We are IntechOpen,
the world's leading publisher of
Open Access books
Built by scientists, for scientists

4,800
Open access books available

122,000
International authors and editors

135M
Downloads

Our authors are among the

154
Countries delivered to

TOP 1%
most cited scientists

12.2%
Contributors from top 500 universities

BOOK CITATION INDEX
CLARIVATE ANALYTICS
INDEXED

WEB OF SCIENCE™

Selection of our books indexed in the Book Citation Index
in Web of Science™ Core Collection (BKCI)

Interested in publishing with us?
Contact book.department@intechopen.com

Numbers displayed above are based on latest data collected.
For more information visit www.intechopen.com

Singular Stochastic Control in Option Hedging with Transaction Costs

Tze Leung Lai and Tiong Wee Lim

Additional information is available at the end of the chapter

1. Introduction

An option written on an underlying asset (e.g., stock) confers on its holder the right to receive a certain payoff before or on a certain (expiration) date T. The payoff $f(\cdot)$ is a function of the price of the underlying asset at the time of exercise (i.e., claiming the payoff), or more generally, a functional of the asset price path up to the time of exercise. We focus here on European options, for which exercise is allowable only at T, which are different from American options, for which early exercise at any time before T is also allowed. For example, the holder of a European call (resp. put) option has the right to buy (resp. sell) the underlying asset at T at a certain (strike) price K. Denoting by S_T the asset price at T, the payoff of the option is $f(S_T)$, with $f(S) = (S - K)^+$ and $(K - S)^+$ for a call and put, respectively.

Black & Scholes [1] made seminal contributions to the theory of option pricing and hedging by modeling the asset price as a geometric Brownian motion and assuming that (i) the market has a risk-free asset with constant rate of return r, (ii) no transaction costs are imposed on the sale or purchase of assets, (iii) there are no limits on short selling, and (iv) trading occurs continuously. Specifically, the asset price S_t at time t satisfies the stochastic differential equation

$$dS_t = \alpha S_t \, dt + \sigma S_t \, dW_t, \quad S_0 > 0, \tag{1}$$

where $\alpha \in \mathbb{R}$ and $\sigma > 0$ are the mean and standard deviation (or volatility) of the asset's return, and $\{W_t, \ t \geq 0\}$ is a standard Brownian motion (with $W_0 = 0$) on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, \ t \geq 0\}, \mathbb{P})$. The absence of transaction fees permits the construction of a continuously rebalanced portfolio consisting of $\mp \Delta$ unit of the asset for every ± 1 unit of the European option such that its rate of return equals the risk-free rate r, where $\Delta = \partial c / \partial S$ (resp. $\partial p / \partial S$) for a call (resp. put) whose price is c (resp. p). By requiring this portfolio to be *self-financing* (in the sense that all subsequent rebalancing is financed entirely by the initial capital and, if necessary, by short selling the risk-free asset) and to perfectly replicate the outcome of the European option at expiration, Black & Scholes [1] have shown that the

"fair" value of the option in the absence of arbitrage opportunities is the initial amount of capital $\hat{E}\{e^{-rT}f(S_T)\}$, where $\hat{E}$ denotes expectation under the equivalent martingale measure (with respect to which S_t has drift $\alpha = r$). Instead of considering geometric Brownian motion $S_t = S_0 \exp\{(r - \sigma^2/2)t + \sigma W_t\}$, it is convenient to work directly with Brownian motion W_t. The fact that σW_t and $W_{\sigma^2 t}$ have the same distribution suggests the change of variables

$$s = \sigma^2(t - T), \quad z = \log(S/K) - (\rho - 1/2)s \tag{2}$$

with $\rho = r/\sigma^2$. The Black-Scholes option pricing formulas $\hat{E}\{e^{-r(T-t)}f(S_T) \mid S_t = S\}$ are given explicitly by $c(s, z) = Ke^{\rho s}[e^{z-s/2}\Phi(z/\sqrt{-s} - \sqrt{-s}) - \Phi(z/\sqrt{-s})]$ for the call and $p(s, z) = Ke^{\rho s}[\Phi(-z/\sqrt{-s}) - e^{z-s/2}\Phi(-\{z/\sqrt{-s} - \sqrt{-s}\})]$ for the put, where Φ is the standard normal distribution function. Correspondingly, the option deltas are $\Delta(s, z) = \pm\Phi(\pm\{z/\sqrt{-s} - \sqrt{-s}\})$ with $+$ for the call and $-$ for the put.

In the presence of transaction costs, perfect hedging of a European option is not possible (since it results in an infinite turnover of the underlying asset and is, therefore, ruinously expensive) and trading in an option involves an essential element of risk. This hedging risk can be characterized as the difference between the realized cash flow from a hedging strategy which uses the initial option premium to trade in the underlying asset and bond, and the desired option payoff at maturity. By embedding option hedging within the framework of portfolio selection introduced by Magill & Constantinides [12] and Davis & Norman [13], Hodges & Neuberger [15] used a risk-averse utility function to assess this shortfall (or "replication error") and formulated the option hedging problem as one of maximizing the investor's expected utility of terminal wealth. This involves the value functions of two singular stochastic control problems, for trading in the market with and without a (short or long) position in the option, and the optimal hedge is given by the difference in the trading strategies corresponding to these two problems. The nature of the optimal hedge is that an investor with an option position should rebalance his portfolio only when the number of units of the asset falls "too far" out of line. For the negative exponential utility function, Davis et al. [14], Clewlow & Hodges [11], and Zakamouline [20] have developed numerical methods to compute the optimal hedge by making use of discrete-time dynamic programming on an approximating binomial tree for the asset price; see Kushner & Dupuis [3] for the general theory of Markov chain approximations for continuous-time processes and their use in the numerical solution of optimal stopping and control problems. More recently, Lai & Lim [18] introduced a new numerical method for solving the singular stochastic control problems associated with utility maximization, yielding a much simpler algorithm to compute the buy and sell boundaries and value functions in the utility-based approach.

The new method is motivated by the equivalence between singular stochastic control and optimal stopping, which was first observed in the pioneering work of Bather & Chernoff [10] on the problem of controlling the motion of a spaceship relative to its target on a finite horizon with an infinite amount of fuel and has since been established for the general class of bounded variation follower problems by Karatzas & Shreve [7, 8], Karatzas & Wang [9] and Boetius [2]. By transforming the original singular stochastic control problem to an optimal stopping problem associated with a Dynkin game, the solution can be computed by applying standard backward induction to an approximating Bernoulli walk. Lai & Lim [18] showed how this backward induction algorithm can be modified, by making use of finite difference methods

if necessary, for the more general singular stochastic control problem of option hedging even though it is not reducible to an equivalent optimal stopping problem because of the presence of additional value functions.

In Section 2, we review the equivalence theory between singular stochastic control and optimal stopping. We also outline the development of the computational schemes of Lai & Lim [18] to solve stochastic control problems that are equivalent to optimal stopping. In Section 3, we introduce the utility-based option hedging problem, outline how the algorithm in Section 2 can be modified to solve stochastic control problems for which equivalence does not exist, and provide numerical examples to illustrate the use of the coupled backward induction algorithm to compute the optimal buy and sell boundaries of a short European call option. We conclude in Section 4.

2. Singular stochastic control and optimal stopping

Bather & Chernoff [10] pioneered the study of singular stochastic control in their analysis of the problem of controlling the motion of a spaceship relative to its target on a finite horizon with an infinite amount of fuel. A key idea in their analysis is the reduction of the stochastic control problem to an optimal stopping problem via a change of variables. This spaceship control problem is an example of a *bounded variation follower problem* and the equivalence between singular stochastic control and optimal stopping has since been established for a general class of bounded variation follower problems by Karatzas & Shreve [7, 8], Karatzas & Wang [9] and Boetius [2]. In this section, we review this equivalence for a particular formulation of the bounded variation follower problem in which the control $\xi^+ - \xi^-$ is not applied additively to the Brownian motion $\{Z_u\}$ and outline the backward induction algorithm for solving the equivalent Dynkin game.

2.1. A bounded variation follower problem and its equivalent optimal stopping problem

Suppose that the state process $\mathbf{S} = \{S_t, \ t \geq 0\}$ representing the underlying stochastic environment (in the absence of control) is given by (1). In our subsequent application to option hedging, $\mathbf{S}$ represents the price of the asset underlying the option, whereas in other applications such as to the problem of reversible investment by Guo & Tomecek [21], $\mathbf{S}$ represents an economic indicator reflecting demand for a certain commodity. A singular control process is given by a pair (ξ^+, ξ^-) of adapted, nondecreasing, LCRL processes such that $d\xi^+$ and $d\xi^-$ are supported on disjoint subsets. We are interested in problems with a finite time horizon, i.e., we consider the time interval $[0, T]$ for some terminal time $T \in (0, \infty)$. Given any times $0 \leq s \leq t \leq T$, $\xi_{t+}^+ - \xi_s^+$ and $\xi_{t+}^- - \xi_s^-$ represent the cumulative increase and decrease, respectively, in control level resulting from the controller's decisions over the time interval $[s, t]$, with $\xi_0^+ = \xi_0^- = 0$. The total control value is therefore given by the finite variation process

$$x_t = x_0 + \xi_t^+ - \xi_t^-. \tag{3}$$

A pair (ξ^+, ξ^-) is an *admissible* singular control if, in addition to the above requirements, $x_t \in \overline{\mathcal{I}}$ for all $t \in [0, T]$, where $\mathcal{I}$ is an open, possibly unbounded interval of $\mathbb{R}$ and $\overline{\mathcal{I}}$ is its closure.

Let $F(t, S, x)$, $\kappa^{\pm}(t, S)$ and $G(S, x)$ be sufficiently smooth functions, with F and G representing the running and terminal reward, respectively, and $\kappa^{\pm}$ the costs of exerting control. The goal of the controller is to maximize an objective function of the form:

$$J_{t,S,x}(\xi^+, \xi^-) = E_{t,S,x}\left\{ \int_t^T e^{-r(u-t)} F(u, S_u, x_u)\, du - \int_{[t,T)} e^{-r(u-t)} \kappa^+(u, S_u)\, d\xi_u^+ \right.$$

$$\left. - \int_{[t,T)} e^{-r(u-t)} \kappa^-(u, S_u)\, d\xi_u^- + e^{-r(T-t)} G(S_T, x_T) \right\},$$

where $E_{t,S,x}$ denotes conditional expectation given $S_t = S$ and $x_t = x$. The value function of the stochastic control problem is

$$V(t, S, x) = \sup_{(\xi^+, \xi^-) \in \mathcal{A}_{t,x}} J_{t,S,x}(\xi^+, \xi^-), \quad (t, S, x) \in [0, T] \times (0, \infty) \times \overline{\mathcal{I}}, \tag{4}$$

where $\mathcal{A}_{t,x}$ denotes the set of all admissible controls which satisfy $x_t = x$.

We derive formally the Hamilton-Jacobi-Bellman equation associated with the stochastic control problem (4), which provides key insights into the nature of the optimal control. Consider, for now, a smaller set $\mathcal{A}_{t,x}^k$ of admissible controls such that $\xi^{\pm}$ are absolutely continuous processes, i.e., $d\xi_t^{\pm} = \ell_t^{\pm} dt$ with $0 \leq \ell_t^+, \ell_t^- \leq k < \infty$. Under this restriction, the value function (4) becomes

$$V^k(t, S, x) = \sup_{(\ell^+, \ell^-) \in \mathcal{A}_{t,x}^k} J_{t,S,x}^k(\ell^+, \ell^-), \quad (t, S, x) \in [0, T] \times (0, \infty) \times \overline{\mathcal{I}},$$

where

$$J_{t,S,x}^k(\ell^+, \ell^-)$$
$$= E_{t,S,x}\left\{ \int_t^T e^{-r(u-t)} \left[F(u, S_u, x_u) - \kappa^+(u, S_u)\ell_u^+ - \kappa^-(u, S_u)\ell_u^- \right] du + e^{-r(T-t)} G(S_T, x_T) \right\}.$$

Since the infinitesimal generator of the stochastic system comprising (1) and $dx_t = (\ell_t^+ - \ell_t^-)\, dt$ (corresponding to (3) for absolutely continuous $\xi^{\pm}$) is

$$\alpha S \frac{\partial}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (\ell^+ - \ell^-) \frac{\partial}{\partial x},$$

the Bellman equation for $V^k(t, S, x)$ is

$$\max_{0 \leq \ell^+, \ell^- \leq k} \left\{ \left[\mathcal{L}_{t,S} + (\ell^+ - \ell^-) \frac{\partial}{\partial x} \right] V^k(t, S, x) + F(t, S, x) - \kappa^+(t, S)\ell^+ - \kappa^-(t, S)\ell^- \right\} = 0,$$

where

$$\mathcal{L}_{t,S} = \frac{\partial}{\partial t} + \alpha S \frac{\partial}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} - r,$$

or equivalently,

$$\max_{0 \le \ell^+, \ell^- \le k} \left\{ \left[\frac{\partial V^k}{\partial x}(t, S, x) - \kappa^+(t, S) \right] \ell^+ - \left[\frac{\partial V^k}{\partial x}(t, S, x) + \kappa^-(t, S) \right] \ell^- \right\}$$

$$+ \mathcal{L}_{t,S} V^k(t, S, x) + F(t, S, x) = 0, \quad (t, S, x) \in [0, T) \times (0, \infty) \times \overline{\mathcal{I}}.$$

Assuming the value function to be an increasing function of x, the optimal control is obtained by considering the following three possible cases (all the other permutations of inequalities are impossible):

(i) If $\partial V^k(t, S, x)/\partial x - \kappa^+(t, S) \ge 0$ and $\partial V^k(t, S, x)/\partial x + \kappa^-(t, S) > 0$, then the maximum is achieved by $\ell^- = 0$ and exerting control ξ^+ at the maximum possible rate $\ell^+ = k$.

(ii) If $\partial V^k(t, S, x)/\partial x - \kappa^+(t, S) < 0$ and $\partial V^k(t, S, x)/\partial x + \kappa^-(t, S) \le 0$, then the maximum is achieved by $\ell^+ = 0$ and exerting control ξ^- at the maximum possible rate $\ell^- = k$.

(iii) If $\partial V^k(t, S, x)/\partial x - \kappa^+(t, S) \le 0$ and $\partial V^k(t, S, x)/\partial x + \kappa^-(t, S) \ge 0$, then the maximum is achieved by not exerting any control, i.e., $\ell^+ = \ell^- = 0$, and $V^k(t, S, x)$ satisfies the partial differential equation (PDE) $\mathcal{L}_{t,S} V^k(t, S, x) + F(t, S, x) = 0$.

Thus, the state space $[0, T] \times (0, \infty) \times \overline{\mathcal{I}}$ is partitioned into three regions, which we denote by $\mathcal{N}$ (corresponding to no control), $\mathcal{B}$ (control ξ^+ is exerted), and $\mathcal{S}$ (control ξ^- is exerted). The boundaries between the no-control region $\mathcal{N}$ and the regions $\mathcal{B}$ and $\mathcal{S}$ are denoted by $\partial \mathcal{B}$ and $\partial \mathcal{S}$.

As $k \to \infty$, the set $\mathcal{A}_{t,x}^k$ of admissible controls becomes the set $\mathcal{A}_{t,x}$ of problem (4) and the state space remains partitioned into the regions $\mathcal{N}$, $\mathcal{B}$ and $\mathcal{S}$. If $(t, S, x) \in \mathcal{B}$ (resp. $\mathcal{S}$), then the control ξ^+ (resp. ξ^-) must be instantaneously exerted to bring the state to the boundary $\partial \mathcal{B}$ (resp. $\partial \mathcal{S}$). Thus, besides an initial jump from $\mathcal{B}$ or $\mathcal{S}$ to the boundary $\partial \mathcal{B}$ or $\partial \mathcal{S}$ (if necessary), the optimal control process acts thereafter only when $(t, S, x) \in \partial \mathcal{B}$ or $\partial \mathcal{S}$ so as to keep the state in $\mathcal{N} \cup \partial \mathcal{B} \cup \partial \mathcal{S}$. Because the optimal process behaves like the *local time* of the (optimally controlled) state process at the boundaries, such a control is termed *singular*. In $\mathcal{B}$, since the optimal control is to increase x by a positive amount δx (up to that required to take the state to $\partial \mathcal{B}$) at the cost of $\kappa^+(t, S)$ per unit increase, the value function satisfies the equation

$$V(t, S, x) = V(t, S, x + \delta x) - \kappa^+(t, S)\delta x \quad (\text{in } \mathcal{B}).$$

Similarly, since the optimal control in $\mathcal{S}$ is to decrease x by a positive amount δx (up to that required to take the state to $\partial \mathcal{S}$) at the cost of $\kappa^-(t, S)$ per unit decrease, the value function satisfies the equation

$$V(t, S, x) = V(t, S, x - \delta x) - \kappa^-(t, S)\delta x \quad (\text{in } \mathcal{S}).$$

Letting $\delta x \to 0$ leads to gradient constraints for the value function in $\mathcal{B}$ and $\mathcal{S}$. In $\mathcal{N}$, $V(t, S, x)$ continues to satisfy the PDE given in (iii) above. From these observations, we obtain the following free boundary problem (FBP) for the value function $V(t, S, x)$:

$$\mathcal{L}_{t,S}V(t,S,x) + F(t,S,x) = 0 \qquad \text{in } \mathcal{N}, \tag{5a}$$

$$\frac{\partial V}{\partial x}(t,S,x) = \kappa^+(t,S) \qquad \text{in } \mathcal{B}, \tag{5b}$$

$$\frac{\partial V}{\partial x}(t,S,x) = -\kappa^-(t,S) \quad \text{in } \mathcal{S}, \tag{5c}$$

$$V(T,S,x) = G(S,x). \tag{5d}$$

It also follows that the Hamilton-Jacobi-Bellman equation associated with (4) is the following variational inequality with gradient constraints:

$$\max\left\{\mathcal{L}_{t,S}V(t,S,x) + F(t,S,x), \frac{\partial V}{\partial x}(t,S,x) - \kappa^+(t,S), -\frac{\partial V}{\partial x}(t,S,x) - \kappa^-(t,S)\right\} = 0, \tag{6}$$

$(t,S,x) \in [0,T) \times (0,\infty) \times \overline{\mathcal{I}}$.

With $\rho = r/\sigma^2$ and $\beta = \alpha/\sigma^2$, a more parsimonious parameterization of (5) can be obtained by considering the change of variables (2) (without K and with ρ replaced by β here since the state process $\mathbf{S}$ has rate of return α under the "physical" measure rather than r under the risk-neutral measure) and $v(s,z,x) = e^{-\rho s}V(t,S,x)$. Applying the chain rule of differentiation yields

$$\frac{\partial V}{\partial S} = \frac{e^{\rho s}}{S}\frac{\partial v}{\partial z}, \quad \frac{\partial^2 V}{\partial S^2} = \frac{e^{\rho s}}{S^2}\left(\frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z}\right), \quad \frac{\partial V}{\partial t} - rV = e^{\rho s}\left[\sigma^2\frac{\partial v}{\partial s} - \left(\alpha - \frac{\sigma^2}{2}\right)\frac{\partial v}{\partial z}\right].$$

We also define $\tilde{F}(s,z,x) = e^{r(T-t)}F(t,S,x)/\sigma^2$ and $\tilde{\kappa}^\pm(s,z) = e^{-\rho s}\kappa^\pm(t,S)$. Upon substitution into (5), we arrive at the FBP

$$\left\{\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial^2}{\partial z^2}\right\}v(s,z,x) + \tilde{F}(s,z,x) = 0 \qquad \text{in } \mathcal{N}, \tag{7a}$$

$$\frac{\partial v}{\partial x}(s,z,x) = \tilde{\kappa}^+(s,z) \qquad \text{in } \mathcal{B}, \tag{7b}$$

$$\frac{\partial v}{\partial x}(s,z,x) = -\tilde{\kappa}^-(s,z) \quad \text{in } \mathcal{S}, \tag{7c}$$

$$v(0,z,x) = G(e^z,x), \tag{7d}$$

Note that $v(s,z,x)$ is the value function of the corresponding singular stochastic control problem for the Brownian motion $\{Z_u, u \leq 0\}$:

$$v(s,z,x) = \sup_{(\xi^+,\xi^-)\in\mathcal{A}_{s,x}} E_{s,z,x}\left\{\int_s^0 \tilde{F}(u,Z_u,x_u)\,du - \int_{[s,0)}\tilde{\kappa}^+(u,Z_u)\,d\xi_u^+\right.$$

$$\left. - \int_{[s,0)}\tilde{\kappa}^-(u,Z_u)\,d\xi_u^- + G(e^{Z_0},x_0)\right\}, \tag{8}$$

where $E_{s,z,x}$ denotes conditional expectation given $Z_s = z$ and $x_s = x$.

We now introduce the change of variables

$$w(s, z, x) = \frac{\partial v}{\partial x}(s, z, x), \quad (s, z, x) \in [-\sigma^2 T, 0] \times \mathbb{R} \times \overline{\mathcal{I}}. \tag{9}$$

From (7) it follows that w solves the FBP

$$\left\{ \frac{\partial}{\partial s} + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right\} w(s, z, x) + \phi(s, z, x) = 0 \qquad \text{in } \mathcal{N}, \tag{10a}$$

$$w(s, z, x) = \tilde{\kappa}^+(s, z) \qquad \text{in } \mathcal{B}, \tag{10b}$$

$$w(s, z, x) = -\tilde{\kappa}^-(s, z) \quad \text{in } \mathcal{S}, \tag{10c}$$

$$w(0, z, x) = g(e^z, x), \tag{10d}$$

where $\phi(s, z, x) = \partial \tilde{F}(s, z, x)/\partial x$ and $g(\cdot, x) = \partial G(\cdot, x)/\partial x$. The FBP (10) can be restated as an optimal stopping problem associated with a Dynkin game, for which $w(s, z, x)$ is the value function. Specifically,

$$w(s, z, x) = \underline{w}(s, z, x) := \sup_{\tau^- \in \mathcal{T}(s,0)} \inf_{\tau^+ \in \mathcal{T}(s,0)} I_{s,z,x}(\tau^+, \tau^-)$$

$$= \overline{w}(s, z, x) := \inf_{\tau^+ \in \mathcal{T}(s,0)} \sup_{\tau^- \in \mathcal{T}(s,0)} I_{s,z,x}(\tau^+, \tau^-), \tag{11}$$

where $\mathcal{T}(a, b)$ denotes the set of stopping times taking values between a and b ($> a$), and

$$I_{s,z,x}(\tau^+, \tau^-) = E_{s,z,x} \left\{ \int_s^{\tau^+ \wedge \tau^-} \phi(u, Z_u, x_u) \, du + \tilde{\kappa}^+(\tau^+, Z_{\tau^+}) I_{\{\tau^+ < \tau^- < 0\}} \right.$$

$$\left. - \tilde{\kappa}^-(\tau^-, Z_{\tau^-}) I_{\{\tau^- < \tau^+ < 0\}} + g(e^{Z_0}, x_0) I_{\{\tau^- = \tau^+ = 0\}} \right\}.$$

The Dynkin game is a "stochastic game of timing" in which there are two players P and M, each of whom chooses a stopping time (τ^+ and τ^-, respectively) in $\mathcal{T}(s, 0)$. As long as the game is in progress, P keeps paying M at the rate $\phi(s, z, x)$ per unit of time. The game terminates as soon as one of the players decides to stop, i.e., at $\tau^+ \wedge \tau^-$. If player M stops first, he pays P the amount $\tilde{\kappa}^-(\tau^-, Z_{\tau^-})$. If player P stops first, he pays M the amount $\tilde{\kappa}^+(\tau^+, Z_{\tau^+})$ (resp. $g(e^{Z_0}, x_0)$) when the game terminates before (resp. at) the end of the time horizon 0. The objective of P is to minimize his expected total payment to M whereas the objective of M is to maximize this quantity.

In addition to the relationship (9) between the value functions v and w, the optimal continuation region of the Dynkin game (11) coincides with the no-control region of the singular stochastic control problem (8) in the sense that if $(\xi^{+,*}, \xi^{-,*})$ is an optimal control of (8) and we define the stopping times $\tau^{+,*} = \inf\{u \in [s, 0) : \xi^{+,*}(u) > 0\}$ and $\tau^{-,*} = \inf\{u \in [s, 0) : \xi^{-,*}(u) > 0\}$ ($\inf \varnothing = 0$), then $(\tau^{+,*}, \tau^{-,*})$ is a *saddlepoint* of the game with the property that $w(s, z, x) = I_{s,z,x}(\tau^{+,*}, \tau^{-,*})$.

2.2. Example: Reversible investment

Before we outline the computational algorithm for solving the Dynkin game (11), we give an example in mathematical economics of a stochastic control problem which has the form (4). In the notation of (4), the problem of reversible investment is one in which a company, by adjusting its production capacity x_t through expansion ξ_t^+ and contraction ξ_t^- according to market fluctuations S_t, wishes to maximize its overall expected net profit $E_{t,S,x} J_{t,S,x}(\xi^+, \xi^-)$ over a finite horizon. The net profit of such an investment depends on the running production function $F(t, S, x)$ of the actual capacity, the benefits of contraction $\kappa^-(t, S) \equiv K^- < 0$, and the cost of expansion $\kappa^+(t, S) \equiv K^+ > 0$, with $K^+ + K^- > 0$. The economic uncertainty about S_t (such as the price or demand for the product) is modeled by geometric Brownian motion (1).

Guo & Tomecek [21] studied the infinite-horizon ($T = \infty$) reversible investment problem and provided an explicit solution to the problem with the so-called Cobb-Douglas production function $F(t, S, x) = S^\lambda x^\mu$, where $\lambda \in (0, n)$, $n = [-(\alpha - \sigma^2/2) + \sqrt{(\alpha - \sigma^2/2)^2 - 2\sigma^2 r}]/\sigma^2 > 0$ and $\mu \in (0, 1]$. The optimal strategy is for the company to increase (resp. decrease) capacity when (S, x) belongs to the investment (resp. disinvestment) region $\mathcal{B}$ (resp. $\mathcal{S}$). Here, $\mathcal{B} = \{(S, x) : x \le X_b(S)\}$ and $\mathcal{S} = \{(S, x) : x \ge X_s(S)\}$, where $X_i(S) = (S/\nu_i)^{\lambda/(1-\alpha)}$, $i = b, s$, ν_b and ν_s are unique solutions to

$$\frac{\alpha}{\lambda - m}(\nu_b^{\lambda-m} - \nu_s^{\lambda-m}) = -\frac{r}{m}(K^+ \nu_b^{-m} + K^- \nu_s^{-m}),$$

$$\frac{\alpha}{n - \lambda}(\nu_b^{\lambda-n} - \nu_s^{\lambda-n}) = \frac{r}{n}(K^+ \nu_b^{-n} + K^- \nu_s^{-n}),$$

and $m = [-(\alpha - \sigma^2/2) - \sqrt{(\alpha - \sigma^2/2)^2 - 2\sigma^2 r}]/\sigma^2 < 0$.

In the case of finite horizon ($T < \infty$), the investment and disinvestment regions have similar forms but are not stationary in time, i.e., $\mathcal{B} = \{(t, S, x) : x \le X_b(t, S)\}$ and $\mathcal{S} = \{(t, S, x) : x \ge X_s(t, S)\}$. It is not possible to express the boundaries $X_i(t, S)$ explicitly. We can solve for them (after applying the change of variables $(t, S, x) \mapsto (s, z, x)$ given by (2)) by making use of the backward induction algorithm described in the next section; for details and numerical results, see Lai et al. [16].

2.3. Computational algorithm for solving the Dynkin game

In view of the equivalence between the stochastic control problem (8) and the Dynkin game (11), which is an optimal stopping problem with (disjoint) stopping regions $\mathcal{B}$ and $\mathcal{S}$ coinciding with the control regions of (8) as well as continuation region $\mathcal{N}$ coinciding with the no-control region of (8), it suffices to solve (11) rather than (8) directly. The backward induction algorithm outlined below is similar to the algorithms studied by Lai et al. [17], for which convergence properties have been established.

Suppose the buy and sell boundaries can be expressed as functions $X_b(s, z)$ and $X_s(s, z)$ such that $\mathcal{B} = \{(s, z, x) : x \le X_b(s, z)\}$ and $\mathcal{S} = \{(s, z, x) : x \ge X_s(s, z)\}$. Whereas $w(s, z, x)$ is given by (10b) and (10c) in the buy and sell regions (i.e., the stopping region in the Dynkin game), the continuation value of the Dynkin game is a solution of the partial differential equation (10a) and can be computed using backward induction on a symmetric Bernoulli random walk which approximates standard Brownian motion. Specifically, let T_{max} denote

the largest terminal date of interest, take small positive δ and ϵ such that $N := \sigma^2 T_{\max}/\delta$ is an integer, and let $\mathbf{Z}_\delta = \{0, \pm\sqrt{\delta}, \pm 2\sqrt{\delta}, \dots\}$ and $\mathbf{X}_\epsilon = \{0, \pm\epsilon, \pm 2\epsilon, \dots\}$. For $i = 1, 2, \dots, N$, with $s_0 = 0, s_i = s_{i-1} - \delta, z \in \mathbf{Z}_\delta, x \in \mathbf{X}_\epsilon$, the continuation value at s_i can be computed using

$$\tilde{w}(s_i, z, x) = \delta\phi(s_i, z, x) + [w(s_i + \delta, z + \sqrt{\delta}, x) + w(s_i + \delta, z - \sqrt{\delta}, x)]/2 \tag{12}$$

with $w(0, z, x) = g(e^z, x)$. The following algorithm allows us to solve for $X_b(s_i, z)$ and $X_s(s_i, z)$ for $z \in \mathbf{Z}_\delta$.

Algorithm 1. Let $w(0, z, x) = g(e^z, x)$ for $z \in \mathbf{Z}_\delta$ and $x \in \mathbf{X}_\epsilon$. For $i = 1, 2, \dots, N$ and $z \in \mathbf{Z}_\delta$:
  (i) Starting at $x_0 \in \mathbf{X}_\epsilon$ with $\tilde{w}(s_i, z, x_0) < \tilde{\kappa}^+(s_i, z)$, search for the first $j \in \{1, 2, \dots\}$ (denoted by j^*) for which $\tilde{w}(s_i, z, x_0 + j\epsilon) \geq \tilde{\kappa}^+(s_i, z)$ and set $X_b(s_i, z) = x_0 + j^*\epsilon$.
  (ii) For $j \in \{1, 2, \dots\}$, let $x_j = X_b(s_i, z) + j\epsilon$. Compute, and store for use at s_{i+1}, $w(s_i, z, x_j) = \tilde{w}(s_i, z, x_j)$ as defined by (12). Search for the first j (denoted by j^*) for which $\tilde{w}(s_i, z, x_j) \geq -\tilde{\kappa}^-(s_i, z)$ and set $X_s(s_i, z) = X_b(s_i, z) + j^*\epsilon$.
  (iii) For $x \in \mathbf{X}_\epsilon$ outside the interval $[X_b(s_i, z), X_s(s_i, z)]$, set $w(s_i, z, x) = \tilde{\kappa}^+(s_i, z)$ or $-\tilde{\kappa}^-(s_i, z)$ according to whether $x \leq X_b(s_i, z)$ or $x \geq X_s(s_i, z)$.

The following backward induction equation summarizes this algorithm:

$$w(s_i, z, x) = \begin{cases} \tilde{\kappa}^+(s_i, z) & \text{if } \tilde{w}(s_i, z, x) < \tilde{\kappa}^+(s_i, z), \\ -\tilde{\kappa}^-(s_i, z) & \text{if } \tilde{w}(s_i, z, x) > -\tilde{\kappa}^-(s_i, z), \\ \tilde{w}(s_i, z, x) & \text{otherwise.} \end{cases} \tag{13}$$

Lai et al. [17] have shown that under suitable conditions discrete-time random walk approximations to continuous-time optimal stopping problems can approximate the value function with an error of the order $O(\delta)$ and the stopping boundary with an error of the order $o(\sqrt{\delta})$, where δ is the interval width in discretizing time for the approximating random walk. To prove this result, they approximate the underlying optimal stopping problem by a recursively defined family of "canonical" optimal stopping problems which depend on δ and for which the continuation and stopping regions can be completely characterized, and use an induction argument to provide bounds on the absolute difference between the boundaries of the continuous-time and discrete-time stopping problems as well as that between the value functions of the two problems. Since the Dynkin game is also an optimal stopping problem (with two stopping boundaries), their result can be extended to the present setting to establish that (13) is able to approximate the value function (11) with an error of the order $O(\delta)$ and Algorithm 1 is able to approximate the stopping boundaries $Z_i(s, x) := X_i^{-1}(s, z)$, $i = b, s$, corresponding to the optimal stopping problem (11) with an error of the order $o(\sqrt{\delta})$. Further refinements to the random walk approximations can be made by correcting for the excess over the boundary when stopping cocurs in the discrete-time problem; for details and other applications, see Chernoff [4], Chernoff & Petkau [5, 6] and Lai et al. [17].

3. Utility-based option theory in the presence of transaction costs

Consider now a risk-averse investor who trades in a risky asset whose price is given by the geometric Brownian motion (1) and a bond which pays a fixed risk-free rate $r > 0$ with the

objective of maximizing the expected utility of his terminal wealth Ω_T^0. The number of units x_t the investor holds in the asset is given by (3), where x_0 denotes the initial asset position and ξ_t^+ (resp. ξ_t^-) represents the cumulative number of units of the asset bought (resp. sold) within the time interval $[0, t]$, $0 \leq t \leq T$. If the investor pays fractions $0 < \lambda < 1$ and $0 < \mu < 1$ of the dollar value transacted on purchase and sale of the asset, the dollar value y_t of his investment in bond is given by

$$dy_t = ry_t \, dt - aS_t \, d\xi_t^+ + bS_t \, d\xi_t^-,$$

with $a = 1 + \lambda$ and $b = 1 - \mu$, or more explicitly,

$$y_T = y_t e^{r(T-t)} - a \int_{[t,T)} e^{r(T-u)} S_u \, d\xi_u^+ + b \int_{[t,T)} e^{r(T-u)} S_u \, d\xi_u^-.$$

Let $U : \mathbb{R} \to \mathbb{R}$ be a concave and increasing (hence risk-averse) utility function. We can express the investor's problem in terms of the value function

$$V^0(t, S, x, y) = \sup_{(\xi^+, \xi^-) \in \mathcal{A}_{t,x,y}} E_{t,S,x,y} \left[U\left(\Omega_T^0\right) \right], \qquad (14)$$

where $\mathcal{A}_{t,x,y}$ denotes the set of all admissible controls which satisfy $x_t = x$ and $y_t = y$, and $E_{t,S,x,y}$ denotes conditional expectation given $S_t = S$, $x_t = x$ and $y_t = y$. For the special case of the negative exponential utility function $U(z) = 1 - e^{-\gamma z}$, which has constant absolute risk aversion (CARA) γ, we can reduce the number of state variables by one by working with

$$H^0(t, S, x) = 1 - V^0(t, S, x, 0)$$

$$= \inf_{(\xi^+, \xi^-) \in \mathcal{A}_{t,x}} E_{t,S,x} \left[\exp\left\{ \gamma \left(\int_{[t,T)} e^{r(T-u)} S_u (a \, d\xi_u^+ - b \, d\xi_u^-) - Z^0(S_T, x_T) \right) \right\} \right], \qquad (15)$$

where

$$Z^0(S, x) = xS(aI_{\{x<0\}} + bI_{\{x\geq0\}}) \qquad (16)$$

denotes the liquidated value of the asset by trading x units of the asset at price S to zero unit.

If the investor is presented with an opportunity to enter into a position in a European call option written on the given asset, with strike price K and expiration date T, the problem can be formulated in the same way as (14) but with Ω_T^0 replaced by Ω_T^i with $i = $ s indicating a short call position and $i = $ b indicating a long call position. The corresponding value functions $V^i(t, S, x, y)$ also admit reductions in dimensionality via $H^i(t, S, x) = 1 - V^i(t, S, x, 0)$, but with $Z^0(S_T, x_T)$ in (15) replaced by $Z^i(S_T, x_T)$. If the option is *asset settled*, then the writer delivers one unit of the asset in return for a payment of K when the buyer exercises the option at maturity T, so

$$Z^i(S, x) = Z^0(S, x - \Delta^i(S)) + K\Delta^i(S), \quad i = \text{s}, \text{b}, \qquad (17)$$

where $\Delta^s(S) = I_{\{S>K\}}$ (short call) and $\Delta^b(S) = -I_{\{S>K\}}$ (long call). In the case of a *cash settled* option, the writer delivers $(S_T - K)^+$ in cash at T, so

$$Z^i(S, x) = Z^0(S, x) - (S - K)\Delta^i(S), \quad i = \text{s}, \text{b}. \qquad (18)$$

As in Section 2.1, we apply the change of variables (2) (with ρ replaced by $\beta = \alpha/\sigma^2$) to (15) and work with the resulting value function $h^i(s, z, x) = H^i(t, S, x)$. Corresponding to the

definitions (16)–(18) of terminal settlement value are

$$A^0(z,x) = xe^z(aI_{\{x<0\}} + bI_{\{x\geq 0\}}),$$

$$A^i(z,x) = A^0(z, x - D^i(z)) + D^i(z) \quad \text{for asset settlement}, \quad i = \text{s, b},$$

$$A^i(z,x) = A^0(z,x) - (e^z - 1)D^i(z) \quad \text{for cash settlement}, \quad i = \text{s, b},$$

with $D^{\text{s}}(z) = I_{\{z>0\}}$ (short call) and $D^{\text{b}}(z) = -I_{\{z>0\}}$ (long call). For use in (22d) below, we define $B^i(z,x) := \partial A^i(z,x)/\partial x$, which are given explicitly by $B^0(z,x) = A^0(z,x)/x$ and, for $i = \text{s, b}$, $B^i(z,x) = B^0(z, x - D^i(z))$ under asset settlement and $B^i(z,x) = B^0(z,x)$ under cash settlement. In the sequel, we fix $i = 0, \text{s, b}$ and drop the superscript i in the value functions (and associated quantities).

3.1. Associated free boundary problems and their solutions

The formulation of the option (pricing and) hedging problem as two stochastic control problems of the form (14) goes back to Hodges & Neuberger [15]. Davis et al. [14] derived the Hamilton-Jacobi-Bellman (HJB) equations associated with the control problems $V(t, S, x, y)$ and $H(t, S, x) = 1 - V(t, S, x, 0)$ in the same way as we did in Section 2.1. By applying the transformation $h(s, z, x) = H(t, S, x)$ to their HJB equations, we obtain the following free boundary problem (FBP) for $h(s, z, x)$:

$$\frac{\partial h}{\partial s} + \frac{1}{2}\frac{\partial^2 h}{\partial z^2} = 0, \qquad x \in [X_{\text{b}}(s,z), X_{\text{s}}(s,z)], \tag{19a}$$

$$\frac{\partial h}{\partial x}(s,z,x) = w_{\text{b}}(s,z,x), \qquad x \leq X_{\text{b}}(s,z), \tag{19b}$$

$$\frac{\partial h}{\partial x}(s,z,x) = w_{\text{s}}(s,z,x), \qquad x \geq X_{\text{s}}(s,z), \tag{19c}$$

$$h(0,z,x) = \exp\{-\gamma K A(z,x)\}, \tag{19d}$$

where

$$w_{\text{b}}(s,z,x) = -a\gamma K e^{z+(\beta-\rho-1/2)s}h(s,z,x), \tag{20a}$$

$$w_{\text{s}}(s,z,x) = -b\gamma K e^{z+(\beta-\rho-1/2)s}h(s,z,x). \tag{20b}$$

Associated with FBP (19) are three regions: $\mathcal{B} = \{(s,z,x) : x \leq X_{\text{b}}(s,z)\}$ where it is optimal to buy the (risky) asset, $\mathcal{S} = \{(s,z,x) : x \geq X_{\text{s}}(s,z)\}$ where it is optimal to sell the asset, and $\mathcal{N} = [-\sigma^2 T, 0] \times \mathbb{R} \times \mathbb{R} \setminus (\mathcal{B} \cup \mathcal{S})$ where it is optimal to not transact. Since $\partial/\partial s + (1/2)\partial^2/\partial z^2$ is the infinitesimal generator of space-time Brownian motion, this means that while (s, Z_s, x_s) is inside the no-transaction region, the dynamics of $h(s, Z_s, x_s)$ is driven by the standard Brownian motion $\{Z_s, s \leq 0\}$. In the buy and sell regions, it follows from (19b) and (19c) that

$$h(s,z,x) = \exp\left\{-a\gamma K e^{z+(\beta-\rho-1/2)s}[x - X_{\text{b}}(s,z)]\right\}h(s,z,X_{\text{b}}(s,z)), \quad x \leq X_{\text{b}}(s,z), \tag{21a}$$

$$h(s,z,x) = \exp\left\{-b\gamma K e^{z+(\beta-\rho-1/2)s}[x - X_{\text{s}}(s,z)]\right\}h(s,z,X_{\text{s}}(s,z)), \quad x \geq X_{\text{s}}(s,z). \tag{21b}$$

Finally, if we let $w(s, z, x) = \partial h(s, z, x)/\partial x$, then $w(s, z, x)$ satisfies the FBP

$$\frac{\partial w}{\partial s} + \frac{1}{2}\frac{\partial^2 w}{\partial z^2} = 0, \qquad x \in [X_b(s, z), X_s(s, z)], \tag{22a}$$

$$w(s, z, x) = w_b(s, z, x), \qquad x \le X_b(s, z), \tag{22b}$$

$$w(s, z, x) = w_s(s, z, x), \qquad x \ge X_s(s, z), \tag{22c}$$

$$w(0, z, x) = -\gamma KB(z, x)h(0, z, x). \tag{22d}$$

If the function $h(s, z, x)$ is known, then by analogy to (10), the FBP (22) is an optimal stopping problem associated with a Dynkin game, and its solution can be computed using the following analog of the backward induction equation (13):

$$w(s_i, z, x) = \begin{cases} w_b(s_i, z, x) & \text{if } \tilde{w}(s_i, z, x) < w_b(s_i, z, x), \\ w_s(s_i, z, x) & \text{if } \tilde{w}(s_i, z, x) > w_s(s_i, z, x), \\ \tilde{w}(s_i, z, x) & \text{otherwise,} \end{cases} \tag{23}$$

where $\tilde{w}(s, z, x)$ is given by (12) with $\phi \equiv 0$. On the other hand, if the boundaries $X_b(s, z)$ and $X_s(s, z)$ are given, then the FBP (19) can also be solved by backward induction: For $z \in \mathbf{Z}_\delta$, compute $h(s_i, z, x)$ using (21) (with s replaced by s_i) if $x \in \mathbf{X}_\epsilon$ is outside the interval $[X_b(s_i, z), X_s(s_i, z)]$, and if $x \in \mathbf{X}_\epsilon \cap [X_b(s_i, z), X_s(s_i, z)]$, let $h(s_i, z, x) = \tilde{h}(s_i, z, x)$ with

$$\tilde{h}(s, z, x) = [h(s + \delta, z + \sqrt{\delta}, x) + h(s + \delta, z - \sqrt{\delta}, x)]/2. \tag{24}$$

By replacing the unknown h in (20a) and (20b) by $\tilde{h}$ and redefining them as $\tilde{w}_b$ and $\tilde{w}_s$, Lai & Lim [18] have developed the coupled backward induction algorithm described below to solve for $X_b(s_i, z)$ and $X_s(s_i, z)$, as well as to compute values of $h(s_i, z, x)$ for $x \in \mathbf{X}_\epsilon \cap [X_b(s_i, z), X_s(s_i, z)]$.

Algorithm 2. Let $h(0, z, x) = \exp\{-\gamma KA(z, x)\}$ and $w(0, z, x) = -\gamma KB(z, x)h(0, z, x)$ for $z \in \mathbf{Z}_\delta$ and $x \in \mathbf{X}_\epsilon$. For $i = 1, 2, \ldots, N$ and $z \in \mathbf{Z}_\delta$:

(i) Starting at $x_0 \in \mathbf{X}_\epsilon$ with $\tilde{w}(s_i, z, x_0) < \tilde{w}_b(s_i, z, x_0)$, search for the first $j \in \{1, 2, \ldots\}$ (denoted by j^*) for which $\tilde{w}(s_i, z, x_0 + j\epsilon) \ge \tilde{w}_b(s_i, z, x_0 + j\epsilon)$ and set $X_b(s_i, z) = x_i + j^*\epsilon$.

(ii) For $j \in \{1, 2, \ldots\}$, let $x_j = X_b(s_i, z) + j\epsilon$. Compute, and store for use at s_{i+1}, $w(s_i, z, x_j) = \tilde{w}(s_i, z, x_j)$ as defined by (12) with $\phi \equiv 0$ and $h(s_i, z, x_j) = \tilde{h}(s_i, z, x_j)$ by (24). Search for the first j (denoted by j^*) for which $\tilde{w}(s_i, z, x_j) \ge \tilde{w}_s(s_i, z, x_j)$ and set $X_s(s_i, z) = X_b(s_i, z) + j^*\epsilon$.

(iii) For $x \in \mathbf{X}_\epsilon$ outside the interval $[X_b(s_i, z), X_s(s_i, z)]$, compute $h(s_i, z, x)$ using (21) and set $w(s_i, z, x) = w_b(s_i, z, x)$ or $w_s(s_i, z, x)$ as defined by (20) according to whether $x \le X_b(s_i, z)$ or $x \ge X_s(s_i, z)$.

It can be established that the convergence property of this algorithm is similar to that of Algorithm 1 even though (22) is not a stopping problem like (10) is. Specifically, the backward inductions (23) as well as (24) and (21) applied to $\{s_N = -\sigma^2 T, s_{N-1}, \ldots, s_1, s_0 = 0\} \times \mathbf{Z}_\delta \times \mathbf{X}_\epsilon$ are able to approximate the value functions $w(s, z, x)$ and $h(s, z, x)$ of the corresponding

continuous-time control problems with an error of the order $O(\delta)$, and Algorithm 2 is able to approximate the buy and sell boundaries $Z_i(s,x) := X_i^{-1}(s,z)$, $i = \text{b,s}$, corresponding to these control problems with an error of the order $o(\sqrt{\delta})$; see Lai et al. [19] for details as well as an extension to the problem of optimal investment and consumption.

3.2. Numerical results

Associated with each of the problems $V^i(t,S,x,y)$, $i = 0$ given by (14) corresponding to an investor with no option position and $i = \text{s,b}$ being analogs of (14) corresponding to an investor with a short or long call position, respectively, is an optimal trading strategy x_t^i ($i = 0, \text{s}, \text{b}$) of the form

$$x_t = \begin{cases} X_\text{b}(t,S_t) & \text{if } x_{t-} < X_\text{b}(t,S_t), \\ X_\text{s}(t,S_t) & \text{if } x_{t-} > X_\text{s}(t,S_t), \\ x_{t-} & \text{if } X_\text{b}(t,S_t) \leq x_{t-} \leq X_\text{s}(t,S_t). \end{cases}$$

The optimal *hedging* strategies for the option writer and buyer are then given by $x_t^\text{s} - x_t^0$ and $x_t^\text{b} - x_t^0$, respectively. In the case of no transaction costs ($\lambda = \mu = 0$), it can be shown that

$$x_t^0 = X_0(t,S) := \frac{e^{-r(T-t)}}{\gamma S} \frac{\alpha - r}{\sigma^2}, \quad x_t^i = \Delta^i(t,S) + X_0(t,S), \quad i = \text{s,b},$$

where $\Delta^\text{b}(t,S)$ (resp. $\Delta^\text{s}(t,S) = -\Delta^\text{b}(t,S)$) denotes the Black-Scholes delta for a long (resp. short) call option, given explicitly (as a function of (s,z) after applying the change of variable (2)) in Section 1. In the case of $\alpha = r$ (risk-neutrality), it can be shown that $x_t^0 \equiv 0$ whether or not there are transaction costs. In particular, if $\alpha = r$ and $\lambda = \mu = 0$, the optimal hedging strategy is to hold Δ shares of stock at all times (see Section 1). Thus, the Black-Scholes option theory is a special case of the more general utility-based option theory.

Whereas Clewlow & Hodges [11] and Zakamouline [20] made use of discrete-time dynamic programming on an approximating binomial tree for the asset price to solve the control problems directly for the optimal hedge, Lai & Lim [18] made use of the simpler Algorithm 2 outlined in Section 3.1. They provided extensive numerical results for the CARA utility function with $\alpha = r$, for which only one pair of boundaries need to be computed (since it is then optimal not to trade in the risky asset when the investor does not have an option position). As an illustration of Algorithm 2, we compute and show in Fig. 1 the optimal buy (lower) and sell (upper) boundaries for a short asset-settled call (solid black lines) with strike price $K = 20$ and for the case of no option (solid red lines) at four different times before expiration ($T - t = 1.5$, 0.5, 0.25, 0.1) when proportional transaction costs are incurred at the rate of $\lambda = \mu = 0.5\%$; the dashed lines correspond to $X_0(t,\cdot)$ and $\Delta^\text{s}(t,\cdot) + X_0(t,\cdot)$ for the case of no transaction costs. Other parameters are: absolute risk aversion $\gamma = 2.0$, risk-free rate $r = 8.5\%$, asset return rate $\alpha = 10\%$ and asset volatility $\sigma = 5\%$. Note that the red boundaries are consistent with the intuitive notion of "buy at the low and sell at the high" when investing only in a risky asset (and bond). However, unlike the case of $\alpha = r$ in which the buy and sell boundaries corresponding to a short asset-settled call always lie between 0 and 1, the black boundaries in this case (where $\alpha \neq r$) do not necessarily take values in the interval $[0,1]$.

1.5 periods from T

0.5 period from T

0.25 period from T

0.1 period from T
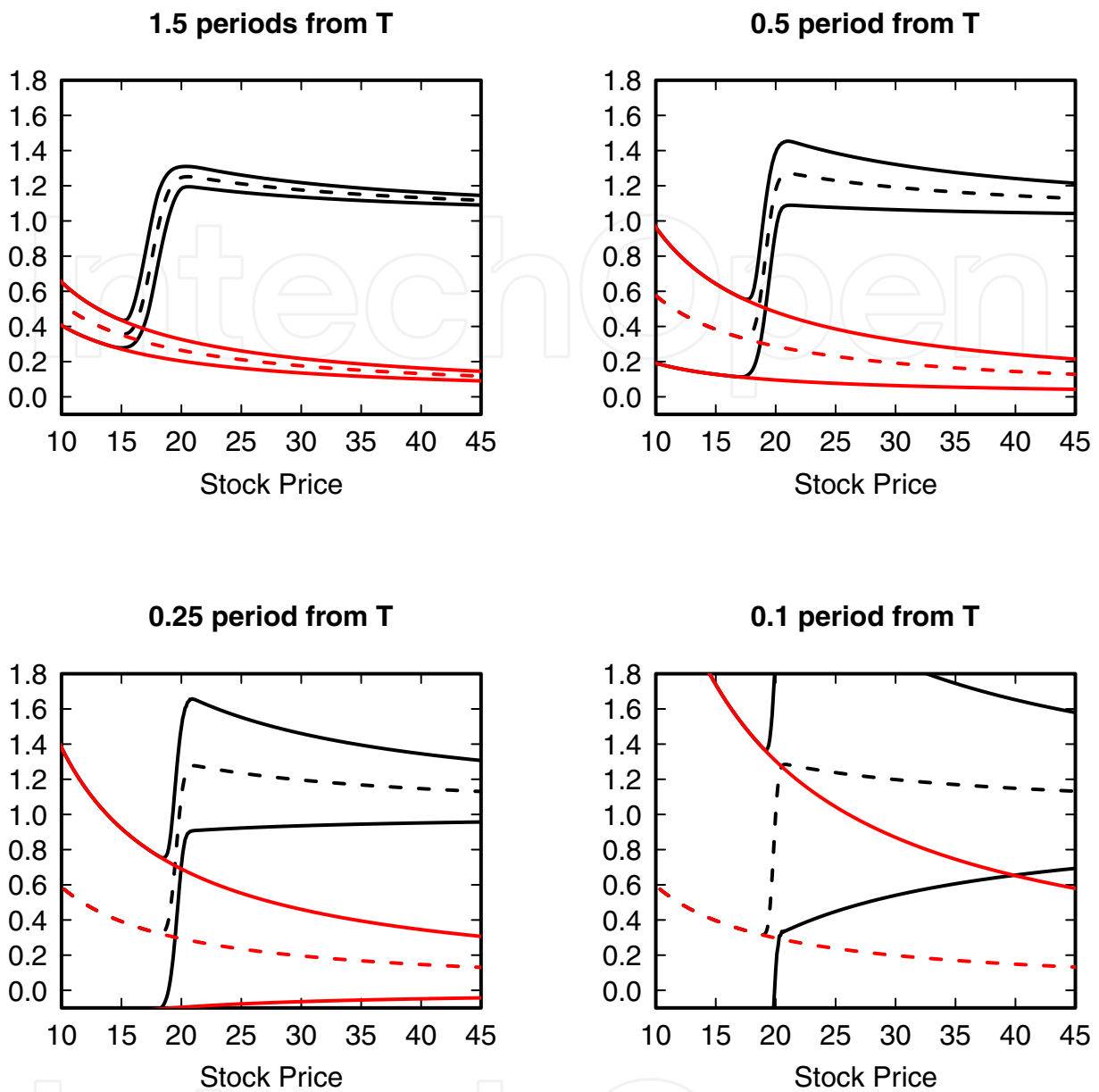
Figure 1. Optimal buy (lower) and sell (upper) boundaries from negative exponential (CARA) utility maximization for a short asset-settled call with strike price $K = 20$ (solid black lines) and for the case of no option (solid red lines), with proportional transaction costs incurred at the rate of $\lambda = \mu = 0.5\%$, absolute risk aversion $\gamma = 2.0$, risk-free rate $r = 8.5\%$, asset return rate $\alpha = 10\%$ and asset volatility $\sigma = 5\%$, at 1.5, 0.5, 0.25 and 0.1 period(s) from expiration T. For each pair of boundaries, the "buy asset" region is below the buy boundary and the "sell asset" region is above the sell boundary; the no-transaction region is between the two boundaries. The dashed lines correspond to the case of no transaction costs.

4. Conclusion

For the so-called bounded variation follower problems, the equivalence between singular stochastic control and optimal stopping can be harnessed to provide a much simpler solution to the control problem by solving the corresponding Dynkin game. This approach can be

used on certain control problems for which there does not exist an equivalent stopping problem. We show how the "standard" algorithm can be modified to provide a coupled backward induction algorithm for solving the utility-based option hedging problem and provide numerical illustrations on the vanilla call option.

Author details

Tze Leung Lai
Department of Statistics
Stanford University, U.S.A.

Tiong Wee Lim
Department of Statistics and Applied Probability
National University of Singapore, Republic of Singapore

5. References

[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.

[2] F. Boetius. Bounded variation singular stochastic control and Dynkin game. *SIAM Journal on Control and Optimization*, 44:1289–1321, 2005.

[3] H.J. Kushner and P.G. Dupuis. *Numerical Methods for Stochastic Control in Continuous Time*. Springer-Verlag, New York, 1992.

[4] H. Chernoff. Sequential tests for the mean of a normal distribution. *Annals of Mathematical Statistics*, 36:55–68, 1965.

[5] H. Chernoff and A.J. Petkau. An optimal stopping problem for sums of dichotomous random variables. *Annals of Probability*, 4:875–889, 1976.

[6] H. Chernoff and A.J. Petkau. Numerical solutions for Bayes sequential decision problems. *SIAM Journal on Scientific and Statistical Computing*, 7:46–59, 1986.

[7] I. Karatzas and S.E. Shreve. Connections between optimal stopping and singular stochastic control I. *SIAM Journal on Control and Optimization*, 22:856–877, 1984.

[8] I. Karatzas and S.E. Shreve. Connections between optimal stopping and singular stochastic control II. *SIAM Journal on Control and Optimization*, 23:433–451, 1985.

[9] I. Karatzas and H. Wang. Connections between bounded-variation control and Dynkin games. In J.L. Menaldi, E. Rofman, and A. Sulem, editors, *Optimal Control and Partial Differential Equations—Innovations and Applications*, pages 353–362. IOS Press, Amsterdam, 2001.

[10] J.A. Bather and H. Chernoff. Sequential decisions in the control of a spaceship. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Volume 3—Physical Sciences*, pages 181–207, Berkeley, 1967. University of California Press.

[11] L. Clewlow and S.D. Hodges. Optimal delta-hedging under transaction costs. *Journal of Economic Dynamics and Control*, 21:1353–1376, 1997.

[12] M.J.P. Magill and G.M. Constantinides. Portfolio selection with transaction costs. *Journal of Economic Theory*, 13:245–263, 1976.

[13] M.H.A. Davis and A.R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15:676–713, 1990.

[14] M.H.A. Davis, V.G. Panas, and T. Zariphopoulou. European option pricing with transaction costs. *SIAM Journal on Control and Optimization*, 31:470–493, 1993.

[15] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8:222–239, 1989.

[16] T.L. Lai, T.W. Lim, and K. Ross. A new approach to solving multi-dimensional singular stochastic control problems with applications to investment theories and queueing networks. Working paper, Stanford University, 2012.

[17] T.L. Lai, Y.-C. Yao, and F. AitSahlia. Corrected random wall approximations to free boundary problems in optimal stopping. *Advances in Applied Probability*, 39:753–775, 2007.

[18] T.L. Lai and T.W. Lim. Option hedging theory under transaction costs. *Journal of Economic Dynamics and Control*, 33:1945–1961, 2009.

[19] T.L. Lai, T.W. Lim, and W. Zhou. Backward induction algorithms for singular stochastic control problems associated with transaction costs. Working paper, Stanford University, 2012.

[20] V.I. Zakamouline. European option pricing and hedging with both fixed and proportional transaction costs. *Journal of Economic Dynamics and Control*, 30:1–25, 2006.

[21] X. Guo and P. Tomecek. Connections between singular control and optimal switching. *SIAM Journal on Control and Optimization*, 47:421–443, 2008.