

The factorization in Eq.(1) carries a symmetry which is not manifestly present in the usual form of the Schrödinger equation. This symmetry is made manifest when Eq.(1) is brought to a matrix form. Defining

$$Q^- \equiv \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, Q^+ \equiv \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \tag{5}$$

we find that we naturally can construct a matrix-valued Hamiltonian H given by

$$H \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix}. \tag{6}$$

It is straightforward to verify that

$$H = Q^-Q^+ + Q^+Q^- \equiv \{Q^-, Q^+\}\,,\ [Q^{\pm}, H] = 0\,,\ (Q^{\pm})^2 = 0\,. \tag{7}$$

This constitutes what is called a super-Lie algebra in contrast to an ordinary Lie algebra which only contains commutators. The commutator in Eq.(7) shows that $Q^{\pm}$ are generators of a symmetry which is left intact under time-translations generated by H. We call this symmetry the supersymmetry of the system.

The matrices above naturally act on a two-dimensional vector space with the natural representation

$$\begin{pmatrix} \psi_n^-(x) \\ \psi_{n-1}^+(x) \end{pmatrix}. \tag{8}$$

It is clear that

$$Q^- \begin{pmatrix} \psi_n^-(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_{n-1}^+(x) \end{pmatrix}, \tag{9}$$

$$Q^+ \begin{pmatrix} 0 \\ \psi_n^+(x) \end{pmatrix} = \begin{pmatrix} \psi_{n+1}^-(x) \\ 0 \end{pmatrix}. \tag{10}$$

Hence, $Q^{\pm}$ relate states with the same eigenvalue of H; the energy states are in other words degenerate. An orthogonal basis can naturally be taken to be states on the form

$$\begin{pmatrix} \alpha(x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \beta(x) \end{pmatrix}. \tag{11}$$

It is customary, due to the intimate relation to supersymmetric quantum field theory, to say that the first vector belongs to the bosonic sector and the other to the fermionic sector, even though no fermions appear in this theory. That $Q^{\pm}$ relate states corresponding to the same energy eigenvalue of the H-operator can also be seen on the level of the $H_{\pm}$ operators by noting that

$$H_+(A^-\psi_n^-) = E_n^-(A^-\psi_n^-)\,. \tag{12}$$

Hence, given an eigenstate ψ_n^- of H_- with energy eigenvalue E_n^-, the state $A^-\psi_n^-$ is an eigenstate of H_+ with energy eigenvalue E_n^-. There is thus a one-to-one correspondence between bosonic and fermionic states with the same energy eigenvalue (Eq.(9-10) above). We call this property the isospectrality of SUSYQM. Much more can be said about SUSYQM, such as the role played by the vacuum in connection with isospectrality. However, for the purpose of this chapter this very brief exposition of some of the basics of SUSYQM is sufficient to fix notation and certain concepts.

3. A novel isospectral deformation chain

In [6] we introduced within the framework of SUSYQM an isospectral deformation on the form

$$W(x) \to \hat{W}_0(x) = F_0(x)W(x)\,, \tag{13}$$

where $W(x)$ is some known superpotential and $F_0(x)$ some function to be determined by the isospectrality condition

$$\hat{V}_+(x) \equiv \hat{W}_0^2(x) + \hat{W}_0'(x) = W^2(x) + W'(x) \equiv V_+(x)\,. \tag{14}$$

It was shown that Eq.(13) includes the only previously explored deformation of this kind, which has the form [5]

$$W(x) \to \hat{W}_0(x) = W(x) + f(x)\,. \tag{15}$$

$f(x)$ is some function which is determined by Eq.(14). In this work we expand the deformation Eq.(13) in various directions and study the implications drawn from the isospectrality condition. We show in particular that the deformation Eq.(13) is the root of an infinitely long and recursively generated chain of deformations. Let us next briefly review some of the findings in [6].

3.1. Base deformations

The deformation Eq.(1) implies the following differential equation for $F_0(x)$ [6][1]

$$\frac{d}{dx}F_0(x) + (\frac{d}{dx}\ln W(x))F_0(x) + W(x)F_0^2(x) = W(x) + \frac{d}{dx}\ln W(x)\,. \tag{16}$$

[1] We will often rewrite fractions on the form $W'(x)/W(x)$ as the logarithmic derivative of $W(x)$ as a formal tool. Caution must of course be exercised when using the corresponding expressions in actual computations.

http://dx.doi.org/10.5772/53845

This is the generalized Riccati equation [7] . If one particular solution $F_{00}(x)$ of Eq.(16) is known another solution is given by [8]

$$F_0(x) = F_{00}(x) + \frac{1}{X_0(x)}\,, \tag{17}$$

where $X_0(x)$ solves the equation

$$\frac{d}{dx}X_0(x) - (\frac{d}{dx}\ln W(x) + 2F_{00}(x)W(x))X_0(x) = W(x)\,. \tag{18}$$

Eq.(18) can be solved by elementary means. The resulting superpotential $\hat{W}_0(x)$ is given by [6]

$$\begin{aligned}\hat{W}_0(x) &= (F_{00}(x) + \frac{1}{X_0(x)})W(x) \equiv \hat{W}_{00}(x) + \frac{1}{X_0(x)}W(x) = \\ &= F_{00}(x)W(x) + \frac{e^{-2\int^x F_{00}(t)W(t)dt}}{C_{01} + \int^x e^{-2\int^u F_{00}(t)W(t)dt}du}\,.\end{aligned} \tag{19}$$

C_{01} is an integration constant, which we will assume to be real. We have explicitly introduced upper integration limits in Eq.(19) in order to avoid sign ambiguities. This explains the difference in the sign in the denominator in Eq.(19) compared with Eq.(2.5) in [6] where the reverse order of integration in one of the integrals was implicitly assumed. We do not specify the lower integration limits in Eq.(19). These are not important, of course, since the values of the integrals there can essentially be absorbed into C_{01}. We can by simple inspection see that the particular solution $F_0(x) = 1$, the *identity deformation*, solves Eq.(16). With $F_{00}(x) = 1$ we identically rederive Eq.(15) and the corresponding expression discussed in [5]. The identity deformation corresponds to the limit $C_{01} \to \infty$ in Eq.(19) with $F_{00}(x) = 1$. In the limit $C_{01} \to \infty$ we generally get $\hat{W}_0(x) = \hat{W}_{00}(x)$. This deformation will play a pivotal role in this work; it will represent the base of a recursive scheme for generating novel isospectral deformations. We will therefore refer to a particular $\hat{W}_{00}(x)$ as *a base deformation* in the following.

In order to expand the space of concrete isospectral deformations further we transform Eq.(16) into an ordinary second order differential equation by the substitution

$$F_0(x) = \frac{1}{W(x)}\frac{d}{dx}\ln U_0(x)\,. \tag{20}$$

This substitution gives rise to the following linear homogeneous second order differential equation

$$-\frac{d^2}{dx^2}U_0(x) + V_+(x)U_0(x) = 0\,. \tag{21}$$

This equation coincides of course with the zero-energy eigenfunction equation. However, keep in mind that U_0 is *not* in general to be identified with the eigenfunction of the system. This is of particular importance to remember in light of Eq.(16). The special solution $F_0(x) = 1$ is generated by the solution

$$U_0(x) \sim e^{\int^x W(t)dt}\,. \tag{22}$$

The particular solutions for $F_0(x)$ stemming from Eq.(21) can be fed into Eq.(19) (as $F_{00}(x)$) and thus expand the space of available concrete deformations. The physical potential $\hat{V}_-(x)$ generated by $\hat{W}_0(x)$ can in general thus be written [6][2]

$$\begin{aligned}\hat{V}_-(x) \equiv \hat{W}_0^2(x) - \hat{W}_0'(x) = \hat{W}_{00}^2(x) - \hat{W}_{00}'(x) + \\ + \frac{4\hat{W}_{00}(x)e^{-2\int^x \hat{W}_{00}(t)dt}}{C_{01} + \int^x e^{-2\int^u \hat{W}_{00}(t)dt}du} + 2\left[\frac{e^{-2\int^x \hat{W}_{00}(t)dt}}{C_{01} + \int^x e^{-2\int^u \hat{W}_{00}(t)dt}du}\right]^2\end{aligned} \tag{23}$$

with

$$\hat{W}_{00}(x) = \frac{d}{dx}\ln U_0(x)\,. \tag{24}$$

3.2. Recursive linear deformations

Although the Riccati equation can be transformed into an ordinary second order differential equation the non-linearity of the Riccati equation allows for a solution space which is larger than the one associated with linear differential equations of second order, as became evident in the previous section. It is therefore natural to ask whether the non-linearity of the Riccati equation implies even more isospectral deformations than the ones we already have deduced [6]. We will explore this question in this and the next section.

3.2.1. The sum

Let us entertain the following idea. Assume that we have derived a particular base deformation $\hat{W}_{00}(x)$ from an explicitly given superpotential $W(x)$. Then assume that we *add* another term $F_1(x)W(x)$ (possibly multiplied with a constant) to that deformation such that we in principle get a novel deformation on the form $\hat{W}(x) = F_{10}(x)W(x) + \hat{W}_{00}(x)$. After determining $F_{10}(x)$ from the isospectrality condition Eq.(14) add yet another term of this kind to the deformation. Let us assume that this process can be repeated indefinitely. Will terms added in this manner give rise to novel deformations? We will in the following show that they do. This represents a recursive deformation scheme.

Following the basic idea, after m iterations we thus have the general recursive linear (in $W(x)$) deformation

$$\hat{W}_{m0}(x) = (\sum_{i=0}^{m}\lambda_i F_{i0}(x))W(x) = \lambda_m F_{m0}(x)W(x) + \hat{W}_{(m-1)0}(x)\ ,\ \lambda_0 \equiv 1\,. \tag{25}$$

[2] Note that the corresponding expression in [6] ((2.14)) is misprinted.

http://dx.doi.org/10.5772/53845

The λ_i's are assumed to be independent real constants. Starting with a known superpotential m consecutive applications of the isospectrality condition yields the following set of equations

$$\begin{cases} F'_{00}(x) + [\ln W(x)]' F_{00}(x) + W(x) F^2_{00}(x) = W(x) + (\ln W(x))' , \\ F'_{10}(x) + [(\ln W(x))' + 2F_{00}(x) W(x)] F_{10}(x) + \lambda_1 W(x) F^2_{10}(x) = 0 , \\ F'_{20}(x) + [(\ln W(x))' + 2(F_{00}(x) + \lambda_1 F_{10}(x)) W(x)] F_{20}(x) + \lambda_2 W(x) F^2_{20}(x) = 0 , \\ \vdots \\ \vdots \\ F'_{m0}(x) + [(\ln W(x))' + 2\hat{W}_{(m-1)0}(x))] F_{m0}(x) + \lambda_m W(x) F^2_{m0}(x) = 0 . \end{cases} \tag{26}$$

The first equation in Eq.(26) coincides of course per definition with Eq.(16). Note that $F_{j0}(x) = 1$ only solves the first equation in Eq.(26). Let us consider an arbitrary iteration level n ($\neq 0$) and make the following substitution in Eq.(26)

$$F_{n0}(x) = \frac{1}{W(x)} (\ln U_n(x))' . \tag{27}$$

The equation for $F_n(x)$ can then be written

$$U''_n(x) + (\lambda_n - 1) \frac{[U'_n(x)]^2}{U_n(x)} + 2\hat{W}_{(n-1)0}(x) U'_n(x) = 0 . \tag{28}$$

This equation corresponds to Eq.(21) in the case when $n = 0$. It reduces in general to an ordinary linear differential equation only when $\lambda_n = 1 , \forall n \neq 0$. We will focus on this special case in this work.

The general solution of Eq.(28) for arbitrary $n \neq 0$, and with λ_n set to unity, can be found by elementary means, and we deduce that

$$\begin{aligned} F_{n0}(x) W(x) &= \frac{C_{n2} e^{-2\int^x \hat{W}_{(n-1)0}(t) dt}}{C_{n1} + C_{n2} \int^x e^{-2\int^u \hat{W}_{(n-1)0}(t) dt} du} = \\ &= \frac{d}{dx} \ln(C_{n1} + C_{n2} \int^x e^{-2\int^u \hat{W}_{(n-1)0}(t) dt} du) . \end{aligned} \tag{29}$$

C_{n1} and C_{n2} are integration constants, which we assume to be real. We can reduce the number of integration constants to one at each iteration level, but we will stick to the habit of explicitly writing down the actual number of constants in order to make it easier to compare the various formulas we deduce, which stem from both second and first order differential equations. We also note that the structure of $F_{n0}(x)$ implies that previous deformations are not regenerated in general. Of course, this does not exclude this possibility to arise, as we will see in Section 5. Hence, m in Eq.(25) has in principle no natural upper bound. From Eq.(25) and Eq.(29) we get the following expression for the superpotential at iteration level m

$$\hat{W}_{m0}(x) = \hat{W}_{00}(x) + \sum_{j=1}^{m} \frac{d}{dx} \ln(C_{j1} + C_{j2} \int^{x} e^{-2\int^{u} \hat{W}_{(j-1)0}(t)dt} du) =$$

$$= \hat{W}_{00}(x) + \frac{d}{dx} \ln \prod_{j=1}^{m} (C_{j1} + C_{j2} \int^{x} e^{-2\int^{u} \hat{W}_{(j-1)0}(t)dt} du) \equiv$$

$$\equiv \hat{W}_{00}(x) + \frac{d}{dx} \ln P_m(x) \,. \tag{30}$$

From Eq.(30) we deduce that

$$e^{-2\int^{x} \hat{W}_{(j-1)0}(t)dt} = P_{j-1}^{-2}(x) e^{-2\int^{x} \hat{W}_{00}(t)dt} \;;\; P_0^2(x) \equiv 1 \,,\, j \neq 0 \,, \tag{31}$$

such that

$$\begin{cases} P_1(x) = C_{11} + C_{12} \int^{x} e^{-2\int^{u} \hat{W}_{00}(t)dt} du \,, \\ P_2(x) = P_1(x)(C_{21} + C_{22} \int^{x} P_1^{-2}(u) e^{-2\int^{u} \hat{W}_{00}(t)dt} du) \,, \\ \vdots \\ \vdots \\ P_m(x) = P_{m-1}(x)(C_{m1} + C_{m2} \int^{x} P_{m-1}^{-2}(u) e^{-2\int^{u} \hat{W}_{00}(t)dt} du) \,. \end{cases} \tag{32}$$

Hence,

$$P_n(x) = \prod_{j=1}^{n} (C_{j1} + C_{j2} \int^{x} P_{j-1}^{-2}(u) e^{-2\int^{u} \hat{W}_{00}(t)dt} du) \,. \tag{33}$$

This last form of the $P_n(x)$ functions neatly exhibits how the base deformation $\hat{W}_{00}(x)$ generates the higher order deformations. Some of the details we have deduced so far are presented in Figure 1.

Make the following substitution at each iteration level in Eq.(25)

$$F_{n0}(x) \to F_{n0}(x) + \frac{1}{X_n(x)} \,. \tag{34}$$

This implies (with the λ_m's reinstated in Eq.(25)) a generalized form $\hat{W}_m(x)$ of the superpotential $\hat{W}_{m0}(x)$

$$\hat{W}_{m0}(x) = \sum_{i=0}^{m} \lambda_i F_{i0} W(x) \Rightarrow \hat{W}_m(x) = \hat{W}_{m0}(x) + \sum_{i=0}^{m} \frac{\lambda_i}{X_i(x)} W(x) \,. \tag{35}$$

$$\begin{cases} W \xrightarrow{\times F_0 = F_{00}} \hat{W}_{00} \xrightarrow{+F_{10}W} \hat{W}_{10} = \hat{W}_{00} + (\ln P_1)' \xrightarrow{+F_{20}W} \cdots \xrightarrow{+F_{m0}W} \hat{W}_{m0} \\ W \xrightarrow{\times F_0 = 1} \hat{W}_{00} = W \xrightarrow{+F_{10}W} \hat{W}_{10} = W + (\ln P_1)' \xrightarrow{+F_{20}W} \cdots \xrightarrow{+F_{m0}W} \hat{W}_{m0} \end{cases}$$

Figure 1. The upper line depicts the solvable deformation chain Eq.(25) to iteration level m. There is no upper bound on m. The $F_{j0}(x)$ functions are given in Eq.(29). The $P_j(x)$ functions are given in Eq.(30) and Eq.(32). They are functions of a base deformation $\hat{W}_{00}(x)$. A base deformation $\hat{W}_{00}(x)$ is generated by the zero-energy Schrödinger equation interacting with the partner potential $V_+(x)$, Eq.(21). The second line depicts the important special case when $F_0(x) = 1$. This particular solution can be derived as a special case of Eq.(19) with $X_0^{-1} = 0$, which can be achieved by $C_{01} \to \infty$, and $F_{00}(x)$ determined by Eq.(20) and the solution Eq.(22). $\hat{W}_{10}(x)$ then coincides with Eq.(19) (when $F_{00}(x) = 1$ and C_{01} is finite in Eq.(19)); Eq.(19) is thus in this particular case regenerated by the scheme at the next recursion level, i.e.

From Eq.(26) we find that $X_n(x)$ satisfies the equation

$$\frac{d}{dx}X_n(x) - (\frac{d}{dx}\ln W(x) + 2\hat{W}_{n0}(x))X_n(x) = \lambda_m W(x)\,. \tag{36}$$

This equation is a generalization of Eq.(18). The n'th deformation term Eq.(29) thus changes into

$$F_{n0}(x)W(x) \to F_{n0}(x)W(x) + \frac{d}{dx}\ln(C_{n3} + \lambda_n \int^x e^{-2\int^u \hat{W}_{n0}(t)dt}du)\,. \tag{37}$$

C_{n3} are integrations constants, which we assume to be real. Eq.(37) implies that the more general expression for the superpotential in Eq.(35) can be written as

$$\hat{W}_m(x) = \hat{W}_{m0}(x) + \frac{d}{dx}\ln Q_m(x)\,, \tag{38}$$

where

$$Q_m(x) \equiv \prod_{i=0}^{m}(C_{i3} + \lambda_i \int^x e^{-2\int^u \hat{W}_{i0}(t)dt}du)\,. \tag{39}$$

$m = 0$ in Eq.(38) ($\lambda_0 \equiv 1$) reproduces Eq.(19). In the special case when $\lambda_m = 1\,, \forall m$ in Eq.(25) we get

$$\hat{W}_m(x) = \hat{W}_{00}(x) + \frac{d}{dx}\ln P_m(x) + \frac{d}{dx}\ln Q_m(x)\,. \tag{40}$$

When we compare the expressions for $(P_m(x))'$ and $(Q_m(x))'$ we find that they differ by just the last term in $(Q_m(x))'$.

3.2.2. The product

What happens if we in Eq.(25) assume a product structure instead of a sum structure ? Let us assume that we have determined a base deformation. Let this be the seed superpotential for the deformation

$$\hat{W}_{00}(x) \to \hat{W}_{10}(x) = F_{10}(x)\hat{W}_{00}(x) = F_{10}(x)F_{00}(x)W(x)\,, \tag{41}$$

where $F_{10}(x)$ is some function to be determined by the isospectrality condition. This product scheme can of course in principle be repeated an arbitrary number m times

$$\hat{W}_{m0}(x) = (\prod_{i=0}^{m} F_{i0}(x))W(x) = F_{m0}(x)\hat{W}_{(m-1)0}\,. \tag{42}$$

This structure gives rise to the following set of equations

$$\begin{cases} F'_{00}(x) + (\ln W(x))' F_{00}(x) + W(x)F^2_{00}(x) = W(x) + (\ln W(x))'\,, \\ F'_{10}(x) + (\ln \hat{W}_{00}(x))' F_{10}(x) + \hat{W}_{00}(x)F^2_{10}(x) = \frac{1}{F_{00}(x)}(W + (\ln W(x))')\,, \\ \vdots \\ \vdots \\ F'_{m0}(x) + (\ln \hat{W}_{(m-1)0}(x))' F_{m0}(x) + \hat{W}_{(m-1)0}(x)F^2_{m0}(x) = \frac{W(x)}{\hat{W}_{(m-1)0}(x)}(W(x) + (\ln W(x))')\,. \end{cases} \tag{43}$$

Clearly, each iteration level depends on all the previous ones, and at each level we are dealing with a non-homogenous non-linear differential equation. Interestingly, by making the following substitution at an arbitrary iteration level $n \neq 0$

$$F_{n0}(x) = \frac{1}{\hat{W}_{(n-1)0}(x)}(\ln U_n(x))'\,, \tag{44}$$

where $U_n(x)$ is some function, the equations Eq.(43) all reduce to Eq.(21). Hence, attempting to generate novel deformations recursively via a product structure, of the kind above, fails. This conclusion was also reached in [6], but at the level of the second order linear differential equation Eq.(21).

3.3. Recursive non-linear deformations

We have so far only considered linear (in the superpotential) deformations. In this section we will briefly consider two non-linear deformation schemes. Let us first consider a polynomial kind of deformation. That is, given a superpotential $\hat{W}_{(i-1)0}(x)$ which we will assume is derived, in some way or another, from some seed superpotential $W(x)$. Consider then the polynomial deformation

$$\hat{W}_{i0}(x) = F_{i0}(x)W^k(x) + \hat{W}_{(i-1)0}(x)\,;\, k \in \{1, 2, 3, \ldots\}\,. \tag{45}$$

http://dx.doi.org/10.5772/53845

The isospectrality condition then implies

$$F'_{i0}(x) + [k(\ln W(x))' + 2\hat{W}_{(i-1)0}(x)]F_{i0}(x) + W^k(x)F^2_{i0}(x) = 0\,. \tag{46}$$

This is a Riccati type equation of the kind we have met earlier in this work. Apparently, different k-values give rise to very different equations to solve. However, and rather intriguingly, all the possible k-values implies the same deformation. This is seen by making the following substitution

$$F_{i0}(x) = \frac{1}{W^k(x)}\frac{U'_k(x)}{U_k(x)}\,, \tag{47}$$

where $U_k(x)$ is some function. This expression inserted into Eq.(46) gives

$$U''_k(x) + 2\hat{W}_{i-1}(x)U'_k(x) = 0\,. \tag{48}$$

Hence, $\hat{W}_{i0}(x)$ is independent of k and we are essentially left with a linear deformation. Clearly, the range of values of k can be expanded to the real numbers.

Another canonical generalization of our work is to consider deformations on the form

$$\hat{W}(x) = H_0(x)\mathcal{F}(W)\,, \tag{49}$$

where $\mathcal{F}$ is any *functional* of the seed superpotential $W(x)$. The isospectrality condition then implies

$$H'_0(x) + (\ln \mathcal{F}(W))'H_0(x) + \mathcal{F}(W)H^2_0(x) = \mathcal{F}(W)^{-1}(W^2(x) + W'(x))\,. \tag{50}$$

Note that $H_0(x) = 1$ does *not* solve this equation unless $\mathcal{F}(W) = W$, since Eq.(50) with $H_0(x) = 1$ implies $\mathcal{F}'(x) + \mathcal{F}^2(x) = V_+(x)$. Since $V_+(x)$ is uniquely given in Eq.(2) any other choice of functional will fail to satisfy the isospectrality condition. Hence the conclusion. The particular solution $H_0(x) = 1$ is not forced upon us. We can in principle do without it. It is easily verified that Eq.(50) can be cast into the form Eq.(21) by the substitution

$$H_0(x) = \frac{1}{\mathcal{F}(W)}(\ln U(x))'\,. \tag{51}$$

We can also look for an expanded solution by writing

$$H_0(x) = H_{00}(x) + \frac{1}{Z_0(x)}\,, \tag{52}$$

where $H_{00}(x)$ is a particular solution of Eq.(50). We then get the equation

$$\frac{d}{dx}Z_0(x) - (\frac{d}{dx}\ln \mathcal{F}(W) + 2H_{00}(x)\mathcal{F}(W))Z_0(x) = \mathcal{F}(W)\,, \tag{53}$$

which is a generalized form of Eq.(18). The reciprocal solution has the general form

$$\frac{1}{Z_0(x)} = \frac{e^{-2\int^x H_{00}(t)\mathcal{F}(W)dt}}{\mathcal{F}(W)(C+\int^x e^{-2\int H_{00}(t)\mathcal{F}(W)dt}du)}\,. \tag{54}$$

C is an integration constant, which we assume to be real. Utilizing that $H_{00}(x) = \mathcal{F}^{-1}(x)(\ln U(x))'$ the resulting deformation coincides with Eq.(19). We thus therefore conclude that non-linear deformations on the form Eq.(49) does not generate additional deformations to the ones already generated by Eq.(13).

3.4. Deforming the Coulomb potential

As a relatively simple application of the linear deformation scheme let us briefly consider deformations of the Coulomb potential. This potential has, within the framework of SUSYQM, been treated in several previous works [5]. The superpotential and the partner potential for the Coulomb potential are given by [5]

$$W(x) = \frac{q^2}{2(l+1)} - \frac{(l+1)}{x}\,, \tag{55}$$

$$V_+(x) = \frac{1}{4}(\frac{q^2}{l+1})^2 - \frac{q^2}{x} + \frac{(l+1)(l+2)}{x^2}\,. \tag{56}$$

q and l in these expressions are the electric charge and the angular momentum quantum numbers, respectively. These potentials result in the following general solution for $U_0(x)$ in Eq.(21) [6]

$$U_0(x) = C_1 M_{l+1,l+\frac{3}{2}}(\frac{q^2x}{l+1}) + C_2 W_{l+1,l+\frac{3}{2}}(\frac{q^2x}{l+1})\,. \tag{57}$$

The $M(x)$- and $W(x)$-functions are the Whittaker functions. The solution Eq.(22) is given by [6]

$$U_0(x) \sim e^{\frac{q^2x}{2(l+1)} - (l+1)\ln(2x)}\,. \tag{58}$$

We will for simplicity assume this solution in the following. We will let $C_{01} \to \infty$ in Eq.(19) such that we deal with the identity deformation $\hat{W}_0(x) = \hat{W}_{00}(x) = W(x)$. We will also ignore the $Q_j(x)$ contributions in the following. Define $A \equiv q^2/(2(l+1))$ and $B \equiv l+1$. It then follows that

$$P_1(x) = C_{11} + C_{12}\int^x t^{2B}e^{-2At}dt\,, \tag{59}$$

such that

$$\hat{W}_{10}(x) = A - \frac{B}{x} + \frac{C_{12}x^{2B}e^{-2Ax}}{C_{11} + C_{12}\int^x t^{2B}e^{-2At}dt}\,. \tag{60}$$

http://dx.doi.org/10.5772/53845

Let us consider the s-state with $l = 0$ in order to get a better grasp on the content buried in Eq.(60). We also set $q \equiv 1$. The expression for $\hat{W}_{10}(x)$ then reduces to

$$\hat{W}_{10}(x) = \frac{1}{2} - \frac{1}{x} + \frac{C_{12}x^2e^{-x}}{C_{11} - C_{12}(x^2+2x+2)e^{-x}} \tag{61}$$

after redefining C_{11} such that the lower integration limit of the integral in Eq.(60) does not appear explicitly in the expression for the potential. We will automatically do such redefinitions in the following when it is appropriate. The corresponding physical potential $\hat{V}_{-1}(x)$ can either be derived from the definition $\hat{V}_{-1}(x) \equiv \hat{W}_{10}^2(x) - \hat{W}_{10}'(x)$ or from Eq.(23) with $\hat{W}_{00}(x) = W(x)$ and C_{01} finite. This is a consequence of a regeneration of Eq.(19) by the recursion scheme which was noted in Figure 1. From the definition it follows that

$$\hat{V}_{-1}(x) = \frac{1}{4} - \frac{1}{x} + \frac{C_{12}x(2x-4)e^{-x}}{C_{11} - C_{12}(x^2+2x+2)e^{-x}} + \frac{2C_{12}^2x^4e^{-2x}}{(C_{11} - C_{12}(x^2+2x+2)e^{-x})^2}\,. \tag{62}$$

In the special case when we set $C_{11} = 0$ the last term in Eq.(61) becomes independent of the exponentials (and C_{12}) and thus reduces to a pure rational function. The physical potential $\hat{V}_{-1}(x)$ generated by $\hat{W}_{10}(x)$ is then given by

$$\hat{V}_{-1}(x) = \frac{1}{4} - \frac{1}{x} + \frac{4x(x+2)}{(x^2+2x+2)^2} \equiv V_-(x) + \frac{4x(x+2)}{(x^2+2x+2)^2}\,. \tag{63}$$

Let us go to the second iteration level starting from the expression for $\hat{W}_{10}(x)$ in Eq.(61) with $C_{11} = 0$, for convenience. It then follows that

$$\hat{W}_{20}(x) = \hat{W}_1(x) + \frac{C_{22}x^2e^x}{C_{21}(x^2+2x+2)^2 + C_{22}(x^2+2x+2)e^x}\,. \tag{64}$$

Note that when $C_{21} = 0$ we get $\hat{W}_{20}(x) = W(x)$. Hence, the deformation scheme allows in general for the possibility that additional iterations in particular cases may regenerate previous potentials in a nontrivial fashion. The expression for the corresponding physical potential is given by

$$\begin{aligned}\hat{V}_{-2}(x) = \hat{V}_{-1}(x) + \left[\frac{C_{22}x^2e^x}{C_{21}(x^2+2x+2)^2 + C_{22}(x^2+2x+2)e^x}\right] \times \\ \times \left[-4\left(\frac{1}{x} + \frac{\frac{1}{2}x^2}{x^2+2x+2}\right) + \frac{2(C_{21}(2x+2) + C_{22}e^x)}{C_{21}(x^2+2x+2) + C_{22}e^x}\right]\,.\end{aligned} \tag{65}$$

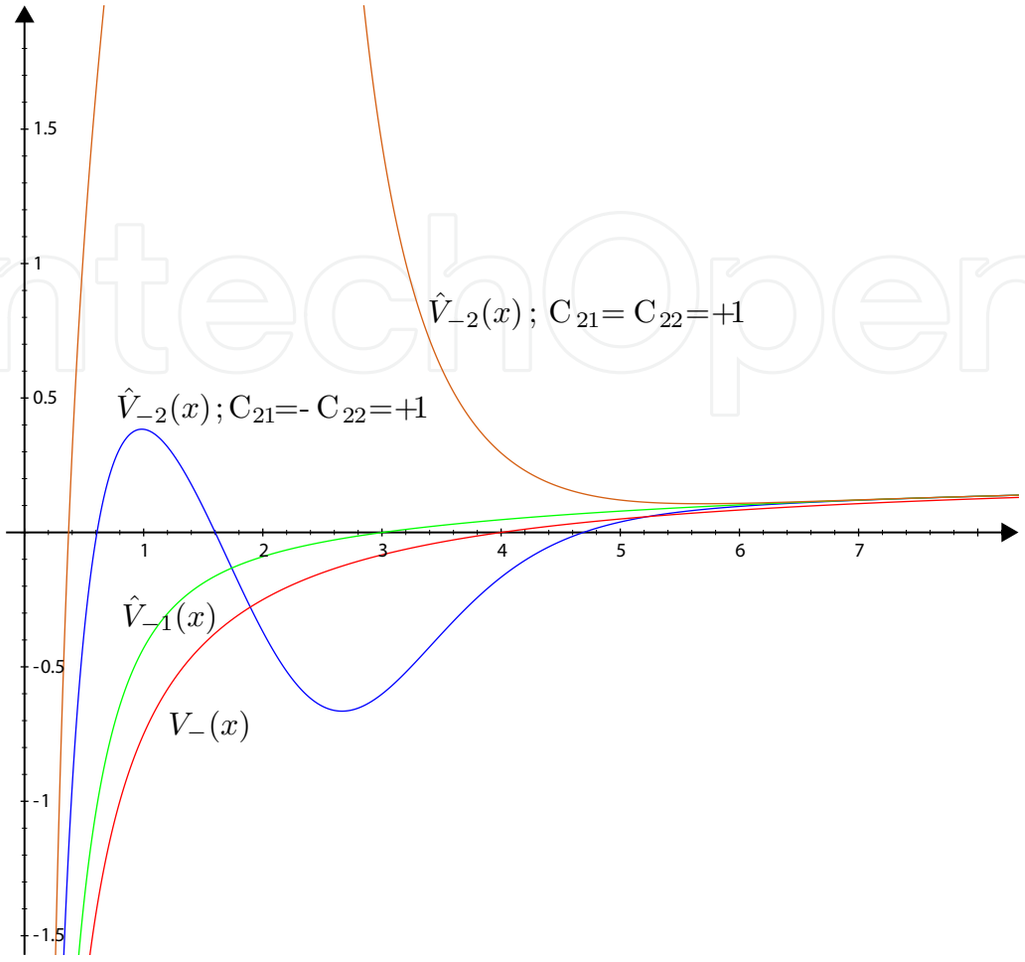

Figure 2. Generic plots depicting $\hat{V}_{-}(x)$, $\hat{V}_{-1}(x)$ and $\hat{V}_{-2}(x)$. The plots for $\hat{V}_{-2}(x)$ show how drastic the nature of a potential might change as the values of the integration constants change.

The superpotential stemming from the third iteration with $\hat{W}_{20}(x)$ in Eq.(64) as the starting point is given by

$$\hat{W}_{30}(x) = \hat{W}_{20}(x) + \frac{\left(\dfrac{C_{32}x^2e^x}{C_{21}(x^2+2x+2)^2 + C_{22}(x^2+2x+2)e^x}\right)}{\left(C_{31} + \dfrac{C_{32}}{C_{22}}\ln\left|\dfrac{C_{21}(x^2+2x+2)+C_{22}e^x}{x^2+2x+2}\right|\right)}. \tag{66}$$

This superpotential introduces the possibility for a logarithmic singularity away from the origin when $C_{22}/C_{21} < 0$. We note that setting $C_{31} = 0$ does not regenerate a previous potential as was possible at the previous iteration level when we correspondingly put $C_{21} = 0$. From Eq.(66) we can deduce the physical potential $\hat{V}_{-3}(x)$ at the third iteration level. We do not reproduce it here due to its complexity. Due to the complicated integrals appearing we are not able to provide the analytical expression for $\hat{W}_{40}(x)$. We leave detailed studies of the Coulomb potential for the future.

http://dx.doi.org/10.5772/53845

4. Conclusion

In a previous paper we showed that isospectral deformations on the form Eq.(15) are contained in the space of deformations generated by isospectral deformations on the form Eq.(13). In this work we have shown that Eq.(13) can be considered as the initial, or base, deformation of a novel infinite recursive isospectral deformation chain. This thus answers to some extend the question by which we ended our previous paper [6]; how does the most general isospectral deformation of the kind considered there (Eq.(13) in this paper) look like. The results in this work do obviously only give a partial answer. We deduced in particular that a class of recursive deformations exists which is generated by the solutions of the *non-linear* differential equations in Eq.(28).

We briefly discussed various ways to construct alternative recursive deformation structures. We considered a linear product structure, polynomial deformations and completely generalized base deformations. They all either failed to provide a recursive structure or they turned out to be identical to the deformation scheme developed in this work.

We applied the linear recursive scheme to the Coulomb potential. We derived novel superpotentials which all per construction satisfy the isospectrality condition. It is straightforward, although very tedious, to check that the corresponding physical potentials V_+ at the various iteration levels all satisfy the isospectrality condition Eq.(26). This application did also demonstrate how easily novel isospectral deformations can be generated in this approach. It did also demonstrate an increased relative complexity of the generated potentials with the number of iterations, as one also naively would expect from the expression Eq.(33).

The results in this work is obviously only a starting point for further research. One issue which needs clarification is the more general implications which can be drawn from Eq.(28). Another obvious issue is the behaviour of the transmission T and reflection R coefficients when the deformation chain is applied to some known initial scattering process. It is known that T and R are invariant under the simple deformation in Eq.(15) [5]. However, it is unclear whether this property is also a property of the general deformation chain. It is also of great interest to study the relation between our chain construction and the conventional operator approaches. One possible strategy one might follow in order to cast some light on this issue is to study the relation between the deformation chain and the concept of intertwining operators. It is known that many exactly solvable potentials are related by intertwining operator transformations including the Darboux transformations which also appear in our context [6]. Clearly, key to our construction in this work is the non-linearity of the Riccati equation. One could contemplate studying the associated JET-space and its deformations. Furthermore, an analysis of the intertwining of the hierarchy of the JET-spaces associated with the system of equations in Eq.(26) might also cast new light on our subject.

Author details

Bjørn Jensen

* Address all correspondence to: bjorn.jensen@hive.no

Vestfold University College, Norway

References

[1] E. Schrödinger, Proc. R. Irish Acad. **A46**, 9 (1940); **A46**, 183 (1940); A47, 53 (1941), L.Infeld and T.E. Hull, Rev. Mod. Phys. 23, 21 (1951) (see also the references in this paper).

[2] E. Witten, Nucl. Phys. B188 513 (1981).

[3] G. Darboux, C.R. Acad. Sci. (Paris) 94, 1456 (1882).

[4] D.L. Pursey, Phys. Rev. D33 2267 (1986).

[5] A. Gangopadhyaya , J.V. Mallow and C. Rasinariu , *Supersymmetric quantum mechanics* World Scientific Publishing Co. (2011).

[6] B. Jensen, JHEP 11, 059 (2011).

[7] E. L. Ince,*Ordinary differential equations* Dover Publications, Inc. (1956).

[8] W.E. Boyce, R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* New York: Wiley, 4th ed. (1986).