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Boolean Petri Nets

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Additional information is available at the end of the chapter

1. Introduction

Petri net is a graphical tool invented by Carl Adam Petri [13]. These are used for describing, designing and studying discrete event-driven dynamical systems that are characterized as being concurrent, asynchronous, distributed, parallel, random and/or nondeterministic. As a graphical tool, Petri net can be used for planning and designing a system with given objectives, more practically effective than flowcharts and block diagrams. As a mathematical tool, it enables one to set up state equations, algebraic equations and other mathematical models which govern the behavior of discrete dynamical systems. Still, there is a drawback inherent in representing discrete event-systems. They suffer from the state explosion problem as what will happen when a system is highly populated, i.e., initial state consists of a large number of places that are nonempty. This phenomenon may lead to an exponential growth of its reachability graph. This makes us to study the safe systems. The aim of this chapter is to present some basic results on 1-safe Petri nets that generate the elements of a Boolean hypercube as marking vectors. Complete Boolean hypercube is the most popular interconnection network with many attractive and well known properties such as regularity, symmetry, strong connectivity, embeddability, recursive construction, etc. For brevity, we shall call a 1-safe Petri net that generates all the binary n-vectors as marking vectors a *Boolean Petri net*. *Boolean Petri nets* are not only of theoretical interest but also are of practical importance, required in practice to construct control systems [1]. In this chapter, we will consider the problems of characterizing the class of Boolean Petri nets as also the class of *crisp* Boolean Petri nets, viz., the Boolean Petri nets that generate all the binary n-vectors exactly once. We show the existence of a disconnected Boolean Petri net whose reachability tree is homomorphic to the n-dimensional complete lattice L_n. Finally, we observe that characterizing a Boolean Petri net is rather intricate.

We begin by showing that a 1-safe *Star Petri net* S_n [5], with $|P| = n$ and $|T| = n + 1$, having a central transition, is a Boolean Petri net; here, P is the set of its places and T is the set of its transitions. Often, it is desirable to have a crisp Boolean Petri net because one may possibly explore for existence of certain sequences of enabled transitions to fire toward initiating and completing a prescribed process that uses specified nodes of the Boolean lattice.

For example, in the design of generalized switches such as those used to control automatic machines [1], suppose that we have a sequence of n terminals each of which can be either at a prescribed low-voltage (denoted by zero '0') or at a prescribed high-voltage (denoted by unity, '1'). It is required to arrange them so that every one of the 2^n sequences of n bits, corresponding to the 2^n binary n-tuples, can appear on the terminals [1]. Now that Q_n, the binary n-cube, is known to be Hamiltonian (in the sense that there exists an all-vertex covering cycle) one can design a "Hamiltonian switch" using a crisp Boolean Petri net that triggers operation of a machine exactly once after 2^n successive switching moves along the prescribed Hamiltonian cycle in Q_n. The 'switch design' may be imagined to be an arbitrary connected graph of order 2^n, where connection between a pair (u, v) of nodes would mean that v is to be the terminal that needs to be turned on after the terminal corresponding to u (which may or may not be in an active state depending on the machine design). Therefore, a good characterization of such Boolean Petri nets is needed. This problem is still open. Many specific classes of such 1-safe Petri nets have been found [5–7]. Also, many fundamental issues regarding Boolean Petri nets emerge from this study.

2. Preliminaries

To keep this chapter self-contained as far as possible, we present some of the necessary definitions and concepts. For standard terminology and notation on Petri net theory and graph theory, we refer the reader to Peterson [12] and Harary [3], respectively. In this chapter, we shall adopt the definition of Jenson [4] for Petri nets:

Definition 1. *A Petri net is a 5-tuple $C = (P, T, I^-, I^+, \mu^0)$, where*

(a) *P is a nonempty set of 'places',*

(b) *T is a nonempty set of 'transitions',*

(c) *$P \cap T = \emptyset$,*

(d) *$I^-, I^+ : P \times T \longrightarrow \mathbf{N}$, where $\mathbf{N}$ is the set of nonnegative integers, are called the negative and the positive 'incidence functions' (or, 'flow functions') respectively,*

(e) *$\forall p \in P, \exists t \in T : I^-(p, t) \neq 0$ or $I^+(p, t) \neq 0$ and*
    $\forall t \in T, \exists p \in P : I^-(p, t) \neq 0$ or $I^+(p, t) \neq 0$,

(f) *$\mu^0 : P \to \mathbf{N}$ is the initial marking.*

In fact, $I^-(p, t)$ and $I^+(p, t)$ represent the number of arcs from p to t and t to p respectively, and some times referred to a 'flow relations'. I^-, I^+ and μ^0 can be viewed as matrices of size $|P| \times |T|$, $|P| \times |T|$ and $|P| \times 1$, respectively.

The quadruple (P, T, I^-, I^+) in the definition of the Petri net is called the *Petri net structure*. The *Petri net graph* is a representation of the Petri net structure, which is essentially a bipartite directed multigraph, in which any pair of symmetric arcs (p_i, t_j) and (t_j, p_i) is called a *self-loop*.

As in many standard books (e.g., see [14]), Petri net is a particular kind of directed graph [3], together with an initial marking μ^0. The underlying graph of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called places and transitions,

where arcs are either from a place to a transition or from a transition to a place. No two of the same kind being adjacent. Hence, Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting places and transitions, to represent the flow relations. The initial marking is represented by placing a token, shown as a black dot, in the circle representing a place p_i, whenever $\mu^0(p_i) = 1$, $1 \le i \le n = |P|$. In general, a *marking* μ is a mapping $\mu : P \longrightarrow \mathbf{N}$. A marking μ can hence be represented as a vector $\mu \in \mathbf{N}^n$, $n = |P|$, such that the ith component of μ is the value $\mu(p_i)$, viz., the number of tokens placed at p_i.

Definition 2. *Let $C = (P, T, I^-, I^+, \mu)$ be a Petri net. A transition $t \in T$ is said to be enabled at μ if and only if $I^-(p, t) \le \mu(p)$, $\forall p \in P$. An enabled transition may or may not 'fire' (depending on whether or not the event actually takes place). After firing at μ, the new marking μ' is given by the rule*

$$\mu'(p) = \mu(p) - I^-(p, t) + I^+(p, t), \text{for all } p \in P.$$

We say that *t fires at μ to yield μ'* (or, that *t fires μ to μ'*), and we write $\mu \xrightarrow{t} \mu'$, whence μ' is said to be *directly reachable* from μ. Hence, it is clear, what is meant by a sequence like

$$\mu^0 \xrightarrow{t_1} \mu^1 \xrightarrow{t_2} \mu^2 \xrightarrow{t_3} \mu^3 \cdots \xrightarrow{t_k} \mu^k,$$

which simply represents the fact that the transitions

$$t_1, t_2, t_3, \ldots, t_k$$

have been successively fired to transform the *initial marking* μ^0 into the *terminal* marking μ^k. The whole of this sequence of transformations is also written in short as $\mu^0 \xrightarrow{\sigma} \mu^k$, where $\sigma = t_1, t_2, t_3, \ldots, t_k$ is called the corresponding *firing sequence*.

A marking μ is said to be *reachable from* μ^0, if there exists a firing sequence of transitions which successively fire to reach the state μ from μ^0. The set of all markings of a Petri net C reachable from a given marking μ is denoted by $M(C, \mu)$ and, together with the arcs of the form $\mu^i \xrightarrow{t_r} \mu^j$, represents what in standard terminology is called the *reachability graph* of the Petri net C, denoted by $R(C, \mu^0)$. In particular, if the reachability graph has no semicycle then it is called the *reachability tree* of the Petri net.

A place in a Petri net is *safe* if the number of tokens in that place never exceeds one. A Petri net is *safe* if all its places are safe.

The *preset* of a transition t is the set of all input places to t, i.e., $^\bullet t = \{p \in P : I^-(p, t) > 0\}$. The *postset* of t is the set of all output places from t, i.e., $t^\bullet = \{p \in P : I^+(p, t) > 0\}$. Similarly, p's preset and postset are $^\bullet p = \{t \in T : I^+(p, t) > 0\}$ and $p^\bullet = \{t \in T : I^-(p, t) > 0\}$, respectively.

Definition 3. *Let $C = (P, T, I^-, I^+, \mu^0)$ be a Petri net with $|P| = n$ and $|T| = m$, the incidence matrix $I = [a_{ij}]$ is an $n \times m$ matrix of integers and its entries are given by $a_{ij} = a_{ij}^+ - a_{ij}^-$ where $a_{ij}^+ = I^+(p_i, t_j)$ is the number of arcs from transition t_j to its output place p_i and $a_{ij}^- = I^-(p_i, t_j)$ is the number of arcs from place p_i to its output transition t_j i.e., in other words, $I = I^+ - I^-$.*

3. 1-safe star Petri net is Boolean

We shall now define *1-safe star* Petri net. The notion of a *star* is from graph theory (see [3]); it is the *complete bipartite graph* $K_{1,n}$ which consists of exactly one vertex c, called the *center*, joined by a single edge cv_i to the pendant vertex v_i (i.e. the degree of v_i is 1) for each $i \in \{1, 2, \ldots, n\}, n \geq 1$. A *1-safe star Petri net* S_n is obtained by *subdividing* every edge of the graph $K_{1,n}, n \geq 1$, so that every subdividing vertex is a place node and the original vertices of $K_{1,n}, n \geq 1$, are the $(n+1)$ transition nodes, $(n+1)^{th}$ being the central node. Further, every arc incident to the central node is directed towards it, and every arc incident to a pendent node is directed towards the pendent node (See Figure 1).
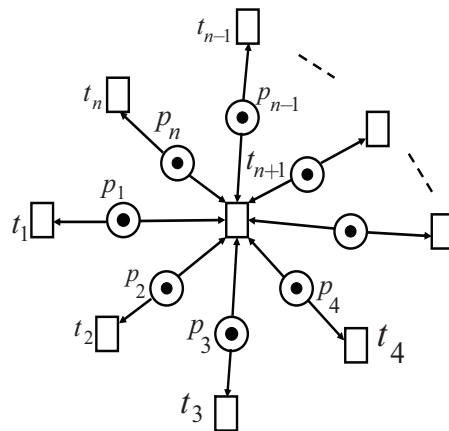

Figure 1. 1-safe star Petri net S_n

Theorem 1. *[5] The reachability tree of S_n with $\mu_0 = (1, 1, 1, 1, \ldots, 1)$ as the initial marking contains every binary n-vector $(a_1, a_2, a_3, \ldots, a_n)$, $a_i \in \{0, 1\}$.*

Proof. We shall prove this result by using the Principle of Mathematical Induction (PMI). Clearly, the reachability tree $R(S_1, \mu^0)$ of S_1 generates both the binary 1-vectors (1) and (0) as shown in Figure 2. Next, consider the 1-safe star Petri net S_2 as shown in Figure 3 and its reachability tree $R(S_2, \mu^0)$ displayed in Figure 4.

It is clear from Figure 4 that $R(S_2, \mu^0)$ has all the $4 = 2^2$, binary 2-vectors (a_1, a_2), $a_1, a_2 \in \{0, 1\}$. We can construct $R(S_2, \mu^0)$ from $R(S_1, \mu^0)$ as follows. Take two copies of $R(S_1, \mu^0)$. In the first copy, augment each vector of $R(S_1, \mu^0)$, by putting a 0 entry at the second position of every marking vector and denote the resulting labeled tree as $R_0(S_1, \mu^0)$. Similarly, in the second copy, augment each vector by putting 1 at the second position of every marking and let $R_1(S_1, \mu^0)$ be the resulting labeled tree (See Figure 5). Now, using the following steps we construct the reachability tree $R(S_2, \mu^0)$ of S_2 from $R_0(S_1, \mu^0)$ and $R_1(S_1, \mu^0)$.
1. Clearly, the set of binary 2-vectors in $R_0(S_1, \mu^0)$ is disjoint with the set of those appearing in $R_1(S_1, \mu^0)$ and together they contain all the binary 2-vectors.
2. In $R_0(S_1, \mu^0)$, transition t_2 does not satisfy the enabling condition, since $I^-(p_i, t) \leq \mu(p_i)$, for each $p_i \in S_1$ is violated. So, we can ignore this transition at this stage.

3. In $R_1(S_1, \mu^0)$, transition t_2 is enabled and the marking obtained after firing of t_2 is actually (1, 0) whereas the augmented vector attached to this node is (0,1). So, we concatenate
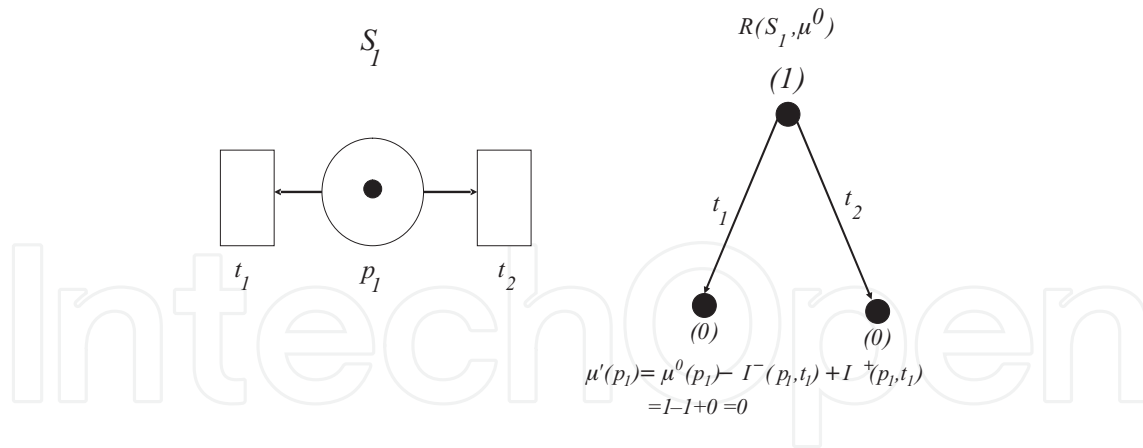
Figure 2. $1-$safe star Petri net S_1 and $R(S_1, \mu^0)$
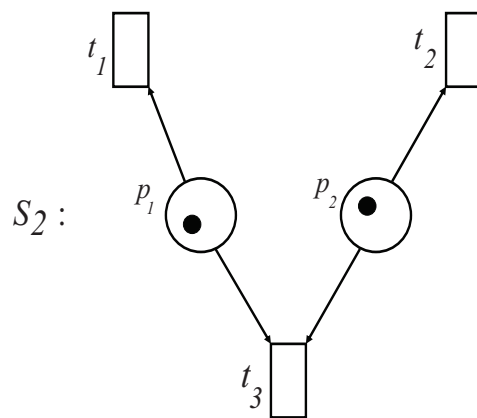

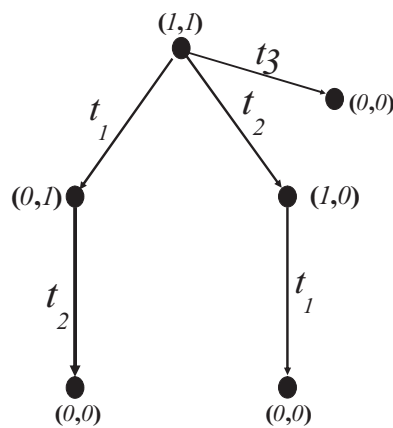**Figure 3.** 1-safe star Petri net S_2

Figure 4. $R(S_2, \mu^0)$, $\mu^0 = (1,1)$

$R_0(S_1, \mu^0)$ by fusing the node labeled $(1, 0)$ with the node labeled $(0, 1)$ in $R_1(S_1, \mu^0)$ and replacing $(0, 1)$ by the label $(1, 0)$ which is the initial marking of $R_0(S_1, \mu^0)$.
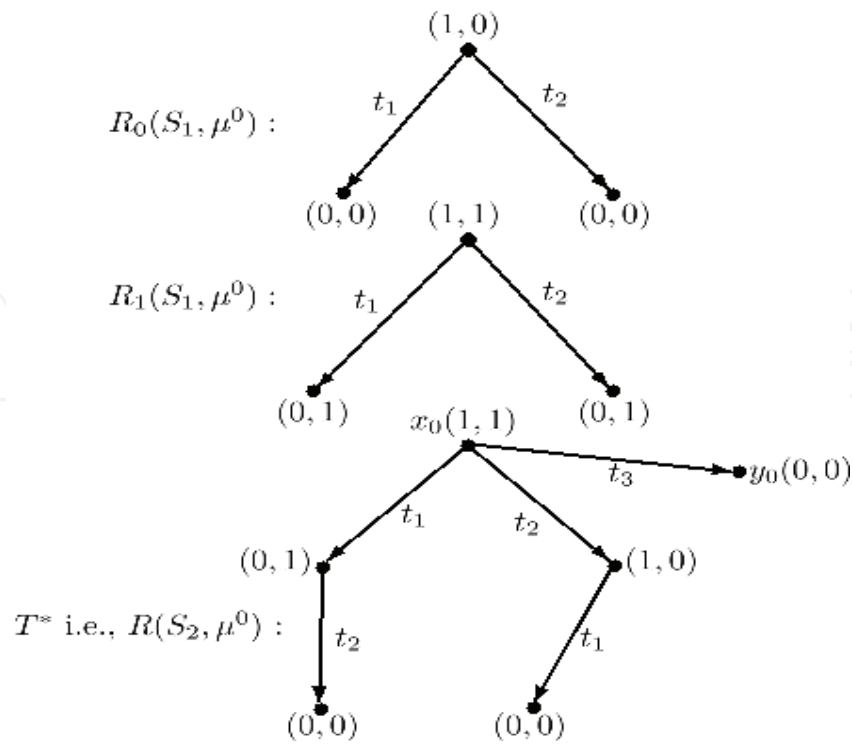
$R_0(S_1, \mu^0)$:

$R_1(S_1, \mu^0)$:

T^* i.e., $R(S_2, \mu^0)$:

Figure 5. Augmented reachability trees and resulting labeled tree T^*

4. We then augment an extra pendent node labeled y_0 joined to the new root node x_0, labeled by the 2-vector (1, 1), by the new arc (x_0, y_0) labeled as t_3 and complete this tree by firing transitions(s) at the marking vector(s) where nonzero components appear, till all the transitions become dead. Then the resulting labeled tree T^* is shown in Figure 5. This has all the binary 2-vectors as its node labels, possibly with repetitions. It remains to show that it is the reachability tree $R(S_2, \mu^0)$ of S_2 with 2-vector (1, 1) as its initial marking μ^0. For this, consider an arbitrary 2-vector $\mu = (a_1, 1)$, where $a_1 \in \{0,1\}$. When transition t_2 is enabled, this yields

$$\mu'(p_i) = \mu(p_i) - I^-(p_i, t_2) + I^+(p_i, t_2)$$

$$= 1 - 1 + 0 = 0$$

Then, we get a new marking $\mu' = (a_1, 0)$, where $a_1 \in \{0, 1\}$. The marking μ' is found in $R_0(S_2, \mu^0)$. If all a_i's are zero then μ' is a dead marking. Hence, suppose some $a_i \neq 0$. In this case, t_i is enabled and in the next new marking μ'', the ith component is reduced to zero. Eventually, this process will lead to a dead marking. Further, the marking vectors of the form $\mu = (a_1, 0)$ are already obtained as a result of firing t_1, t_2, through some subsequences. Thus, T is indeed the reachability tree $R(S_2, \mu^0)$ of S_2.

Now, we assume that the result is true for all the 1-safe star Petri nets S_k having k places, $k \leq n$. We will prove the result for the 1-safe star Petri net S_{n+1} having $(n + 1)$ places. For this purpose, consider two copies of the reachability tree $R(S_n, \mu^0)$ of S_n. In the first copy, we extend each vector by augmenting a 0 entry at the $(n+1)^{th}$ position and let $R_0(S_n, \mu^0)$ denote the resulting labeled tree. Next, in the second copy of $R(S_n, \mu^0)$, we augment the entry 1 to the $(n + 1)^{th}$ position in every marking vector and let $R_1(S_n, \mu^0)$ be the resulting labeled

tree. Hence, using the following steps we construct the reachability tree of the 1-safe star Petri net S_{n+1} having $(n+1)$ places.

1. Clearly, the set of binary $(n+1)$-vectors in $R_0(S_n, \mu^0)$ is disjoint with the set of those appearing in $R_1(S_n, \mu^0)$ and together they contain all the binary $(n+1)$-vectors.

2. In $R_0(S_n, \mu^0)$, transition t_{n+1} does not satisfy the enabling condition, $I^-(p_i, t) \leq \mu(p_i)$, for each $p_i \in S_n$. So, we can ignore this transition for the moment.

3. In $R_1(S_n, \mu^0)$, transition t_{n+1} is enabled and the marking obtained after firing of t_{n+1} is actually $(1, 1, \ldots, 0)$. So we concatenate $R_0(S_n, \mu^0)$ at this node with the $(n+1)$-vector $(0, 0, \ldots, 1)$ replaced by the actual marking $(1, 1, \ldots, 0)$ being the initial marking of $R_0(S_n, \mu^0)$.

4. We then augment an extra pendent node labeled y_0 joined to the new root node x_0, labeled by the $(n+1)$-vector $(1, 1, \ldots, 1)$ by the new arc (x_0, y_0) labeled as t_{n+2} and complete this tree by firing transition(s) at the marking vector(s) where nonzero components appear, till all the transitions become dead. In this way, the tree T^* so obtained has all the binary $(n+1)$-vectors as its node labels, possibly with repetitions. It remains to show that T^* is indeed the reachability tree $R(S_{n+1}, \mu^0)$ of S_{n+1} with binary $(n+1)$-vector $(1, 1, 1, \ldots, 1)$ as its initial marking μ^0. For this, consider an arbitrary $(n+1)$-vector $\mu = (a_1, a_2, a_3, \ldots, a_n, 1)$, where $a_i \in \{0, 1\}, \forall\, i$. When transition t_{n+1} is enabled, this yields

$$\mu'(p_i) = \mu(p_i) - I^-(p_i, t_{n+1}) + I^+(p_i, t_{n+1}) = 1 - 1 + 0 = 0$$

Then, we get a new marking $\mu' = (a_1, a_2, a_3, \ldots, a_n, 0)$, where $a_i \in \{0, 1\}$. The marking μ' is found in $R_0(S_{n+1}, \mu^0)$. If all a_i's are zero, then μ' is a dead marking. Hence, suppose some $a_i \neq 0$. In this case, t_i is enabled and in the next new marking μ'', the ith component is reduced to zero. Eventually, this process will lead to a dead marking. Further, the marking vectors of the form $\mu = (a_1, a_2, a_3, \ldots, a_n, 0)$ are already obtained as a result of firing $t_1, t_2, t_3, \ldots, t_n$ through some subsequences by virtue of the induction hypothesis. Thus, T^* is precisely the reachability tree $R(S_{n+1}, \mu^0)$ of S_{n+1}. Hence, the result follows by PMI. □

4. Some general questions and a necessary condition

The above theorem opens not only the general problem of determining all such Petri nets but also raises the question of determining such optimal Petri nets ; for example, one can ask

1. Precisely which Petri nets produce the set of all binary n-vectors with minimum repetitions?

2. Precisely which Petri nets produce all the binary n-vectors in the smallest possible number of steps? As pointed out, these questions could be quite important from practical application point of view.

3. Do there exist Petri nets that generate every binary n-vector exactly once?

4. Is it not possible to take any marking other than $(1, 1, 1, \cdots, 1)$ as an initial marking for such a Petri net?

The following proposition and theorem answer the last two questions.

Proposition 1. *[6] If a Petri net is Boolean then $\mu^0(p) = 1, \forall p \in P$.*

Proof. Suppose $C = (P, T, I^-, I^+, \mu^0)$ is a Petri net which is Boolean and $\mu^0(p_i) \neq 1$ for some $p_i \in P$. By the definition of a Petri net, no place can be isolated. Therefore p_i has to be connected to some $t_i \in T$. Now, three cases arise for consideration:

Case-1: $p_i \in t_i^\bullet$,
Case-2: $p_i \in {}^\bullet t_i \cap t_i^\bullet$, and
Case-3: $p_i \in {}^\bullet t_i$

In Case 1, since the given Petri net C is safe, ${}^\bullet t_i \neq \varnothing$ [2]. Therefore, $\exists\, p_j \in {}^\bullet t_i$ for some $p_j \in P$. p_j will have either one token or no token. If p_j has one token then t_i is enabled and hence fires. After firing of t_i, p_j will have no token and p_i will receive one token. So, both the places cannot have one token simultaneously. Hence, we will not get the marking vector whose components are all equal to 1. Again, if p_j has no token then t_i cannot fire, whence p_i will never receive a token, which contradicts the assumption of the case.

Case 2 follows from the arguments given for Case 1 above since, in particular, $p_i \in t_i^\bullet$.

Also, in Case 3, as in the proof of Case 1 $p_i \in {}^\bullet t_i$ implies that we cannot have the marking vector whose components are all equal to 1.

Thus, if a Petri net generates all the binary n-vectors then $\mu^0(p_i) = 1 \,\forall\, p_i \in P$. $\square$

5. Crisp Petri nets

Theorem 2. *[6] There exists a 1-safe Petri net with the initial marking $\mu^0(p) = 1, \forall p \in P$ which generates each of the 2^n binary n-vectors*

$$(a_1, a_2, a_3, \cdots, a_n),\ a_i \in \{0, 1\},\ n = |P|,$$

as one of its marking vectors, exactly once.

Proof. We shall prove this result again by using the PMI on $n = |P|$.

For $n = 1$, we construct a Petri net C_1 as shown in Figure 6. In this Petri net C_1,
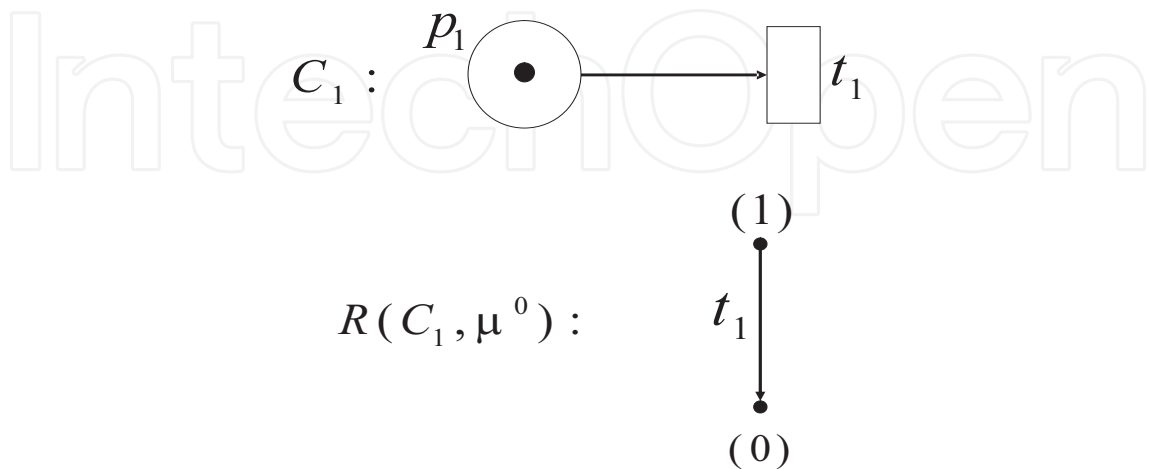

Figure 6. Petri net C_1 and $R(C_1, \mu^0)$

the total number of transitions $= 2^1 - 1 = 1$,

$|p_1^\bullet| = 2^1 - 1 = 1$,

$|^\bullet p_1| = 2^{1-1} - 1 = 0$,

$|^\bullet t_1| = 1$.

Total number of transitions whose post-sets having no element $= {}^1C_0 = 1$ and this transition is t_1. Clearly, $R(C_1, \mu^0)$ of C_1 generates both the binary 1-vectors (1) and (0) as shown in Figure 6 in the first step and after this step, transition becomes dead.

Next, for $n = 2$, the Petri net C_2 shown in Figure 7 has two places. In C_2, we have
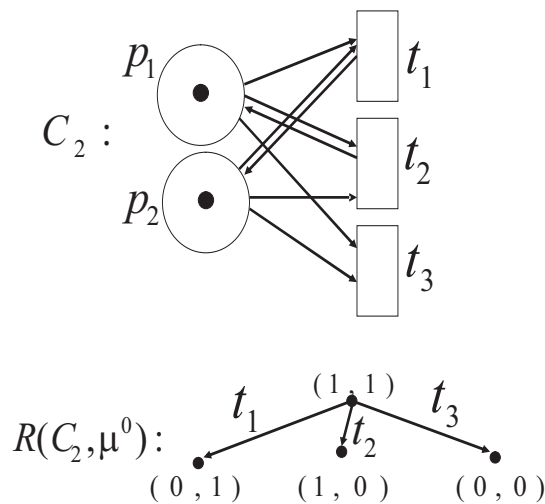

Figure 7. Petri net C_2 and $R(C_2, \mu^0)$

the total number of transitions $= 2^2 - 1 = 4 - 1 = 3$,

$|p^\bullet| = 2^2 - 1 = 3, \ \forall \ p$,

$|^\bullet p| = 2^{2-1} - 1 = 1, \ \forall \ p$,

$|^\bullet t| = 2, \ \forall \ t$.

The total number of transitions whose post-sets have one element $= {}^2C_1 = 2$ and these transitions are t_1, t_2.

The total number of transitions whose post-sets have no element $= {}^2C_0 = 1$ and this transition is t_3.

It is clear from Figure 7 that $R(C_2, \mu^0)$ has exactly $4 = 2^2$ binary 2-vectors (a_1, a_2), $a_1, a_2 \in \{0, 1\}$ in the first step and after this step, all the transitions become dead.

We can construct $R(C_2, \mu^0)$ from $R(C_1, \mu^0)$ as follows: Take two copies of $R(C_1, \mu^0)$. In the first copy, augment each vector of $R(C_1, \mu^0)$ by the adjunction of a '0' entry at the second coordinate of every marking vector and denote the resulting labeled tree as $R_0(C_1, \mu^0)$. Similarly, in the second copy, augment each vector by the adjunction of a '1' at the second coordinate of every marking vector and let $R_1(C_1, \mu^0)$ be the resulting labeled tree (see Figure 8). Now, using
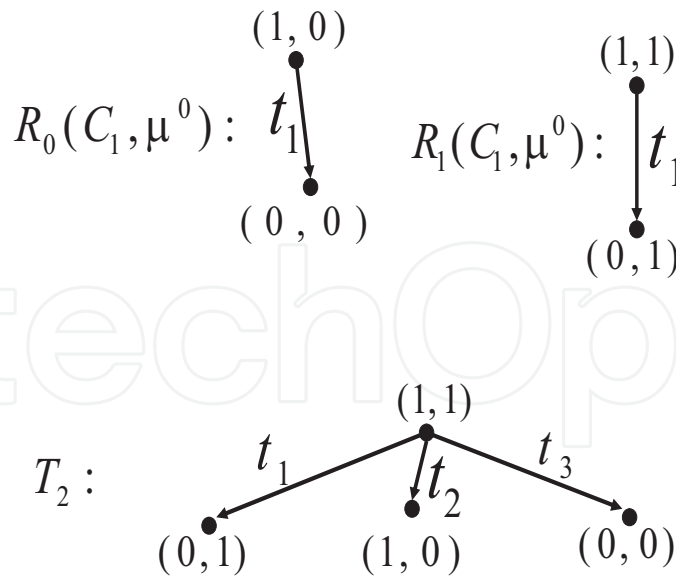
Figure 8. Augmented reachability trees and resulting labeled tree T_2

the following steps we construct the reachability tree $R(C_2, \mu^0)$ of C_2 from $R_0(C_1, \mu^0)$ and $R_1(C_1, \mu^0)$.

Step-1. Clearly, the binary 2-vectors in $R_0(C_1, \mu^0) \cup R_1(C_1, \mu^0)$ are all distinct and are exactly $2^2 = 4$ in number.

Step-2. In $R_0(C_1, \mu^0)$, none of the transitions t_j is enabled at $(1, 0)$.

Step-3. In $R_0(C_1, \mu^0)$, the root node $(1, 0)$ has the marking obtained after firing of transition t_2 in C_2. Hence, we join the root node $(1, 0)$ of $R_0(C_1, \mu^0)$ to the root node $(1, 1)$ of $R_1(C_1, \mu^0)$ by an arc labeled t_2 so that $(1, 0)$ would become the 'child node' obtained by firing t_2 in C_2. Next, we join the child node $(0, 0)$ of $R_0(C_1, \mu^0)$ to the root node $(1, 1)$ of $R_1(C_1, \mu^0)$ by an arc labeled t_3 so that $(0, 0)$ would become the child node obtained by firing t_3 in C_2. Then, the resulting labeled tree T_2 has exactly 2^2 binary 2-vectors as its set of nodes. T_2 is indeed the reachability tree of C_2 because in C_2 all the transitions t_1, t_2 and t_3 are enabled at the initial marking $(1, 1)$ and fire. Further, after firing of each transition, the new markings obtained by the rule

$$\mu'(p_i) = \mu^0(p_i) - I^-(p_i, t_j) + I^+(p_i, t_j)$$

are $(0, 1), (1, 0)$ and $(0, 0)$ respectively and no further firing takes place as the enabling condition fails to hold for these marking vectors; i.e., we get exactly $2^2 = 4$ binary 2-vectors in the first step only.

Next, suppose this result is true for $n = k$. That is, C_k is the 1-safe Petri net having k-places and $2^k - 1$ transitions $t_1, t_2, t_3, \cdots$, generating each of the 2^k binary k-vectors exactly once and having the structure as schematically shown in Figure 9 which has the following parameters:

$|p^\bullet| = 2^k - 1, \forall p,$
$|^\bullet p| = 2^{k-1} - 1, \forall p,$
$|^\bullet t| = k, \forall t.$

The total number of transitions whose post-sets have $k - 1$ elements $= {}^kC_{k-1} = {}^kC_1 = k$ and these transitions are $t_1, t_2, t_3, \cdots, t_k$.
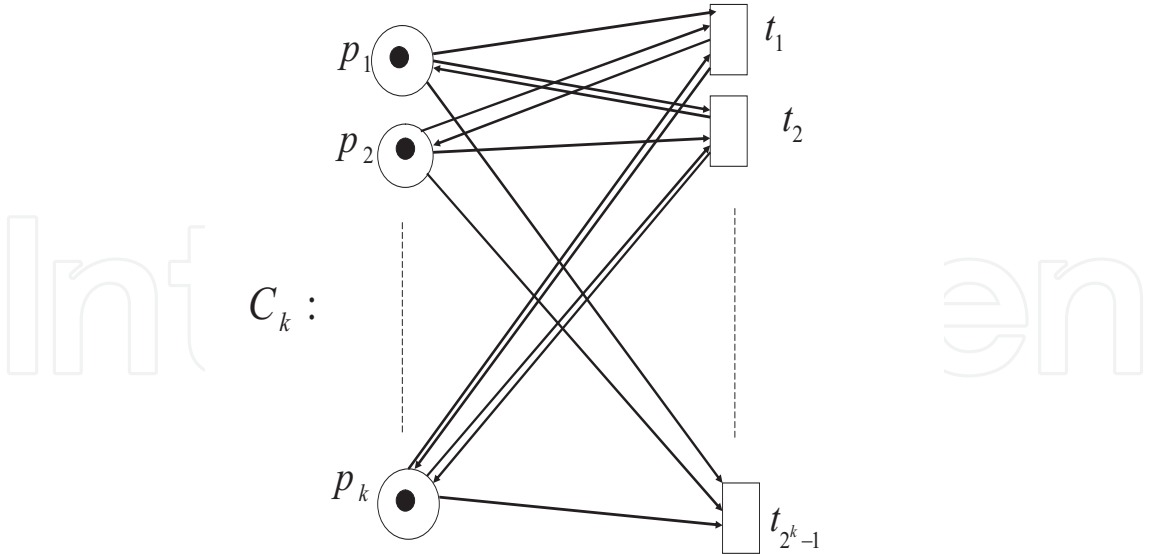
Figure 9. Petri net C_k for k places

The total number of transitions whose post-sets have $k-2$ elements $= {}^kC_{k-2} = {}^kC_2 = \frac{k(k-1)}{2}$ and these transitions are $t_{k+1}, t_{k+2}, t_{k+3}, \cdots, t_{\frac{k^2+k}{2}}$.

The total number of transitions whose post-sets have $k-3$ elements $= {}^kC_{k-3} = {}^kC_3 = \frac{k(k-1)(k-2)}{6}$ and these transitions are $t_{\frac{k^2+k+2}{2}}, t_{\frac{k^2+k+4}{2}}, t_{\frac{k^2+k+6}{2}}, \cdots, t_{\frac{k^3+5k}{6}}$.

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

The total number of transitions whose post-sets have one element $= {}^kC_1 = k$ and these transitions are $t_{2^k-k-1}, t_{2^k-k}, t_{2^k-k+1}, \cdots, t_{2^k-2}$.

The total number of transitions whose post-sets have no element $= {}^kC_0 = 1$ and this transition is t_{2^k-1}.

We will now prove the result for the 1-safe Petri net C_{k+1} having $k+1$ places and $t_{2^{k+1}} - 1$ transitions and having the structure shown schematically in **Figure-9**. For this purpose, take two copies of $R(C_k, \mu^0)$. In the first copy, augment each vector of $R(C_k, \mu^0)$ by the adjunction of a '0' entry at the $(k+1)^{th}$ coordinate of every marking vector and denote the resulting labeled tree as $R_0(C_k, \mu^0)$. Similarly, in the second copy, augment each vector by the adjunction of a '1' at the $(k+1)^{th}$ coordinate of every marking vector and let $R_1(C_k, \mu^0)$ be the resulting labeled tree. Now, using the following steps we construct the reachability tree $R(C_{k+1}, \mu^0)$ of C_{k+1} from $R_0(C_k, \mu^0)$ and $R_1(C_k, \mu^0)$.

Step-1. The induction hypothesis implies that the binary $(k+1)$-vectors in $R_0(C_k, \mu^0) \cup R_1(C_k, \mu^0)$ are all distinct and they are exactly $2^k + 2^k = 2^{k+1}$ in number.

Step-2. In $R_0(C_k, \mu^0)$, none of the transitions is enabled at $(1, 1, 1, \cdots, 0)$.

Step-3. In $R_0(C_k, \mu^0)$, the root node $(1, 1, 1, \cdots, 0)$ is the marking obtained after firing of transition t_{k+1} in C_{k+1}. Hence, we join the root node $(1, 1, 1, \cdots, 0)$ of $R_0(C_k, \mu^0)$ to the

root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$ by an arc labeled t_{k+1} so that $(1, 1, 1, \cdots, 0)$ would become the child node obtained by firing t_{k+1} in C_{k+1} and in $R_1(C_k, \mu^0)$ the child node $(0, 0, 0, \cdots, 1)$ is the marking obtained after firing of the transition t_{k+2} at the root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$; so, we replace the arc labeled as t_{k+1} by t_{k+2} in $R_1(C_k, \mu^0)$. Next, we join the remaining $(2^{k+1} - 1) - \overline{k+2}$ child nodes $(0, 1, 0, \cdots, 0)$, $(1, 0, 0, \cdots, 0)$, $\cdots, (0, 0, 0, \cdots, 0)$ of $R_0(C_k, \mu^0)$ to the root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$ by an arc each, labeled $t_{k+3}, t_{k+4}, t_{k+5}, \cdots, t_{2^k-1}$ respectively, so that $(0, 1, 0, \cdots, 0)$, $(1, 0, 0, \cdots, 0)$, $\cdots$, $(0, 0, 0, \cdots, 0)$ would become the marking vector obtained after firing of t_{k+3}, t_{k+4}, t_{k+5}, $\cdots, t_{2^k-1}$ respectively in C_{k+1}. Then the resulting labeled tree T_{k+1} has exactly 2^{k+1} binary $(k+1)$-vectors. T_{k+1} is indeed the reachability tree of C_{k+1} because in C_{k+1} all the transitions are enabled at the initial marking $(1, 1, 1, \cdots, 1)$ and fire. After firing, the new markings obtained by the rule

$$\mu'(p_i) = \mu^0(p_i) - I^-(p_i, t_j) + I^+(p_i, t_j)$$

are

$$(0, 1, 1, \cdots, 1), (1, 0, 1, \cdots, 1), (1, 1, 0, \cdots, 1), \cdots, (0, 0, 0, \cdots, 0)$$

respectively and no further firing takes place as the enabling condition fails to hold for these marking vectors; i.e., we get exactly 2^{k+1} binary $(k+1)$-vectors, each generated exactly once in the first step itself.

It is clear that the Petri net constructed above generates each of the 2^n binary n-vectors exactly once in the very first step and, hence, is the smallest number of steps because no firing will take place after that step.

Hence, the result follows by the PMI.   □

Hence, we shall call a Boolean Petri net *crisp* if it generates every binary $n-$vector exactly once.

It may be observed from the above proof that the Petri net constructed therein yields all the binary n-vectors as marking vectors in the least possible number of steps. Such a Boolean Petri net will be called *optimal*.

6. Uniqueness of minimal crisp Petri net

The problem of characterizing 1-safe Petri nets generating all the 2^n binary n-vectors as marking vectors exactly once is an open problem [6]. We completely settle a part of this problem, viz., to determine minimal such Petri nets, 'minimal' in the sense that the depth of their reachability tree is minimum possible, where the *depth* of a rooted tree is defined as the maximum distance of any vertex in it from the root. In fact, we show here that such a 1-safe Petri net has a unique structure.

Theorem 3. *[8] The Petri net $C=(P, T, I^-, I^+, \mu^0)$ constructed in theorem 2 is the only minimal Crisp Petri net and the underlying graph of its reachability tree is isomorphic to the star $\downarrow K_{1, 2^n-1}$, where '$\downarrow$' indicates the fact that arcs of the reachability tree of C are oriented downward from its root which is the center of $K_{1, 2^n-1}$.*

Proof. The existence of C has already been established in Theorem 2. We will establish here the uniqueness of C. Suppose there exists a Petri net

$$C' = (P', T', I'^{-}, I'^{+}, \mu^0)$$

satisfying the hypothesis of the theorem. This implies, in particular that the reachability graph $R(C', \mu^0)$ of C' is isomorphic to the reachability graph $R(C, \mu^0)$ of C, i.e.,

$$R(C', \mu^0) \cong R(C, \mu^0) \cong\downarrow K_{1,2^n-1}.$$

Now, we need to show that $C' \cong C$.

Toward this end, define a map $\psi : P' \cup T' \longrightarrow P \cup T$ satisfying $\psi(p'_i) = p_i$ and $\psi(t'_i) = t_i$. Clearly, ψ is a bijection. We shall now show that it preserves the directed adjacency of C' onto C. For this, consider any isomorphism $\varphi : M(C', \mu^0) \longrightarrow M(C, \mu^0)$ from the reachability set $M(R(C', \mu^0))$ of C' onto the reachability set of C; this has the property that

$$(\mu^0, \mu^i) \in \mathcal{A}(R(C', \mu^0)) \Leftrightarrow (\varphi(\mu^0), \varphi(\mu^i)) \in \mathcal{A}(R(C, \mu^0)), \tag{1}$$

where $\mathcal{A}(D)$ denotes the set of arcs of any digraph D (in this case, D is the reachability graph of the corresponding Petri net).

Let (p'_i, t'_j) be an arc in C', we will show then that $(\psi(p'_i), \psi(t'_j))$ is an arc in C. Suppose, on the contrary $(\psi(p'_i), \psi(t'_j)) = (p_i, t_j)$ is not an arc in C. This implies in C that the marking vector μ^i whose i^{th} component is zero does not get generated by firing t_j or when $t_j^{\bullet} = \varnothing$ the marking vector obtained by firing t_j is repeated. The latter case does not arise due to the hypothesis that every marking vector is generated exactly once in C. But, then the former statement implies $\varphi(\mu^0)$ does not form the arc $(\varphi(\mu^0), \varphi(\mu^i))$ in $R(C, \mu^0)$ and hence, from (1), it follows that (μ^0, μ^i) does not form an arc in the reachability tree $R(C', \mu^0)$. This is a contradiction to our assumption that C' generates all the binary n-vectors exactly once. Similarly, one can arrive at a contradiction by assuming $(\psi(t'_j), \psi(p'_i))$ is not an arc in C. Thus, $C' \cong C$ follows, because the choice of the arcs (p'_i, t'_j) and (t'_j, p'_i) was arbitrary in each case. ☐

7. A Boolean Petri net whose reachability graph is homomorphic to the complete lattice

As mentioned already, Boolean Petri nets generating all the 2^n binary n-vectors as their marking vectors are not only of theoretical interest but also are of practical importance. We demonstrate the existence of a disconnected 1-safe Petri net whose reachability tree is homomorphic to the n-dimensional complete lattice L_n. This makes the problem of characterizing the crisp Boolean Petri nets appear quite intricate.

Definition 4. *Given any graph $G = (V, E)$, by a homomorphism of G we mean a partition $\{V_1, V_2, \dots, V_t\}$ of its vertex-set $V(G) := V$ such that for any $i \in \{1, 2, \dots, t\}$ no two distinct vertices in V_i are adjacent; in other words, V_i is an independent set of G. In general, given any partition $\pi = \{V_1, V_2, \dots, V_t\}$ (not necessarily a homomorphism) of G, the partition graph with respect to π of G, denoted $\pi(G)$, is the graph whose vertex-set is π and any two vertices V_i and V_j are adjacent whenever there exist vertices $x \in V_i$ and $y \in V_j$ such that x and y are adjacent in G, that is, whenever, $xy \in E(G)$. If, in particular, π is a homomorphism then $\pi(G)$ is called a homomorphic image of G; further, a graph H is homomorphic to a graph G if there exists a homomorphism π of H such that $\pi(H) \cong G$ (read as "$\pi(H)$ is 'isomorphic to' G").*

Theorem 4. *[7] Let $C_n = (P, T, I^-, I^+, \mu^0)$ be the 1-safe Petri net consisting of n connected components, each isomorphic to $C^\star := \odot \longrightarrow \square$. Then the reachability tree of C_n is homomorphic to the n-dimensional complete lattice L_n.*

Proof. We prove this result by using the PMI on the number of connected components each isomorphic to $C^\star$.

Let $n = 1$. That is, C_1 has only one connected component $C^\star$, whence $C_1 = C^\star$. Then, the reachability tree of C_1 is the 1-dimensional complete lattice L_1, in which the direction of the 'link' (or, 'arc') $((1), (0))$ between the two 1-dimensional marking vectors (1) and (0) is shown as the 'vertical' one, as in Figure 10. The arc $((1), (0))$ in L_1, labeled as t_1, signifies the fact that the transition t_1 fires at (1), moving the only token out of the place p_1 resulting in the next state of the Petri net in which t_1 is 'dead' in the sense that it no longer fires at (0). Therefore, the next state of the Petri net C_1 is determined by the *zero vector* (0) as the marking vector of C_1. Thus, C_1 has just two states, viz., the 'active' one represented by the 1-dimensional 'unit vector' (1) and the 'dead' one represented by the 1-dimensional zero vector (0). Hence, the entire 'dynamics' of C_1 is completely represented by L_1. Next, consider $n = 2$. That is, we
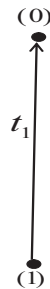

Figure 10. Lattice L_1

have the 1-safe Petri net C_2, consisting of two connected components, each isomorphic to $C^\star$ as shown in Figure 11, along with its reachability tree (seen as a connected acyclic digraph) that is isomorphic to the 2-dimensional complete lattice L_2. In C_2, the transitions t_1 and t_2 are
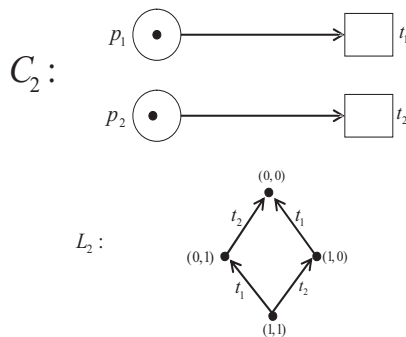

Figure 11. Petri net C_2 and its directed reachability tree L_2

both enabled. After firing t_1 and t_2 at the node $(1, 1)$ successively, in the first step, we get the marking vectors $(0, 1)$ and $(1, 0)$ respectively. Here, we fix the direction of t_2 to be 'orthogonal' to that of t_1 in L_2. Further, at $(0, 1)$ the transition t_2 is enabled, which fires in the same direction

as in the first step giving the marking $(0,0)$. Subsequently, at $(1,0)$ transition t_1 is enabled, which fires in the same direction as in its previous step of firing giving the marking $(0,0)$. After this second step of firing, both the transitions t_1 and t_2 become dead at $(0,0)$. Thus, it is clear from Figure 11 that the reachability tree L_2 of C_2, seen as a connected acyclic digraph, has exactly $4 = 2^2$ binary vectors (a_1, a_2), $a_1, a_2 \in \{0,1\}$ as the marking vectors of C_2.

We can construct L_2 tactically from L_1 as follows.

Step1. Take two copies of L_1. In the first copy, augment each vector of L_1 by one extra coordinate position on the right by putting a 0 entry in that position and denote the resulting labeled copy of L_1 as L_1^0. Similarly, in the second copy, augment each vector by one extra coordinate position on the right by filling it with 1 and denote the resulting labeled copy of L_1 as L_1^1.

Step2. Take the union $L_1^0 \cup L_1^1$ and augment the new 'edges' (i.e., undirected line segments) joining those pairs of nodes whose marking vectors are at unit Hamming distance from each other. Direct each of these edges from the node, represented by its marking vector, at which t_2 fires to the node whose label (i.e., marking vector) gives the result of that firing. Accordingly, label each of such arcs by the label t_2.

Thus, the directed arcs labeled t_2 join every node of L_1^1 to exactly one node in L_1^0 in a bijective manner as shown in Figure 12. In this way, we see that the resulting discrete structure L_2^* has $4 = 2^2$ nodes which correspond to 2^2 binary vectors (a_1, a_2), $a_1, a_2 \in \{0,1\}$. Clearly, L_2^* is nothing but the reachability tree L_2 of C_2, seen as a connected acyclic digraph. Next, consider
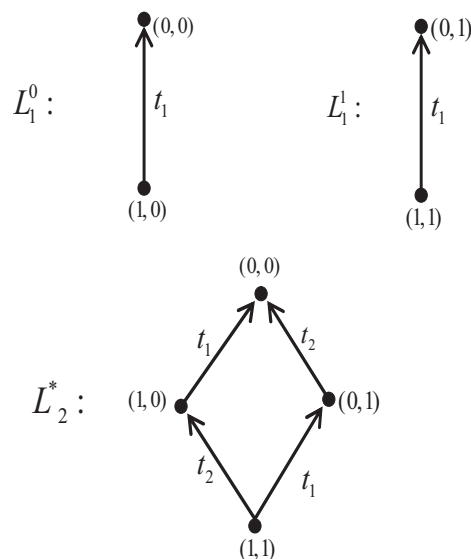

Figure 12. Augmented lattices and resulting complete lattice for 2 places

$n = 3$. That is, we have the 1-safe Petri net C_3 consisting of three connected components each isomorphic to C^* as shown in Figure 13. In C_3, all the three transitions t_1, t_2, t_3 are enabled. Hence, after firing the transitions t_1, t_2 and t_3 successively at $(1,1,1)$ we get the marking vectors $(0,1,1), (1,0,1)$ and $(1,1,0)$ respectively, in first step. Right here, we fix the directions of t_1, t_2 and t_3 so as to be orthogonal to each other. Further, at $(0,1,1)$ the transitions
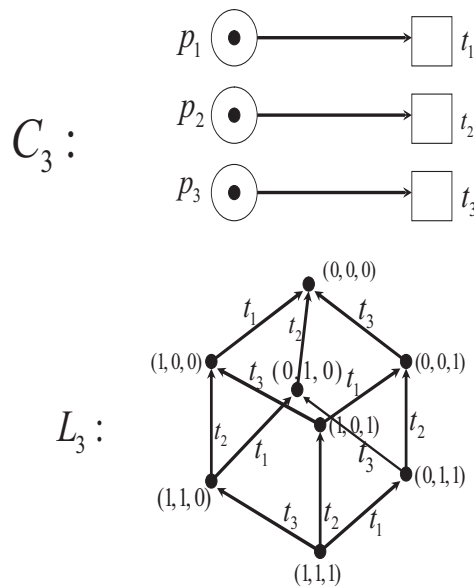
Figure 13. Petri net C_3 with 3 places and its complete lattice

t_2 and t_3 are enabled. After firing them successively we get the marking vectors $(0,0,1)$ and $(0,1,0)$, respectively. Subsequently, firing t_1 and t_3 at $(1,0,1)$ will give the marking vectors $(0,0,1)$ and $(1,0,0)$ respectively, whereas the firing of t_1 and t_2 at $(1,1,0)$ give the marking vectors $(0,1,0)$ and $(1,0,0)$. On continuing the process of firing in the next (i.e., the third) step we get the marking vector $(0,0,0)$ at which no transition is enabled. So, we have the reachability tree L_3 of C_3, which is isomorphic to the 3-dimensional complete lattice L_3, seen as a connected acyclic digraph.

We can construct L_3 from L_2 as follows.

Step1. Take two copies of L_2. In the first copy, augment each vector of L_2 by one extra coordinate position on the extreme right by putting a 0 entry in that position; denote the resulting labeled copy of L_2 as L_2^0. Similarly, in the second copy, augment each vector by one extra coordinate position on the extreme right by filling it with 1; denote the resulting labeled copy of L_2 as L_2^1.

Step2. Take the union $L_2^0 \cup L_2^1$ and augment the new 'edges' (i.e., undirected line segments) joining those pairs of nodes whose marking vectors are at unit Hamming distance from each other. Direct each of these edges from the node, represented by its marking vector, at which t_3 fires, to the node whose label (i.e., marking vector) gives the result of that firing. Accordingly, label each of such arcs by the label t_3.

Thus, the directed arcs labeled t_3 join every node of L_2^1 to exactly one node in L_2^0 in a bijective manner as shown in Figure 14. In this way, we see that the resulting discrete structure L_3^* has $8 = 2^3$ nodes which correspond to 2^3 binary vectors (a_1, a_2, a_3), $a_1, a_2, a_3 \in \{0,1\}$. Clearly, L_3^* is nothing but the reachability tree L_3 of C_3, seen as a connected acyclic digraph. Hence, let us assume that the result is true for $n = k$. That is, we have the 1-safe Petri net C_k consisting of k connected components each isomorphic to C^* and L_k is isomorphic to the reachability
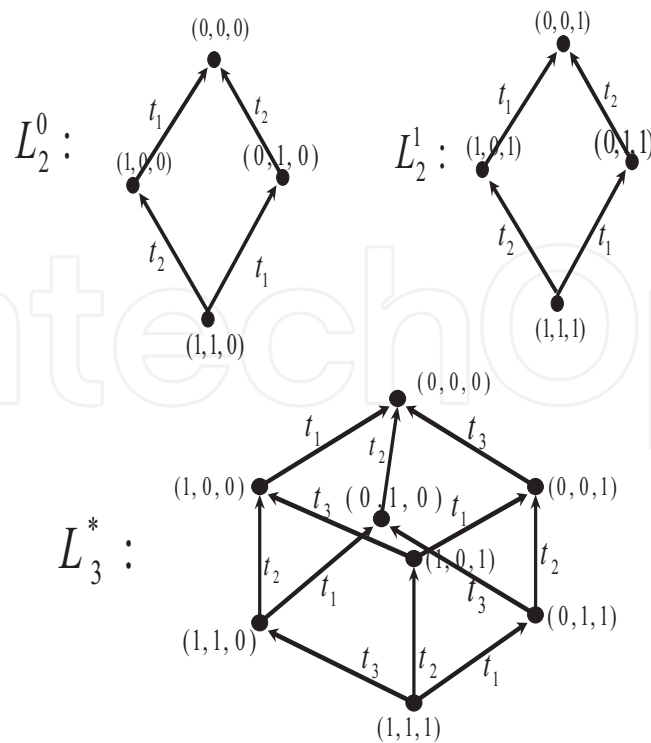
Figure 14. Augmented lattices and resulting complete lattice

tree of C_k, seen as a connected acyclic digraph. Now, we will prove that the result is true for $n = k + 1$.

Note that, in C_k, all the k transitions $t_1, t_2, \ldots, t_k$ are enabled at the k-dimensional unit vector $(1, 1, \ldots, 1)$.

Step1. Take two copies of L_k. In the first copy, augment each vector of L_k by one extra coordinate position on the extreme right by putting a 0 entry in that position; denote the resulting labeled copy of L_k as L_k^0. Similarly, in the second copy, augment each vector by one extra coordinate position on the extreme right by filling it with 1; denote the resulting labeled copy of L_k as L_k^1.

Step2. Take the union $L_k^0 \cup L_k^1$ and augment the new 'edges' (i.e., undirected line segments) joining those pairs of nodes whose marking vectors are at unit Hamming distance from each other. Direct each of these edges from the node, represented by its marking vector, at which t_{k+1} fires, to the node whose label (i.e., marking vector) gives the result of that firing. Accordingly, label each of such arcs by the label t_{k+1}.

It is now enough to show that L_{k+1} is indeed isomorphic to the reachability tree of C_{k+1}, seen as a connected acyclic digraph. Towards this end, consider the sets $A_0 = \{(a_1, a_2, \ldots, a_{k+1}) : a_{k+1} = 0\}$ and $A_1 = \{(a_1, a_2, \ldots, a_{k+1}) : a_{k+1} = 1\}$ Clearly, the subdigraph induced by A_0 is 'label-isomorphic' to L_k^0 whose nodes are labeled by the 2^k $(k+1)$-dimensional vectors in A_0 and the subdigraph induced by A_1 is 'label-isomorphic' to L_k^1 whose nodes are labeled by the 2^k $(k+1)$-dimensional vectors in A_1. Every arc in $L_k^0 \cup L_k^1$ is labeled by one of the transitions $t_1, t_2, \ldots, t_k$ in such a way that for any two indices $i, j \in \{1, 2, \ldots, k\}$ t_i and t_j are in orthogonal or parallel directions in each of L_k^0 and L_k^1 according to whether $i \neq j$ or $i = j$; thus, by Step 1

and the induction hypothesis, each of the $k2^{k-1}$ arcs in L_k^0 (respectively, in L_k^1) represents one of the transitions $t_1, t_2, \ldots, t_k$ fired in accordance with the firing rule, yielding the next state marking vector from its previous state marking vector that is at unit Hamming distance from the former. Now, by Step 2, the edges joining those pairs of nodes whose marking vectors are at unit Hamming distance from each other are directed in such a way that each of the resulting arcs represents the firing of a new transition t_{k+1} at the node, represented by its marking vector in the reachability tree of C_{k+1}, yielding the node of the reachability tree of C_{k+1} whose label (i.e., marking vector) gives the result of that firing. Accordingly, label each of such arcs by the label t_{k+1}. Since no two marking vectors in A_0 or in A_1 are interconnected by an arc labeled t_{k+1} in the above scheme, it follows that every arc labeled t_{k+1} has its initial node in A_1 and terminal node in A_0, signifying the fact that t_{k+1} fires at its initial node in A_1 and yields the next state marking vector that belongs to A_0. Further, no two of these arcs have a common node (whence we say that they are *independent*). Also, every node in A_1 is joined to a unique node in A_0 at Hamming distance one by an arc labeled t_{k+1} in a bijective way and, therefore, the number of such arcs is 2^k.

Next, consider the node labeled $(0, 0, \ldots, 0)$ in L_{k+1}. The only arcs incoming at this node are from the nodes that are at unit Hamming distance from it, viz., those that are labeled by the elementary coordinate vectors $(1, 0, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $(0, 0, 1, \ldots, 0)$, $\ldots$, $(0, 0, 0, \ldots, 1)$ and, hence, the corresponding arcs are labeled $t_1, t_2, \ldots, t_k, t_{k+1}$. Consequently, all these transitions become dead at the node labeled by the $(k + 1)$-dimensional zero vector $(0, 0, \ldots, 0)$ as its marking vector.

The foregoing arguments imply that L_{k+1} indeed represents the reachability tree of C_{k+1}, being a connected acyclic digraph, invoking the PMI.

Now, for any arbitrary positive integer n, construct the partition π_H of the reachability tree $R(C_n, \mu^0)$ by defining its 'parts' (which are subsets of the nodes of $R(C_n, \mu^0)$) by letting $V_0 = \{(1, 1, \ldots, 1)\}$ and $V_i = \{(a_1, a_2, \ldots, a_n) : d_H((1, 1, \ldots, 1), (a_1, a_2, \ldots, a_n)) = i\}$, for each $i \in \{1, 2, \ldots, n\}$, where $d_H(A, B)$ denotes the Hamming distance between the vectors A and B of the same dimension. Clearly, $|V_i| = {}^nC_i$ for each $i \in \{0, 1, 2, \ldots, n\}$, where ${}^nC_k = \frac{n!}{k!(n-k)!}$ will in general denote the number of ways in which k objects can be selected out of n given objects. Now, consider the mapping $\eta^{\mu^0} : V(R(C_n, \mu^0)) \to \{0, 1\}^n$ that assigns to each node $u \in R(C_n, \mu^0)$ the marking vector derived by the sequence of transitions fired starting from the initial marking vector μ^0 as specified by the unique path from the root vertex u_0 whose marking vector is μ^0. Hence, we consider the *refinement* π'_H of π_H defined as follows: For each $i \in \{0, 1, 2, \ldots, n\}$ and for each $(a_1, a_2, \ldots, a_n) \in V_i$, let $U_{(a_1, a_2, \ldots, a_n)}^i = \{u \in V(R(C_n, \mu^0)) : \eta^{\mu^0}(u) = (a_1, a_2, \ldots, a_n)\}$. Then, clearly, U^i's form a partition of the set V_i for each $i \in \{0, 1, 2, \ldots, n\}$. It may be easily verified that no two marking vectors in V_i are adjacent in $R(C_n, \mu^0)$, whence π'_H is a homomorphism of $R(C_n, \mu^0)$. Further, the homomorphic image $\pi'_H(R(C_n, \mu^0)) \cong L_n$ because (V_i, V_{i+1}) is an arc in L_n if and only if there is an arc from a marking vector in V_i to a marking vector in V_{i+1} in $R(C_n, \mu^0)$.

This completes the proof.   □

In the above theorem, we have shown that by fixing the sequence of transitions in a Petri net for firing tactfully, one can produce the complete Boolean lattice as a homomorphic image of its reachability tree. One can perhaps produce many such interesting results like, for instance,

getting the vertices of a regular polyhedron in terms of marking vectors and using them for analysis of Boolean circuits with given properties. This raises another new question, viz., to characterize 1-safe Petri nets whose reachability trees have a given property $\mathcal{P}$.

Note that given any 1-safe Petri net C of order n, its reachability tree $R(C, \mu)$, with an arbitrary binary n-dimensional vector μ as its root, is essentially finite (cf.: [11]) and has all its nodes labeled by the function η^μ into the vertex set of the Boolean complete lattice L_n, possibly with repetitions of marking vectors. Consider the *Hamming distance partition* $\pi_H := \pi_H(V(R(C, \mu))$ of the vertex-set of $R(C, \mu)$ defined by letting $V_0 = \{\mu\}$ and

$$V_i = \{(a_1, a_2, \ldots, a_n) : d_H(\mu, (a_1, a_2, \ldots, a_n)) = i\},$$

for each $i \in \{1, 2, \ldots\}$. Then, its refinement $\pi'_H = \{V_0, V_1, V_2, \ldots, V_n\}$ is a homomorphism of $R(C, \mu)$ into a connected sublattice of L_n. Thus, we are lead to the problem of determining 1-safe Petri nets whose reachability trees are homomorphic to a given sublattice of L_n. First of all, we have the question whether for any arbitrarily given connected sublattice L of L_n there exists a 1-safe Petri net whose usual reachability tree is homomorphic to L. We have a conjecture that the answer to this question is in the affirmative.

Next, given a connected sublattice L of L_n, let $\mathcal{C}_L^n$ denote the set of all 1-safe Petri nets of order n whose reachability trees are homomorphic to L. Let $\mathcal{L}_n$ denote the set of all pairwise nonisomorphic connected sublattices of L_n. Clearly, $\{\mathcal{C}_L^n : L \in \mathcal{L}_n\}$ is a partition of the set $\mathcal{S}_1$ of all 1-safe Petri nets of order n. In other words, 1-safe Petri nets in any one of the sets in $\mathcal{C}_L^n$ are all 'equivalent' in the sense that the dynamics of any two of them are in accordance with the given connected sublattice L of L_n; thus, it is enough to pick any one of them so that we have an option to choose the 'required' one as per our practical constraints.

8. Towards characterizing Boolean Petri nets

We discuss here some necessary and sufficiency conditions for a 1-safe Petri net to be Boolean.

Lemma 1. *[9] If a 1-safe Petri net* $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ *is Boolean then* $p \in t^\bullet \Rightarrow p \in {}^\bullet t$.

Proof. Suppose $p \in t^\bullet$ and $p \notin {}^\bullet t$. Since C is 1-safe, ${}^\bullet t \neq \varnothing \ \forall \ t \in T$ (see [2]). This means that there exists at least one place $p_i \in P$ such that $p_i \in {}^\bullet t$. Further, since C is Boolean, $\mu^0(p)$=1 $\forall \ p \in P$ (Proposition 1) and, therefore, every transition is enabled. In particular, t is enabled. After firing of t the place p will receive 2 tokens ($\because \mu'(p) = \mu(p) - I^-(p, t) + I^+(p, t)$), which contradicts the fact that the Petri net is 1-safe. Hence, $p \in t^\bullet \Rightarrow p \in {}^\bullet t$. $\square$

Lemma 2. *[9] If a 1-safe Petri net* $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ *is Boolean then* $|P| \leq |T|$.

Proof. Since C generates all the binary n-vectors, it generates the marking vectors of the type $(0, 1, \cdots, 1), (1, 0, 1, \cdots, 1), \cdots, (1, 1, \cdots, 0)$, each having the Hamming distance 1 from the initial marking vector μ^0=$(1, 1, \cdots, 1)$. These n marking vectors can be obtained only in the very first step of firing because the marking vector whose Hamming distance is 1 from the initial marking cannot be obtained from any other marking vector whose Hamming distance is greater than or equal to 2 from the initial marking. These n marking vectors can be generated only if for every place $p_i \in P$, $i = 1, 2, 3, \cdots, n$, there exist n distinct transitions say $t_1, t_2, t_3, \cdots, t_n$ such that $p_i \in {}^\bullet t_i$ and $p_i \notin t_i^\bullet, \forall \ i = 1, 2, 3, \cdots, n$. Hence, $|P| \leq |T|$. $\square$

Lemma 3. *[9] If a 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ is Boolean then the incidence matrix I of C contains $-I_n$, the identity matrix of order n as a submatrix.*

Proof. Since C is Boolean, $\mu^0(p)=1 \; \forall \; p \in P$ (Proposition 1). Again, because of the generation of all the binary n-vectors, the vectors of the type $(0, 1, \cdots, 1), (1, 0, 1, \cdots, 1), \cdots, (1, 1, \cdots, 0)$ each at a Hamming distance 1 from the initial marking, have also been generated. These vectors can be obtained only in the first step of firing, as shown in Lemma 2. Therefore, $\forall$ $p_i \in P$, $i = 1, 2, 3, \cdots, n$, there exist n distinct transitions, say $t_1, t_2, t_3, \cdots, t_n$ such that $p_i \in {}^\bullet t_i$ and $p_i \notin t_i^\bullet$ and hence $I^-(p_i, t_j)=1$ if $i=j$ and 0 if $i \neq j$ and also $I^+(p_i, t_i) = 0 \; \forall \; i = 1, 2, 3, \cdots, n$. Since $I=I^+-I^-$, I contains $-I_n$ as a submatrix. $\qquad\square$

Lemma 4. *[9] If a 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ is Boolean then there exists at least one transition t such that $t^\bullet = \varnothing$.*

Proof. Suppose, under the hypothesis, there does not exist any $t \in T$ such that $t^\bullet = \varnothing$; i.e., $t^\bullet \neq \varnothing$ for every $t \in T$. Since $t^\bullet \neq \varnothing$, $p \in t^\bullet$ for some $p \in P$. Then, by Lemma 1, $p \in {}^\bullet t$. Then, at p, the number of tokens remains one throughout the dynamic states of C. This implies that the vector $(0, 0, \cdots, 0)$ would never occur as a marking vector, a contradiction to the hypothesis. Therefore, the lemma follows by contraposition. $\qquad\square$

Now, we will study necessary and sufficient conditions for a 1-safe Petri net that generates all the binary n-vectors as its marking vectors.

Theorem 5. *[9] A 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ with $t^\bullet = \varnothing \; \forall \; t \in T$ is Boolean if and only if*

1. $\mu^0(p) = 1 \; \forall \; p \in P$
2. $|P| \leq |T|$
3. *The incidence matrix I of C contains $-I_n$ as a submatrix.*

Proof. Necessity: This follows from Proposition 1, Lemma 2 and Lemma 3 above.

Sufficiency: Given the hypothesis and conditions (1), (2) and (3), we claim that C is Boolean. Since $I = I^+ - I^-$ and $t^\bullet = \varnothing, \; \forall \; t \in T$, $I^+ = 0$. This implies that $I = -I^-$. Since I contains $-I_n$ as a submatrix, $\forall \; p_i \in P$, $\exists \; t_i \in T$ such that $p_i \in {}^\bullet t_i \; \forall \; i = 1, 2, \cdots, n$. Also, $\mu^0(p) = 1 \; \forall p \in P$. Therefore, all the n transitions $t_1, t_2, \cdots, t_n$ are enabled and fire. After firing, we get distinct ${}^n C_1 = n$ marking vectors whose Hamming distance is 1 from the initial marking vector. At these n new marking vectors, $n - 1$ transitions are enabled and give at least ${}^n C_2$ distinct marking vectors, each of whose Hamming distance is 2 from the initial marking. Therefore this set of new vectors contains at least ${}^n C_2$ new distinct binary $n-$vectors.

In general, at any stage j, $3 \leq j \leq n$, we get a set of at least ${}^n C_j$ new distinct binary n-vectors whose Hamming distance is j from the initial marking, which are also distinct from the sets of ${}^n C_r$ distinct marking vectors for all r, $2 \leq r \leq j - 1$. Therefore, at the nth stage we would have obtained at least ${}^n C_1 + {}^n C_2 + \cdots + {}^n C_n = 2^n - 1$ distinct binary n-vectors. Together with the initial marking $(1, 1, \cdots, 1)$, we thus see that all the 2^n binary n-vectors would have been obtained as markings vectors, possibly with repetitions. $\qquad\square$

Theorem 6. *[9] A 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ with $I^-(p_i, t_j) = 1 \; \forall \; i, j$ is Boolean if and only if there exist at least ${}^nC_r$, $r = 1, 2, \cdots, n$ distinct transitions $t \in T$ such that $|t^\bullet| = n - r$, where r is the Hamming distance of any binary n-vector from the initial marking $(1, 1, \cdots, 1)$.*

Proof. Necessity: Since C generates all the binary n-vectors, we have binary n-vectors $(0, 1, 1, \cdots, 1)$, $(1, 0, 1, \cdots, 1)$, $(1, 1, 0, \cdots, 1)$, $\cdots$, $(1, 1, 1, \cdots, 0)$, whose Hamming distance is 1 from the initial marking $(1, 1, 1, \cdots 1)$. They are n in number. Since $I^-(p_i, t_j) = 1 \; \forall i, j$, these vectors can be obtained only if $I^+(p_i, t_j) = 0$ for $i = j$ and 1 for $i \neq j$, $1 \leq j \leq n$. This implies that there are at least ${}^nC_1$ distinct transitions say $t_1, t_2, \cdots, t_n$ such that $|t^\bullet| = n - 1$. After firing, they become dead. Further, we also have the binary n-vectors $(0, 0, 1, \cdots, 1)$, $(1, 0, 0, 1, \cdots, 1)$, $(1, 0, 1, 0, 1, \cdots, 1)$, $\cdots$, $(1, 1, \cdots, 1, 0, 0)$ whose Hamming distance is 2 from $(1, 1, 1, \cdots 1)$, $r = 1, 2, \ldots, n$. These vectors are ${}^nC_2$ in number and can be obtained only if there exist at least ${}^nC_2$ distinct transitions with $|t^\bullet| = n - 2$. In general, there are at least ${}^nC_r$ distinct transitions t such that $|t^\bullet| = n - r$, that yield ${}^nC_r$ binary n-vectors at Hamming distance r from $(1, 1, 1, \cdots 1)$, $r = 1, 2, \ldots, n$.

Sufficiency: Since $\mu^0(p) = 1 \; \forall p$, all the transitions are enabled and fire. After firing they all become dead as $I^-(p_i, t_j) = 1 \; \forall \; i, j$. This implies that the matrix I^- is of order $n \times m$ where $m \geq 2^n - 1$ and the matrix I^+ gets constructed as follows. By hypothesis, there are at least ${}^nC_1 = n$ distinct transition in C say $t_1, t_2, \cdots, t_n$ which on firing generate all the binary n-vectors each having exactly one zero because $|t_i^\bullet| = n - 1$ (w.l.o.g., we assume that there is no arc from t_i to p_i i.e., $I^+(p_i, t_i) = 0$ for $i = 1, 2, \cdots, n$). Thus, we place the transpose of these binary n-vectors as the first n-columns in I^+ matrix. Next, by hypothesis, we have ${}^nC_2$ distinct transitions, say $t_{n+1}, t_{n+2}, \cdots, t_{{}^nC_2}$, such that $|t_j^\bullet| = n - 2$. Since they all become dead after firing and $|t_j^\bullet| = n - 2$ for all $n + 1 \leq j \leq {}^nC_2$ these must generate all the distinct binary n-vectors each having exactly two zeros. Hence, the transpose of these ${}^nC_2$ vectors are placed as columns in the matrix I^+ immediately after the previous $n = {}^nC_1$ columns. We are thus enabled by the hypothesis to construct the submatrix H of order $n \times (2^n - 1)$ of I^+ which contains all the $2^n - 1$ distinct binary n-vectors, the last column of H being the all zero n-vector. We may augment to H the initial all-one n-vector as a column either on the extreme left or on the extreme right of H in I^+. Let the so augmented submatrix of I^+ have more columns. That means, each one of them is a repetition of some column in H. Thus, we see that the Petri net C generates all the binary n-vectors as its marking vectors. $\square$

Definition 5. *A Petri net $C = (P, T, I^-, I^+, \mu^0)$ is said to have a **Strong chain cycle** (SCC) Z if Z is a subnet satisfying $|{}^\bullet t| = 2$, $|p^\bullet| = 2$ and $|t^\bullet| = 1 \; \forall \; p, t \in Z$ (See Figure 15). Any SCC is said to become a **strong chain** after the removal of the arcs of any one of its self loops.*

Theorem 7. *[9] A 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ having an SCC Z, covering all the places, is Boolean if and only if*

1. $\mu^0(p) = 1 \; \forall \; p \in P$

2. *there exists at least one transition t outside Z such that $t^\bullet = \varnothing$.*

Proof. Necessity: Suppose that the 1-safe Petri net C with an SCC covering all the places is Boolean.
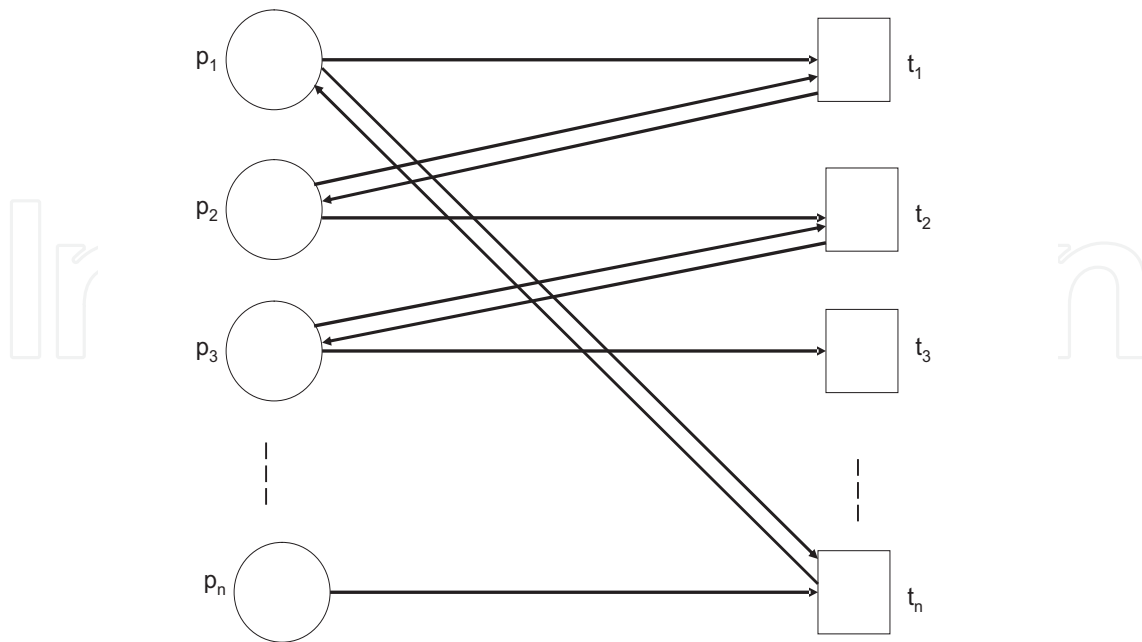
Figure 15. Strong Chain Cycle

In part 1 of the statement of the theorem, let $\mu^0(p_i) \neq 1$ for some $p_i \in P$. Then, $\mu^0(p_i) = 0$. Since C has an SCC, we cannot get one token in the place p_i. So, we cannot get all the binary n-vectors, which is a contradiction to the hypothesis.

In part 2 of the theorem, since C generates all the binary n-vectors, we have $(0, 0, \cdots, 0)$ as a marking vector. This vector can be obtained only if there is a transition t such that $t^\bullet = \varnothing$, by virtue of Lemma 4. We claim that such a transition t does not belong to Z. Suppose $t \in Z = (p_1, t_1, p_2, t_2, p_3, t_3, \cdots, p_{n-1}, t_{n-1}, p_n, t_n, p_1)$ where (p_i, t_i) is a single arc in C and (t_i, p_{i+1}) is a symmetric arc in C. This means, $t = t_i$ for some i, $1 \leq i \leq n$. This implies, $|t^\bullet| = |t_i^\bullet| = 1$, which is a contradiction to our assumption that $t^\bullet = \varnothing$. Therefore, t does not belong to any SCC.

Sufficiency: Since $\mu^0(p) = 1 \ \forall \ p \in P$, all the transitions belonging to Z are enabled and fire. After firing, they give ${}^nC_1$ distinct binary n-vectors $a_1 = (0, 1, 1, \cdots, 1)$, $a_2 = (1, 0, 1, \cdots, 1)$, $\cdots$, $a_i = (1, 1, \cdots, 1, 0, 1, \cdots, 1)$, $\cdots$, $a_n = (1, 1, \cdots, 1, 0)$, whose Hamming distance is 1 from μ^0, since $p_i^\bullet = \{t_{i-1}, t_i\}$ for $i > 1$ and $p_1^\bullet = \{t_1, t_n\}$, at each of these vectors a_i, exactly $(n-2)$ remaining transitions on Z are enabled and fire. After firing them, we get at least ${}^nC_2$ distinct marking vectors each of whose Hamming distance from μ^0 is 2 because in this second stage of firing there are $n(n-2)$ binary n-vectors in each of which there are exactly two zeros. In the third stage, at least ${}^nC_3$ distinct marking vectors are obtained by firing the above n-vectors obtained in the second stage and each of these vectors contains exactly three zeros. Continuing in this manner in the rth stage we get at least ${}^nC_r$ distinct marking vectors, each containing exactly r zeros, by firing all the n-vectors obtained in the $(r-1)$ stage. Since r ranges from $1, 2, \cdots, (n-1)$, we thus obtain at least ${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_{n-1}$ which is equal to $2^n - {}^nC_0 - {}^nC_n = 2^n - 2$ distinct n-marking vectors. Since all of them are distinct from μ^0 as well as from the zero vector $(0, 0, \cdots, 0)$, which is obtained due to the hypothesis that there exists

a transition t outside Z such that $t^\bullet = \emptyset$. Thus, we see that all the 2^n distinct binary n-vectors are generated by C. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma 5. *Let $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ be a 1-safe Petri net with $\mu^0(p) = 1 \,\forall\, p \in P$ and let Z be an SCC that passes through all the n places. Then any Petri net C' obtained from C by the deletion of any of the self loops belonging to Z generates all the binary n-vectors.*

Proof. First, we note that the removal of any self-loop from C results in a Petri net C' with $\mu^0(p) = 1 \,\forall\, p \in P$ and a transition t such that $t^\bullet = \emptyset$. Now, if C' has an SCC Z' then by Theorem 7, C' is Boolean and hence there is nothing to prove. Hence, without loss of generality, we may assume that the given Petri net C has an SCC, say Z. We shall then prove the result by invoking the PMI on $n = |P|$. First, let $n = 1$. Then C does not contain any SCC and, therefore $C' = C$. Further, it is easy to verify that C' generates all the binary 1-vectors, namely (1), (0) as shown in Figure 16. Next, let $n = 2$. Then C' contains the following

$$p_1 \odot \longrightarrow \square\, t_1$$

Figure 16. Petri net $C' =:\cong C$ for 1 place

structure, shown in Figure 17. Here, $t_2^\bullet = \emptyset$. Since $\mu^0(p) = 1 \,\forall\, p \in P$, t_2 is enabled and it
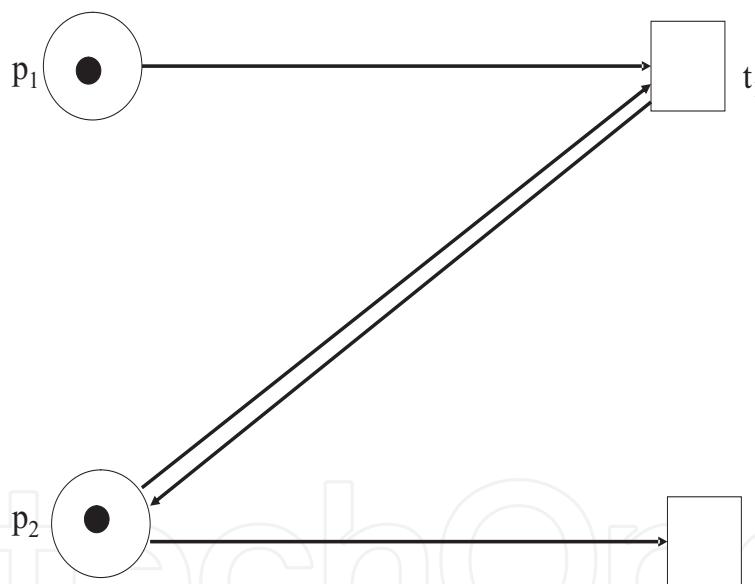

Figure 17. Petri net C' for 2 places

fires. After firing t_2, we get the marking vector $\mu^1 = (1, 0)$. Since t_1 fires simultaneously with t_2, we get the marking vector $\mu^2 = (0, 1)$. At this stage, t_2 is enabled and after firing, it gives the marking vector $\mu^3 = (0, 0)$. Hence we can obtain these marking vectors procedurally by taking two marking vectors obtained in the previous case namely, (1), (0) as follows.

Step-1. Augment 1 in the second position (corresponding to the case $t_2^\bullet = \emptyset$) in each of these vectors.

Step-2. In each case, t_2 fires and after firing, we get the marking vectors $(1, 0)$ and $(0, 0)$, respectively.

Next, take $n = k \geq 3$ and assume the validity by the above procedure to get all the 2^k binary k-vectors.

Let $n = k + 1$. Then, apply the following procedure.

Step-I. Augment 1 to each of the marking vectors $(a_1, a_2, \cdots, a_k)$ at its right most end to get the $(k+1)$-vector $(a_1, a_2, \cdots, a_k, 1)$.

Step-II. In this case, t_{k+1} fires and after firing we get the marking vectors $(a_1, a_2, \cdots, a_k, 0)$.

By the induction hypothesis, we have all the 2^k marking vectors to which Step-I has been applied to obtain the 2^k binary $(k+1)$-vectors as marking vectors (because of firing at each stage), having 1 in the $(k+1)^{th}$ coordinate. Further, each of these 2^k binary $(k+1)$-vectors has been fired using Step-II to obtain the binary $(k+1)$-vectors of the form $(a_1, a_2, \cdots, a_k, 0)$ which are all distinct from those obtained in Step-I. Together, therefore, we have obtained $2^k + 2^k = 2^{k+1}$ binary $(k+1)$-vectors as marking vectors from C'. Thus, the proof follows by PMI. □

9. An embedding theorem and complexity

The following is a "frustration theorem" due to the negative fact it reveals, to the effect that one cannot hope to have a "forbidden subgraph characterization" of a Boolean Petri net.

Theorem 8. *[10]: Every 1-safe Petri net $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ with $\mu^0(p) = 1 \ \forall \ p \in P$ can be embedded as an induced subnet of a Boolean Petri net.*

Proof. Let $C = (P, T, I^-, I^+, \mu^0)$, $|P| = n$ be a 1-safe Petri net. If C is a Boolean Petri net then there is nothing to prove. Hence, assume that C is not a Boolean Petri net. Then, we have the following steps to obtain a Boolean Petri net C' in which C is one of its induced subnets.

Step-1: First of all, find those places in C each of whose postsets has single distinct sink transitions (if the postset of a place has more than one distinct sink transitions then choose only one transition giving K_2). Suppose such places are $p_1, p_2, \cdots, p_k$, $1 \leq k < n$. If there is no sink transition in C, then augment one sink transition to each place in C.

Step-2: Augment $n - k$ new transitions and join each of them to the remaining $n - k$ places in C by an arc from a place to a new transition creating $n - k$ new active transitions.

Step-3: Thus, in C' we have n-copies of K_2 as its subgraph. Since $\mu^0(p) = 1 \ \forall p \in P$, all the transitions are enabled. Firing of n transitions forming n 'pendant transitions' will produce $^nC_1$ distinct binary n-vectors whose Hamming distance is 1 from the initial marking vector. At these marking vectors, $n - 1$ transitions out of those n transitions are enabled, and after firing give at least $^nC_2$ distinct marking vectors, each of whose Hamming distance is 2 from the initial marking.

In general at any stage j, $3 \leq j \leq n$, we get a set of at least ^nC_j new distinct binary n-vectors whose Hamming distance is j from the initial marking, which are also distinct from the sets of ^nC_r distinct marking vectors for all r, $2 \leq r \leq j - 1$. Therefore, at the nth stage we would have obtained at least $^nC_1 +^n C_2 + \cdots +^n C_n = 2^n - 1$ distinct binary n-vectors. Together with the initial marking $(1, 1, \cdots, 1)$, we thus see that all the 2^n binary n-vectors would have been obtained as marking vectors, possibly with repetitions. Thus C' is Boolean.

Therefore, every 1-safe Petri net C can be embedded as an induced subgraph of a Boolean Petri net.   □

10. Scope for research

Precisely which Petri nets generate all the binary n-vectors as their marking vectors, or the so-called Boolean Petri net? This has been a hotly pursued research problems. We have shown in this chapter some necessary and sufficient conditions to characterize a Boolean Petri net, containing an SCC. However, the general problem of characterizing such a 1-safe Petri net C, when C does not contain an SCC or a strong chain, is still open. A Petri net containing an SCC is strongly connected, in the graph-theoretical sense that any two nodes in it are mutually reachable. However, the converse is not true; that is, if the underlying digraph of a Petri net is strongly connected, it need not contain an SCC. So, even a characterization of strongly connected Boolean Petri net is an open problem. Further, in general, characterizing crisp Boolean Petri nets is open too. If we relax the condition on the depth of the reachability tree in our original definition of minimality of a 'minimal' crisp Boolean Petri net and require instead that the number of enabled transitions be kept at minimum possible, the reachability graphs of such Petri nets may not have their underlying graph structures isomorphic to $K_{1,2^n-1}$, whence they would all be trees of the same order 2^n. Since they would be finite in number, determination of the structures of such Petri nets and their enumeration would be of potential practical interest. It involves orienting trees of order 2^n (in general, for theoretical purposes, trees of any order as such) that admit an orientation of their edges to make them the reachability trees of minimal $1-$safe crisp Boolean Petri nets.

11. Concluding remarks

As pointed out, many fundamental issues regarding Boolean Petri nets emerge from the above study. For example, it is found and established that the reachability tree of a 1-safe Petri net can be homomorphically mapped on to the n-dimensional complete Boolean lattice, thereby yielding new techniques to represent the dynamics of these Petri nets. One can expect to bring out in the near future some salient features of 1-safe Petri nets in general as a part of a theory that is likely to emerge even in our work.

Following our first discovery of an infinite class of 1-safe star Petri nets that are Boolean, we came across crisp Boolean Petri nets, viz., that generate every binary n-vector as marking vector exactly once. This motivated us to move towards a characterization of such 1-safe Petri nets in general. Our work towards this end revealed to our surprise that there can be even such disconnected 1-safe Petri nets. We demonstrated the existence of a disconnected 1-safe Petri net which was obtained by removing the central transition from the star Petri net S_n, whose reachability tree can be tactically represented as an n-dimensional complete lattice L_n [7]. In this disconnected Petri net, the firing of transitions in a particular way (which we may regard as a 'tact' or 'strategy'), gives exactly 2^n marking vectors, repetitions occurring possibly only within a level, that can be arranged as a homomorphic image of the reachability tree of the Petri net, forming the n-dimensional complete Boolean lattice L_n.

The results of this chapter can perhaps be used gainfully in many purely theoretical areas like mathematics, computer science, universal algebra and order theory, the extent and effectiveness of its utility in solving the practical problem requiring the design of

multi-functional switches for the operation of certain discrete dynamical systems of common use such as washing machines and teleprinters (e.g., see [1]).

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12. References

[1] Acharya, B.D (2001). *Set-Indexers of a Graph and Set-Graceful Graphs*, Bull. Allahabad Math. Soc. 16, pp. 1-23.

[2] Best, E. and Thiagarajan, P.S. (1987). Some Classes of Live and Safe Petri Nets, Concurrency and Nets, Vol. 25, pp. 71-94.

[3] Harary, F. (1969).Graph Theory, Addison-Wesley, Reading, Massachusettes.

[4] Jensen, K. (1986). Coloured Petri nets, Lecture Notes in Computer Science, Vol. 254, Springer-Verlag, Berlin, pp. 248-299.

[5] Kansal, S., Singh, G.P. and Acharya, M. (2010). On Petri Nets Generating all the Binary n-Vectors, Scientiae Mathematicae Japonicae, 71, No. 2, pp. 209-216.

[6] Kansal, S., Singh, G.P. and Acharya, M. (2011). 1-Safe Petri Nets Generating Every Binary $n-$Vectors Exactly Once, Scientiae Mathematicae Japonicae, 74, No. 1, pp. 29-36.

[7] Kansal, S., Singh, G.P. and Acharya, M. (2011). A Disconnected 1-Safe Petri Net Whose Reachability Tree is Homomorphic to a Complete Boolean Lattice, proceeding of PACC-2011, IEEE Xplore, Catalog Number: CFP1166N-PRT ISBN: 978-1-61284-762-7.

[8] Kansal, S., Acharya, M. and Singh, G.P. (2012). Uniqueness of Minimal 1-Safe Petri Net Generating Every Binary n-Vectors as its Marking Vectors Exactly Once, Scientiae Mathematicae Japonicae, pp., e-2012, 75-78.

[9] Kansal, S., Acharya, M. and Singh, G.P. (2012). On the problem of characterizing 1-safe Petri nets that generate all the binary n-vectors as their marking vectors, Preprint.

[10] Singh, G.P., Kansal, S. and Acharya, M. (2012). Embedding an Arbitrary 1-Safe Petri Net in a Boolean Petri Net, Research Report, Deptartment of Applied Mathematics, Delhi Technological University, Delhi, India.

[11] Nauber, W. (2010). *Methods of Petri Net Analysis.* Ch.5 In: Lectures on Design and Analysis With Petri Nets, <http://www.tcs.inf.tu-dresden.de/nauber/dapn.shtml>

[12] Peterson, J.L. (1981). Petri Net Theory and the Modeling of Systems, Prentice-Hall, Inc., Englewood Cliffs, NJ.

[13] Petri, C.A. (1962). Kommunikation Mit Automaten, Schriften des Institutes fur Instrumentelle Mathematik, Bonn.

[14] Reisig, W. (1985). Petri nets, Springer-Verleg, New York.