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A Computational Approach to Vibration Control of Vehicle Engine-Body Systems

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1. Introduction

In recent years, the noise and vibration of cars have become increasingly important [20, 23, 29, 30, 35]. A major comfort aspect is the transmission of engine-induced vibrations through powertrain mounts into the chassis (see Figure 1). Engine and powertrain mounts are usually designed according to criteria that incorporate a trade-off between the isolation of the engine from the chassis and the restriction of engine movements. The engine mount is an efficient passive means to isolate the car chassis structure from the engine vibration. However, the passive means for isolation is efficient only in the high frequency range. However the vibration disturbance generated by the engine occurs mainly in the low frequency range [8, 19, 23, 30]. These vibrations are result of the fuel explosion in the cylinder and the rotation of the different parts of the engine (see Figure 2). In order to attenuate the low frequency disturbances of the engine vibration while keeping the space and price constant, active vibration means are necessary.

A variety of control techniques, such as Proportional-Integral-Derivative (PID) or Lead-Lag compensation, Linear Quadratic Gaussian (LQG), H_2, H_∞, μ-synthesis and feedforward control have been used in active vibration systems [1, 3, 4, 10, 11, 15, 24, 26, 31, 32, 34, 35]. The main characteristic of feedforward control is that information about the disturbance source is available and is usually realised with the Filtered-X Least-Mean-Squares (Fx-LMS) algorithms. However, the disturbance source is assumed to be unknown in feedback control, then different strategies of feedback control for vibration attenuation of unknown disturbance exist ranging from classical methods to a more advanced methods. Recently, the performance result obtained by H_∞ feedback controller with the result obtained by feedforward controller using Fx-LMS algorithms for vehicle engine-body vibration system was compared in [30, 35].

On the other hand, wavelet theory is a relatively new and an emerging area in mathematical research [2]. It has been applied in a wide range of engineering disciplines such as signal

processing, pattern recognition and computational graphics. Recently, some of the attempts are made in solving surface integral equations, improving the finite difference time domain method, solving linear differential equations and nonlinear partial differential equations and modelling nonlinear semiconductor devices [5, 6, 7, 13, 16, 17, 18, 21, 27].

chassis subframe

engine mount point
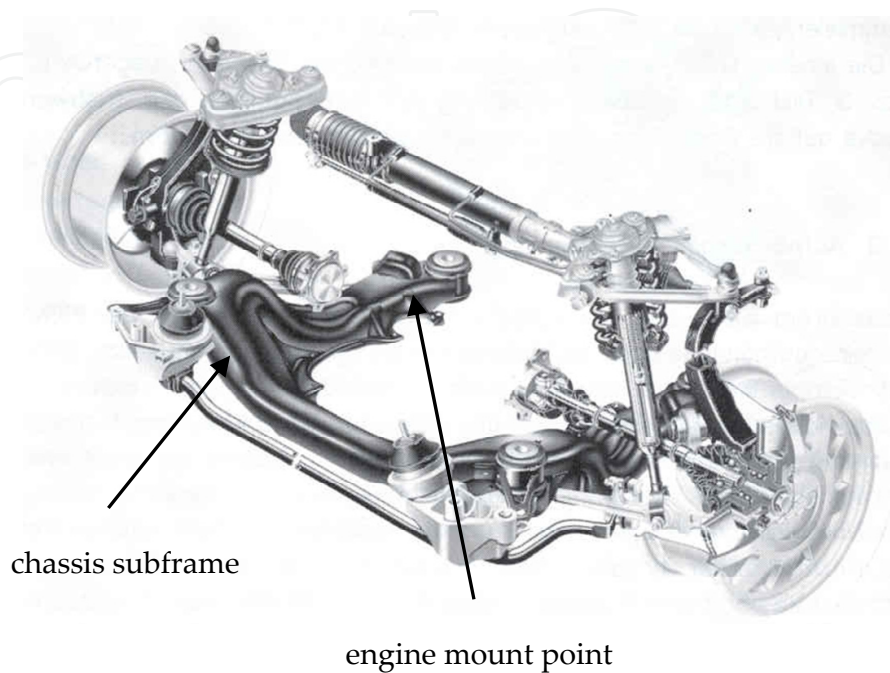
Figure 1. Front axis of AUDI A 8 from [22, 30] (Werkbild Audi AG).
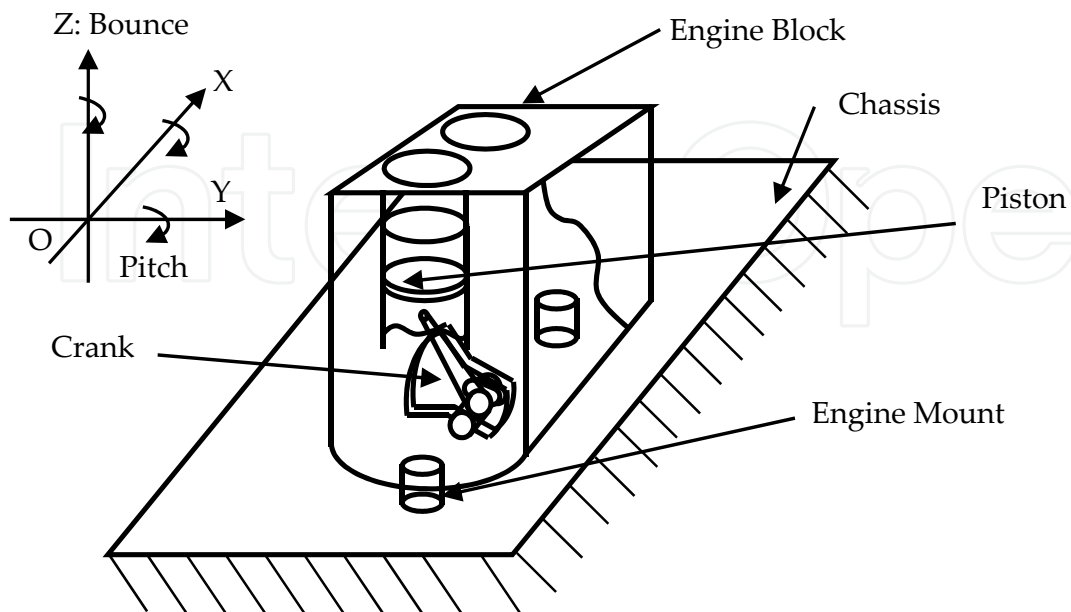
Figure 2. Chassis excited by the engine vibration.

Orthogonal functions like Haar wavelets (HWs) [13, 16], Walsh functions [7], block pulse functions [27], Laguerre polynomials [14], Legendre polynomials [5], Chebyshev functions [12] and Fourier series [28], often used to represent an arbitrary time functions, have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations for the solution of problems described by differential equations, such as analysis of linear time-invariant, time-varying systems, model reduction, optimal control and system identification. Thus, the solution, identification and optimisation procedure are either greatly reduced or much simplified accordingly. The available sets of orthogonal functions can be divided into three classes such as piecewise constant basis functions (PCBFs) like HWs, Walsh functions and block pulse functions; orthogonal polynomials like Laguerre, Legendre and Chebyshev as well as sine-cosine functions in Fourier series [21].

In the present paper, we, for the first time, introduce a computational solution to the finite-time robust optimal control problem of the vehicle engine-body vibration system based on HWs. To this aim, mathematical model of the engine-body vibration structure is presented such the actuators and sensors used to investigate the robust optimal control are selected to be collocated. Moreover, the properties of HWs, Haar wavelet integral operational matrix and Haar wavelet product operational matrix are given and are utilized to provide a systematic computational framework to find the approximated robust optimal trajectory and finite-time H_∞ control of the vehicle engine-body vibration system with respect to a H_∞ performance by solving only the linear algebraic equations instead of solving the differential equations. One of the main advantages is solving linear algebraic equations instead of solving nonlinear differential Riccati equation to optimize the control problem of the vehicle engine-body vibration system. We demonstrate the applicability of the technique by the simulation results.

The rest of this paper is organized as fallows. Section 2 introduces properties of the HWs. Mathematical model of the engine-body vibration structure is stated in Section 3. Algebraic solution of the engine-body system is given in Section 4 and Haar wavelet-based optimal trajectories and robust optimal control are presented in Sections 5 and 6, respectively. Simulation results of the robust optimal control of the vehicle engine-body vibration system are shown in Section 7 and finally the conclusion is discussed.

The notations used throughout the paper are fairly standard. The matrices I_r, 0_r and $0_{r \times s}$ are the identity matrix with dimension $r \times r$ and the zero matrices with dimensions $r \times r$ and $r \times s$, respectively. The symbol $\otimes$ and $tr(A)$ denote Kronecker product and trace of the matrix A, respectively. Also, operator $vec(X)$ denotes the vector obtained by putting matrix X into one column. Finally, given a signal $x(t)$, $\left\| x(t) \right\|_2$ denotes the L_2 norm of $x(t)$; i.e.,

$$\left\| x(t) \right\|_2^2 = \int_0^\infty x(t)^T x(t) \, dt .$$

2. Properties of Haar Wavelets

Properties of HWs, which will be used in the next sections, are introduced in this section.

2.1. Haar Wavelets (HWs)

The oldest and most basic of the wavelet systems is named Haar wavelet that is a group of square waves with magnitude of . ± 1 . in the interval $\left[0,1\right)$ [6]. In other words, the HWs are defined on the interval $\left[0,1\right)$ as

$$\psi_0(t) = 1, \qquad t \in \left[0,1\right),$$

$$\psi_1(t) = \begin{cases} 1, & for \quad t \in \left[0, \frac{1}{2}\right), \\ -1, & for \quad t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{1}$$

and $\psi_i(t) = \psi_1(2^j t - k)$ for $i \geq 1$ and we write $i = 2^j + k$ for $j \geq 0$ and $0 \leq k < 2^j$. We can easily see that the $\psi_0(t)$ and $\psi_1(t)$ are compactly supported, they give a local description, at different scales j, of the considered function.

2.2. Function approximation

The finite series representation of any square integrable function $y(t)$ in terms of an orthogonal basis in the interval $\left[0,1\right)$, namely $\hat{y}(t)$, is given by

$$\hat{y}(t) = \sum_{i=0}^{m-1} a_i \psi_i(t) := a^T \Psi_m(t) \tag{2}$$

where $a := \left[a_0 \, a_1 \cdots a_{m-1}\right]^T$ and $\Psi_m(t) := \left[\psi_0(t) \, \psi_1(t) \cdots \psi_{m-1}(t)\right]^T$ for $m = 2^j$ and the Haar coefficients a_i are determined to minimize the mean integral square error $\varepsilon = \int_0^1 (y(t) - a^T \Psi_m(t))^2 \, dt$ and are given by

$$a_i = 2^j \int_0^1 y(t) \, \psi_i(t) \, dt \tag{3}$$

Remark 1. The approximation error, $\Xi_y(m) := y(t) - \hat{y}(t)$, is depending on the resolution m and is approaching zero by increasing parameter of the resolution.

The matrix H_m can be defined as

$$H_m = \left[\Psi_m(t_0), \Psi_m(t_1), \cdots, \Psi_m(t_{m-1})\right] \tag{4}$$

where $i/_m \le t_i < {}^{i+1}/_m$ and using (2), we get

$$\left[\hat{y}(t_0)\ \hat{y}(t_1) \dots \hat{y}(t_{m-1}) \right] = a^T H_m .$$ (5)

The integration of the vector $\Psi_m(t)$ can be approximated by

$$\int_0^t \Psi_m(t)\, dt = P_m\, \Psi_m(t)$$ (6)

where the matrix

$$P_m = \left< \int_0^t \Psi_m(\tau)\, d\tau,\, \Psi_m(t) \right> = \int_0^1 \int_0^t \Psi_m(r)\, dr\ \Psi_m^T(t)\, dt$$

represents the integral operator matrix for PCBFs on the interval $\left[0, 1 \right)$ at the resolution m. For HWs, the square matrix P_m satisfies the following recursive formula [13]:

$$P_m = \frac{1}{2m} \begin{bmatrix} 2m\, P_{\frac{m}{2}} & -H_{\frac{m}{2}} \\ H_{\frac{m}{2}}^{-1} & 0_{\frac{m}{2}} \end{bmatrix}$$ (7)

with $P_1 = \frac{1}{2}$ and $H_m^{-1} = \frac{1}{m} H_m^T\, diag\, (r)$ where the matrix H_m defined in (4) and also the vector r is represented by

$$r := (1, 1, 2, 2, 4, 4, 4, 4, \cdots, \underbrace{(\frac{m}{2}), (\frac{m}{2}), \cdots, (\frac{m}{2})}_{(\frac{m}{2})\ elements})^T$$

for $m > 2$. For example, at resolution scale $j = 3$, the matrices H_8 and P_8 are represented as

$$H_8 = \begin{bmatrix} \psi_0(t_0)\ \psi_0(t_1) & \psi_0(t_7) \\ \psi_1(t_0)\ \psi_1(t_1) & \psi_1(t_7) \\ \psi_2(t_0)\ \psi_2(t_1) & \psi_2(t_7) \\ \psi_3(t_0)\ \psi_3(t_1) & \psi_3(t_7) \\ \psi_4(t_0)\ \psi_4(t_1) & \psi_4(t_7) \\ \psi_5(t_0)\ \psi_5(t_1) & \psi_5(t_7) \\ \psi_6(t_0)\ \psi_6(t_1) & \psi_6(t_7) \\ \psi_7(t_0)\ \psi_7(t_1) & \psi_7(t_7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

and

$$P_8 = \frac{1}{16} \left[\begin{array}{ccc|c} \begin{matrix} 8 & -4H_1 \\ \hline 4H_1^{-1} & 0 \end{matrix} & -2H_2 & \\ \hline & & -H_4 \\ 4H_2^{-1} & 0 & \\ \hline & H_4^{-1} & 0 \end{array} \right] = \frac{1}{16} \begin{bmatrix} 16P_4 & -H_4 \\ H_4^{-1} & 0 \end{bmatrix}$$

$$= \frac{1}{64} \begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & -4 & 0 & 0 & 0 & 0 & -4 & 4 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for further information see [13, 25].

2.3. The product operational matrix

In the study of time-varying state-delayed systems, it is usually necessary to evaluate the product of two Haar function vectors [13]. Let us define

$$R_m(t) := \Psi_m(t) \Psi_m^T(t) \tag{8}$$

where $R_m(t)$ satisfies the following recursive formula

$$R_m(t) = \frac{1}{2m} \begin{bmatrix} R_{\frac{m}{2}}(t) & H_{\frac{m}{2}} \, diag(\Psi_b(t)) \\ (H_{\frac{m}{2}} \, diag(\Psi_b(t)))^T & diag(H_{\frac{m}{2}}^{-1} \Psi_a(t)) \end{bmatrix} \tag{9}$$

with $R_1(t) = \psi_0(t)\psi_0^T(t)$ and

$$\begin{cases} \Psi_a(t) := \left[\psi_0(t), \psi_1(t), \cdots, \psi_{\frac{m}{2}-1}(t) \right]^T = \Psi_{\frac{m}{2}}(t) \\ \Psi_b(t) := \left[\psi_{\frac{m}{2}}(t), \psi_{\frac{m}{2}+1}(t), \cdots, \psi_{m-1}(t) \right]^T. \end{cases} \tag{10}$$

Moreover, the following relation is important for solving optimal control problem of time-varying state-delayed system:

$$R_m(t) a_m = \tilde{a}_m \Psi_m(t) \tag{11}$$

where $\tilde{a}_1 = a_0$ and

$$\tilde{a}_m = \begin{bmatrix} \tilde{a}_{\frac{m}{2}} & H_{\frac{m}{2}} diag(a_b) \\ diag(a_b)H_{\frac{m}{2}}^{-1} & diag(a_a^T H_{\frac{m}{2}}) \end{bmatrix} \tag{12}$$

with

$$\begin{cases} a_a := \begin{bmatrix} a_0, a_1, \cdots, a_{\frac{m}{2}-1} \end{bmatrix}^T = a_{\frac{m}{2}}(t) \\ a_b := \begin{bmatrix} a_{\frac{m}{2}}(t), a_{\frac{m}{2}+1}(t), \cdots, a_{m-1}(t) \end{bmatrix}^T. \end{cases} \tag{13}$$
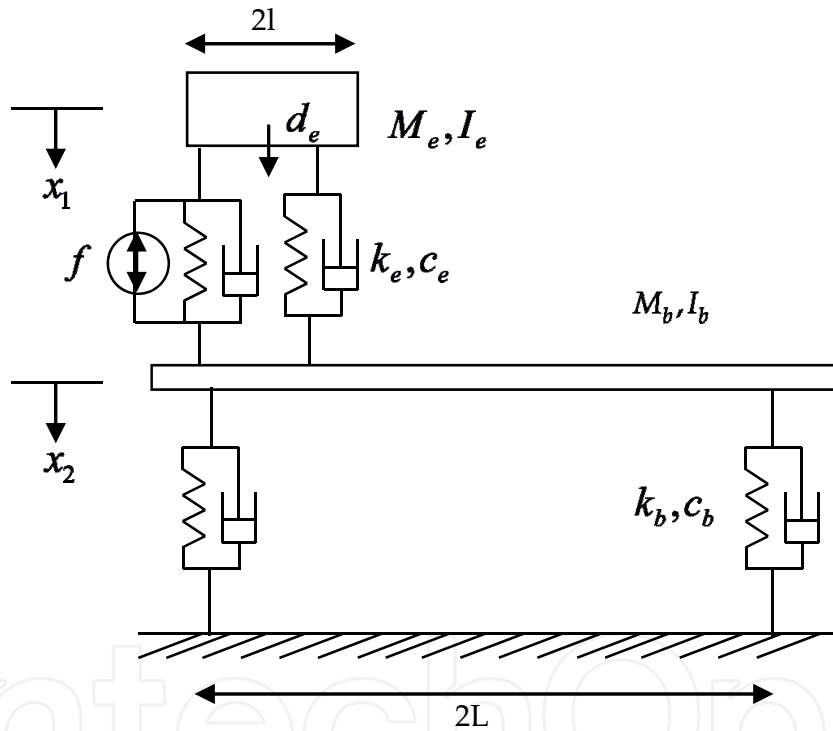

Figure 3. The sketch of engine-body vibration system

3. Mathematical model description

A schematic of the vehicle engine-body vibration structure is shown in Figure 3. The actuator and sensor used to this control framework are selected to be collocated, since this arrangement is ideal to ensure the stability of the closed loop system for a slightly damped structure [26]. In our study, only the bounce and pitch vibrations in the engine and body are considered [35]. The engine with mass M_e and inertia moment I_e is mounted in the body by the engine mounts k_e and c_e. The front mount is the active mount, the output force of

which can be controlled by an electric signal. The active mount consists of a main chamber where an oscillating mass (inertia mass) is moving up and down. The inertia mass is driven by an electro-magnetic force generated by a magnetic coil which is controlled by the input current.

The vehicle body with mass M_b and inertia moment I_b is supported by front and rear tires, each of which is modeled as a system consisting of a spring k_b and a damping device c_b. Therefore, a four degree-of-freedom vibration suspension model shown in Figure 3 can be described by the following equations

$$
\begin{cases}
M_e \ddot{x}_1 + 2c_e \dot{x}_1 + 2k_e x_1 - 2c_e \dot{x}_2 - 2k_e x_2 - 2(L-l)c_e \dot{x}_4 - 2(L-l)k_e x_4 = f(t) + d_e(t) \\[2mm]
M_b \ddot{x}_2 + 2(c_e + c_b)\dot{x}_2 + 2(k_e + k_b)x_2 - 2c_e \dot{x}_1 - 2k_e x_1 + 2(L-l)c_e \dot{x}_4 + 2(L-l)k_e x_4 = -f(t) \\[2mm]
I_e \ddot{x}_3 + 2l^2 c_e \dot{x}_3 + 2l^2 k_e x_3 - 2l^2 c_e \dot{x}_4 - 2l^2 k_e x_4 = l\,f(t) \\[2mm]
I_b \ddot{x}_4 + ((L^2 + (L-2l)^2)c_e + 2L^2 c_b)\dot{x}_4 + ((L^2 + (L-2l)^2)k_e + 2L^2 k_b)x_4 - 2l^2 c_e \dot{x}_3 \\[1mm]
\quad - 2l^2 k_e x_3 - 2l c_e \dot{x}_1 - 2l k_e x_1 + 2(L-l)c_e \dot{x}_2 + 2(L-l)k_e x_2 = -L\,f(t)
\end{cases}
\tag{14}
$$

where the states $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ are the bounces and pitches of the engine and body, respectively, where displacement of the chassis $(x_2(t))$ is usually taken as an output. Input force, $f(t)$, is used as the active force to compensate the vibration transmitted to vehicle body. Moreover, engine disturbance $d_e(t)$ can be the excitation, generated by the motion up/down of the different parts inside the engine;

The system Eq. (14) can be represented in the following state-space form

$$
\begin{cases}
M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = B_f\,f(t) + B_d\,d_e(t), \qquad t \in \left[0, T_f\right] \\[3mm]
z(t) = \begin{bmatrix} C_1 x(t) \\ C_2 \dot{x}(t) \\ C_3 f(t) \end{bmatrix}
\end{cases}
\tag{15}
$$

where $x(t) \in \Re^4$ is the state; $f(t) \in \Re$ is the control input; $d_e(t) \in \Re$ is the disturbance input which belongs to $L_2[0, \infty)$; and $z(t) \in \Re^3$ is the controlled output with $C_1 \in \Re^{1\times4}$, $C_2 \in \Re^{1\times4}$ and C_3 is a positive scalar. The state-space matrices are also defined as

$$
M = \begin{bmatrix} M_e & 0 & 0 & 0 \\ 0 & M_b & 0 & 0 \\ 0 & 0 & I_e & 0 \\ 0 & 0 & 0 & I_b \end{bmatrix}, \quad
C = \begin{bmatrix} 2c_e & -2c_e & 0 & -2(L-l)c_e \\ -2c_e & 2(c_e + c_b) & 0 & 2(L-l)c_e \\ 0 & 0 & 2l^2 c_e & -2l^2 c_e \\ -2l c_e & 2(L-l)c_e & -2l^2 c_e & 0 \end{bmatrix},
$$

$$B_f = \begin{bmatrix} 1 \\ -1 \\ l \\ -L \end{bmatrix}, \ B_d = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$K = \begin{bmatrix} 2k_e & -2k_e & 0 & -2(L-l)k_e \\ -2k_e & 2(k_e + k_b) & 0 & 2(L-l)k_e \\ 0 & 0 & 2l^2 k_e & -2l^2 k_e \\ -2lk_e & 2(L-l)k_e & -2l^2 k_e & (L^2 + (L-2l)^2)k_e + 2L^2 k_b \end{bmatrix}.$$

Taking displacement of the chassis $(x_2(t))$ as an output then a comparison of the displacement response respect to the input force $f(t)$ and the external disturbance $d_e(t)$ in the frequency range up to 1 KHz is depicted in Figure 4a) and 4b). Three relevant modes occur around the frequencies 1, 5 and 9 Hz, respectively, which represent the dynamics of the main degrees of freedom (DOFs) of the system.

4. Algebraic solution of system equations

In this section, we study the problem of solving the second-order differential equations of the engine-body system (14) in terms of the input control and exogenous disturbance using HWs and develop appropriate algebraic equations.

Based on HWs definition on the interval time $\begin{bmatrix} 0,1 \end{bmatrix}$, we need to rescale the finite time interval $\begin{bmatrix} 0, T_f \end{bmatrix}$ into $\begin{bmatrix} 0,1 \end{bmatrix}$ by considering $t = T_f \sigma$; normalizing the system Eq. (15) with the time scale would be as follows

$$M \ddot{x}(\sigma) + C \dot{x}(\sigma) + K x(\sigma) = B_f f(\sigma) + B_d d_e(\sigma) \tag{16}$$

Now by integrating the system above in an interval $\begin{bmatrix} 0, \sigma \end{bmatrix}$, we obtain

$$M(x(\sigma) - x(0)) + T_f C \int_0^\sigma x(\tau)\,d\tau + T_f^2 K \int_0^\sigma \int_0^\xi x(\tau)\,d\tau\,d\xi = T_f^2 B_f \int_0^\sigma \int_0^\xi f(\tau)\,d\tau\,d\xi + T_f^2 B_d \int_0^\sigma \int_0^\xi d_e(\tau)\,d\tau\,d\xi$$
$$+ \int_0^\sigma (M \dot{x}(0) + T_f C x(0))\,d\xi. \tag{17}$$

By using the Haar wavelet expansion (2), we express the solution of Eq. (15), input force $f(\sigma)$ and engine disturbance $d_e(\sigma)$ in terms of HWs in the forms

$$x(\sigma) = X \Psi_m(\sigma), \tag{18}$$

$$f(\sigma) = F \Psi_m(\sigma), \tag{19}$$

$$d_e(\sigma) = D_e \, \Psi_m(\sigma),\tag{20}$$

where $X \in \Re^{4 \times m}$, $F \in \Re^{1 \times m}$ and $D_e \in \Re^{1 \times m}$ denote the wavelet coefficients of $x(\sigma)$, $f(\sigma)$ and $d_e(\sigma)$, respectively. The initial conditions of $x(0)$ and $\dot{x}(0)$ are also represented by $x(0) = X_0 \, \Psi_m(\sigma)$ and $\dot{x}(0) = \bar{X}_0 \, \Psi_m(\sigma)$, where the matrices $\{X_0, \bar{X}_0\} \in \Re^{4 \times m}$ are defined, respectively, as

$$X_0 := \left[x(0) \; \underbrace{0_{4 \times 1} \; \cdots \; 0_{4 \times 1}}_{(m-1)} \right]\tag{21}$$

$$\bar{X}_0 := \left[\dot{x}(0) \; \underbrace{0_{4 \times 1} \; \cdots \; 0_{4 \times 1}}_{(m-1)} \right]\tag{22}$$

Therefore, using the wavelet expansions (18)-(20), the relation (17) becomes

$$M(X - X_0) + \mathrm{T}_f C X P_m + \mathrm{T}_f^2 K X P_m^2 = \mathrm{T}_f^2 B_f F P_m^2 + \mathrm{T}_f^2 B_d D_e P_m^2 + (M \bar{X}_0 + \mathrm{T}_f C X_0) \, P_m\tag{23}$$

For calculating the matrix X, we apply the operator $vec(.)$ to Eq. (23) and according to the property of the Kronecker product, i.e. $vec(ABC) = (C^T \otimes A) \, vec(B)$, we have:

$$\begin{aligned}
(I_m \otimes M)(vec(X) - vec(X_0)) &+ \mathrm{T}_f (P_m^T \otimes C) vec(X) + \mathrm{T}_f^2 (P_m^{2T} \otimes K) vec(X) \\
&= \mathrm{T}_f^2 (P_m^{2T} \otimes B_f) vec(F) + \mathrm{T}_f^2 (P_m^{2T} \otimes B_d) vec(D_e) \\
&+ \mathrm{T}_f (P_m^T \otimes C) vec(X_0) + (P_m^T \otimes M) vec(\bar{X}_0).
\end{aligned}\tag{24}$$

Solving Eq. (24) for $vec(X)$ leads to

$$vec(X) = \Delta_1 vec(F) + \Delta_2 \, vec(D_e) + \Delta_3 \, vec(X_0) + \Delta_4 \, vec(\bar{X}_0)\tag{25}$$

where the matrices $\{\Delta_1, \Delta_2\} \in \Re^{4m \times m}$ and $\{\Delta_3, \Delta_4\} \in \Re^{4m \times 4m}$ are defined as

$$\begin{cases}
\Delta_1 = \mathrm{T}_f^2 (\mathrm{T}_f (P_m^T \otimes C) + \mathrm{T}_f^2 (P_m^{2T} \otimes K) + I_m \otimes M)^{-1} (P_m^{2T} \otimes B_f) \\
\Delta_2 = \mathrm{T}_f^2 (\mathrm{T}_f (P_m^T \otimes C) + \mathrm{T}_f^2 (P_m^{2T} \otimes K) + I_m \otimes M)^{-1} (P_m^{2T} \otimes B_d) \\
\Delta_3 = (\mathrm{T}_f (P_m^T \otimes C) + \mathrm{T}_f^2 (P_m^{2T} \otimes K) + I_m \otimes M)^{-1} (I_m \otimes M + \mathrm{T}_f P_m^T \otimes C) \\
\Delta_4 = (\mathrm{T}_f (P_m^T \otimes C) + \mathrm{T}_f^2 (P_m^{2T} \otimes K) + I_m \otimes M)^{-1} (P_m^T \otimes M).
\end{cases}\tag{26}$$

Consequently, using (25) and (26) and the properties of the Kronecker product, the solution of system (15) is

$$x(\sigma) = (\Psi_m^T(\sigma) \otimes I_4) \, vec(X)\tag{27}$$

and it is also clear that to find the approximated solution of the system, we have to calculate the inverse of the matrix $T_f(P_m^T \otimes C) + T_f^2(P_m^{2T} \otimes K) + I_m \otimes M$ with dimension $4m \times 4m$ only once.
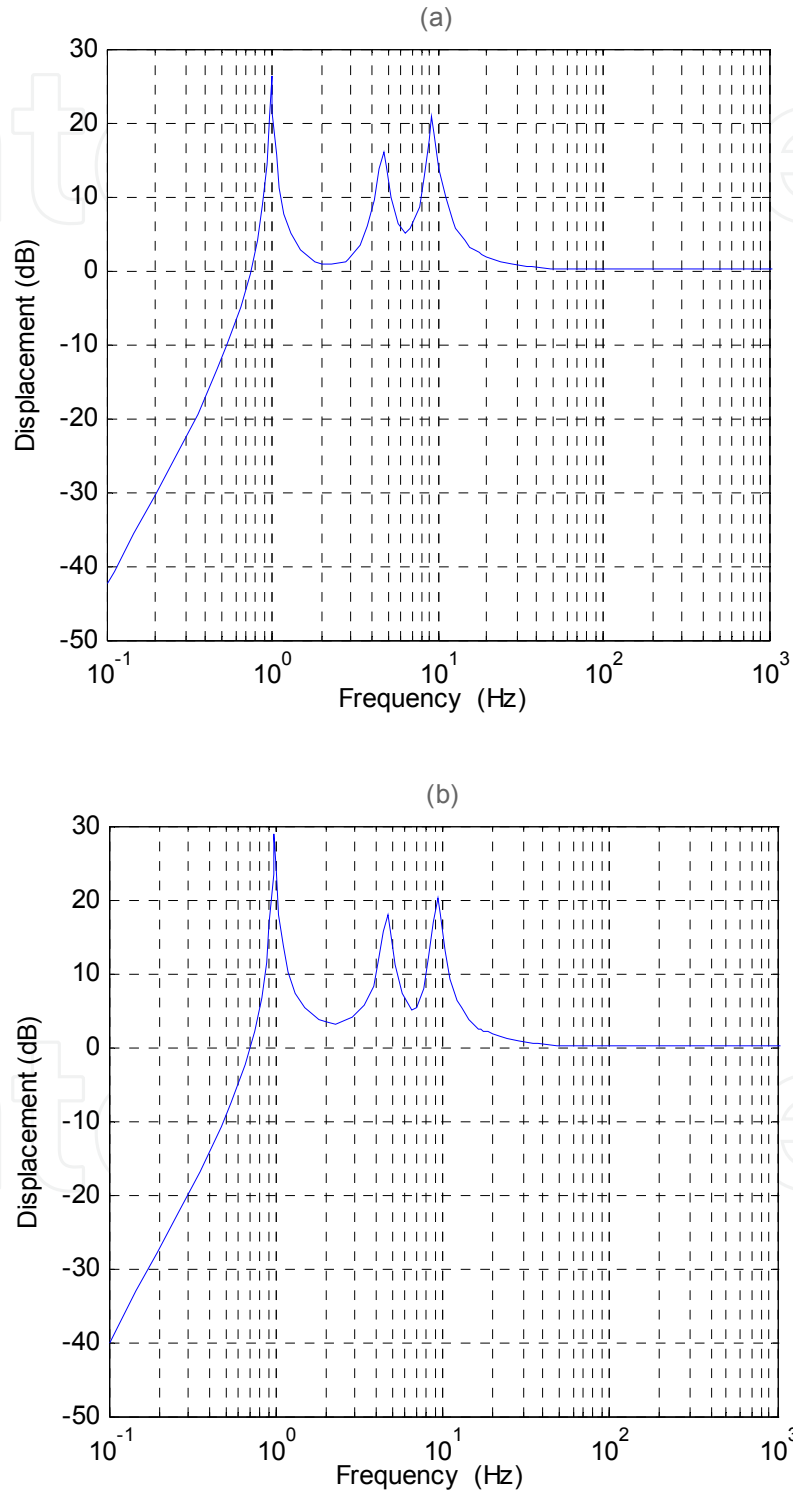
Figure 4. Displacement of the chassis respect to $f(t)$ (a) and $d_e(t)$ (b).

5. Optimal control design

The control objective is to find the optimal control $f(t)$ with respect to a quadratic cost functional approximately such acts as the active force to compensate the vibration transmitted to vehicle body. The quadratic cost functional weights the states and their derivatives with respect to time in the cost function as follows:

$$J = \frac{1}{2} x^T(\mathrm{T}_f) S_1 x(\mathrm{T}_f) + \frac{1}{2} \dot{x}^T(\mathrm{T}_f) S_2 \dot{x}(\mathrm{T}_f) + \frac{1}{2} \int_0^{\mathrm{T}_f} (x^T(t) Q_1 x(t) + \dot{x}^T(t) Q_2 \dot{x}(t) + R f(t)^2) \, dt \quad (28)$$

where $S_1 : 4 \times 4$, $S_2 : 4 \times 4$, $Q_1 : 4 \times 4$ and $Q_2 : 4 \times 4$ are positive-definite matrices and R is a positive scalar. We can rewrite the cost function (28) as follows:

$$J = \frac{1}{2} [x^T(1) \quad \mathrm{T}_f^{-1} \dot{x}^T(1)] \tilde{S} \begin{bmatrix} x(1) \\ \mathrm{T}_f^{-1} \dot{x}(1) \end{bmatrix} + \frac{\mathrm{T}_f}{2} \frac{1}{2} \int_0^1 ([x^T(\sigma) \quad \mathrm{T}_f^{-1} \dot{x}^T(\sigma)] \tilde{Q} \begin{bmatrix} x(\sigma) \\ \mathrm{T}_f^{-1} \dot{x}(\sigma) \end{bmatrix} + R f(\sigma)^2) \, d\sigma . \quad (29)$$

where $\tilde{S} = diag(S_1, S_2)$ and $\tilde{Q} = diag(Q_1, Q_2)$ with the time scale $t = \mathrm{T}_f \, \sigma$.

From (15) and the relation $\dot{x}(\sigma) = \bar{X} \, \Psi_m(\sigma)$, where $\bar{X} : 4 \times m$ denotes the wavelet coefficients of $\dot{x}(\sigma)$ after its expansion in terms of HFs, we read

$$\begin{bmatrix} x(\sigma) \\ \mathrm{T}_f^{-1} \dot{x}(\sigma) \end{bmatrix} = \begin{bmatrix} X \\ \mathrm{T}_f^{-1} \bar{X} \end{bmatrix} \Psi_m(\sigma) := X_{aug} \, \Psi_m(\sigma) \quad (30)$$

where $X_{aug} = \begin{bmatrix} X \\ \mathrm{T}_f^{-1} \bar{X} \end{bmatrix}$ and

$$vec(X_{aug}) = \begin{bmatrix} vec^T(X) & \mathrm{T}_f^{-1} vec^T(\bar{X}) \end{bmatrix}^T \quad (31)$$

Remark 2. By substituting $\dot{x}(\sigma) = \bar{X} \, \Psi_m(\sigma)$ into $x(\sigma) - x(0) = \int_0^{\sigma} \dot{x}(t) \, dt$, we have:

$$X \, \Psi_m(\sigma) - X_0 \, \Psi_m(\sigma) = \int_0^{\sigma} \bar{X} \, \Psi_m(\tau) \, d\tau , \quad (32)$$

and using (4), we read $X - X_0 = \bar{X} P_m$. Then, by applying the operator of $vec(.)$ and according to the properties of Kronecker product in Appendix A1, we obtain

$$vec(X) - vec(X_0) = (P_m^T \otimes I_n) \, vec(\bar{X}) \quad (33)$$

By substituting the definition (31) in (33) and using the properties of the operator $tr(.)$ in Appendix A1, the cost function (28) is given by

$$J = \frac{1}{2}(vec^T(X_{aug})\Pi_{m1}\,vec(X_{aug}) + vec^T(F)\Pi_{m2}\,vec(F)) \tag{34}$$

where the matrices $\Pi_{m1} : 8m \times 8m$ and $\Pi_{m2} : m \times m$ are defined as

$$\Pi_{m1} = M_f^T \otimes \tilde{S} + T_f(M^T \otimes \tilde{Q}) \text{ and } \Pi_{m2} = R\,T_f\,M_m \text{, respectively,}$$

and the matrices $M_m : m \times m$ and $M_{mf} : m \times m$ are defined as

$$M_m := \int_0^1 \Psi_m(\sigma)\Psi_m^T(\sigma)\,d\sigma \text{ and } M_{mf} := \Psi_m(1)\Psi_m^T(1) \text{, respectively.}$$

It is clear that the cost function of $J(.)$ is a function of $\frac{i}{m} \le \sigma_i < \frac{i+1}{m}$, then for finding the optimal control law, which minimizes the cost functional $J(.)$, the following necessary condition should be satisfied

$$\frac{\partial J}{\partial vec(F)} = 0 \tag{35}$$

By considering $vec(X_{aug})$, which is a function of $vec(F)$, and using the properties of derivatives of inner product of Kronecker product in Appendix A2, we find

$$\frac{\partial J}{\partial vec(F)} = [\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1} \otimes I_4)]\,\Pi_{m1}\,vec(X_{aug}) + \Pi_{m2}\,vec(F) \tag{36}$$

Then the wavelet coefficients of the optimal control law will be in vector form as

$$vec(F) = -\Pi_{m2}^{-1}[\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1} \otimes I_4)]\,\Pi_{m1}\,vec(X_{aug}) \tag{37}$$

Consequently, the optimal vectors of $vec(X)$ and $vec(F)$ are found, respectively, in the following forms

$$vec(X) = (I_{4m} + \Delta_1(\Pi_{m2}^{-1}[\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1} \otimes I_4)]\,\Pi_{m1}\begin{bmatrix} I_{4m} \\ T_f^{-1}(P_m^T \otimes I_4)^{-1} \end{bmatrix})^{-1}(\Delta_2\,vec(D_e) + (\Delta_1\,\Pi_{m2}^{-1}$$

$$\times [\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1} \otimes I_4)]\,\Pi_{m1}\begin{bmatrix} 0_{4m} \\ T_f^{-1}(P_m^T \otimes I_4)^{-1} \end{bmatrix} + \Delta_3)\,vec(X_0) + \Delta_4\,vec(\bar{X}_0)), \tag{38}$$

and

$$vec(F) = -\Pi_{m2}^{-1}[\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1}\otimes I_4)]\ \Pi_{m1}\left\{\begin{bmatrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1} \end{bmatrix}\right.$$

$$\times(I_{4m} + \Delta_1\Pi_{m2}^{-1}[\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1}\otimes I_4)]\ \Pi_{m1}\begin{bmatrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1} \end{bmatrix})^{-1}$$

$$\times(\Delta_2\, vec(D_e)+(\Delta_1\Pi_{m2}^{-1}[\Delta_1^T \quad T_f^{-1}\Delta_1^T(P_m^{-1}\otimes I_4)]\ \Pi_{m1}\begin{bmatrix} 0_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1} \end{bmatrix}+\Delta_3)\, vec(X_0)$$

$$+\Delta_4\, vec(\bar{X}_0)) - \begin{bmatrix} 0_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1} \end{bmatrix} vec(X_0)\Big\}.$$

(39)

Finally, the Haar function-based optimal trajectories and optimal control are obtained approximately from Eq. (27) and $f(t) = \Psi_m^T(t)\, vec(F)$.

6. Robust optimal control design

In this section, an optimal state feedback controller is to be determined computationally such that the following requirements are satisfied:

i.   the closed-loop system is asymptotically stable;

ii. under zero initial condition, the closed-loop system satisfies $\left\|z(t)\right\|_2 < \gamma\left\|d_e(t)\right\|_2$ for any non-zero $d_e(t)\in[0,\infty)$ where $\gamma > 0$ is a prescribed scalar.

The control objective is to find the approximated robust optimal control $f(t)$ with H_∞ performance such $f(t)$ acts as the active force to compensate the vibration transmitted to vehicle body, i.e. guarantees desired L_2 gain performance. Next, we shall establish the H_∞ performance of the system (15) under zero initial condition. To this end, we introduce

$$J = \tfrac{1}{2}x^T(T_f)S_1 x(T_f) + \frac{1}{2}\dot{x}^T(T_f)S_2\dot{x}(T_f) + \tfrac{1}{2}\int_0^{T_f}(z^T(t)z(t) - \gamma^2 d_e^2(t))\,dt.$$

(40)

It is well known that a sufficient condition for achieving robust disturbance attenuation is that the inequality $J < 0$ for every $d_e(t)\in L_2[0,\infty)$ [33, 36]. Therefore, we will establish conditions under which

$$\underset{vec(F)\ vec(D_e)}{Inf\quad Sup}\ J(vec(F),vec(D_e))\le 0$$

(41)

From (15), the Eq. (40) can be represented as

$$J = \frac{1}{2}(x^T(1) \quad \frac{1}{T_f}\dot{x}^T(1))\tilde{S}\begin{pmatrix} x(1) \\ T_f^{-1}\dot{x}(1) \end{pmatrix}$$

$$+ \frac{T_f}{2}\int_0^1 ((x^T(\sigma) \quad T_f^{-1}\dot{x}^T(\sigma))\tilde{C}\begin{pmatrix} x(\sigma) \\ T_f^{-1}\dot{x}(\sigma) \end{pmatrix} + C_3^2 f^2(\sigma) - \gamma^2 d_e^2(\sigma))\, d\sigma \tag{42}$$

where $t = T_f\,\sigma$, $\tilde{S} = diag(S_1, S_2)$ and $\tilde{C} = diag(C_1^T C_1, C_2^T C_2)$.

Using the relation $\dot{x}(\sigma) = \overline{X}\,\Psi_m(\sigma)$, we read

$$\begin{bmatrix} x(\sigma) \\ T_f^{-1}\dot{x}(\sigma) \end{bmatrix} = \begin{bmatrix} X \\ T_f^{-1}\overline{X} \end{bmatrix}\Psi_m(\sigma) := X_{aug}\,\Psi_m(\sigma) \tag{43}$$

where $X_{aug} = \begin{bmatrix} X \\ T_f^{-1}\overline{X} \end{bmatrix}$ and

$$vec(X_{aug}) = \begin{bmatrix} vec^T(X) & T_f^{-1}vec^T(\overline{X}) \end{bmatrix}^T \tag{44}$$

Moreover, according to Remark 2 in [18], the following relation is already satisfied between $vec(X)$ and $vec(\overline{X})$

$$vec(X) - vec(X_0) = (P_m^T \otimes I_4)\,vec(\overline{X}) \tag{45}$$

By using the definition (44) in Eq. (45), we have

$$J = \frac{1}{2}(tr(M_{mf} X_{aug}^T \tilde{S} X_{aug})) + \frac{T_f}{2}(tr(M_m X_{aug}^T \tilde{C} X_{aug}) + tr(C_3^2 M_m F^T F) - \gamma^2 tr(M_m D_e^T D_e)) \tag{46}$$

Using the property of the Kronecker product, i.e. $tr(ABC) = vec^T(A^T)(I_p \otimes B)\,vec(C)$, $(A \otimes C)(D \otimes B) = AD \otimes CB$ and $vec(ABC) = (C^T \otimes A)\,vec(B)$, we can write (42) as

$$J = \frac{1}{2}(vec^T(X_{aug})\Pi_{m1}\,vec(X_{aug}) + C_3^2\,vec^T(F)\Pi_{m2}\,vec(F) - \gamma^2\,vec^T(D_e)\Pi_{m2}\,vec(D_e)) \tag{47}$$

where the matrices $\Pi_{m1} \in \mathfrak{R}^{8m \times 8m}$, $\Pi_{m2} \in \mathfrak{R}^{m \times m}$ are defined as $\Pi_{m1} = M_{mf} \otimes \tilde{S} + \frac{T_f}{2}(M_m \otimes \tilde{C})$ and $\Pi_{m2} = \frac{T_f}{2} M_m$, respectively.

It is easy to show that the worst-case disturbance in Eq. (47) occurs when

$$vec^*(D_e) = \gamma^{-2}\Pi_{m2}^{-1}\begin{bmatrix} \Delta_2^T & T_f^{-1}\Delta_2^T(P_m^{-1} \otimes I_4) \end{bmatrix}\Pi_{m1}\,vec(X_{aug}) := \gamma^{-2}\Pi_{md}\,vec(X_{aug}) \tag{48}$$

By substituting Eq. (48) into Eq. (47) we obtain

$$\underset{vec(F)}{Inf}\ \underset{vec(D_e)}{Sup}\ J(vec(F),vec(D_e)) = \underset{vec(F)}{Inf}\ J(vec(F),vec^*(D_e)) \tag{49}$$

Minimizing the right-hand side of Eq. (49) results in the algebraic relation between wavelet coefficients of the robust optimal control and of the optimal state trajectories in the following closed form

$$vec(F) = -C_3^{-2}\Pi_{m2}^{-1}\left[\Delta_1^T\quad T_f^{-1}\Delta_1^T(P_m^{-1}\otimes I_4)\right](\Pi_{m1} - \gamma^{-2}\Pi_{md}^T\Pi_{m2}\Pi_{md})\ vec(X_{aug}) \tag{50}$$
$$:=\Pi_{mf}\ vec(X_{aug}).$$

As a result we have

$$\underset{vec(F)}{Inf}\ \underset{vec(D_e)}{Sup}\ J(vec(F),vec(D_e)) \le vec^T(X_{aug})\,(\Pi_{m1} + R\Pi_{mf}^T\Pi_{m2}\Pi_{mf} - \gamma^2\Pi_{md}^T\Pi_{m2}\Pi_{md})vec(X_{aug}) \tag{51}$$

Consequently, if there exists positive scalar γ to the matrix inequality

$$\Pi_{m1} + C_3^2\Pi_{mf}^T\Pi_{m2}\Pi_{mf} - \gamma^2\Pi_{md}^T\Pi_{m2}\Pi_{md} \le 0 \tag{52}$$

then inequality (41) is concluded.

From the relations above we obtain the robust optimal vectors of $vec(X)$ and $vec(F)$ after some matrix calculations, respectively, in the following forms

$$vec(X) = (I_{4m} - (\Delta_1\Pi_{mf} + \gamma^{-2}\Delta_2\Pi_{md})\left[\begin{matrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right])^{-1}((\Delta_3 - (\Delta_1\Pi_{mf} + \gamma^{-2}\Delta_2\Pi_{md}) \tag{53}$$
$$\times\left[\begin{matrix} 0_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right])\ vec(X_0) + \Delta_4\ vec(\bar{X}_0)),$$

and

$$vec(F) = \Pi_{mf}\ \{(\left[\begin{matrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right]((I_{4m} - (\Delta_1\Pi_{mf} + \gamma^{-2}\Delta_2\Pi_{md})\left[\begin{matrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right])^{-1}$$
$$\times(\Delta_3 - (\Delta_1\Pi_{mf} + \gamma^{-2}\Delta_2\Pi_{md})\left[\begin{matrix} 0_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right]) - \left[\begin{matrix} 0_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right])\ vec(X_0) \tag{54}$$
$$+\left[\begin{matrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right](I_{4m} - (\Delta_1\Pi_{mf} + \gamma^{-2}\Delta_2\Pi_{md})\left[\begin{matrix} I_{4m} \\ T_f^{-1}(P_m^T\otimes I_4)^{-1}\end{matrix}\right])^{-1}\Delta_4\ vec(\bar{X}_0))\}$$

Finally, the Haar wavelet-based robust optimal trajectories and robust optimal control are obtained approximately from Eq. (27) and $f(t) = \Psi_m^T(t)\ vec(F)$, respectively.

7. Numerical results

In this section the proposed computational methodology is applied to the vehicle engine-body vibration system (15) such the exogenous disturbance $d_e(t)$ is assumed to be a $Sin\,(.)$ function at the frequency of $10\,Hz$. The system parameters, used for the design and simulation are given in Tables 1 and 2 in the *Appendix B*. Table 3 in the *Appendix* gives the pole-zero locations of 8th –order model of the vehicle engine-body vibration system. It is clear that the vehicle engine-body vibration system is unstable and has the nonminimum phase property. The objective is to find the approximated robust optimal displacement of the chassis and robust optimal input force with H_∞ performance using HWs at the finite time interval $[0,1]$. Moreover, the matrices $\{S_1, S_1\} \in \mathfrak{R}^{4\times4}$ and the vectors C_1, C_2 and the scalar C_3 in the controlled output $z(t)$ in Eq. (15) are chosen as $S_1 = S_2 = 0_4$, $C_1 = [0, 1, 1, 2]$, $C_2 = [3, -1, 0, 1]$ and $C_3 = 1$.
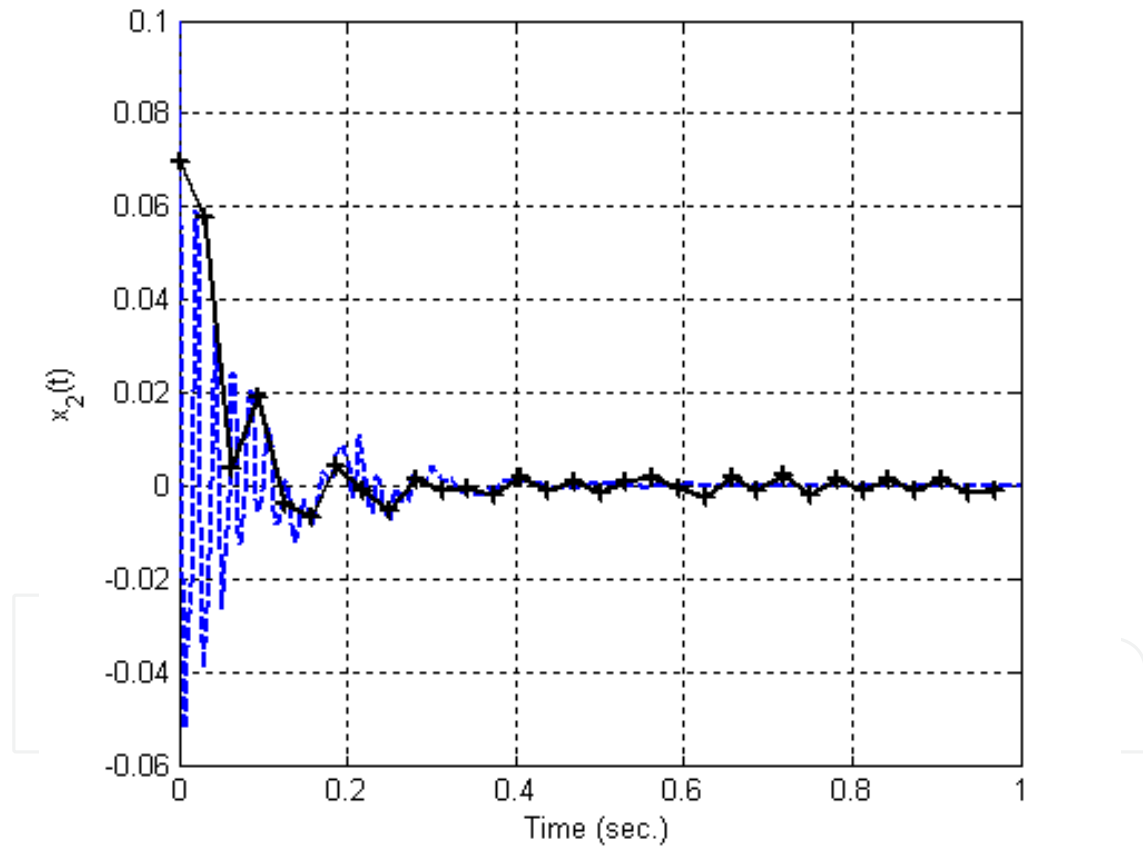
Figure 5. Comparison of displacement of the chassis found by HWs at resolution level $j = 5$ (solid) and by analytical solution (dashed).

To compare the approximate solutions $x_2(t)$ and $f(t)$, found by HWs, to the analytical solution found by Theorem 1 in the Appendix C, we choose the performance bound and the resolution level equal to 3.15 and 5, respectively, i.e. $\gamma = 3.15$ and $j = 5$. The time curves found are plotted in Figures 5 and 6. It is clear that the effect of the engine

disturbance is attenuated onto the displacement of the chassis as the output as well. In other words, $f(t)$ compensates the vibration transmitted to the chassis. Compare the Haar wavelet based solutions to the continuous solutions using the differential Riccati equation, the approximate solutions (53) and (54) deliver both, robust control $f(t)$ and state trajectory $x(t)$ in one step by solving linear algebraic equations instead of solving nonlinear differential Riccati equation, while accuracy can easily be improved by increasing the resolution level j.
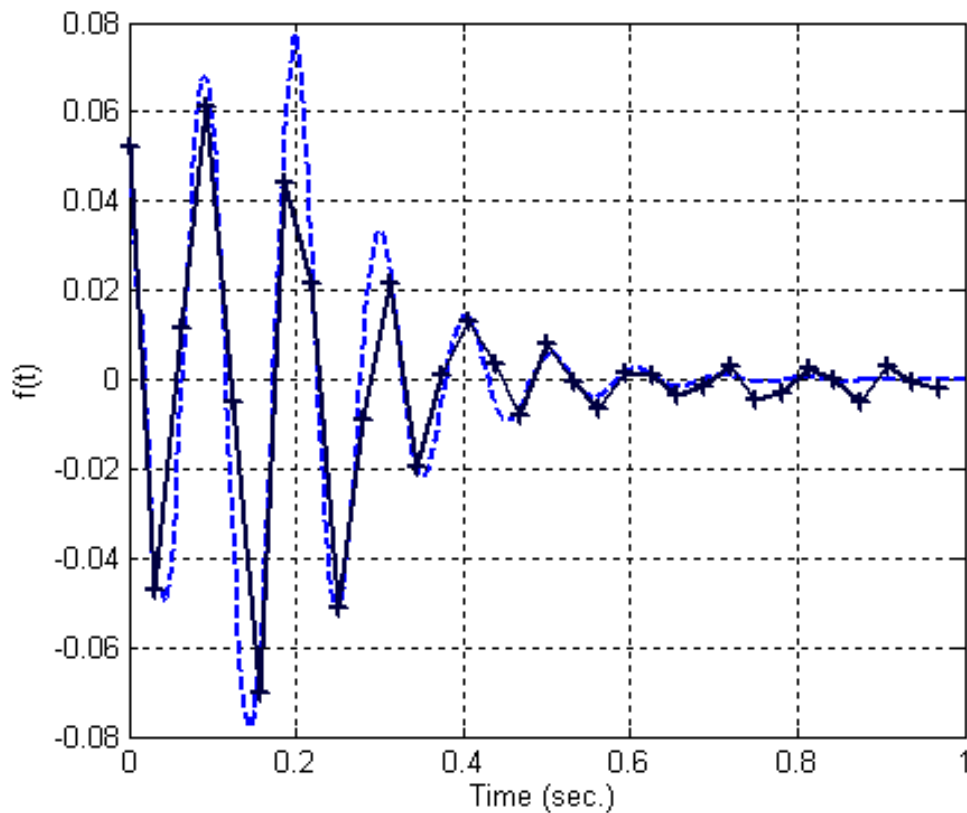
Figure 6. Comparison of input force found by HWs at resolution level $j = 5$ (solid) and by analytical solution (dashed).

8. Conclusion

This paper presented the modelling of engine-body vibration structure to control of bounce and pitch vibrations using HWs. To this aim, the Haar wavelet-based optimal control for vibration reduction of the engine-body system was developed computationally. The Haar wavelet properties were introduced and utilized to find the approximate solutions of optimal trajectories and robust optimal control by solving only algebraic equations instead of solving the Riccati differential equation. Numerical results were presented to illustrate the advantage of the approach.

9. Appendix

9.1. Appendix A

A1. *Some properties of Kronecker product*

Let $A : p \times q$, $B : q \times r$, $C : r \times s$ and $D : q \times t$ be fixed matrices, then we have:

$$vec(ABC) = (C^T \otimes A)\,vec(B),$$
$$tr(ABC) = vec^T(A^T)\,(I_p \otimes B)\,vec(C),$$
$$tr(ABC) = vec^T(A^T)\,(I_p \otimes B)\,vec(C),$$
$$(A \otimes C)(D \otimes B) = A D \otimes C B.$$

A2. *Derivatives of inner products of Kronecker product*

Let $A : n \times n$ be fixed constants and $x : n \times 1$ be a vector of variables. Then, the following results can be established:

$$\frac{\partial(Ax)}{\partial x} = vec(A),$$

$$\frac{\partial(Ax)}{\partial x^T} = A,$$

$$\frac{\partial(x^T A x)}{\partial x} = A x + A^T x$$

A3. *Chain rule for matrix derivatives using Kronecker product*

Let Z be a $p \times q$ matrix whose entries are a matrix function of the elements of $Y : s \times t$, where Y is a function of matrix $X : m \times n$. That is, $Z = H_1(Y)$, where $Y = H_2(X)$. The matrix of derivatives of Z with respect to X is given by

$$\frac{\partial Z}{\partial X} = \left\{ \frac{\partial vec^T(Y)}{\partial X} \otimes I_p \right\} \left\{ I_n \otimes \frac{\partial Z}{\partial vec(Y)} \right\}.$$

9.2. Appendix B

Parameters	Values
M_b	1000 [kg]
I_b	810 [$kg\ m^2$]
k_b	20000 [N/m]
c_b	300 [$N/m/s$]
L_b	2.5 [m]

Table 1. The vehicle body parameter.

Parameters	Values
M_e	250 [kg]
I_e	8.10 [$kg\ m^2$]
k_e	200000 [N/m]
c_e	200 [$N/m/s$]
L_e	0.5 [m]

Table 2. The engine parameters.

Poles	Zeros
$-6.2313 \pm i\ 111.62$	$-6.23 \pm i\ 111.69$
$-1.09 \pm i\ 58.17$	$-0.97 \pm i\ 62.59$
$0.14 \pm i\ 29.48$	$0.03 \pm i\ 26.86$
$-0.29 \pm i\ 6.19$	-1.10

Table 3. Pole-zero locations of the 8th-order model.

9.3. Appendix C

Theorem 1 (State Feedback) [9]. Consider dynamical system

$$\begin{cases} \dot{x}(t) = A\,x(t) + B_1\,u(t) + B_2\,w(t) \\ z(t) = C\,x(t) + D\,u(t) \end{cases}$$

under assumption (A, B_1, C) is stabilizable. For a given $\gamma > 0$, the differential Riccati equation

$$\dot{X} = A^T\,X + X\,A + X(\gamma^{-2}B_2\,B_2^T - B_1\,B_1^T)X + C^T C$$

has a positive semi-definite solution $X(t)$ such that $A - (B_1\,B_1^T - \gamma^{-2}B_2\,B_2^T)X(t)$ is asymptotically stable. Then the control law $u(t) = -B_1^T X(t)x(t) := K(t)\,X(t)$ is stabilizing and satisfies $\left\| z(t) \right\|_2 < \gamma \left\| w(t) \right\|_2$.

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