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Output Tracking Control for Fuzzy Systems via Static-Output Feedback Design

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1. Introduction

Over the past decades, many advances have been made in the field of control theory which rely on state-space theory. The control design methodology that has been most investigated for the state-feedback control, see for example [1, 2] and the references therein. The state-feedback control design supposes that all the system states are available, which is not always possible in realistic applications. Instead, one has to deal with the absence of full-state information by using observers. From the control point of view, observers can be used as part of dynamical controllers. This observer-based design has been extensively studied in the literature [3, 4]. However, it leads to high-order controllers. As a matter of fact, one has to solve a large problem, which increases numerical computations for large scale systems. Other difficulties may arise, if we consider additional performances, such as disturbance rejection, time delays, uncertainties, etc. Hence, it is more suitable to develop methodologies which involve a design with a low dimensionality. In this context, intensive efforts have been devoted to design low-order controllers [3, 5–7]. In particular, it has been shown that designing reduced order stabilizing controllers can be cast as a static output-feedback stabilization problem. Also, it is recognized that, in general, the static output-feedback control design may not exist for certain systems. Note that an important advantage of these controllers is that they are easy to implement without significant numerical burden.

In general, the synthesis of static output-feedback stabilizing controllers is known to be a hard task [5–7]. The main difficulty rises from its nonconvexity. In the literature, some convexification techniques and iterative algorithms have been proposed to handle this problem [3, 5, 7]. A comprehensive survey on static output-feedback stabilization can be found in [6]. The authors show that despite the considerable efforts devoted to solve this problem, there is yet no methodology that can solve it exactly, so it is still an important open topic. However, it has been shown that for SISO (Single-Input Single-Output) systems, this problem can be solved exactly based on an algebraic characterization [8, 9]. Unfortunately, these approaches are valid only for SISO case and cannot be used to take into account additional constraints on the system. In any case, the investigation of this topic within the field of fuzzy control is continuously increasing and leading to many approaches. A most

efficient approach is based on the Linear Matrix Inequality (LMI) technique: see for example [10–12]. Indeed, since the developed interior-point methods [13], LMIs can be solved in polynomial-time, using numerical algorithms [14]. Recently, other approach, based on a projective algorithm has been proposed [15]. Notice that the existing LMI tools have opened an important research area in system and control theory and tackled numerous unsolved problems [14]. Therefore, our main focus in this chapter is the design of static output-feedback controllers using LMI theory for a class of nonlinear systems described by Takagi-Sugeno (T-S) fuzzy models.

Recently, the study of T-S fuzzy models has attracted the attention of many of researchers: see [16] and references therein. Fuzzy models have local dynamics (i.e., dynamics in different state space regions), that are represented by local linear systems. The overall model of a fuzzy system is then obtained by interpolating these linear models through nonlinear fuzzy membership functions. Unlike conventional modeling techniques, which use a single model to describe the global behavior of a nonlinear system, fuzzy modeling is essentially a multi-model approach, in which simple local linear submodels are designed in the form of a convex combination of local models in order to describe the global behavior of the nonlinear system. This kind of models has proved to be a good representation for a certain class of nonlinear dynamic systems.

Since the work by [17] on stability analysis and state feedback stabilization for fuzzy systems, the Parallel Distributed Compensation (PDC) procedure has extensively been used for the control of such systems: for more details see [16]. The basic idea of this procedure is to design a feedback gain for each local model, and then to construct a global controller from these local gains, so that the global stability of the overall fuzzy system can be guaranteed. The most interesting of this concept is that the obtained stability conditions do not depend on the nonlinearities (membership functions), so that this makes possible to use linear system techniques for nonlinear control design.

Up to now, the stabilization control design for T-S systems is successfully investigated based on state-feedback or static/dynamic output-feedback [18, 19]. However, the design of a controller which guarantees an adequate tracking performance for finite-dimensional systems is more general problem than the stabilization one, and is still attract considerable attentions due to demand from practical dynamical processes in electric, mechanics, agriculture, ... One of our main interest in this chapter is solving the static output-feedback tracking problem. Due to the fact that the T-S fuzzy models aggregate a set of local linear subsystems, blended together through nonlinear scalar functions, the static output-feedback control problem can be very complicated to solve. With regard to the literature of fuzzy control, a few recent approaches have dealt with the tracking control design problem for nonlinear systems described by T-S fuzzy model. Generally speaking, the incorporation of linearization techniques and adaptive schemes usually needs system's perfect knowledge and leads to complicated adaptation control laws. In [20], the author has been shown that the use of the feedback linearization strategy [21] may lead to unbounded controllers, since their stability is not guaranteed. To overcome these drawbacks, LMI-based methodologies have been developed for tracking control problem, using observer-based fuzzy controller to deal with the absence of full-state information [22].

In this context, this chapter will tackle the static output-feedback fuzzy tracking control problem, focusing on an H_∞ tracking performance, related to an output tracking error for all bounded references inputs. The presented results are an extension of already published works for the stabilization case [12, 23]. In fact, to solve the nonconvexity problem, inherent

to static output-feedback control synthesis, a cone complementarity formulation [7] for T-S fuzzy systems is used combined with an iterative algorithm. This algorithm has to optimize a linear objective function subject to a set of LMIs in each iteration. Thus, controllers are derived that not only ensure stability of the closed-loop system, but also provide a prescribed level of output tracking error attenuation.

The main contribution of this chapter is the purpose of a simple procedure reflected by an efficient LMI-based iterative algorithm to solve the fuzzy tracking control problem for nonlinear systems described by discrete-time T-S models. Therefore, since the proposed fuzzy tracking controllers have a low-order character, they are suitable for industrial application. Furthermore, this chapter shows an application to a relevant practical problem, in power engineering and drives field, of the proposed design procedure: guaranteeing a good tracking of the output voltage of DC-DC buck converter [24–26].

2. Problem formulation and preliminaries

Consider a nonlinear system which is approximated by a T-S fuzzy model of the following form:

$i^{th}\mathcal{R}ule$: IF $z_1(k)$ is μ_1^i and $\ldots$ and $z_p(k)$ is μ_p^i,

$$\text{THEN} \begin{cases} x(k+1) = (A_i + \Delta A_i(k))x(k) + (B_i + \Delta B_i(k))u(k) + E_i w(k), \\ \\ y(k) = C_i x(k), i = 1, \ldots, N, \end{cases} \tag{1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^{n_u}$ is the input vector, $w(k) \in \Re^{n_w}$ comprises the bounded external disturbances and $y(k) \in \Re^{n_y}$ is the system output. N is the number of IF-THEN rules. $z_1(k), \ldots z_p(k)$ are the premise variables (that comprises states and/or inputs) and μ_j^i ($i = 1, \ldots, N, j = 1, \ldots, p$) are the fuzzy sets. A_i, B_i, C_i and E_i are known constant matrices of appropriate size, $\Delta A_i(k)$, $\Delta B_i(k)$ are unknown matrices representing time-varying parameter uncertainties, and are assumed to be as follows:

$$[\Delta A_i(k) \ \Delta B_i(k)] = [M_1 F(k) N_{1i} \ M_2 F(k) N_{2i}], \ \ i = 1, 2, \ldots, N, \tag{2}$$

where M_i, N_{1i} and N_{2i} are known real constant matrices. $F(k)$ is the uncertainty function that satisfies the classical bounded condition:

$$F(k)^T F(k) \leq I, \quad \forall k. \tag{3}$$

Thus, the global T-S model is an interpolation of all uncertain subsystems through nonlinear functions [16]:

$$\begin{aligned} x(k+1) &= \frac{\displaystyle\sum_{i=1}^{N} \theta_i(z) \left[(A_i + \Delta A_i(k))x(k) + (B_i + \Delta B_i(k))u(k) + E_i w(k) \right]}{\displaystyle\sum_{i=1}^{N} \theta_i(z)}, \\ &= \sum_{i=1}^{N} \alpha_i(z) \left[(A_i + \Delta A_i(k))x(k) + (B_i + \Delta B_i(k))u(k) + E_i w(k) \right], \\ y(k) &= \sum_{i=1}^{N} \alpha_i(z) C_i x(k), \end{aligned} \tag{4}$$

where $\theta_i, i = 1, \ldots, N$, is the membership function corresponding to system rule i, and $\alpha_i(z) = \theta_i(z) / \sum_{i=1}^{N} \theta_i(z)$, fulfills the convex property: $0 \leq \alpha_i(z) \leq 1$ and $\sum_{i=1}^{N} \alpha_i(z) = 1$.

Note that using the so-called sector of nonlinearity approach, a wide number of nonlinear systems can be represented exactly by T-S models in a compact set of the state space. However, with the growing complexity of nonlinear systems, it is useful to take into account the approximations in the dynamical process. Thus, the main objective of the next paragraph is to provide stability conditions that ensure the tracking performance for the uncertain T-S model (4).

3. H_∞ output tracking performance analysis

This section gives sufficient stability conditions which ensure an H_∞ output tracking performance of the uncertain system (4) using a fuzzy Lyapunov function. We recall the following lemma which will be used in this section.

Lemma 3.1. [27] *Let $\mathcal{A}, \mathcal{D}, \mathcal{S}, \mathcal{W}$ and $\mathcal{F}$ be real matrices of appropriate dimension such that $\mathcal{W} > 0$ and $\mathcal{F}\mathcal{F}^T \leq I$. Then, for any scalar $\epsilon > 0$ such that $\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^T > 0$, we have $(\mathcal{A} + \mathcal{D}\mathcal{F}\mathcal{S})^T\mathcal{W}^{-1}(\mathcal{A} + \mathcal{D}\mathcal{F}\mathcal{S}) \leq \mathcal{A}^T(\mathcal{W} - \epsilon\mathcal{D}\mathcal{D}^T)^{-1}\mathcal{A} + \epsilon^{-1}\mathcal{S}^T\mathcal{S}$.*

Suppose that the desired trajectory can be generated by the following reference model as follows:

$$\begin{cases} x_d(k+1) = \mathbb{A}x_d(k) + \mathbb{B}r(k), \\ y_d(k) = \mathbb{C}x_d(k), \end{cases} \tag{5}$$

where, $y_d(k)$ has the same dimension as $y(k)$, $x_d(k)$ and $r(k) \in \Re^{n_r}$ are respectively the reference state and the bounded reference input, $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ are appropriately dimensional constant matrices with $\mathbb{A}$ Hurwitz.

Since we deal with the static output-feedback control design problem, the fuzzy controller can incorporates information from $y(k)$ and $y_d(k)$. Thus, the control law which is based on the classical structure of the Parallel Distributed Compensation (PDC) concept [17, 28] shares the same fuzzy sets as the T-S system and can be given as follows:

$$i^{th}\mathcal{R}ule: \text{IF} \quad z_1(k) \text{ is } \mu_1^i \text{ and } \ldots \text{ and } z_p(k) \text{ is } \mu_p^i,$$

$$\text{THEN} \quad u(k) = K_i(y(k) - y_d(k)), \tag{6}$$

where the the controller gain K_i is to be chosen. The overall static output-feedback control law is thus inferred as:

$$u(k) = \sum_{i=1}^{N} \alpha_i(z)K_i(y(k) - y_d(k)). \tag{7}$$

The advantages of the static output-feedback controller (7), is well discussed in the literature [3], [6]. This fact motivates us to use such type of control law avoiding the complex control schemes with an additional observer.

Combining (4), (5) and (7), the following augmented closed-loop system is obtained

$$\tilde{x}(k+1) = \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z) \left[(G_{1ijs} + G_{2ijs}(k))\tilde{x}(k) + W_i\tilde{w}(k) \right],$$ (8)

where

$$G_{1ijs} = \begin{bmatrix} A_i + B_iK_jC_s & -B_iK_j\mathbb{C} \\ 0 & \mathbb{A} \end{bmatrix},$$

$$G_{2ijs}(k) = \begin{bmatrix} \Delta A_i(k) + \Delta B_i(k)K_jC_s & -\Delta B_i(k)K_j\mathbb{C} \\ 0 & 0 \end{bmatrix},$$ (9)

$$W_i = \begin{bmatrix} E_i & 0 \\ 0 & \mathbb{B} \end{bmatrix}, \tilde{x} = \begin{bmatrix} x(k) \\ x_d(k) \end{bmatrix}, \tilde{w} = \begin{bmatrix} w(k) \\ r(k) \end{bmatrix},$$

Hence, to meet the required tracking performance, the effect of $\tilde{w}(k)$ on the tracking error $y(k) - y_d(k)$ should be attenuated below a desired level in the sense of [29]:

$$\sum_{k=0}^{k_f} (y(k) - y_d(k))^T (y(k) - y_d(k)) \leq \gamma^2 \sum_{k=0}^{k_f} \tilde{w}(k)^T \tilde{w}(k),$$ (10)

$\forall k_f \neq 0$, and $\forall \tilde{w}(k) \in l_2$, k_f is the control final time.

The following theorem shows that H_∞ output tracking performances can be guaranteed if there exist some matrices satisfying certain conditions.

Theorem 3.1. *The augmented closed-loop system in (8) achieves the H_∞ output tracking performance γ, if there exists matrices $P_1 > 0, \ldots, P_N > 0$ and controller gains $K_1, \ldots, K_N$ such that the following conditions hold:*

$$\begin{bmatrix} -P_r^{-1} & 0 & 0 & G_{1ijs} & W_i & \tilde{M} \\ 0 & -\epsilon I & 0 & \tilde{N}_{ijs} & 0 & 0 \\ 0 & 0 & -I & H_i & 0 & 0 \\ G_{1ijs}^T & \tilde{N}_{ijs}^T & H_i^T & -P_i & 0 & 0 \\ W_i^T & 0 & 0 & 0 & -\gamma^2 I & 0 \\ \tilde{M}^T & 0 & 0 & 0 & 0 & -\epsilon^{-1}I \end{bmatrix} < 0, \quad 1 \leq i,j,s,r \leq N,$$ (11)

where

G_{1ijs} *and* W_i *are defined in (9),* $H_i = [C_i \quad - \mathbb{C}]$, $\tilde{M} = \begin{bmatrix} M_1 & M_2 \\ 0 & 0 \end{bmatrix}$ *and* $\tilde{N}_{ijs} =$

$\begin{bmatrix} N_{1i} & 0 \\ N_{2i}K_jC_s & -N_{2i}K_j\mathbb{C} \end{bmatrix}$.

Proof. Consider the following fuzzy Lyapunov function $V(\tilde{x}, k)$ given by

$$V(\tilde{x}, k) = \tilde{x}(k)^T \sum_{i=1}^{N} \alpha_i(z) P_i \tilde{x}(k).$$

The stability of (8) is ensured, under zero initial condition, with guaranteed H_∞ performance (10) if [29]:

$$\Delta V(\tilde{x}, k) + (y(k) - y_d(k))^T (y(k) - y_d(k)) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) < 0 \tag{12}$$

where $\Delta V(\tilde{x}, k)$ is the rate of V along the trajectory:

$$\Delta V(\tilde{x}, k) = V(\tilde{x}(k+1)) - V(\tilde{x}(k)). \tag{13}$$

By substituting (13) in(12), we have:

$$\tilde{x}(k+1)^T P^+ \tilde{x}(k+1) - \tilde{x}(k)^T P_z \tilde{x}(k) + (y(k) - y_d(k))^T (y(k) - y_d(k)) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) < 0 \tag{14}$$

where

$$P_z = \sum_{i=1}^{N} \alpha_i(z) P_i \text{ and } P^+ = \sum_{i=1}^{N} \alpha_i(z(k+1)) P_i.$$

Now, let

$$G_z(k) = \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z) G_{1ijs} + \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z) G_{2ijs}(k),$$

$$W_z = \sum_{i=1}^{N} \alpha_i(z) W_i. \tag{15}$$

Then, the inequality (14) can be rewritten as follows

$$[G_z(k)\tilde{x}(k) + W_z\tilde{w}(k)]^T P^+ [G_z(k)\tilde{x}(k) + W_z\tilde{w}(k)] - \tilde{x}(k)^T P_z \tilde{x}(k) - \gamma^2 \tilde{w}(k)^T \tilde{w}(k) +$$

$$(y(k) - y_d(k))^T (y(k) - y_d(k)) < 0. \tag{16}$$

By consequence, (16) leads to:

$$\begin{bmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{bmatrix}^T (\mathcal{M}_1 - \mathcal{M}_2) \begin{bmatrix} \tilde{x}(k) \\ \tilde{w}(k) \end{bmatrix} < 0, \tag{17}$$

where

$$
\mathcal{M}_1 = \begin{bmatrix} G_z(k)^T P^+ G_z(k) & G_z(k)^T P^+ W_z \\ W_z^T P^+ G_z(k) & W_z^T P^+ W_z \end{bmatrix},
$$

$$
\mathcal{M}_2 = \begin{bmatrix} P_z - H_z^T H_z & 0 \\ 0 & \gamma^2 \end{bmatrix},
$$

$$
H_z = \sum_{i=1}^{N} \alpha_i(z) H_i.
$$

(18)

Thus, to proof (12), it is sufficient to show that

$$
\mathcal{M}_1 - \mathcal{M}_2 < 0.
$$

(19)

The first part of (19) can also be rewritten as

$$
\mathcal{M}_1 - \mathcal{M}_2 = (\tilde{G}_z + \tilde{M} F(k) \mathcal{N}_z)^T P^+ (\tilde{G}_z + \tilde{M} F(k) \mathcal{N}_z),
$$

(20)

where

$$
\tilde{G}_z = \begin{bmatrix} G_{1z} & W_z \end{bmatrix}, G_{1z} = \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z) G_{1ijs},
$$

$$
and \quad \mathcal{N}_z = \begin{bmatrix} \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z) \tilde{N}_{1ijs} & 0 \end{bmatrix}.
$$

(21)

On the other hand, pre- and post-multiplying (11) by $diag\{P_r, I, I, I, I, I\}$ gives

$$
\Gamma_{ijs}^r \equiv \begin{bmatrix} -P_r & 0 & 0 & P_r G_{1ijs} & P_r W_i & P_r \tilde{M} \\ 0 & -\epsilon I & 0 & \tilde{N}_{ijs} & 0 & 0 \\ 0 & 0 & -I & H_i & 0 & 0 \\ G_{1ijs}^T P_r & \tilde{N}_{ijs}^T & H_i^T & -P_i & 0 & 0 \\ W_i^T P_r & 0 & 0 & 0 & -\gamma^2 I & 0 \\ \tilde{M}^T P_r & 0 & 0 & 0 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0, \quad 1 \le i,j,s,r \le N.
$$

(22)

Since $\sum_{i=1}^{N} \alpha_i(z) = \sum_{r=1}^{N} \alpha_r(k+1) = 1$, (22) can be written as

$$\sum_{r=1}^{N} \alpha_r(k+1) \sum_{i,j,s=1}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z)\Gamma_{ijs}^r \equiv \begin{bmatrix} -P^+ & 0 & 0 & P^+G_{1z} & P^+W_z & P^+\tilde{M} \\ 0 & -\epsilon I & 0 & \tilde{N}_z & 0 & 0 \\ 0 & 0 & -I & H_z & 0 & 0 \\ G_{1z}^T P^+ & \tilde{N}_z^T & H_z^T & -P_z & 0 & 0 \\ W_z^T P^+ & 0 & 0 & 0 & -\gamma^2 I & 0 \\ \tilde{M}^T P^+ & 0 & 0 & 0 & 0 & -\epsilon^{-1} I \end{bmatrix} < 0, \text{(23)}$$

Applying Schur complement on (23), it is straightforward to verify that the condition (23) is equivalent to the following inequalities:

$$(P^+\tilde{G}_z)^T \left(P^+ - \epsilon P^+\tilde{M}\tilde{M}^T P^+\right)^{-1} P^+\tilde{G}_z + \epsilon^{-1}\mathcal{N}_z^T\mathcal{N}_z - \mathcal{M}_2 < 0 \text{ and}$$
$$P^+ - \epsilon P^+\tilde{M}\tilde{M}^T P^+ > 0. \tag{24}$$

Using (20), (24) and Lemma 3.1, we have

$$\mathcal{M}_1 - \mathcal{M}_2 = (\tilde{G}_z + \tilde{M}F(k)\mathcal{N}_z)^T P^+(\tilde{G}_z + \tilde{M}F(k)\mathcal{N}_z)$$
$$\leq (P^+\tilde{G}_z)^T \left(P^+ - \epsilon P^+\tilde{M}\tilde{M}^T P^+\right)^{-1} P^+\tilde{G}_z + \epsilon^{-1}\mathcal{N}_z^T\mathcal{N}_z - \mathcal{M}_2 \tag{25}$$
$$< 0.$$

By consequence

$$\sum_{k=0}^{k_f} (y(k) - y_d(k))^T (y(k) - y_d(k)) < \gamma^2 \sum_{k=0}^{k_f} \tilde{w}(k)^T \tilde{w}(k).$$

Hence, H_∞ output tracking performance is achieved with the prescribed attenuation level γ. On the other hand, it follows from (11) and (25) that $\Delta V(\tilde{x}) < 0$ for $\tilde{w}(k) = 0$, which leads that the uncertain system (8) with $\tilde{w}(k) = 0$ is robustly asymptotically stable. □

4. H_∞ fuzzy tracking controller synthesis

In this section, a cone complementary formulation [7] is used to solve the bilinearity involved in (11). The idea is based on converting the conditions (11) to convex and nonconvex parts and then casting them into an optimization problem subject to some LMIs. For this, first recall the following lemma, which generalizes the result of [7].

Lemma 4.1. *[12] Let $P_i \in \Re^{n \times n}$, $Q_i \in \Re^{n \times n}$, $i = 1, \ldots, N$ be any symmetric positive definite matrices, then the following statements are equivalent:*

$(a): P_i Q_i = I, \; i = 1, \ldots, N.$

$(b):$
$$
\begin{cases}
\displaystyle\sum_{i=1}^{N} \mathbf{Tr}(P_i Q_i) = N \times n, \\[3mm]
\begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0, \; 1 \leq i \leq N.
\end{cases}
$$

Using $P_r = Q_r^{-1}$, the stability condition (11) can be rewritten as follows:

$$
\Omega_{ijs}^{r} \equiv \begin{bmatrix}
-Q_r & 0 & 0 & G_{1ijs} & W_i & \tilde{M} \\
0 & -\epsilon I & 0 & \tilde{N}_{ijs} & 0 & 0 \\
0 & 0 & -I & H_i & 0 & 0 \\
G_{1ijs}^{T} & \tilde{N}_{ijs}^{T} & H_i^{T} & -P_i & 0 & 0 \\
W_i^{T} & 0 & 0 & 0 & -\gamma^2 I & 0 \\
\tilde{M}^{T} & 0 & 0 & 0 & 0 & -\epsilon^{-1}I
\end{bmatrix} < 0, \quad 1 \leq i,j,s,r \leq N, \tag{26}
$$

$$
P_r Q_r = I, \; 1 \leq r \leq N. \tag{27}
$$

Before giving the final formulation of the problem in hand, we suggest to relax the LMIs (26) from the point of view number of LMIs to be satisfied, for this, we suggest to use the following lemma.

Lemma 4.2. *[12] Consider the following matrix* $\bar{\mathbf{A}} = \displaystyle\sum_{i,j,s=1}^{N} \alpha_{ijs} \mathbf{A}_{ijs}$, *where* $\alpha_{ijs} = \alpha_i \alpha_j \alpha_s$ *and* $\displaystyle\sum_{i=1}^{N} \alpha_i = 1$. *Then,* $\bar{\mathbf{A}}$ *can be expressed as follows*

$$
\bar{\mathbf{A}} = \sum_{i=1}^{N} \alpha_i^3 \mathbf{A}_{iii} + \sum_{s>j\geq i}^{N} \alpha_{ijs}(\mathbf{A}_{ijs} + \mathbf{A}_{jsi} + \mathbf{A}_{sij}) + \sum_{s\geq j>i}^{N} \alpha_{ijs}(\mathbf{A}_{sji} + \mathbf{A}_{isj} + \mathbf{A}_{jis}),
$$

Moreover,

$$
\sum_{ijs=1}^{N} \alpha_{ijs} = \sum_{i=1}^{N} \alpha_i^3 + 3\sum_{s>j\geq i}^{N} \alpha_{ijs} + 3\sum_{s\geq j>i}^{N} \alpha_{ijs} = 1.
$$

Hence, using Lemma 4.2, (26) can be rewritten as follows:

$$
Y_{iii}^{r} < 0, \; 1 \leq i,r \leq N,
$$

$$
\Phi_{ijs}^{r} \leq 0, \; 1 \leq i \leq j < s \leq N, \; 1 \leq r \leq N, \tag{28}
$$

$$
\Psi_{ijs}^{r} \leq 0, \; 1 \leq i < j \leq s \leq N, \; 1 \leq r \leq N,
$$

where,

$$
\Upsilon_{iii}^r \equiv
\begin{bmatrix}
-Q_r & 0 & 0 & G_{1iii} & W_i & \tilde{M} \\
0 & -\epsilon I & 0 & \tilde{N}_{iii} & 0 & 0 \\
0 & 0 & -I & H_i & 0 & 0 \\
G_{1iii}^T & \tilde{N}_{iii}^T & H_i^T & -P_i & 0 & 0 \\
W_i^T & 0 & 0 & 0 & -\gamma^2 I & 0 \\
\tilde{M}^T & 0 & 0 & 0 & 0 & -\epsilon^{-1}I
\end{bmatrix},
$$

$$
\Phi_{ijs}^r \equiv
\begin{bmatrix}
-3Q_r & 0 & 0 & G_{ijs}+G_{jsi}+G_{sij} & \mathcal{W} & 3\tilde{M} \\
0 & -3\epsilon I & 0 & \tilde{N}_{ijs}+\tilde{N}_{jsi}+\tilde{N}_{sij} & 0 & 0 \\
0 & 0 & -3I & H_i+H_j+H_s & 0 & 0 \\
* & * & * & -(P_i+P_j+P_s) & 0 & 0 \\
* & 0 & 0 & 0 & -3\gamma^2 I & 0 \\
* & 0 & 0 & 0 & 0 & -3\epsilon^{-1}I
\end{bmatrix},
$$

$$
\Psi_{ijs}^r \equiv
\begin{bmatrix}
-3Q_r & 0 & 0 & G_{sji}+G_{isj}+G_{jis} & \mathcal{W} & 3\tilde{M} \\
0 & -3\epsilon I & 0 & \tilde{N}_{sji}+\tilde{N}_{isj}+\tilde{N}_{jis} & 0 & 0 \\
0 & 0 & -3I & H_i+H_j+H_s & 0 & 0 \\
* & * & * & -(P_i+P_j+P_s) & 0 & 0 \\
* & 0 & 0 & 0 & -3\gamma^2 I & 0 \\
* & 0 & 0 & 0 & 0 & -3\epsilon^{-1}I
\end{bmatrix},
$$

where $\mathcal{W} = W_i + W_j + W_s$.

From Lemma 4.2, It is only sufficient to see that [12]

$$
\sum_{i=1}^{N} \alpha_i(z)\Omega_{ijs}^r = \sum_{i=1}^{N} \alpha_i^3(k)\Upsilon_{iii}^r + \sum_{i\le j<s}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z)\Phi_{ijs}^r + \sum_{i<j\le s}^{N} \alpha_i(z)\alpha_j(z)\alpha_s(z)\Psi_{ijs}^r.
$$

It should be noted that, Lemma 4.2 is very useful in reducing the number of LMIs to be satisfied. Indeed, (26) leads to N^4 LMIs to be satisfied. In contrast, by using Lemma 4.2, this number decreases to $(N^2(N^2+2))/3$.

Now, back to our main problem. We suggest to use Lemma 4.1 to handle the nonconvexity involved in (27), as it is clearly shown by the following theorem:

Theorem 4.1. *Given a weight $\beta > 0$ and $\epsilon > 0$. The augmented closed-loop system in (8) achieves the H_∞ output tracking performance γ, if there exists positive definite matrices $P_1 > 0, \ldots, P_N > 0$, $Q_1 > 0, \ldots, Q_N > 0$ and controller gains $K_1, \ldots, K_N$ such that the following optimization problem is solvable and equal to $n_{\tilde{x}} \times N$:*

$$
\begin{cases}
\underset{K_i, P_i, Q_i, \gamma}{minimize} \quad \beta \sum_{i=1}^{N} \mathbf{Tr}(P_i Q_i) + (1 - \beta)\gamma \\[2ex]
\text{subject to:} \\[2ex]
(28) \; and \; \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0, \; 1 \leq i \leq N.
\end{cases}
\tag{29}
$$

The following iterative algorithm [7, 12] can be used to linearize the objective function of the optimization problem (29).

Algorithm 4.1

give a weight β, fix a tolerance ε (for example $\varepsilon = 10^{-6}$) and execute the following steps:

- Step 1: Set $P_i^0 = I$ and $Q_i^0 = I$, for $i = 1, \ldots, N$.
- Step 2: Solve the following LMI optimization:

$$
\underset{K_i, P_i, Q_i, \gamma}{minimize} \quad \beta \sum_{i=1}^{N} \mathbf{Tr}(P_i^* Q_i + Q_i^* P_i) + (1 - \beta)\gamma
$$
$$
subject \; to :
$$
$$
(28) \; and \; \begin{bmatrix} P_i & I \\ I & Q_i \end{bmatrix} \geq 0, 1 \leq i \leq N.
$$

- Step 3: If $\|P_i - Q_i^{-1}\| \leq \varepsilon$.
  While $\|P_i - Q_i^{-1}\| \leq \varepsilon$,
  Select $\beta = \beta - 0.01$ and repeat from step 1. Else
  Set $P_i^* \longleftarrow P_i, Q_i^* \longleftarrow Q_i$ and repeat from step 2.

Remark 4.1. *In the optimization problem (29), the attenuation level γ is also included in the optimization function. Thus, a multi-objective optimization problem is solved by the Algorithm 4.1.*

5. Illustrative example

In this section, the proposed tracking control scheme is applied to regulate the output voltage of DC-DC converter. The model of a buck converter is described in Fig. 1. Using the Kirchoff laws, the converter of Fig. 1 can be represented by the following discrete-time nonlinear model [24]:
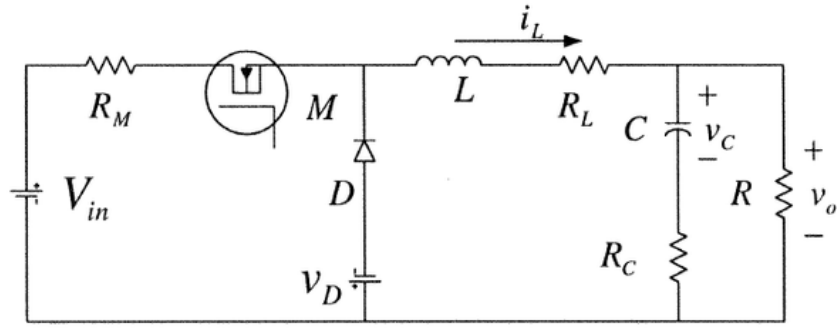
Figure 1. Buck converter circuit.

$$x(k+1) = \begin{bmatrix} \frac{-T_s}{L}\left(R_L + \frac{R(k)R_c}{R(k)+R_c}\right) + 1 & \frac{-T_s R(k)}{L(R(k)+R_c)} \\ \\ \frac{T_s R(k)}{C(R(k)+R_c)} & \frac{-T_s}{C(R(k)+R_c)} + 1 \end{bmatrix} x(k) +$$

$$\begin{bmatrix} \frac{-T_s}{L}(R_M i_L(k) - V_{in}(k) - V_D) \\ \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} \frac{-T_s V_D}{L} \\ \\ 0 \end{bmatrix}, \tag{30}$$

$$y(k) = \begin{bmatrix} \frac{R(k)R_c}{(R(k)+R_c)} & \frac{R(k)}{(R(k)+R_c)} \end{bmatrix} x(k),$$

where $x(k) = [i_L(k) \; v_c(k)]^T$ is the state vector, $u(k)$ is the control vector i.e. the duty cycle of the switched **M**, $y(k)$ is the output vector i.e. the output voltage and T_s is the sampling period $T_s = 0.001 \times 1/f_0$, with f_0 is the resonance frequency of the buck converter (30). $R(k)$ and $V_{in}(k)$ are uncertain parameters satisfying $R(k) \in [\underline{R}(k), \overline{R}(k)]$, $V_{in}(k) \in [\underline{V_{in}}(k), \overline{V_{in}}(k)]$. Table (1) gives the parameter values of the buck converter (Fig. 1). Similar to [24], we assume that the inductor current belongs in a compact set: $i_L(k) \in [\underline{i_L}, \overline{i_L}]$, and select the membership functions as follows

$$\alpha_1(k) = \frac{-i_L(k) + \overline{i_L}}{\overline{i_L} - \underline{i_L}}, \quad \alpha_2(k) = 1 - \alpha_1(k). \tag{31}$$

The nonlinear system (30) can be represented by the following uncertain T-S model:

$\mathcal{R}\text{ule}^i$ If $i_L(k)$ is μ_i

$$\text{Then} \begin{cases} x(k+1) = (A_{noi} + \Delta A_i(k))x(k) + (B_{noi} + \Delta B_i(k))u(k) + E_i w(k), & (32) \\ \\ y(k) = C_i x(k), \quad i = 1, 2, \end{cases}$$

where
$$A_{no1} = A_{no2} = \frac{\overline{A_1} + \underline{A_1}}{2}, B_{no1} = \frac{\overline{B_1} + \underline{B_1}}{2}, B_{no2} = \frac{\overline{B_2} + \underline{B_2}}{2}, \text{ with}$$

$$\overline{A_1} = \overline{A_2} = \begin{bmatrix} -\frac{T_s}{L}(R_L + \frac{\overline{R}R_c}{(\overline{R}+R_c)}) + 1 & -\frac{T_s\overline{R}}{L(\overline{R}+R_C)} \\[2mm] \frac{T_s\overline{R}}{C(\overline{R}+R_C)} & -\frac{T_s}{C(\overline{R}+R_c)} + 1 \end{bmatrix},$$

$$\underline{A_1} = \underline{A_2} = \begin{bmatrix} -\frac{T_s}{L}(R_L + \frac{\underline{R}R_c}{(\underline{R}+R_c)}) + 1 & -\frac{T_s\underline{R}}{L(\underline{R}+R_C)} \\[2mm] \frac{T_s\underline{R}}{C(\underline{R}+R_C)} & -\frac{T_s}{C(\underline{R}+R_c)} + 1 \end{bmatrix},$$

$$\overline{B_1} = \begin{bmatrix} -\frac{T_s}{L}(R_M \underline{i}_L - \overline{V_{in}} - V_D) \\[2mm] 0 \end{bmatrix}, \quad \overline{B_2} = \begin{bmatrix} -\frac{T_s}{L}(R_M \overline{i}_L - \overline{V_{in}} - V_D) \\[2mm] 0 \end{bmatrix},$$

$$\underline{B_1} = \begin{bmatrix} -\frac{T_s}{L}(R_M \underline{i}_L - \underline{V_{in}} - V_D) \\[2mm] 0 \end{bmatrix}, \quad \underline{B_2} = \begin{bmatrix} -\frac{T_s}{L}(R_M \overline{i}_L - \underline{V_{in}} - V_D) \\[2mm] 0 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} \frac{RR_c}{R+R_c} & \frac{R}{R+R_c} \end{bmatrix}, \text{ and } E_1 = E_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$\Delta A_1(k)$, $\Delta A_2(k)$, $\Delta B_1(k)$ and $\Delta B_2(k)$ can be represented in the form of (2) with $M_1 = 0.1$, $M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $N_{11} = 10\frac{\overline{A_1} - \underline{A_1}}{2}, N_{12} = N_{11}, N_{21} = \frac{\overline{B_1} - \underline{B_1}}{2}, N_{22} = \frac{\overline{B_2} - \underline{B_2}}{2}.$

In this example, the objective is to make the output voltage of the buck converter, i.e. v_o follow a desired signal to meet the H_∞ tracking performance of the uncertain system (30). The reference system matrices of (5) is selected as follows

$$\mathbb{A} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \mathbb{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbb{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{33}$$

Let $\beta = 0.99$ and $\epsilon = 1$, using the Algorithm 4.1, the following feasible solution is obtained after only 41 iterations:

$$P_1 = \begin{bmatrix} 0.122328 & 0.72818 & 0 & -0.070451 \\ 0.72818 & 7.378511 & 0 & -0.550637 \\ 0 & 0 & 1 & 0 \\ -0.070451 & -0.550637 & 0 & 2.846761 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.124832 & 0.7441565 & 0 & -0.09171 \\ 0.7441565 & 7.455168 & 0 & -0.661751 \\ 0 & 0 & 1.116002 & 0 \\ -0.09171 & -0.661751 & 0 & 3.00123 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 19.852249 & -1.950697 & 0 & 0.113988 \\ -1.950697 & 0.329190 & 0 & 0.015398 \\ 0 & 0 & 1 & 0 \\ 0.113988 & 0.015398 & 0 & 0.357075 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 19.869471 & -1.967944 & 0 & 0.173243 \\ -1.967944 & 0.3317250 & 0 & 0.013007 \\ 0 & 0 & 0.896056 & 0 \\ 0.173243 & 0.013007 & 0 & 0.341358 \end{bmatrix},$$

$$K_1 = -6.0943; \quad K_2 = -7.1963,$$

and the H_∞ output tracking performance index: $\gamma = 2.52$. Hence, according to (7), the static output-feedback control law that ensures the desired trajectory tracking for (30) is given as follows:

$$u(k) = (\alpha_1(k)K_1 + \alpha_2(k)K_2)(y(k) - y_d(k)). \tag{34}$$

Fig. 2 shows the evolution of the output signal of the nonlinear system (30), using the fuzzy controller, with an external disturbance input $w(k)$ defined as $w(k) = \frac{r_o}{1+15(k+1)} - T_s V_D / L$, where, r_o is a random number taken from a uniform distribution over $[0,\ 2]$, the uncertain parameters are as follow

$$R(k) = \frac{\overline{R}+\underline{R}}{2} + \frac{\overline{R}-\underline{R}}{2}cos(k\pi/T_s),$$

$$V_{in}(k) = \frac{\overline{V_{in}}+\underline{V_{in}}}{2} + \frac{\overline{V_{in}}-\underline{V_{in}}}{2}cos(k\pi/T_s), \tag{35}$$

and the reference signal $r(k)$, are supposed to be

$$\begin{cases} r(k) = 12V & \text{for}\ \ 0 \leq k \leq 0.005s, \\ r(k) = 6V & \text{for}\ \ 0.005 < k \leq 0.01s, \\ r(k) = 24V & \text{for}\ \ k > 0.01s, \end{cases} \tag{36}$$
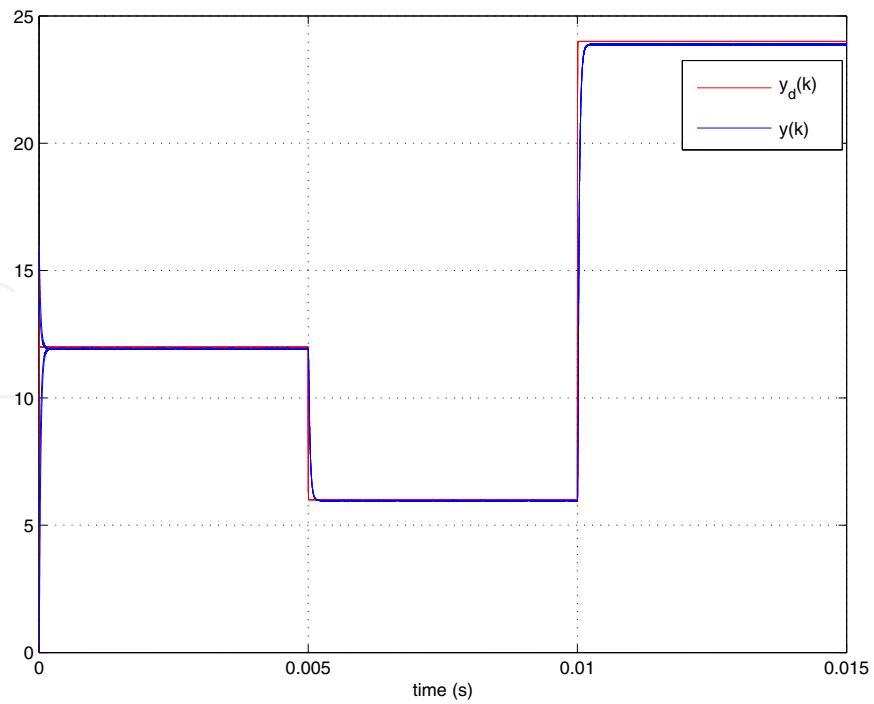
Figure 2. Response of $y(k)$ and $y_d(k)$.

Fig. 3 and Fig. 4 depict a zoom of Fig. 2 at 0 s and between 5 ms and 10 ms respectively. It can be seen that the designed fuzzy static output-feedback controller ensures the robust stability of the nonlinear system (30) and guarantees an acceptable H_∞ trajectory tracking performance level.
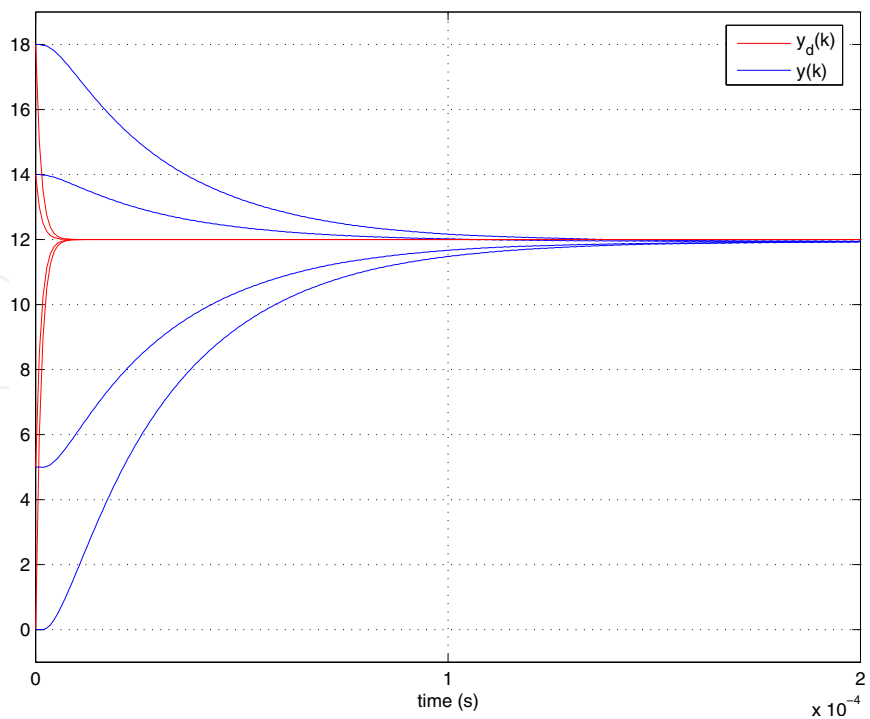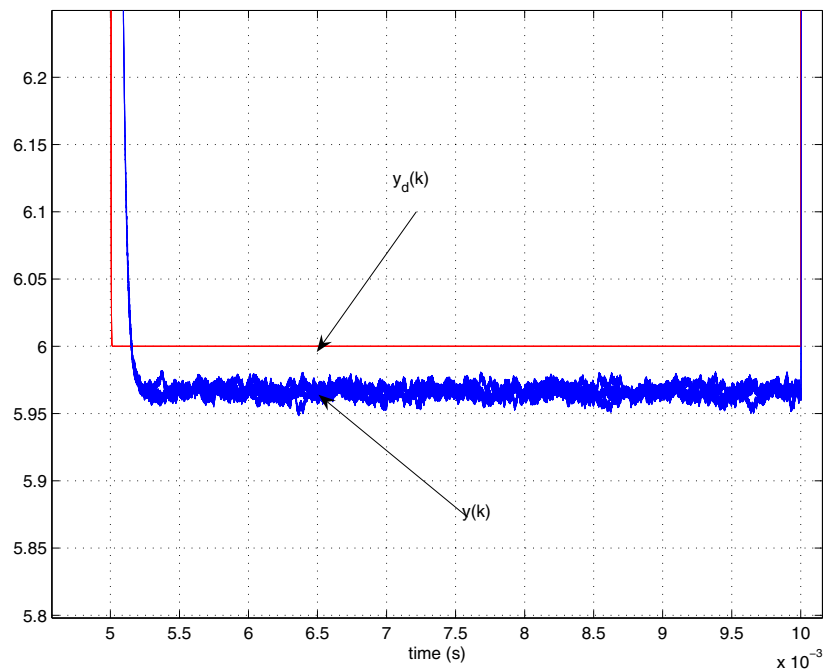

Figure 3. Zoom on Fig. 2 at 0 sec.

Figure 4. Zoom on Fig. 2 between 5 msec and 10 msec.

Parameter	Value	Unity
Input voltage, $V_{in}(k)$	$V_{in}(k) \in [10, \ 30]$	V
Current in the inductance , i_L	-8 - 8	A
Inductance, L	98.58	μH
Parasitic resistance of L, R_L	48.5	$m\Omega$
Capacitor, C	202.5	μF
Parasitic resistance of C, R_c	162	$m\Omega$
Resistance of Switch, R_M	0.27	Ω
Diode voltage, V_D	0.82	V
Load resistance, $R(k)$	$R(k) \in [2, \ 10]$	Ω

Table 1. Parameter values of the buck converter.

6. Conclusion

In this chapter, the problem of model reference tracking control with a guaranteed H_∞ performance is solved for uncertain discrete-time fuzzy systems. Based on the fuzzy Lyapunov function and cone complementary formulation, a fuzzy static output controller is calculated to make small as possible as the tracking output error and reject disturbances.

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