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Microwave Open Resonator Techniques – Part I: Theory

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1. Introduction

This chapter is devoted to the theory of open resonators. It is well known that lasers uses open resonators as an oscillatory system. In the simplest case, this consists of two mirrors facing each other. This is the first but not the only application of open resonators, whose salient features consist in the fact that their dimensions are much larger than the wavelength and the spectrum of their eigenvalues is much less dense than that of close cavity.

The origin of Open Resonators can be dated to the beginning of the twentieth century when two French physicists developed the classical Fabry-Perot interferometer or Etalon [1]. This novel form of interference device was based on multiple reflection of waves between two closely spaced and highly reflecting mirrors.

In [2] and [3] a theory was developed for resonators with spherical mirrors and approximated the modes by wave beams. The concept of electromagnetic wave beams was also introduced in [5, 12] where was investigated the sequence of lens for the guided transmission of electromagnetic waves.

The use of open resonators either in the microwave region, or at higher frequencies (optical region) has taken place over a number of decades. The related theory and its applications have found a widespread use in several branches of optical physics and today is incorporated in many scientific instruments [6].

In microwave region open resonators have also been proposed as cavities for quasi-optical gyrotrons[16] and as an open cavity in a plasma beat wave accelerator experiment [9].

The use of Open Resonators as microwave Gaussian Beam Antennas [10, 11, 18] provides a very interesting solution as they can provide very low sidelobes level. They are based on the result that the field map at the mid section of an open resonator shows a gaussian distribution that can be used to illuminate a metallic grid or a dielectric sheet.

For microwave applications a reliable description of the coupling between the cavity and the feeding waveguide is necessary. Several papers deal with the coupling through a small hole or a rectangular waveguide taking into account only for the fundamental cavity and waveguide mode [7, 8, 17].

In [4] a complete analysis of the coupling between a rectangular waveguide and an open cavity has been developed taking into account for all the relevant eigenfunctions in the waveguide and in the cavity.

In this paper we review the general theory of Open Resonators and propose to study the coupling between a feeding coupling aperture given by a rectangular or circular waveguide.

Starting from the paraxial approximation of the wave equation, we derive the modal expansion of the field into the cavity taking into account for the proper coordinate system. The computation of the modal coefficients takes into account for the characteristics of the mirrors, the ohmic and diffraction losses and coupling.

2. Open resonator theory

2.1. Parabolic approximation to wave equation

A parabolic equation was first introduced into the analysis of electromagnetic wave propagation in [13] and [14]. Since then, it has been used in diffraction theory to obtain approximate (asymptotic) solutions when the wavelength is small compared to all characteristic dimensions. As open resonators usually satisfy this condition, the parabolic equation finds wide application in developing a theory of open resonators.

A rectangular field component of a coherent wave satisfies the scalar wave equation:

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

where $k = 2\pi/\lambda$ is the propagation constant in the medium.

For a wave traveling in the zeta forward direction, assuming an $e^{j\omega t}$ time dependance, we put:

$$u = \psi(x, y, z)e^{-jkz} \tag{2}$$

where ψ is a slowly varying function which represents the deviation from a plane wave. By inserting (2) into (1) and assuming that ψ varies so slowly with z that its second derivative can be neglected with respect to $\left| k\frac{\partial\psi}{\partial z} \right|$, one obtains the well know parabolic approximation to the wave equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2jk\frac{\partial\psi}{\partial z} = 0 \tag{3}$$

The differential equation (3), similar to the Schrodinger equation, has solution of the type:

$$\psi = e^{-j\left(P + \frac{k}{2q}r^2\right)} \tag{4}$$

where:

$$r^2 = x^2 + y^2 \tag{5}$$

The parameter $P(z)$ represents a complex phase shift associated to the propagation of the beam along the z axis, $q(z)$ is the complex parameter which describe the Gaussian beam intensity with the distance r from the z axis.

The insertion of (4) in (3) gives the relations:

$$\frac{dq}{dz} = 1 \tag{6}$$

$$\frac{dP}{dz} = -\frac{j}{q} \tag{7}$$

The integration of (6) yields:

$$q(z_2) = q(z_1) + z \tag{8}$$

which relates the intensity in the plane z_2 with the intensity in the plane z_1.

A wave with a Gaussian intensity profile, as (4), is one the most important solutions of equation (3) and is often called *fundamental mode*.
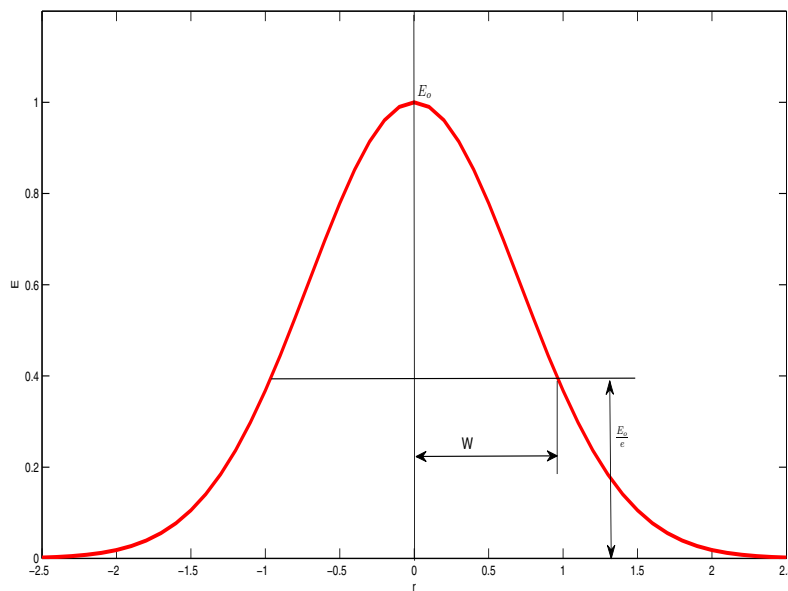

Figure 1. Amplitude distribution of cavity fundamental mode

Two real beam parameters, R and w, are introduced in relation to the above complex parameters q by

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2} \tag{9}$$

Introducing (9) in the solution (4), we obtain:

$$\psi = e^{-j\left(P + \frac{\pi}{\lambda}\frac{r^2}{R}\right)} e^{-\frac{r^2}{w^2}} \tag{10}$$

Now the physical meaning of these two parameters becomes clear:

- R(z) is the curvature radius of the wavefront that intersects the axis at z;

- w(z) is the decrease of the field amplitude with the distance r from the axis.

Parameter w is called *beam radius* and the term 2w *beam diameter*. The Gaussian beam contracts to a minimum diameter $2w_0$ at beam waist where the phase is plane. The beam parameter q at waist is given by:

$$q_0 = j\frac{\pi w_0}{\lambda} \tag{11}$$

and, using (8), at distance z from the waist:

$$q = q_0 + z = j\frac{\pi w_0}{\lambda} + z \tag{12}$$

Combining (12) and (10), we have:

$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right] \tag{13}$$

and

$$w^2(z) = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right] \tag{14}$$

where z_R is the Rayleigh distance:

$$z_R = \frac{\pi w_0^2}{\lambda} \tag{15}$$

The beam contour is an hyperbola with asymptotes inclined to the axis at an angle:

$$\theta = \frac{\lambda}{\pi w_0} \tag{16}$$

In (14) w is the beam radius, w_0 is the minimum beam radius (called beam waist) where one has a plane phase front at $z = 0$ and R is the curvature radius of of the phase front at z. It should be noted that the phase front is not exactly spherical; therefore, its curvature radius is exactly equal to R only on the z-axis. The parameter of the Gaussian beam are illustrated in Fig. 2.
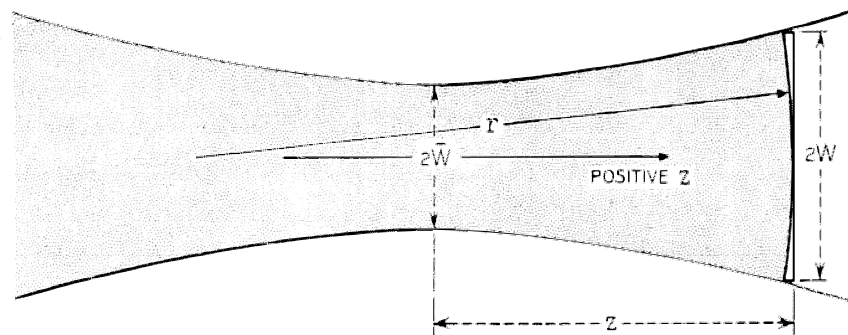

Figure 2. Parameters of Gaussian beam

Dividing (14) by (13), the useful relation is obtained:

$$\frac{\lambda}{\pi w_0^2} = \frac{\pi w^2}{\lambda R} \tag{17}$$

The expression (17) is used to express w_0 and z in terms of w and R:

$$w_0^2 = \frac{w^2}{1 + \left(\frac{\pi w^2}{\lambda R}\right)^2} \tag{18}$$

$$z = \frac{R}{1 + \left(\frac{\lambda R}{\pi w^2}\right)^2} \tag{19}$$

Inserting (11) in (7) we obtain the complex phase shift at distance z from the waist:

$$\frac{dP}{dz} = -\frac{j}{z + j\frac{\pi w_0^2}{\lambda}} \tag{20}$$

Integration of (20) yields

$$jP(z) = \lg\left[1 - j\left(\frac{\lambda z}{\pi w_0^2}\right)\right] = \lg\sqrt{1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2} - j\arctan\left(\frac{\lambda z}{\pi w_0^2}\right) \tag{21}$$

The real part of P represent the phase shift difference Φ between the Gaussian beam and an ideal plane wave, while the imaginary part produces an amplitude factor $\frac{w_0}{w}$ which gives the decrease of intensity due to the expansion of the beam. Now we can write the fundamental Gaussian beam:

$$u(r,z) = \frac{w_0}{w}e^{\left\{-j(kz-\Phi)-r^2\left(\frac{1}{w^2}+\frac{jk}{2R}\right)\right\}} \tag{22}$$

where:

$$\Phi = \arctan\left(\frac{\lambda z}{\pi w_0^2}\right) \tag{23}$$

2.2. Stability of open resonator

A resonator with spherical mirrors of unequal curvature is representable as a periodic sequence of lens which can be stable or unstable. The stability condition assumes the form:

$$0 < \left(1 - \frac{2l}{R_1}\right)\left(1 - \frac{2l}{R_2}\right) < 1 \tag{24}$$

The above expression was previously derived in [3] from geometrical optics approach based on equivalence of the resonator and a periodic sequence of parallel lens and independently in [5] solving the integral equation for the field distribution of the resonant modes in the limit of infinite Fresnel numbers.

To show graphically which type of resonator is stable and which is unstable, it is useful to plot a stability diagram on which each type of resonator type is represented by a point (Fig. 3).
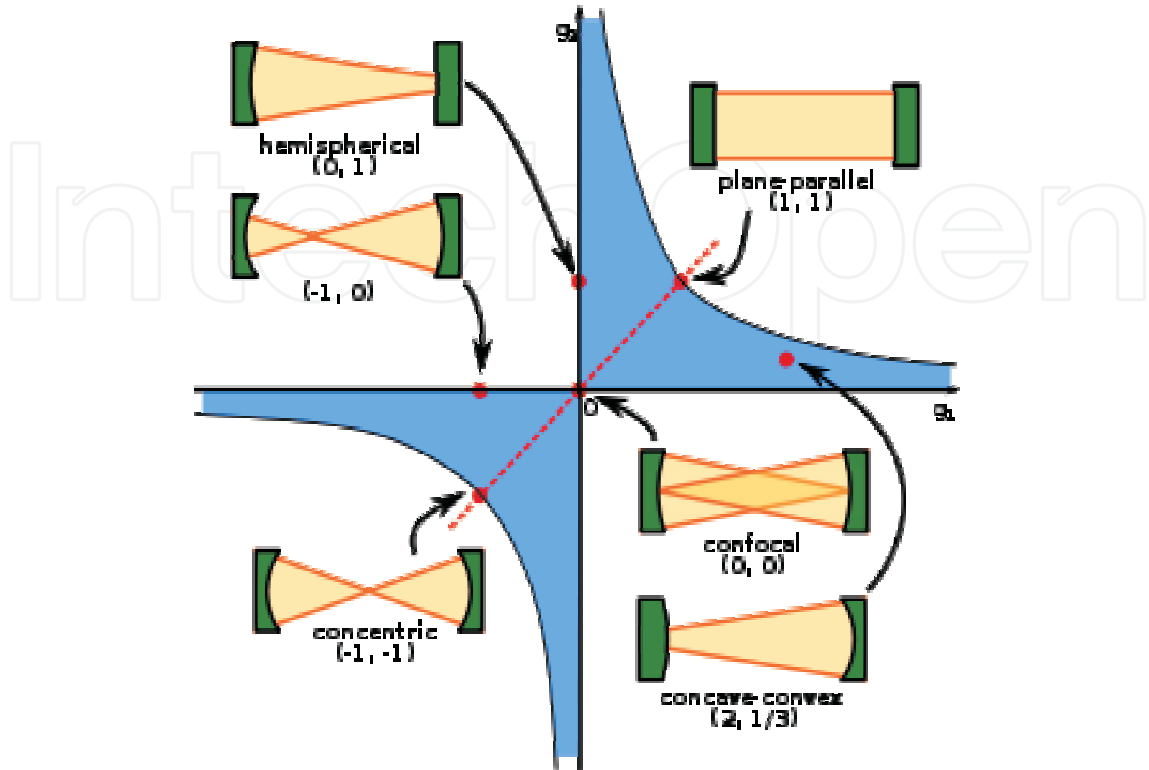

Figure 3. Stability diagram. $g_1 = 1 - \frac{2l}{R_1}$, $g_2 = 1 - \frac{2l}{R_2}$.

2.3. Spherical cavity in cartesian coordinates

In cartesian (x, y, z) coordinates, the separate solutions for (3) are [12]:

$$\psi_{mp}(x, y, z) = \phi_{mp}(x, y, z) exp \left[j(m + p + 1) \tan^{-1} \frac{z}{z_R} - j \frac{\pi}{\lambda} \frac{r^2}{R(z)} \right] \tag{25}$$

where[1]

$$\phi_{mp}(x, y, z) = \frac{1}{w(z)} \sqrt{\frac{2}{\pi 2^{m+p} m! p!}} H_m \left(\sqrt{2} \frac{x}{w} \right) H_p \left(\sqrt{2} \frac{y}{w} \right) exp \left[-\frac{r^2}{w^2(z)} \right] \tag{26}$$

H_m is a Hermite polynomial of order m (Appendix A).

Note that both the ϕ_{mp} and ψ_{mp} functions are orthonormal on the transverse planes z=cost.

When we assume that the mirrors are sufficiently large to permit the total reflection of the gaussian beams of any relevant order, we can put:

$$\Psi_{mpq} = u_{mpq}^{(+)} + u_{mpq}^{(-)} \tag{27}$$

---

[1] To be consistent with the parabolic approximation the condition $|m + n + 1| << (kw_0)^2$ must be satisfied.

where $u_{mpq}^{(+)}$ and $u_{mpq}^{(-)}$ represent Hermite Gauss beams propagating from left to right and from right to left,respectively.

Resonance occurs when the phase shift from one mirror to the other is a multiple of π. Using (2), (4) and (9) this condition leads to:

$$k_{mpq}2l - 2(m + p + 1)\tan^{-1}\left(\frac{l}{z_R}\right) = \pi(q + 1) \tag{28}$$

where q is the number of nodes of the axial standing wave pattern and $2l >> z_R$ the distance between the mirrors (Fig.4).

The fundamental beat frequency Δf_0, i.e. the frequency spacing between successive longitudinal resonances, is given by:

$$\Delta f_0 = \frac{c}{4l} \tag{29}$$

where c is the velocity of light. From (10) the resonant frequency f of a mode can be expressed as:

$$\frac{f_{mpq}}{\Delta f_0} = q + 1 + \frac{1}{\pi}(m + p + 1)\cos^{-1}\left(1 - \frac{2l}{R}\right) \tag{30}$$

The combined use of eqs. (2),(4) and (9) yields:

$$\Psi_{mpq}(x, y, z) = \phi_{mp}(x, y, z)\cos\left[k_{mpq}z - (m + s + 1)\tan^{-1}\frac{z}{z_R} + \frac{\pi}{\lambda}\frac{r^2}{R(z)} + \frac{q\pi}{2}\right] \tag{31}$$
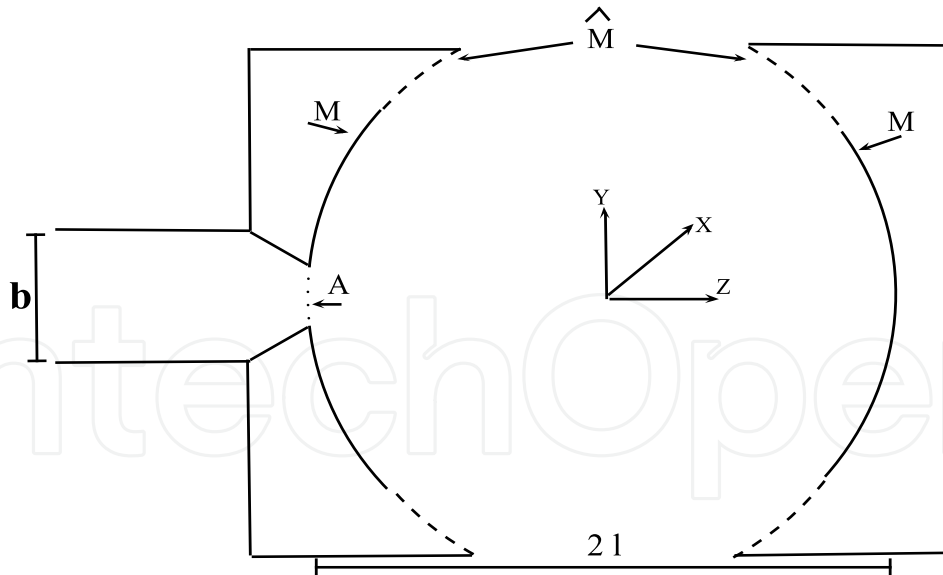

Figure 4. Spherical Open Cavity

In the paraxial approximation the eigenfunctions Ψ_{spq} satisfy the normalization relation

$$\iiint\limits_{cavity} \Psi_{mpq}\Psi_{nst}^* dxdydz = \begin{cases} l, & mpq \equiv nst; \\ 0, & mpq \neq nst. \end{cases} \tag{32}$$

3. Coupling to feeding waveguide

Because the expressions for the solution of scalar wave equation (3) with the boundary condition $\Phi = 0$ on the mirrors is given by (31) the electromagnetic field inside the cavity can be expressed in terms of (quasi) transverse electromagnetic modes (TEM):

$$\mathbf{E} = \sum_n V_n \mathbf{e}_n \tag{33}$$

$$\mathbf{H} = \sum_n I_n \mathbf{h}_n \tag{34}$$

According to the results reported in sect. 2, the expressions for the $\hat{\mathbf{y}}$ polarized modes are:

$$\mathbf{e}_n = \phi_{mp}(x,y,z) \cos\left[k_{mpq}z - (m+p+1)\tan^{-1}\frac{z}{z_R} + \frac{\pi}{\lambda}\frac{r^2}{R(z)} + \frac{q\pi}{2} \right] \hat{\mathbf{y}}$$

$$\mathbf{h}_n = -\phi_{mp}(x,y,z) \sin\left[k_{mpq}z - (m+p+1)\tan^{-1}\frac{z}{z_R} + \frac{\pi}{\lambda}\frac{r^2}{R(z)} + \frac{q\pi}{2} \right] \hat{\mathbf{x}}$$

and similarly for the $\hat{\mathbf{x}}$ polarized ones.

In (33) the index n summarizes the indexes (mpq).

From Maxwell equations we get for the mode vectors [15]:

$$k_n \mathbf{h}_n = \nabla \times \mathbf{e}_n \tag{35}$$

and for the coefficients V_n, I_n

$$I_n = \frac{j\omega\varepsilon_0}{k^2 - k_n^2}\frac{1}{l}\iint_S \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS \tag{36}$$

$$V_n = \frac{k_n}{k^2 - k_n^2}\frac{1}{l}\iint_S \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS = -j\frac{\zeta_0\omega_n}{\omega}I_n \tag{37}$$

where S is the cavity surface, $*$ denotes the complex conjugate and ζ_0 is the free space impedance. Note explicitly that the tangential electric field appearing in expression (36) and (37) is the actual field over S. This is not given by expression (33), which, at variance with expression (34), does not provide a representation for the tangential components uniformly valid up to the cavity boundaries. Let us divide the surface S into three parts: the coupling aperture A, the mirrors M and the (ideal) cavity boundary external to the the mirrors, $\hat{M}$ (see Fig. 4). Hence:

$$I_n = \frac{j\omega\varepsilon_0}{k^2 - k_n^2}\frac{1}{l} \cdot \left\{ \iint_A \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS + \iint_M \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS + \iint_{\hat{M}} \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS \right\} \tag{38}$$

Strictly speaking, in one of the integrals over the mirrors surfaces the coupling aperture should be deleted. However, the error that we make in extending the integration to the whole mirror is negligible provided that the waveguide dimension is much smaller than that of the mirrors.

The Leontovič boundary condition:

$$\hat{\mathbf{n}} \times \mathbf{E} = \frac{1+j}{\sigma\delta} \hat{\mathbf{n}} \times \mathbf{H} \times \hat{\mathbf{n}} \tag{39}$$

wherein σ is the electric conductivity of the mirrors and $\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$ is the penetration depth, can be applied to express the (tangential) electric field over the mirrors in terms of magnetic one, given by (16). Hence:

$$\iint_M \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS = \sum_m I_m \frac{1+j}{\sigma\delta} \iint_M \mathbf{h}_m \cdot \mathbf{h}_n^* dS = 2\frac{1+j}{\sigma\delta} \sum_m \alpha_{nm} I_m \tag{40}$$

being:

$$\alpha_{nm} = \frac{1}{2} \iint_M \mathbf{h}_n^* \cdot \mathbf{h}_m dS \tag{41}$$

Outside the mirrors we can assume that the field is an outgoing locally plane wave (Fresnel-Kirchhoff approximation), so that on $\hat{M}$:

$$\hat{\mathbf{n}} \times \mathbf{E} = \zeta_o \hat{\mathbf{n}} \times \mathbf{H} \times \hat{\mathbf{n}} \tag{42}$$

Accordingly, we have for the last integral in (38):

$$\iint_{\hat{M}} \hat{\mathbf{n}} \times \mathbf{E} \cdot \mathbf{h}_n^* dS = \zeta_o \sum_m I_m \iint_{\hat{M}} \mathbf{h}_m \cdot \mathbf{h}_n^* dS = 2\zeta_o \sum_m I_m (\delta_{nm} - \alpha_{nm}) \tag{43}$$

The last equality in (43) follows from the fact that:

$$\iint_{M \cup \hat{M}} \mathbf{h}_m \cdot \mathbf{h}_n^* dS \simeq \iint_{z=-l} \mathbf{h}_m \cdot \mathbf{h}_n^* dxdy + \iint_{z=l} \mathbf{h}_m \cdot \mathbf{h}_n^* dxdy = 2\delta_{nm} \tag{44}$$

according to the orthogonality condition satisfied by the ϕ functions.

The cavity quality factor Q_n for the n-th mode is defined as:

$$Q_n = \omega_n \frac{W_n}{P_n} \tag{45}$$

where W_n is the mean electromagnetic energy stored in the cavity and P_n is the power lost when only the nth mode is excited at the resonance pulsation ω_n. The power is lost due to diffraction and ohmic losses.

By taking (32) into account, we can express the diagonal term in (40) as a function of the quality factor for the ohmic losses, Q_{rn}:

$$2\alpha_{nn} = \iint_M |\mathbf{h}_n|^2 dS = \frac{\omega_n \mu_0}{Q_{rn}} \sigma \delta_n l \tag{46}$$

being δ_n the skin depth at the resonant frequency.

The diffraction losses of a cavity can be calculated by taking into account for the diffraction effects produced by the finite size of the mirrors. Under the simplifying assumption of

quasi-optic nature of the problem (dimensions of the resonator large compared to wavelength and quasi-transverse electromagnetic fields) the Fresnel-Kirchhoff formulation can be invoked for the diffracted field from the mirrors. Hence we have for the diffraction loss of the nth mode:

$$P_{dn} = Re\left(\iint_{\hat{M}} \frac{1}{2}\mathbf{E}_n \times \mathbf{H}_n^* \cdot \hat{\mathbf{n}}dS\right) = \frac{1}{2}\zeta_0 |I_n|^2 \iint_{\hat{M}} |\mathbf{h}_n|^2 ds \tag{47}$$

and for the corresponding quality factor for the diffraction losses, Q_d,:

$$\frac{\zeta_o}{l}\iint_{\hat{M}} |\mathbf{h}_n|^2 dS = \frac{\omega_n \mu_0}{Q_{dn}} \tag{48}$$

By using (40, 41,44, 45) and taking into account that $\sigma\delta\zeta_0 >> 1$ and $\delta_n/\delta \simeq 1$ for all relevant modes, equation (38) becomes:

$$I_n = \frac{j\omega\varepsilon_0}{\left(k^2 - k_n^2 + \frac{kk_n}{Q_{rn}}\right) - j\frac{kk_n}{Q_{Tn}}} \frac{1}{l}\left\{\iint_A \mathbf{E} \times \mathbf{h}_n^* \cdot \hat{\mathbf{n}}dS - 2\zeta_0 \sum_m' I_m \alpha_{nm}\right\} \tag{49}$$

wherein $\sum' \equiv \sum_{n\neq m}$ and:

$$\frac{1}{Q_{Tn}} = \left(\frac{1}{Q_{dn}} + \frac{1}{Q_{rn}}\right) \tag{50}$$

A metallic waveguide is assumed to feed the cavity. The waveguide field on the coupling aperture A is represented as:

$$\mathbf{E}^g = \sum_n V_n^g \mathbf{e}_n^g \tag{51}$$

$$\mathbf{H}^g = \sum_n I_n^g \mathbf{h}_n^g \tag{52}$$

where $\mathbf{e}^g$ and $\mathbf{h}^g$ are TE electromagnetic modes of the waveguide:

Assuming that the mirror curvature can be neglected over the extension of the coupling aperture, fields (51, 52) verify the following orthonormality relation:

$$\iint_A \mathbf{e}_n^g \times \mathbf{h}_m^g \cdot \hat{\mathbf{z}}dS = \delta_{nm} \tag{53}$$

Expressing the field over the coupling aperture A by means of expression (52) we obtain from (3):

$$I_n + 2F_n \sum_m' \alpha_{nm} I_m = \frac{F_n}{\zeta_0}\sum_m \beta_{nm} V_m^g \tag{54}$$

where:

$$F_n = \frac{jk/l}{\left(k^2 - k_n^2 + \frac{kk_n}{Q_{rn}}\right) - j\frac{kk_n}{Q_{Tn}}} \tag{55}$$

and:

$$\beta_{nm} = \iint_A \mathbf{e}_m^g \times \mathbf{h}_n^* \cdot \hat{\mathbf{n}}dS = -\iint_A \mathbf{h}_n^* \cdot \mathbf{h}_m^g dS. \tag{56}$$

By introducing the matrices $\underline{A}$ and $\underline{B}$, whose elements are:

$$a_{nm} = \begin{cases} \frac{1}{F_n} & \text{n=m} \\ 2\alpha_{nm} & n \neq m. \end{cases} \tag{57}$$

and β_{nm} respectively [2], and the vectors $\underline{I} \equiv \{I_n\}$ and $\underline{V}^g \equiv \{V_n^g\}$, relation (3) can be written in a compact form as:

$$\zeta_0 \underline{\underline{A}} \cdot \underline{I} = \underline{\underline{B}} \cdot \underline{V}^g = \underline{\underline{B}} \cdot \left(\underline{V}^+ + \underline{V}^-\right) \tag{58}$$

wherein $\underline{V}^+$ and $\underline{V}^-$ are the vectors of the incident and reflected waveguide mode amplitudes respectively. By enforcing the continuity of the magnetic field tangential component over the coupling aperture, we get:

$$-\underline{\underline{B}}^+ \cdot \underline{I} = \underline{I}^g = \frac{1}{\zeta_0} \underline{\underline{\zeta}}^{-1} \cdot \left(\underline{V}^+ - \underline{V}^-\right) \tag{59}$$

wherein $\underline{\underline{B}}^+$ is the adjoint (i.e., the transpose, being $\underline{\underline{B}}$ a real matrix) of $\underline{\underline{B}}$ and $\underline{\underline{\zeta}}$ is the diagonal matrix whose elements are the modes characteristic impedances, normalized to ζ_0. From (48) and (49) we immediately obtain:

$$\left(\underline{\underline{I}} - \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}} \cdot \underline{\underline{\zeta}} \cdot \underline{\underline{B}}^+\right) \cdot \underline{I} = \frac{2}{\zeta_0} \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}} \cdot \underline{V}^+ \tag{60}$$

$$\left(\underline{\underline{I}} - \underline{\underline{\zeta}} \cdot \underline{\underline{B}}^+ \cdot \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}}\right) \cdot \underline{V}^- = \left(\underline{\underline{I}} + \underline{\underline{\zeta}} \cdot \underline{\underline{B}}^+ \cdot \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}}\right) \cdot \underline{V}^+ \tag{61}$$

wherein $\underline{\underline{I}}$ is the unit matrix and $\underline{\underline{A}}^{-1}$ the inverse of the matrix $\underline{\underline{A}}$.

Solution of eq. (60) and (61) provides the answer to our problem. In particular, from (61) we get the (formal) expression for the feeding waveguide scattering matrix $\mathcal{S}$:

$$\underline{\underline{S}} = \left(\underline{\underline{I}} - \underline{\underline{\zeta}} \cdot \underline{\underline{B}}^+ \cdot \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}}\right)^{-1} \cdot \left(\underline{\underline{I}} + \underline{\underline{\zeta}} \cdot \underline{\underline{B}}^+ \cdot \underline{\underline{A}}^{-1} \cdot \underline{\underline{B}}\right) \tag{62}$$

4. Field on the coupling aperture

In order to solve the system (60-61) a suitable description of the field on the aperture A is necessary. Any device, able to support electromagnetic field matching the cavity field on the mirror (33-34), can be used to feed the cavity. In the following, the case of metallic and circular waveguide will be treated in detail.

4.1. Modes in rectangular waveguide

A rectangular metallic waveguide, with transverse dimensions $a \times b$, is assumed to feed the cavity. The waveguide TE electromagnetic modes of the metallic rectangular waveguide, on the coupling aperture A, is represented as:

$$\mathbf{h}_n = \mathbf{h}_{pq} = \frac{1}{k_{tpq}} \sqrt{\frac{4\varepsilon_p \varepsilon_q}{ab}} \cdot \tag{63}$$

$$\left\{ \frac{p\pi}{a} \sin \frac{p\pi}{a}\left(x + \frac{a}{2}\right) \cos \frac{q\pi}{b}\left(y + \frac{b}{2}\right) \hat{\mathbf{x}} + \frac{q\pi}{b} \cos \frac{p\pi}{a}\left(x + \frac{a}{2}\right) \sin \frac{q\pi}{b}\left(y + \frac{b}{2}\right) \hat{\mathbf{y}} \right\}$$

---

[2] Note explicitly that α_{nm} and β_{nm} are real quantities, as both the cavity and waveguide mode vectors are real. Moreover $\alpha_{nm} = \alpha_{mn}$ so that the matrix $\underline{\underline{A}}$ is symmetric.

$$\mathbf{e}_{pq} = \mathbf{h}_{pq} \times \hat{\mathbf{z}} \tag{64}$$

$$\varepsilon_p = \begin{cases} 1, & p \neq 0; \\ \frac{1}{2}, & p = 0. \end{cases} \tag{65}$$

$$k_{tpq}^2 = \left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{b}\right)^2 \tag{66}$$

and the TM electromagnetic modes:

$$\mathbf{e}_n = \mathbf{e}_{pq} = -\frac{1}{k_{tpq}}\sqrt{\frac{4\varepsilon_p\varepsilon_q}{ab}} \cdot \tag{67}$$

$$\left\{ \frac{p\pi}{a}\cos\frac{p\pi}{a}\left(x+\frac{a}{2}\right)\sin\frac{q\pi}{b}\left(y+\frac{b}{2}\right)\hat{\mathbf{x}} + \frac{q\pi}{b}\sin\frac{p\pi}{a}\left(x+\frac{a}{2}\right)\cos\frac{q\pi}{b}\left(y+\frac{b}{2}\right)\hat{\mathbf{y}} \right\}$$

$$\mathbf{h}_{pq} = \hat{\mathbf{z}} \times \mathbf{e}_{pq} \tag{68}$$

Note explicitly that in expressions (63-68) the index n summarizes the indexes (pq).

4.2. Modes in circular waveguide

When a circular waveguide , with radius a, is assumed to feed the cavity, the TE electromagnetic modes are:

$$\mathbf{h}_n = \mathbf{h}_{pr} = -\sqrt{\frac{\xi_p}{\pi}}\frac{1}{\sqrt{q_{pr}'^2 - p^2}}\frac{1}{J_p(q_{pr}')} \cdot$$

$$\left\{ \left[\frac{q_{pr}'}{a}J_p'(k_{tpr}'\rho)\cos\phi \begin{Bmatrix} \cos(p\phi) \\ \sin(p\phi) \end{Bmatrix} - \frac{p}{\rho}J_p(k_{tpr}'\rho)\sin\phi \begin{Bmatrix} \sin(p\phi) \\ -\cos(p\phi) \end{Bmatrix} \right]\hat{\mathbf{x}} \right.$$

$$\left. + \left[\frac{q_{pr}'}{a}J_p'(k_{tpr}'\rho)\sin\phi \begin{Bmatrix} \cos(p\phi) \\ \sin(p\phi) \end{Bmatrix} + \frac{p}{\rho}J_p(k_{tpr}'\rho)\cos\phi \begin{Bmatrix} \sin(p\phi) \\ -\cos(p\phi) \end{Bmatrix} \right]\hat{\mathbf{y}} \right\} \tag{69}$$

$$\mathbf{e}_{pr} = \mathbf{h}_{pr} \times \hat{\mathbf{z}} \tag{70}$$

and the TM electromagnetic modes:

$$\mathbf{e}_n = \mathbf{e}_{pr} = -\sqrt{\frac{\xi_p}{\pi}}\frac{1}{J_{p+1}'(q_{pr})} \cdot$$

$$\left\{ \left[\frac{1}{a}J_p'(k_{tpr}\rho)\cos\phi \begin{Bmatrix} \cos(p\phi) \\ \sin(p\phi) \end{Bmatrix} + \frac{p}{q_{pr}\rho}J_p(k_{tpr}\rho)\sin\phi \begin{Bmatrix} \sin(p\phi) \\ -\cos(p\phi) \end{Bmatrix} \right]\hat{\mathbf{x}} \right.$$

$$\left. + \left[\frac{1}{a}J_p'(k_{tpr}\rho)\sin\phi \begin{Bmatrix} \cos(p\phi) \\ \sin(p\phi) \end{Bmatrix} - \frac{p}{q_{pr}\rho}J_p(k_{tpr}\rho)\cos\phi \begin{Bmatrix} \sin(p\phi) \\ -\cos(p\phi) \end{Bmatrix} \right]\hat{\mathbf{y}} \right\} \tag{71}$$

$$\mathbf{h}_{pr} = \hat{\mathbf{z}} \times \mathbf{h}_{pr} \tag{72}$$

where:

$$k_{tpr}^2 = \left(\frac{q_{pr}}{a}\right)^2; \quad k_{tpr}'^2 = \left(\frac{q_{pr}'}{a}\right)^2 \tag{73}$$

q_{or} is the r-mo zero of Bessel function of order p and q'_{or} r-mo zero of the derivative of Bessel function of order p.

$$\xi_p = \begin{cases} 1, \ p = 0; \\ 2, \ p \neq 0. \end{cases} \tag{74}$$

Note explicitly that in expressions (69, 71) the index n summarizes the indexes (pr).

5. Equivalent circuit

Let us consider systems (60) and (61) under the following assumptions:

1. Negligible intercoupling between cavity modes, i.e. :

$$(\underline{\underline{A}})_{pq} = \frac{1}{F_p}\delta_{pq} \iff (\underline{\underline{A}}^{-1})_{pq} = F_p\delta_{pq} \tag{75}$$

2. Single cavity mode approximation, i.e., -see (3)-:

$$F_p = \delta_{p0}F_0 \tag{76}$$

3. Beam diameter at the mirror much larger than the waveguide dimension, i.e. :

$$w = w(l) \gg a \tag{77}$$

Putting

$$V_n^g = V_1^+\delta_{1n} + V_n^- \tag{78}$$

and taking (75-76) into account, equations (60-61) became

$$-\zeta_0\zeta_n\beta_{0n}I_0 = V_1^+\delta_{0n} - V_n^- \tag{79}$$

$$\left(1 - F_0\sum_n \beta_{0n}^2\zeta_n\right)I_0 = \frac{2F_0}{\zeta_0}\beta_{01}V_1^+ \tag{80}$$

From (79,80) we immediately get:

$$V_n^- = \left(\delta_{1n} + \frac{2F_0\zeta_n\frac{\beta_{0n}}{\beta_{01}}}{1 - F_0\sum_k \beta_{0k}^2\zeta_k}\right)V_1^+ \tag{81}$$

Hence

$$\Gamma = \frac{V_1^-}{V_1^+} = \frac{1 + F_0\zeta_1\beta_{01}^2 - F_0\sum_{k\neq1}\zeta_k\beta_{0k}^2}{1 - F_0\sum_k\zeta_k\beta_{0k}^2} \tag{82}$$

From (82) we get for the equivalent terminal impedance relative to the fundamental mode:

$$Z = \zeta_0\zeta_1\frac{1+\Gamma}{1-\Gamma} = -\frac{\zeta_0}{\beta_{01}^2 F_0} + \zeta_0\sum_{k\neq1}\left(\frac{\beta_{0k}}{\beta_{01}}\right)^2\zeta_k \tag{83}$$

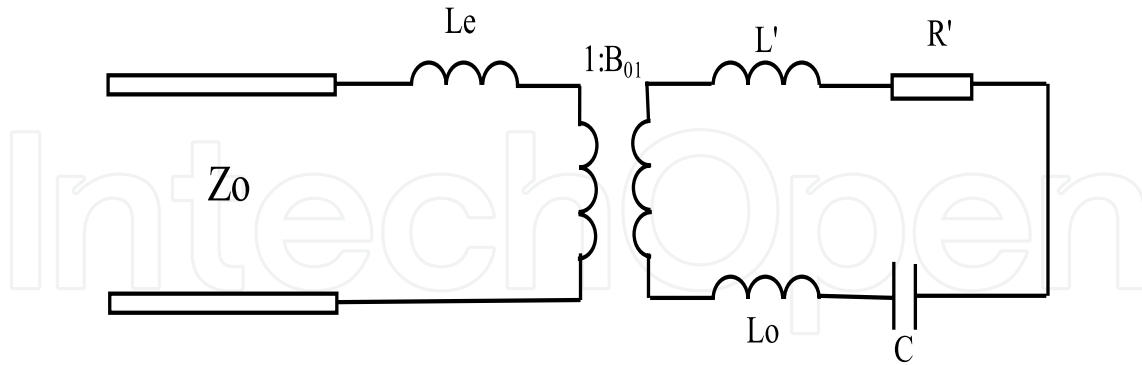Taking into account for the expression (55) for F_0, we get the equivalent circuit representation of Fig. 5.

Figure 5. Equivalent Circuit

where the explicit expression for its elements are collected under Table 1.

L_0	$\mu_0 l$	(H)	C	$\varepsilon l/(k_0 l)^2$ (F)
L'	$\mu_0 \delta$	(H)	R'	$2/\sigma\delta$ (Ω)

Table 1. Expressions for the circuit elements of Fig.5

The value of L_e depends on the feeding waveguide and is reported in the following subsections for rectangular and circular waveguides.

5.1. Rectangular waveguide

The cavity is assumed to be excited by the incident fundamental TE_{10} mode. From expressions (51-52) and (63) follows that in the cavity are excited only the (0,0,q) mode, in the feeding waveguide are excited the modes TE_{n0}.

When the cavity is fed by a rectangular waveguide under the approximation 3), the expression (56) for β_{0k} can be explicitly evaluated, leading to:

$$\beta_{0n} = \frac{4}{\pi}\frac{1}{n}\sqrt{\frac{ab}{w^2}} \quad n = 1, 3, \cdots \tag{84}$$

Hence, for the sum at the right hand side of (83) we have:

$$\zeta_0 \sum_{k=3,5,\cdots} \left(\frac{\beta_{0k}}{\beta_{01}}\right)^2 \zeta_k = j\zeta_0 \sum_{k=0}^{\infty} \frac{1}{(2k+3)^2} \frac{1}{\sqrt{\left(\frac{2k+3}{2a}\lambda\right)^2 - 1}} \simeq$$

$$\simeq j\zeta_0 \left(\frac{2a}{\lambda}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+3)^3} = j\omega\mu a \frac{1}{\pi}\left[\left(1-\frac{1}{8}\right)\zeta(3) - 1\right] = j\omega L_e \tag{85}$$

wherein $\zeta(\cdot)$ denotes the Rieman zeta function. From equation (85), we have the value of L_e:

$$L_e = 16.5 \; 10^{-3}\mu_0 \, a \; [henry] \tag{86}$$

5.2. Circular waveguide

The cavity is assumed to be excited by the incident fundamental TE_{11} mode. From expressions (51, 52) and (69) follows that in the cavity are excited the $(0,0,q)$ mode, in the feeding waveguide are excited the modes TE_{1r}.

When the cavity is fed by a circular waveguide under the approximation 3), the expression (56) for β_{0k} can be evaluated (Appendix A), leading to:

$$\beta_{0r} = -\frac{2}{\pi} \frac{1}{\sqrt{q_{pr}'^2 - p^2}} \frac{1}{J_p(q_{pr}')} \frac{1}{w(l)} \cdot [I_1 - I_2] \tag{87}$$

$$I_1 = \pi \frac{q_{pr}'}{a} \int_0^a J_1'(k_{t1r}'\rho) e^{\left[-\frac{\rho^2}{w^2(l)}\right]} \sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right] \rho\, d\rho \tag{88}$$

$$I_2 = \pi \int_0^a J_1(k_{t1r}'\rho) e^{\left[-\frac{\rho^2}{w^2(l)}\right]} \sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right] d\rho \tag{89}$$

allowing the computation of L_e.

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Appendix

A. Hermite polynomials

The Hermite polynomials are defined as:

$$H_n(x) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2} \tag{90}$$

The differential equation:

$$\frac{\partial^2 y}{\partial x^2} - 2x\frac{\partial y}{\partial x} + 2y = 0 \tag{91}$$

admits as solution the Hermite polynomial $H_n(x)$. For the the Hermite polynomials the following orthogonal relation holds:

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} H_n(x)H_m(x) = \begin{cases} \sqrt{\pi}2^n n!, & \text{n=m}; \\ 0, & \text{n} \neq \text{m}. \end{cases} \tag{92}$$

In the following some particular values with the recursion relation are reported:

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$--- \quad -------$$
$$H_{n+1} = 2xH_n(x) - 2nH_{n-1}(x)$$

B. Coupling cavity - circular waveguide

The x component of the magnetic field, for TE modes, in the waveguide is:

$$\mathbf{h}_{\mathbf{x}pr}^g = -\sqrt{\frac{\xi_p}{\pi}} \frac{1}{\sqrt{q_{pr}'^2 - p^2}} \frac{1}{J_p(q_{pr}')} \cdot \tag{93}$$

$$\left[\frac{q_{pr}'}{a} J_p'(k_{tpr}'\rho) \cos\phi \cos(p\phi) - \frac{p}{\rho} J_p(k_{tpr}'\rho) \sin\phi \sin(p\phi) \right]$$

The x component of the magnetic field in the cavity is:

$$\mathbf{h}_{\mathbf{x}n} = -\phi_{mp}(x,y,z) \sin\left[k_{mpq}z - (m+p+1)\tan^{-1}\frac{z}{z_R} + \frac{\pi}{\lambda}\frac{r^2}{R(z)} + \frac{q\pi}{2} \right] \tag{94}$$

where

$$\phi_{mp}(x,y,z) = \frac{1}{w(z)} \sqrt{\frac{2}{\pi 2^{m+p} m! p|}} H_m\left(\sqrt{2}\frac{x}{w}\right) H_p\left(\sqrt{2}\frac{y}{w}\right) exp\left[-\frac{r^2}{w^2(z)} \right] \tag{95}$$

According to the position of sect.4 we consider the mode (0,0,q) in the cavity, so the equation (95) reduces to:

$$\phi_{00}(x,y,z) = \frac{1}{w(z)} \sqrt{\frac{2}{\pi}} exp\left[-\frac{r^2}{w^2(z)} \right] \tag{96}$$

and the equation (94) on the mirror (for z=l):

$$\mathbf{h}_{\mathbf{x}n} = -\frac{1}{w(l)} \sqrt{\frac{2}{\pi}} e^{\left[-\frac{r^2}{w^2(l)} \right]} \sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{r^2}{R(l)} + \frac{q\pi}{2} \right] \tag{97}$$

$$\beta_{0m} = -\iint_A \mathbf{h}_0^* \cdot \mathbf{h}_m^g dS \tag{98}$$

$$= -\frac{2}{\pi} \frac{1}{\sqrt{q_{pr}'^2 - p^2}} \frac{1}{J_p(q_{pr}')} \frac{1}{w(l)} \cdot \tag{99}$$

$$\iint_A \left[\frac{q_{pr}'}{a} J_p'(k_{tpr}'\rho) \cos\phi \cos(p\phi) - \frac{p}{\rho} J_p(k_{tpr}'\rho) \sin\phi \sin(p\phi) \right] \cdot \tag{100}$$

$$e^{\left[-\frac{\rho^2}{w^2(l)} \right]} \sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2} \right] \rho d\rho d\phi \tag{101}$$

$$\beta_{0r} = -\frac{2}{\pi}\frac{1}{\sqrt{q_{pr}'^2 - p^2}}\frac{1}{J_p(q_{pr}')}\frac{1}{w(l)} \cdot [I_1 - I_2] \qquad (102)$$

where:

$$I_1 = p\frac{\sin(2p\pi)}{p^2-1}\frac{q_{pr}'}{a}\int_0^a J_p'(k_{tpr}'\rho)e^{\left[-\frac{\rho^2}{w^2(l)}\right]}\sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right]\rho d\rho \quad (103)$$

$$I_2 = \frac{\sin(2p\pi)}{p^2-1}\int_0^a \frac{p}{\rho}J_p(k_{tpr}'\rho)e^{\left[-\frac{\rho^2}{w^2(l)}\right]}\sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right]\rho d\rho \quad (104)$$

that are not equal to zero for p=1, giving:

$$I_1 = \pi\frac{q_{pr}'}{a}\int_0^a J_1'(k_{t1r}'\rho)e^{\left[-\frac{\rho^2}{w^2(l)}\right]}\sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right]\rho d\rho \qquad (105)$$

$$I_2 = \pi\int_0^a J_1(k_{t1r}'\rho)e^{\left[-\frac{\rho^2}{w^2(l)}\right]}\sin\left[k_{00q}l - \tan^{-1}\frac{l}{z_R} + \frac{\pi}{\lambda}\frac{\rho^2}{R(l)} + \frac{q\pi}{2}\right]d\rho \qquad (106)$$

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